Occasional Ratcheting: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living

Philip H. Dybvig*

First draft: June, 1993

Abstract

A constraint that consumption cannot fall over time (or can fall only at a limited rate) can arise directly from preferences or indirectly from internal production considerations. For example, much of a university's budget includes expenditures that must be maintained because of implicit and explicit long-term contracts. Similarly, expenditures on consumer durables are also long-term commitments. This paper analyses optimal consumption and investment for agents whose consumption cannot fall over time. The preferences are time-separable with constant relative risk aversion conditional on consumption never falling, and they can be represented as non-time-separable extended-real-valued von Neumann-Morgenstern preferences unconditionally. Optimal consumption increases each time that wealth achieves a new maximum. Optimal investment at a point in time is similar to constant proportions portfolio insurance, but the dynamics are different due to the form of consumption dynamics. Extensions include bounding the rate of decrease in consumption with a constant other than zero and the placement of a bound on the maximal holding of the risky asset as a proportion of wealth.

*Washington University in Saint Louis. I am grateful for helpful comments and suggestions from George Constantinides, Jon Ingersoll, Andy Lo, and especially Kerry Back and Eric Reiner. I alone am responsible for any errors or deficiencies.

Introduction

Models of dynamic consumption and investment have important positive and normative implications. Positively, they help us to understand the testable relationship between pricing of assets and consequent portfolio demand and consumption, and more generally they are building blocks for testable equilibrium models of security markets and the macroeconomy. Normatively, these models can be used to prescribe policies for managing portfolios. The purpose of this paper is to investigate a new class of portfolio and consumption strategies, characterized by "occasional ratcheting," which are optimal in a standard model with a new constraint imposed that says that consumption can never fall. In these models, consumption is constant most of the time and increases proportionately as wealth achieves a new maximum. These preferences are in effect a simple and extreme version of habit-formation preferences, and share their property that consumption is much smoother than in time-separable models, a property that seems to be consistent with the empirical evidence. In addition, the simple explicit solution for portfolio choice and consumption withdrawal makes the model and its close relatives attractive prospects for use in practice.

Conditional on consumption never falling, the assumed preferences are time-separable von Neumann-Morgenstern preferences with a constant pure rate of time discount. Alternatively, if consumption ever falls, utility is defined to be minus infinity (or a strict global minimum). Taking the two cases together, these are extended-real-valued von Neumann-Morgenstern preferences that are not time-separable. There are two obvious motivations for such preferences. Most directly, they represent a limiting case of habit-formation preferences for which declines in standard-of-living are not tolerated. Indirectly, they can be the derived preferences when consumption involves pre-commitment, as for purchases of durables without secondary markets or other long-term commitments. For example, it may be a good model of a university or foundation that at least some of the expenditures cannot be decreased quickly, due to implicit and explicit long-term commitments to faculty and due to the commitments to donors about the use of buildings or equipment. The preferences in the model are faced with security returns following the standard continuous-time model with a riskless asset bearing a constant interest rate and a single risky asset following a lognormal diffusion. The model is analyzed both with an unconstrained portfolio choice and with a portfolio choice that is arbitrarily constrained by some proportion or multiple of wealth.

The form of the optimal consumption and portfolio policy in the model is simple and available in closed form. Consumption is either left unchanged or is increased as necessary to maintain consumption of at least some proportion (which depends on preference and security market parameters) of wealth. Ratcheting of consumption occurs only when consumption achieves a new maximum, which is a set of times of measure zero, which is the reason for calling the policy one of "occasional ratcheting." The portfolio choice has a form similar to the "constant proportions portfolio insurance" strategy advocated by Fischer Black at Goldman Sachs, in that at any point in time, a "fixed amount" of wealth is invested in the riskless asset and fixed proportions of any additional wealth is allocated to risky and riskless assets. However, this policy differs in its dynamics. The "fixed amount" of wealth is what is needed to maintain current consumption indefinitely, and the interest on this fixed amount goes into current consumption rather than being re-invested in the portfolio. Furthermore, the "fixed amount" is actually not fixed, since it ratchets up whenever wealth achieves a new maximum.

The paper contains several generalizations of the model. One generalization replaces the nondecreasing wealth constraint with a constraint that consumption can fall no faster than at some given rate (where the given rate for the original problem is implicitly zero). A simple change of numeraire converts this problem back into the original problem, so the original solution can be applied directly. A second generalization has two consumption goods, one constrained never to decline and the other unconstrained. Additive separability allows us to solve the model by combining the solution of our basic model and the solution of Merton [1971] for the unconstrained model. A more difficult generalization puts an upper bound on the proportion of wealth that can be placed in the risky asset. The resultant problem must be solved numerically, but the form of the solution is the same as before except that there is an additional region of the state space in which the constraint is binding and the portfolio choice equals the constraint.

The original motivation of the paper was to provide an alternative to the analysis of Grossman and Zhou [1993], who model ratcheting of risky investments by assuming a draw-down constraint that precludes wealth from falling below some proportion of the previous maximum in a model without intermediate consumption withdrawal. Their model is less than satisfying because the constraint seems arbitrary: if interpreted in terms of preferences, the agent cares about market prices (through wealth) and not just about the consumption process or the arrival of information about consumption. Except in one generalization, the present analysis has no exogenous constraint that cannot be interpreted in terms of preferences over consumption.

While not the original motivation, the model in this paper is closest to the models of habit formation such as those of Constantinides [1990], Detemple and Zapatero [1991], or Ingersoll [1992]. The main difference is in the rapidity (immediate) of the habit formation and the severity (lexicographic) of the agent's preferences for maintaining a new standard of living. In this paper, the marginal utility of consumption at a point in time is discontinuous when consumption equals the previous maximum, and in fact the marginal utility is infinite for any decrease in consumption but finite for any increase.

Part of the mathematical analysis is the "smooth-pasting" of the value function (equality of the value function and its first two derivatives) implied by optimal choice of a free boundary. The smooth-pasting condition is familiar to financial economists as an important part of analyzing the pricing and optimal exercise of American put options (as described in Merton [1973], Section 7, with some of the history). Smooth pasting also occurs in options to irreversible investments or shut-downs, as described by McDonald and Siegel [1985, 1986], or in the presence of transaction costs (as in Davis and Norman [1990]). These tools have also seen wide usage recently in the literature on international trade (see Dixit [1992] and the papers referenced there).

Section 1 contains the specification of the basic consumption and investment problem with no decline in consumption allowed, with a theorem characterizing its solution and a sketch of its proof. Section 2 contains an analysis of the three generalizations of the basic problem. Section 3 contains the formal proof of the theorem characterizing the solution of the basic problem. Section 4 briefly closes the paper.

1 A Simple Model of Optimal Consumption and Investment

Consider an agent with strong (lexicographic) preferences not to have a decline in fortune, defined in terms of consumption. Formally, we assume that the von Neumann-Morgenstern utility function is given by

$$(1) \qquad \int_{t=0}^{\infty} u(c_t)e^{-\delta t}dt$$

when the consumption rate c is a nonincreasing function of time and is $-\infty$ otherwise. Note that although this is a standard time-separable utility function conditional on nonincreasing consumption, the conditioning makes it non-time-separable.[1]

Now consider the portfolio problem faced by the agent when given an initial endowment of W_0 and faced with a riskless asset and a single risky asset following the usual lognormal returns. The process for the reinvested stock price is governed by

$$(2) \qquad \frac{dS}{S} = \mu dt + \sigma dZ$$

subject to some given initial stock price S_0, where μ and $\sigma > 0$ are constants, and Z is a standard Wiener process. The instantaneous riskless rate is r. To simplify interpretation later, we assume without loss of generality that $\mu > r$, so that the risky asset is an attractive investment. The negative

[1] For example, standard time-separable preferences would assign the same utility level to the following two lotteries, but the preferences we are studying will typically assign finite utility to the first lottery and negative infinite utility to the second lottery. The first lottery has a 50-50 chance of 1 for all time and a value or 2 for all time. The second lottery has a 50-50 chance of 1 until time 1 with 2 thereafter or 2 until time 1 with 1 thereafter. For the preferences we are studying, the value of the first lottery is $.5((u(1)/\delta) + (u(2)/\delta)) > -\infty$, while the value of the second lottery is $.5((u(1)/\delta + (u(2) - u(1))e^{-\delta}/\delta) + (-\infty)) = -\infty$.

infinite utility when consumption ever declines can be represented in the choice problem as a constraint. We write the choice problem in the following form.

Problem 1 *Choose adapted* $\{c_t\}_{t=0}^{\infty}$ *and* $\{\alpha_t\}_{t=0}^{\infty}$*, both right-continuous with left limits, to maximize* $E[\int_{t=0}^{\infty} u(c_t)e^{-\delta t}dt]$ *subject to*

$$(3) \quad c_0 \geq c_{0-} \equiv C_{0-} > 0,$$ [2]

$$(4) \quad (\forall t > s) \quad c_t \geq c_s,$$

and

$$(5) \quad (\forall t) \quad w_t \geq 0,$$

where $\{w_t\}_{t=0}^{\infty}$ *solves*

$$(6) \quad dw_t = w_t r dt + \alpha_t(\mu dt + \sigma dZ_t - r dt) - c_t dt$$

subject to the initial condition $w_0 = W_0$*.*

In Problem 1, the choice variables are the process for the consumption rate c and the process for ownership of the risky asset α. The initial conditions are given by W_0, the initial wealth, and C_{0-}, the inherited previous largest

[2]Allowing $C_{0-} = 0$ would not pose insurmountable problems, but the slight additional generality does not justify the additional complexity of the statement and proof of the results. For example, $W_0 = C_{0-} = 0$ would then result in a feasible problem for constant relative risk aversion less than one, but not greater. This type of feature makes the statements of the theory messier without any real gain in economic content.

consumption. The first two constraints ensure that consumption is nonincreasing (or else expected utility would be at a global minimum), the third constraint rules out negative net worth (or else there would be arbitrage by borrowing indefinitely to consume or by following a doubling strategy), and the final constraint defines the wealth process.

Note that the problem is stationary with the natural analogues c_{t-} and w_t of C_{0-} and W_0 as state variables. (Note that for $t > 0$, $c_{t-} = \sup_{0 \leq s < t}\{c_s\}$, and is therefore the right state variable given (4), i.e., for feasible consumption, the constraint past consumption places on future consumption is summarized by c_{t-}.) Given these state variables, the continuation from time t has the same form as the original problem. Therefore, the value function for Problem 1 can be written as $V(w, c_-)$ which is the value of Problem 1 given that $W_0 = w$ and $C_{0-} = c_-$. To simplify the problem, we assume that preferences are scale independent, which makes the optimal choices of α and c change proportionally given multiplication of w and c_- by the same constant.

Scale independence (or homotheticity) of preferences implies constant relative risk aversion. The Arrow-Pratt coefficient of relative risk aversion $R(c) \equiv -cu''(c)/u'(c)$ is a scale-independent measure of the amount of compensation an agent needs to make the agent just indifferent about taking on a small gamble (see Arrow [1965] and Pratt [1964]). Constant relative risk aversion says that risk preferences are scale-independent. Constant relative risk aversion includes both power utility and logarithmic utility. For power utility, we have $u(c) = c^{1-R}/(1-R)$ for $R > 0$, $R \neq 1$, where division by $1 - R$ ensures $u(c)$ is increasing. For logarithmic utility, $R = 1$, and we have $u(c) = \log(c)$.[3] Scale-independence of Problem 1 implies that the value function is of the form $V(w, c_-) = \log(c_-)/\delta + V(w/c_-, 1)$ in the logarithmic case ($R = 1$) or $V(w, c_-) = c_-^{1-R} V(w/c_-, 1)$ in the power case ($R \neq 1$). Exploiting scale-independence greatly simplifies the task of solving Problem 1. The general solution given constant relative risk aversion is given in Theorem 1.

[3]As is well known, these preferences are only determined up to an increasing affine transform. The particular choice is a matter of convenience; for example, $u(c) = 10 + 3\pi \log(c)$ would be an equally valid representation of the logarithmic utility preferences.

Theorem 1 The Solution of Problem 1: *Let* $u(c) = \log(c)$ *(*$R = 1$*) or* $u(c) = c^{1-R}/(1-R)$ *for* $R > 1$, $R \neq 0$. *Then Problem 1 is feasible if and only if both* $C_{0-} \leq rW_0$ *and* $r > 0$. *Given feasibility, Problem 1 has an optimum if and only if* $\delta > 0$ *and either* $C_{0-} = rW_0$ *or* $R > R^*$, *where*

$$(7) \qquad 0 < R^* \equiv \frac{\sqrt{(\delta+\kappa-r)^2 + 4r\kappa} - (\delta+\kappa-r)}{2r} < 1$$

and

$$(8) \qquad \kappa \equiv \frac{(\mu-r)^2}{2\sigma^2}.$$

The solution to Problem 1 is characterized up to equivalence on a null set of times and states by risky investment

$$(9) \qquad \alpha = \frac{\mu - r}{R^*\sigma^2}(w - c/r).$$

and a nondecreasing and right-continuous consumption process regulated to be no smaller than r^*w *(with the singular control expressed in the spirit of Harrison [1985, Chapter 2]):*

$$(10) \quad (\forall t) \int_{s=0}^{t} (c_s - r^* w_s)^+ dc_s = 0,$$

$$(11) \quad (\forall t) c_t \geq r^* w_t,$$

and

$$(12) \quad (\forall t) \int_{s=0}^{t} (dc_s)^- = 0,$$

where

$$(13)\quad r^* \equiv r\frac{R-R^*}{R}$$

and x^+ *(resp.* x^-*) denotes the positive part (resp. negative part) of* x*. For feasible* w *and* c_- *(i.e.* $c_- \leq r^*w$*), the value function is given by*

$$(14)\quad V(w,c_-) = \frac{u(\max(c_-,r^*w))}{\delta} + \gamma \max(c_-,r^*w)^{1-R}\left(\frac{w}{\max(c_-,r^*w)} - \frac{1}{r}\right)^{1-R^*}$$

where

$$(15)\quad \gamma \quad \equiv \quad \frac{r}{\delta}\left(\frac{1}{r^*} - \frac{1}{r}\right)^{R^*}.$$

Proof The formal proof is based on standard martingale techniques with a few new wrinkles and is given in Section 3. Here is an intuitive derivation of the result without most of the technical details.

Because the form of the problem remains the same when we multiply W_0, C_{0-}, $\{c_t\}$, and $\{\alpha_t\}$ by the same constant, the nature of the solution depends on the relative size of W_0 and C_{0-}. For $C_{0-} > rW_0$, the problem is infeasible because the interest flow is insufficient to maintain consumption (or equivalently the perpetuity value of maintaining consumption at its previous level is larger than the initial wealth). For $C_{0-} = rW_0$, wealth is just sufficient to maintain consumption at its previous level, but only if all investment is in the riskless asset and none is in the risky asset. In the next region, $r^*W_0 < C_{0-} < rW_0$ for some critical rate r^* of consumption as a fraction of wealth, consumption is maintained at the previous level and the value function satisfies the Bellman equation which says that the expected change in value (including proper allowance for the value of consumption withdrawal) is maximized at zero by the optimal choice of α. The Bellman equation is

$$(16)\quad 0 \quad = \quad \max_{\alpha}\{(e^{-\delta t}u(c)dt + e^{-\delta(t+dt)}E[V(w+dw,c)]$$

$$
\begin{aligned}
& - e^{-\delta t} V(w, c_-))/dt\} \\
= \quad & e^{-\delta t} \max_{\alpha} \{\log(c) - \delta V + (wr + \alpha(\mu - r) - c_-)V_w \\
& + \frac{1}{2} V_{ww} \alpha^2 \sigma^2\}
\end{aligned}
$$

where we have used Itôs Lemma or a Taylor series expansion of V and we have eliminated terms of order dt and higher.[4] The first-order condition for optimal α gives that $\alpha = -(\mu - r)V_w/(\sigma^2 V_{ww})$. Substituting this result back into (16) yields the differential equation

$$
(17) \quad 0 = u(c) - \delta V + (wr - c)V_w - \frac{(\mu - r)^2 (V_w)^2}{2\sigma^2 V_{ww}}.
$$

It is easy to verify that the claimed value function satisfies this differential equation, and that the optimal α and c in this region are as claimed. The boundary conditions are given by the value at $w = c_-/r$ computed above to be $u(c)/\delta$ and the familiar "smooth-pasting condition" (continuity of V, V_w, and V_{ww}) at the boundary $c_- = r^* w$ with the region we have yet to analyze.[5] At $w = c_-/r^*$, the "smooth-pasting condition" eliminates any "local time" term from the stochastic integral and allows us to use the familiar "simple" form of Itô's lemma. The "local time" term from a discontinuous first derivative cannot be present because otherwise it would dominate the expected change in value and contradict the condition that the expected value conditional on information at time t is a martingale. A discontinuous second derivative would imply a discontinuity in the expected change in value at the boundary, also keeping the expected value from being zero everywhere. In the claimed value function, smooth pasting can be verified directly by differentiation.

The remaining region is $C_{0-} < r^* W_0$. In this region, the value function does not depend on c_- and the optimal strategy is to increase consumption

[4]If we do not take c as given, c is also determined by the Bellman equation: there would be a term in (16) of the form $V_{c_-} dc$. However, in this region $V_{c_-} < 0$ so it never pays to increase c, and a decrease is infeasible.

[5]Another boundary, in some directions of increasing w and c_-, is given by a growth bound inferred from the scale independence of the problem.

immediately to the level r^*w. Consequently, the value function in this region satisfies $V(w, c_-) = V(w, r^*w)$. In the original derivation, the location of the boundary was determined by this condition and smooth pasting.

As mentioned before, the formal proof is given in Section 3. ∎

Interpreting the Solution

The optimal strategy for Problem 1 characterized by Theorem 1 is illustrated by Figure 1, in which are plotted the time series of several variables from a particular simulated random draw of the optimal strategy's evolution over time. Wealth is assumed to start at 100 with initial consumption chosen between rw and r^*w, and other parameters are set to values intended to be representative of what is relevant for practice. (The random draw itself should not be thought of as representative, since the random draw and initial consumption were chosen from the results of several draws as an "interesting" draw that illustrates the qualitative possibilities.) Consumption increases immediately to r^*w and from then on consumption is regulated as wealth rises to maintain $c \geq r^*w$. The risky portfolio allocation α is proportional to $w - c/r$, which is the excess of wealth over the perpetuity value of current consumption. Therefore, α decreases as wealth decreases, approaching 0 as wealth approaches the bottom of the band. This ensures that wealth never falls below c/r (which in the absence of risky investment is sufficient to maintain consumption indefinitely). Conversely, as wealth increases, α increases, reaching the largest proportion of wealth when reaching the boundary $c = r^*w$. At this boundary, the proportion of wealth invested in the risky asset is the same as for the unconstrained problem studied by Merton [1971]. Consumption is implicit in the plot, since the top and bottom of the band are both proportional to consumption. Consumption increases only as wealth reaches a new maximum, at which time the the top and bottom of the band increase in proportion.

The level curves of the value functions for the two cases $R = 2.5$ and $R = 1$ are shown in Figure 2. The value function increases in w given c_- (since

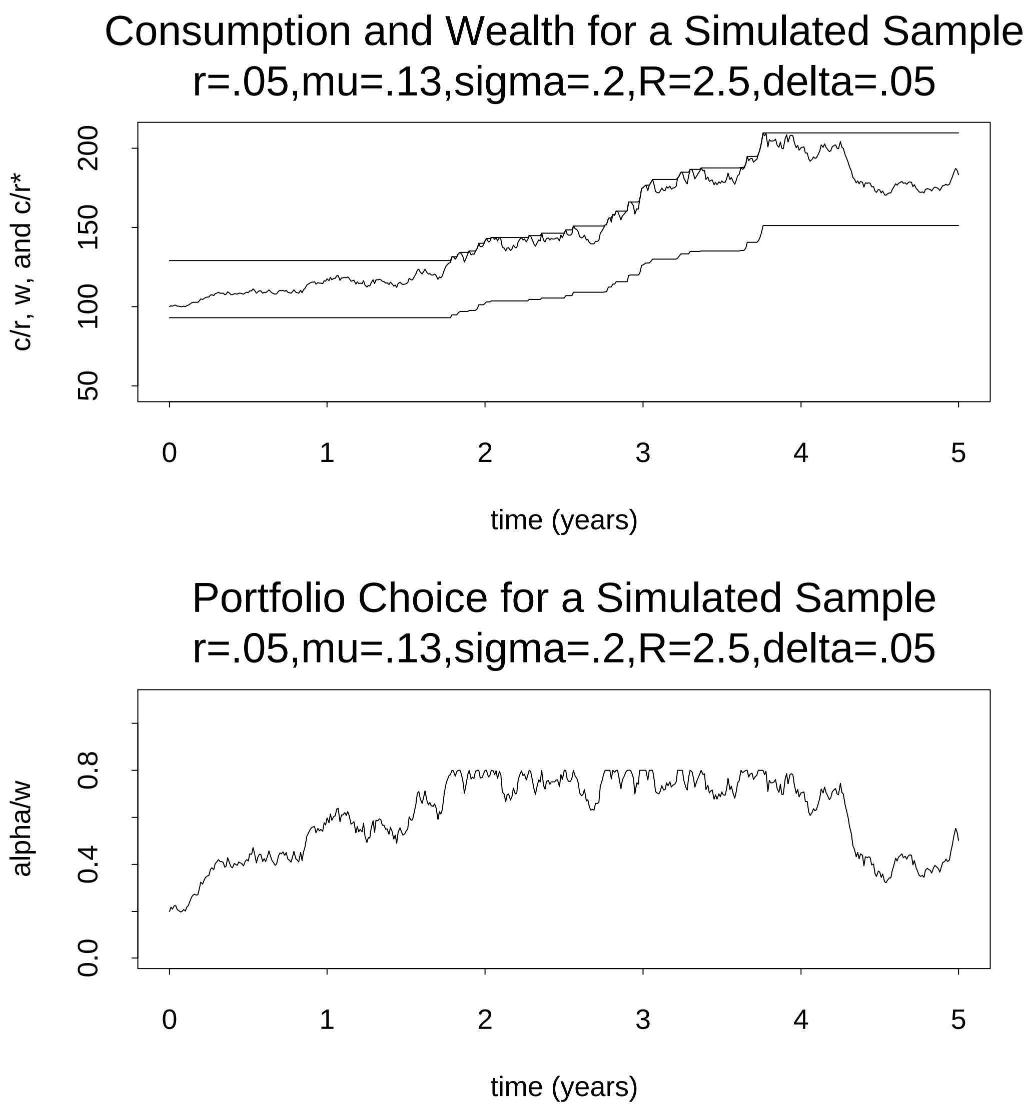

Figure 1: A simulated random draw illustrates the dynamics of the optimal strategy. Wealth stays within the band $c/r \le w \le c/r^*$, and the value α invested in the risky asset is proportional to $w - c/r$, which is the excess of wealth over the perpetuity value of maintaining current consumption.

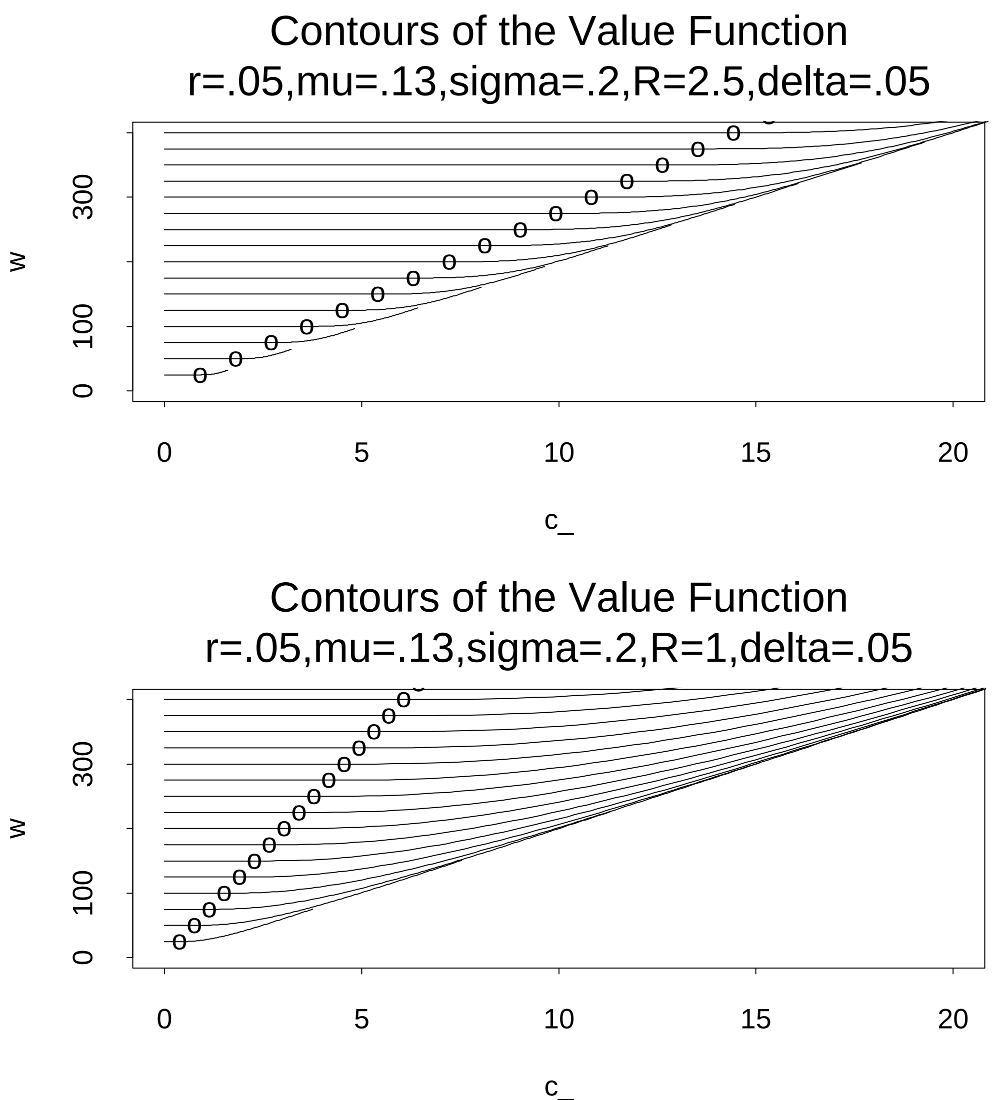

Figure 2: This figure shows level curves for the value function. The optimization problem is feasible only for $c_- \leq rw$, which is why there are no level curves in the lower right part of each figure where $c_- \geq rw$. The circles mark the points of smooth pasting at which $c_- = r^* w$. Below this level of prior consumption, it is optimal to increase consumption immediately to the level $c = r^* w$, which is why the level curves are flat on the upper left.

more wealth is better) and decreases or is constant in c_- given w (since more commitment cannot be strictly helpful in this problem). The value function is undefined when $c_- > rw$, where the problem is infeasible. For $c_- < r^*w$, committed consumption is lower than would be preferred, and the optimal strategy increases consumption immediately to $c = r^*w$. At this boundary (in the circles), the functional form changes but the value and first and second derivatives match, which is the "smooth pasting" property. Curiously, while the critical rate r^* depends on the risk aversion parameter, the shape of the value function for $w/c_- \in (1/r, 1/r^*)$ is the same (up to affine transform) independent of the relative risk aversion parameter R.

2 Extensions of the Simple Model

There are several extensions of the simple model that can be solved easily. The first extension we consider generalizes the criterion for making the utility of consumption $-\infty$ by bounding the relative rate of decline. The second extension considers a model with two commodities, one of which cannot be decreased and the other of which is unconstrained over time. The third extension imposes an upper bound on the risky portfolio position like the one faced by some managers. Other obvious extensions are possible. For example, it would be simple to combine intolerance beyond some rate of decline with a bound on the risky asset position. Therefore, the section is an indication of how the model can be extended rather than an attempt at a general list.

Intolerance Beyond Some Rate of Decline

In the simple model, any decline in consumption triggers infinite disutility. In our first extension, infinite disutility is triggered when the proportional rate of consumption decline is greater than some critical value D (with $D = 0$ implicit in the original analysis, and $D < 0$ if it is actually necessary to increase consumption over time). This new problem is formalized as Problem 2.

Problem 2 *Choose adapted* $\{c_t\}_{t=0}^{\infty}$ *and* $\{\alpha_t\}_{t=0}^{\infty}$*, both right-continuous with left limits, to maximize* $E[\int_{t=0}^{\infty} u(c_t)e^{-\delta t}dt]$ *subject to*

(18) $c_0 \geq c_{0-} \equiv C_{0-} > 0,$

(19) $(\forall t > s) \quad c_t \geq e^{-D(t-s)}c_s,$

and

(20) $(\forall t) \quad w_t \geq 0,$

where $\{w_t\}_{t=0}^{\infty}$ *solves*

(21) $dw_t = w_t r dt + \alpha_t(\mu dt + \sigma dZ_t - r dt) - c_t dt.$

subject to the initial condition $w_0 = W_0$.

Problem 2 is identical to Problem 1 except that the constraint $(\forall t > s) \quad c_t \geq c_s$ has become $(\forall t > s) \quad c_t \geq e^{-D(t-s)}c_s$.

To transform Problem 2 into the form of Problem 1 requires only a simple change of variables. Let $\tilde{c}_t \equiv e^{Dt}c_t$, $\tilde{\alpha}_t \equiv e^{Dt}\alpha_t$, $\tilde{w}_t \equiv e^{Dt}w_t$, $\tilde{r} \equiv r + D$, $\tilde{\mu} = \mu + D$, $\tilde{\sigma} = \sigma$, $\tilde{R} = R$, and $\tilde{\delta} = \delta + (1 - R)D$. These substitutions into Problem 2 change it into the form of Problem 1 in the variables and parameters with tildes. Therefore, after the transformation, we will have the same solution as before, and it is a simple matter to invert the transformation to obtain the solution to Problem 2.

Two Consumption Goods

In general, financial economists do not use models with many commodities, primarily because such models can be difficult to solve. However, when markets are complete (as they are here[6]) and preferences are additively separable across commodities. The same is true of our basic problem. For example, the following problem has consumption of a "durable" good "1" that cannot fall over time and a "nondurable" good "2" that is not so constrained.

Problem 3 *Choose adapted* $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *and* $\{\alpha_t\}_{t=0}^{\infty}$*, both right-continuous with left limits, to maximize* $E[\int_{t=0}^{\infty}(u_1(c_t^1) + u_2(c_t^2))e^{-\delta t}dt]$ *subject to*

(22) $c_0^1 \geq c_{0-}^1 \equiv C_{0-} > 0,$

(23) $(\forall t > s) \quad c_t^1 \geq e^{-D(t-s)}c_s^1,$

(24) $(\forall t) \quad c_t^2 \geq 0,$

and

(25) $(\forall t) \quad w_t \geq 0,$

where $\{w_t\}_{t=0}^{\infty}$ *solves*

(26) $dw_t = w_t r dt + \alpha_t(\mu dt + \sigma dZ_t - r dt) - c_t^1 dt - c_t^2 dt.$

subject to the initial condition $w_0 = W_0$*.*

[6] Completeness of the market can be proven applying a martingale representation theorem (like that of Clark [1970]) to a transform of the consumption process similar to $\mathcal{M}_0^a$ used in the proof of Lemma 1 in Section 3.

Because of the separability noted above, the solution of this choice problem is in two parts: first, write down the solution and value function for the single-good choice problem for each good separately. For the constrained good, this is the solution to Problem 1 as given by Theorem 1. For the unconstrained problem, this is the seminal model of Merton [1971]. Then write an indirect value function to this problem that allocates wealth to the two problems, i.e. $V(w^1, w^2) = V_1(w^1, c_-) + V_2(w^2)$, where V_1 and V_2 are the value functions for the individual commodities. Then solve the one-dimensional problem of maximizing $V(w^1, w^2)$ subject to $w^1 + w^2 = W_0$. This is the value function for Problem 3, and the optimal policy has consumption as in the subproblems and risky portfolio choice the sum of the optimal choice in the two subproblems. Because of the additive separability, the Bellman equation is additively separable in the the two types of consumption and looks like the sum of the Bellman equations for the two problems, which leads naturally to a proof of optimality along the same lines as above.[7]

Qualitatively, we can think of wealth at any point in time as allocated to three things: maintaining current durable consumption forever, investing for future growth in durable consumption, and wealth earmarked for nondurable consumption now and in the future. Since only a fraction of wealth is allocated to the nondurable consumption and since durable consumption cannot decrease, nondurable consumption can be very sensitive to falls in the stock price.

It is a trivial matter to extend this solution to more goods, durable or not, or to a bound on the rate of decline of wealth as in Problem 2. However, some other simple changes, like the bound on asset position, would destroy the separability and make the problem harder to solve.

[7] To prove a verification theorem for the two-commodity problem will require essentially a verification theorem for Merton's problem. Such a verification theorem is given by Davis and Norman [1990], but has an additional constraint that α/w is bounded. I believe that the proof in this paper can extend Davis and Norman [1990] to do without that assumption; if not, that assumption should be imposed here.

Bounded Risky Asset Position

A more difficult extension adds to Problem 1 a constraint that the risky portfolio holding can be no more than M times wealth for some constant M, i.e. $\alpha \leq Mw$. This type of constraint is motivated by constraints imposed on actual fund managers. For example, a fund that is precluded from borrowing would be constrained to have $\alpha \leq w$, whereas a fund that is precluded from borrowing more than 50% of the value of its equity position would face $\alpha \leq 2w$ or $M = 2$. These constraints are a bit problematic conceptually, since they seem arbitrary and do damage whenever they are binding. To the extent that there is a rational reason for the constraints, we should be including that in our model. However, such an inquiry is beyond the scope of the paper.

Formally, the choice problem given the constraint is of the form given in Problem 4, which is the same as Problem 1, except for the additional constraint $(\forall t)\alpha_t \leq Mw$.

Problem 4 *Choose adapted $\{c_t\}_{t=0}^{\infty}$ and $\{\alpha_t\}_{t=0}^{\infty}$, both right-continuous with left limits, to maximize $E[\int_{t=0}^{\infty} u(c_t)e^{-\delta t}dt]$ subject to*

(27) $c_0 \geq c_{0-} \equiv C_{0-} > 0,$

(28) $(\forall t > s) \quad c_t \geq c_s,$

(29) $(\forall t) \quad \alpha_t \leq Mw,$

and

(30) $(\forall t) \quad w_t \geq 0,$

where $\{w_t\}_{t=0}^{\infty}$ *solves*[8]

$$(31) \quad dw_t = w_t r dt + \alpha_t(\mu dt + \sigma dZ_t - r dt) - c_t dt.$$

subject to the initial condition $w_0 = W_0$.

In some cases, the new constraint is not binding, and then we obviously have the same solution as described in Theorem 1. For example, if $M = 1$ (the simple no-borrowing constraint), then the constraint is never violated for the example with $R = 2.5$ whose optimal portfolio strategy was illustrated in Figure 1. Therefore, the solution to that problem would not be altered by introducing the constraint. Of course, the constraint with $M = 1$ would be binding in the logarithmic case in Figure 1.

In general, the constraint can bind and the solution can be different. Before deriving the solution, let us examine the optimal strategy for the logarithmic example graphed earlier, but with the constraint $\alpha \leq w$ (or $M = 1$ in Problem 4), as illustrated by Figure 3. The associated value function contours are shown in Figure 4. This is comparable to the contours for $R = 1$ in Figure 2. (For $R = 2.5$, the constraint would never be binding and the value contours would be as before.) The value function is the same for $c_- = rw$ and less elsewhere in this case (because the choice problem is more constrained), and both $\underline{\omega}$ and $\overline{\omega}$ are significantly less than the corresponding $1/r^*$ in the unconstrained case. Putting more of wealth into committed consumption and less into discretionary investment reflects the decreased value of investment due to the constraint on the portfolio position.

Note that each of Figures 1 and 3 reveals an upper bound on α/w. However, the two cases are qualitatively different. In Problem 1 (and the corresponding Figure 1), the upper bound comes only from the regulation of consumption to satisfy $c \leq r^* w$, whereas in Problem 4 (and the corresponding Figure 3), the upper bound comes directly from the constraint. This implies that achieving

[8]We require a strong solution. Therefore, there is an implicit constraint on the processes α and c that they must be consistent with the existence of a strong solution to the wealth equation.

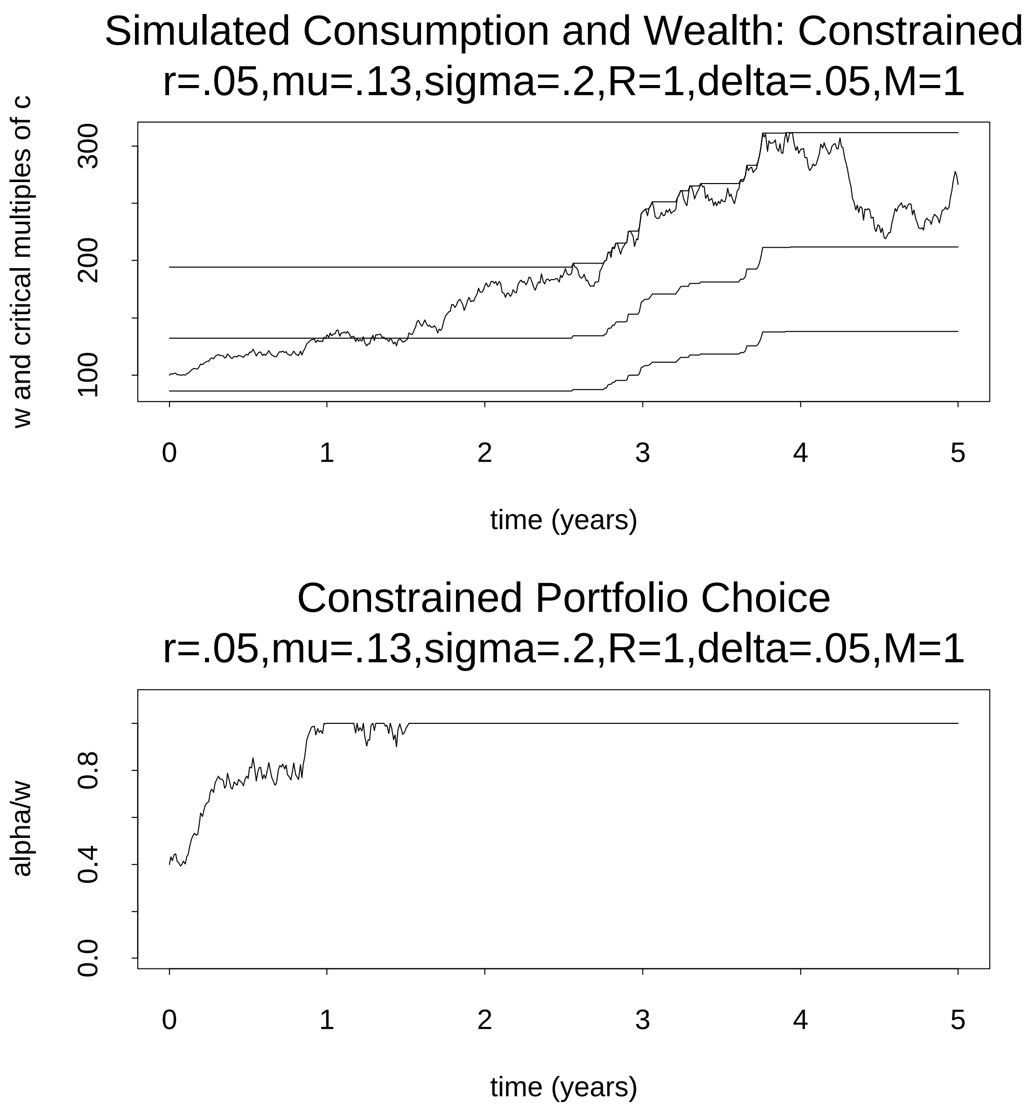

Figure 3: When the constraint $\alpha \leq w$ is added to the original problem with logarithmic utility, a complete analytic solution is not available but a simple numerical integration provides the solution. The solution is similar to the previous solution but with a new interior region in which the portfolio choice is at its constraint and consumption is maintained at its previous level.

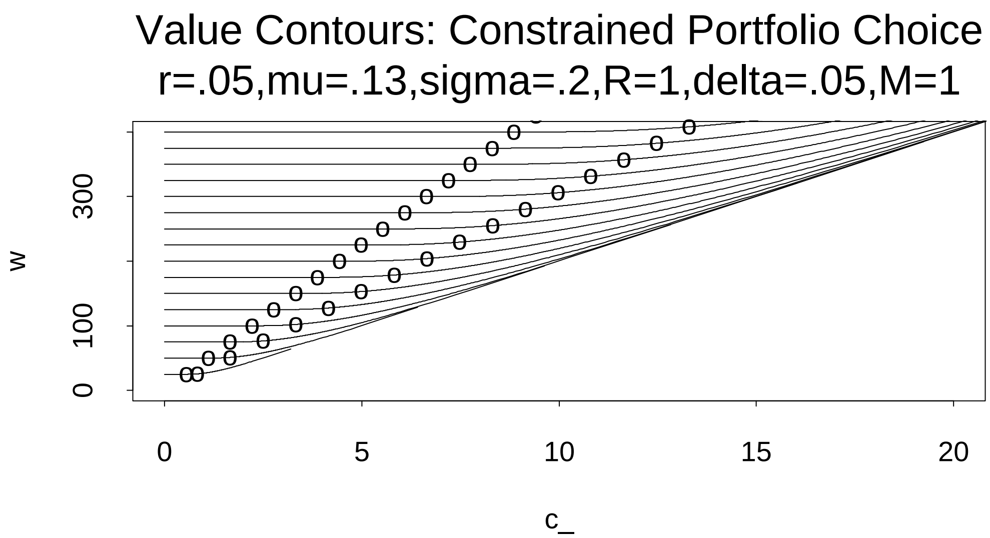

Figure 4: Here are the value contours for the problem with the constraint on the maximum risky portfolio position (here equal to wealth). The value function satisfies a smooth pasting condition at two boundaries $w/c_{-} = \underline{\omega}$ and $w/c_{-} = \overline{\omega}$ indicated by the circles.

the upper bound is a "rare" event occuring on a set of times of measure zero in Problem 1, but occuring on a set of times of positive measure if at all in Problem 4. This is a consequence of the fact that the set of times of regulation is like the set of times of increase of local time and has the property of being, with probability one, either empty or an uncountable set of zero Lebesgue measure. By contrast, the constraint is binding in a whole region of consumption and wealth values, and with probability one that region is either not entered or is entered for a finite span of time.

Now we turn to the derivation of the solution to Problem 4. From now on, our analysis will assume that the new constraint is strictly binding for some w and c_-. Then there are three regions, not two. The middle region has $\alpha = Mw$, $c = c_-$, and a value function satisfying $E[\Delta V]/dt = 0$ when we fix $\alpha = Mw$. The other two regions are as before, and smooth pasting at both boundaries determines the solution. The "reduced" value function $v(w/c_-) \equiv V(w/c_-, 1)$ in the middle region satisfies the following ordinary differential equation in $\omega \equiv w/c_-$:

$$(32) \quad 0 = u(1) - \delta v(\omega) + (\omega(r + M(\mu - r)) - 1)v'(\omega) + \frac{M^2\sigma^2}{2}\omega^2 v''(\omega),$$

which is the condition for $E[\Delta V]/dt = 0$ given $\alpha = Mw$, written in a form that is valid both for the logarithmic case and for the power case. The actual value function is then given by $V(w, c_-) = c_-{}^{1-R}v(w/c_-)$ in the power case and $V(w, c_-) = \log(c_-)/\delta + v(w/c_-)$ in the logarithmic case. The ordinary differential equation (32) is the Bellman equation given $\alpha = Mw$ and given the expression for V in terms of v. Even lacking a closed-form solution for (32), we know enough to characterize and compute the optimal strategy. We will let $\underline{\omega}$ and $\overline{\omega}$ be the two boundary values of ω. Both $\underline{\omega}$ and $\overline{\omega}$ have the same units as $1/r^*$ or $1/r$ in the solution of Problem 1.

In the first region, $1/r \leq w/c_- \leq \underline{\omega}$, the Bellman equation and its solution have the same form

$$(33) \quad v(\omega) = u(1)/\delta + \gamma^*(\omega - 1/r)^{1-R^*},$$

as for Problem 1, where R^* is the same as before, but γ^* is now a free parameter. It can be seen that the portfolio choice α is not dependent on γ^*, and therefore the portfolio choice will agree with that of Problem 1 in this region.

The boundary $w/c_- = \underline{\omega}$ is determined by smooth pasting, which turns out to be the same as matching α's. Since the portfolio choice for $1/r \leq w/c_- \leq \underline{\omega}$ is the same as in Problem 1, this turns out to be the same as substituting $\alpha = Mw$ into (9), from which we obtain

$$(34) \quad \underline{\omega} = \frac{1}{r} + \frac{1}{\frac{\mu - r}{\sigma^2 R^* M} - 1}.$$

Note that our assumption that the new constraint is strictly binding for some ω ensures that the denominator of the second term is positive, since $(\mu - r)/\sigma^2 R^*$ is the proportion of wealth in excess of c_-/r placed in the risky asset: if this is not greater than M, than neither can be α/w.

Now we are very close to a solution, and it is a simple matter to complete the solution numerically. Here is what we know at this point. For $1/r \leq \omega \equiv w/c_- \leq \underline{\omega}$, $c = c_-$, $\alpha = (w - c_-/r)(\mu - r)/\sigma^2 R^*$, and $v(\omega) = u(1)/\delta + \gamma^*(\omega - 1/r)^{1-R^*}$. The value of R^* is given by (7) and the value of $\underline{\omega}$ is given by (34). At this point, γ^* is an unknown positive constant. For $\underline{\omega} \leq \omega \equiv w/c_- \leq \overline{\omega}$, $c = c_-$, $\alpha = Mw$, and $v(\omega)$ solves (32). Given γ^*, there is a unique solution to (32) on this region that satisfies smooth pasting at $\underline{\omega}$. Furthermore, multiplying γ^* by some factor implies multiplying $v(\omega) - v(1/r) = v(\omega) - u(1)/\delta$ by the same factor for all of $1/r \leq \omega \leq \overline{\omega}$: this follows from the form of (33) and the linearity of (32) in $v(\omega) - u(1)/\delta$. Finally, for $\overline{\omega} \leq w/c_- \equiv \omega$, $c = w/\overline{\omega}$, $\alpha = Mw$, and $v(\omega) = v(\overline{\omega}) + (\log(\omega) - \log(\overline{\omega}))/\delta$ (logarithmic case) or $v(\omega) = v(\overline{\omega})(\omega/\overline{\omega})^{1-R}$ (power case). (The arguments for this form of the value function are essentially the same as before.)

At this point, the optimal strategy is determined except for the value of $\overline{\omega}$. In both logarithmic and power cases, smooth pasting at $\overline{\omega}$ implies that $-\overline{\omega}v''(\overline{\omega})/v'(\overline{\omega}) = R$. Since this expression is independent of the scale of v, we can use it to compute $\overline{\omega}$ without knowing the correct γ^*. We simply pick γ^*

arbitrarily (say $= 1$), we use smooth pasting at $\underline{\omega}$ and numerical integration of (32) to compute $v(\omega)$ on an interval increasing from $\underline{\omega}$, stopping when the scale-independent condition for $\overline{\omega}$ is satisfied. The true value of γ^* can then be computed by smooth pasting of either the first or second derivative, after which the whole value function will be known. Note that while this derivation has been informal, a formal proof of the validity of this construction would not be all that different from the proof of Theorem 1. The main modifications would be the verification of the Bellman equation and smooth pasting in the middle region, which would require a standard existence proof for differential equations relying on a Lipschitz property of (32) and arguments signing the derivatives to ensure concavity of the value function in this region.

3 The Verification Theorem

We turn now to the formal proof of Theorem 1, which can be viewed as a *verification theorem* because it verifies that the solution to the Bellman equation as analyzed in the informal derivation earlier is actually optimal. In the proof, feasibility if and only if $W_0 \geq C_{0-}/r$ and $r \geq 0$ is based on a standard type of pricing result, while the verification of the proposed optimum and the value function is based on an integral of the Bellman equation as in Fleming and Rishel [1975], with modifications for our particular problem, similar to those of Davis and Norman [1990]. Both parts of the proof rely on standard martingale techniques, a knowledge of which is assumed of the reader.

Besides the analysis of a slightly different problem, the main conceptual difference between this proof and the proof in Davis and Norman [1990] is the use here of the nonnegative wealth constraint (as in Dybvig and Huang [1988]) rather than a bound on α/w to rule out doubling strategies and the like. This is a bit more work, but it does admit a more general class of strategies. Since Davis and Norman [1990] impose nonnegative wealth (to rule out borrowing forever without repayment), a bit of work should yield a generalization of the result of Davis and Norman [1990] without the restriction to strategies with bounded α/w. The restriction to bounded α/w seems at least somewhat arbitrary since there does not seem to be any pragmatic limit on the scale of α/w given that wealth does not go negative and given that one can buy options and futures. It is nice (although arguably not of great importance) to know that the restriction is superfluous.

To prepare for the proof, we consider first a lemma containing a standard type of pricing result and a corollary that feasible consumption is no greater than the interest flow from wealth. The pricing result is stated in terms of an inequality, since it is feasible to consume less than the full value or to destroy value as by following a doubling strategy in reverse. The corollary depends critically on nondecreasing consumption, while the lemma depends only on consumption nonnegative (which follows from initial consumption positive and nondecreasing consumption).

Lemma 1 Pricing of consumption streams: *For all strategies satisfying constraints (3), (4), and (6) of Problem 1,*

$$(35) \quad (\forall \tau \geq 0) \int_{t=\tau}^{\infty} c_\tau \frac{\rho_t}{\rho_\tau} dt \leq w_\tau,$$

where

$$(36) \quad \rho_t \equiv \exp\left(-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) t - \frac{\mu - r}{\sigma} Z_t\right)$$

is the state-price density*, which is a positive process.*

Proof   Clearly, ρ_t is well-defined and positive, and Itô's lemma implies that

$$(37) \quad \frac{d\rho_t}{\rho_t} = -r dt - \frac{\mu - r}{\sigma} dZ_t.$$

Now fix $\tau \geq 0$. For $t \geq \tau$, define the process

$$(38) \quad \mathcal{M}_t^a \equiv \int_{s=\tau}^{t} \frac{\rho_s}{\rho_\tau} c_s ds + \frac{\rho_t}{\rho_\tau} w_t.$$

Based on Itô's Lemma, (6), (37), and (38),

$$\begin{aligned}
(39) \quad d\mathcal{M}_t^a &= \frac{\rho_t}{\rho_\tau} c_t dt + \frac{\rho_t}{\rho_\tau}(wrdt + \alpha((\mu - r)dt + \sigma dZ_t - c_t dt) \\
&\quad + \frac{\rho_t w_t}{\rho_\tau}(-rdt - \frac{\mu - r}{\sigma} dZ_t) + \frac{\rho_t}{\rho_\tau}(-\frac{\mu - r}{\sigma}\alpha\sigma)dt \\
&= \frac{\rho_t}{\rho_\tau}\left(\sigma - \frac{\mu - r}{\sigma}\right) dZ_t.
\end{aligned}$$

Therefore, $\mathcal{M}_t^a$ is a local martingale. But, feasible wealth and consumption are non-negative and ρ is always positive. Therefore, $(\forall t \geq \tau)\mathcal{M}_t^a \geq 0$. But any positive local martingale is a supermartingale. Therefore,

$$\begin{aligned}
(40) \quad E_\tau[\int_{s=\tau}^{\infty} \frac{\rho_s}{\rho_\tau} c_s ds] &\leq \lim_{t=\infty} E_\tau[\int_{s=\tau}^{t} \frac{\rho_s}{\rho_\tau} c_s ds] \\
&\leq \lim_{t=\infty} E_\tau[\int_{s=\tau}^{t} \frac{\rho_s}{\rho_\tau} c_s ds + \frac{\rho_t}{\rho_\tau} w_t] \\
&= \lim_{t=\infty} E_\tau[\mathcal{M}_t^a] \\
&\leq \mathcal{M}_\tau^a = w_\tau.
\end{aligned}$$

∎

Corollary 1 *For all strategies satisfying constraints (3), (4), and (6) of Problem 1, $r > 0$ and*

$$(41) \quad (\forall \tau \geq 0) w_\tau \geq c_{\tau-}/r.$$

Consequently, Problem 1 is feasible if and only if $W_0 \geq C_{0-}/r$ and $r > 0$.

Proof Fix $\tau \geq 0$. From Lemma 1 and (4),

$$(42) \quad w_\tau \geq E[\int_{t=\tau}^{\infty} \frac{\rho_t}{\rho_\tau} c_t dt]$$

$$
\begin{aligned}
&\geq E[\int_{t=\tau}^{\infty} \frac{\rho_t}{\rho_\tau} c_{\tau-} dt] \\
&= \int_{t=\tau}^{\infty} E[\exp\left(-\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)(t-\tau)\right. \\
&\left. -\frac{\mu-r}{\sigma}(Z_t - Z_\tau)]c_{\tau-} dt\right) \\
&= \int_{t=\tau}^{\infty} \exp(-r(t-\tau))c_{\tau-} dt \\
&= \begin{cases} c_{\tau-}/r & \text{for } r > 0 \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Now, consider feasibility of Problem 1. If $W_0 \geq C_{0-}/r$ and $r > 0$, then the problem is feasible given the choices $c_t \equiv C_{0-}$ and $\alpha = 0$, which are consistent with $w_t = C_{0-}/r + (W_0 - C_{0-})\exp(rt) > 0$. Conversely, by (42) it follows that Problem 1 is infeasible if $W_0 \geq C_{0-}/r$ or $r > 0$. ∎

The remainder of the proof of Theorem 1 is divided into pieces to make the logic clearer. Like the proofs of verification theorems by Fleming and Rishel [1975] and Davis and Norman [1990] on which the proof is based, the focus of the proof is a process that values current an past consumption by the realization and future consumption according to the proposed value function. For any feasible strategy $(\{\alpha_t\}, \{c_t\})$, define

$$
(43) \quad \mathcal{M}_t^b \equiv \int_{s=0}^{t} e^{-\delta s} u(c_s) ds + e^{-\delta t} V(w_t, c_t)
$$

for $t \geq 0-$, where $V(\cdot,\cdot)$ is the proposed value function defined in (14) in the statement of Theorem 1. For the optimal strategy, $\mathcal{M}_t^b$ should be the expected value of the problem, conditional on information available at time t, which it will be if $\mathcal{M}_t^b$ is a martingale for $t \in [0-,\infty)$ and the residual part vanishes ($\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t,c_t)] = 0$). Any other strategy will be known to do worse if it makes $\mathcal{M}_t^b$ a supermartingale with a vanishing or positive residual part ($\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t,c_t)] \geq 0$). The Bellman equation is the first-order condition for $\mathcal{M}_t^b$ to be a local martingale for the optimal strategy and a local supermartingale for all strategies; the proofs exploit some boundedness or similar regularity to show that the local supermartingales and local martingale are respectively supermartingales and a martingale.

The next piece of the proof shows that $\mathcal{M}_t^b$ is a local supermartingale for all strategies and a local martingale for the proposed optimum. Note that the proposed optimum is feasible for the parameter values over which Problem 1 is claimed to be feasible and bounded.

Lemma 2 *Given any feasible strategy, the process $\mathcal{M}^b$ defined in (43) is a local supermartingale. For the proposed optimal strategy, $\mathcal{M}^b$ is a local martingale.*

PROOF We are studying $\mathcal{M}^b$ only for feasible strategies. From Corollary 1, all feasible strategies satisfy $(\forall t) w_t \geq c_{t-}/r$. Furthermore, if at any time τ, $w_\tau = c_{\tau-}/r$, the only feasible continuation satisfies $\alpha_t \equiv 0$ and $c_t \equiv c_{\tau-}$ for all $t \geq \tau$: given that feasibility implies $c_t \geq c_{\tau-}$, any other feasible strategy would imply from (6) that $dw = \alpha_t((\mu - r)dt + \sigma dZ_t) - (c_t - c_{t-})dt$, which implies a positive probability of violating $w_t \geq c_{t-}/r$ soon after if $\alpha_t \neq 0$ or $c_t > c_{t-}$ (and $c_t < c_{t-}$ is not feasible).

We consider the local dynamics of $\mathcal{M}_t^b$ for three cases: $c_{t-} = rw_t$, $r^* w_t \leq c_{t-} < rw_t$, and $c_{t-} < r^* w_t$. (The case $c_{t-} > rw_t$ cannot arise for any feasible strategy.) Taking first $c_{t-} = rw_t$, we have that $\alpha = 0$ and $c = c_{t-}$ and therefore (6) implies $dw_t = 0$ and consequently (43) and (14) imply that

$$\begin{aligned} (44) \quad d\mathcal{M}_t^b &= e^{-\delta t} u(c_{t-})dt - \delta e^{-\delta t}(u(c_{t-})/\delta)dt \\ &= 0, \end{aligned}$$

and therefore $\mathcal{M}^b$ is a local martingale (and in fact a constant) in this case. Since every local martingale is a local supermartingale, this verifies the proposition for this case.

In the remaining cases, we will use the observation that $V(w, c_-)$ is twice continuously differentiable in w for $w \in (c_-/r, \infty)$. This can be verified directly from taking the first two derivatives of V as defined in (14). At $w = c_-/r^*$, this is the familiar "smooth-pasting condition," without which the expected change would be dominated by a local time term or would be discontinuous at the boundary.

Consider now the second case of the region $r^* w_t \leq c_{t-} < r w_t$. On the interior of this region, it follows from elementary (but in some cases surprisingly messy) calculus and algebra that $V_w > 0$, $V_{ww} < 0$, $V_{c_-} < 0$, and $V_{c_- c_-} > 0$. On the boundary $w_t = \omega^* c_{t-}$, $V_w > 0$ and $V_{ww} < 0$, but $V_{c_-} = V_{c_- c_-} = 0$. In this region, discontinuous changes in $\mathcal{M}_t^b$ can arise from discontinuous upward jumps in c (decreases are not feasible). At a point of discontinuous increase in c,

$$(45) \quad \Delta \mathcal{M}_t^b = e^{-\delta t}(V(w_t, c_t) - V(w_t, c_{t-})),$$

which is either negative or zero since $V_{c_-} < 0$ on the interior of this region, and is zero for the proposed strategy. Continuous change can be due to consumption withdrawal, continuous change in w and c, and to discounting. From (43), (14), (6), and Itô's lemma,

$$\begin{aligned}
(46) \quad d\mathcal{M}_t^b &= e^{-\delta t} u(c_{t-}) dt - \delta e^{-\delta t} V(w_t, c_{t-}) + e^{-\delta t} V_w(w, c_{t-}) dw \\
&\quad + \frac{1}{2} e^{-\delta t} V_{ww} dw^2 + V_{c_-} dc_t, \\
&= e^{-\delta t} u(c_{t-}) dt - \delta e^{-\delta t} V(w_t, c_{t-}) \\
&\quad + e^{-\delta t} V_w(w_t, c_{t-})(w_t r dt + \alpha_t((\mu - r)dt + \sigma dZ) - c_t dt) \\
&\quad + \frac{1}{2} e^{-\delta t} V_{ww} dw_t^2 + V_{c_-} dc_t, \\
&= (u(c_{t-}) - \delta V + (w_t r + \alpha_t(\mu - r) - c_t) V_w + \frac{1}{2} \alpha^2 \sigma^2 V_{ww}) e^{-\delta t} dt \\
&\quad + V_{c_-} dc_t + e^{-\delta t} \alpha_t \sigma V_w dZ_t.
\end{aligned}$$

Because a decrease in c is infeasible and $V_{c_-} < 0$, the term in dc_t is maximized at zero by the proposed optimal policy. Furthermore, $V_{ww} < 0$ implies that the coefficient to dt is maximized for $\alpha_t = -(\mu - r) V_w / (\sigma^2 V_{ww})$. Substitution of this choice of α_t and the derivatives of V into the coefficient yields a drift of zero. Since this is the maximum, all other possible strategies must have zero or negative drift. Furthermore, it is simple to verify that the proposed α_t is the maximal one. For the region $r^* w_t \leq c_{t-} < r w_t$, we have shown that $\mathcal{M}_t^b$ is a local supermartingale for every feasible strategy and a local martingale for the proposed optimal strategy.

The final region to consider is the case $c_{t-} < r^* w_t$. According to the proposed optimal strategy, in this region it is optimal to move immediately to the boundary. Such a jump has an expected (and realized) change of zero in the value function, since the value function does not depend on c_- in the region. To show it is optimal to jump to the boundary, we want to show that the expected change in value is negative for a strategy of remaining in the region for a positive length of time. In this region, $V_w > 0$, $V_{ww} < 0$, and $V_{c_-} = V_{c_- c_-} = 0$. Now, if we remain in the interior, $d\mathcal{M}_t^b$ is as described in (46) except with the derivatives in the value function computed as at the boundary (due to the $\max(c, r^* w)$ in (14)), and we can compute that the largest possible drift is negative and occurs for $\alpha_t = -(\mu - r)V_w/(\sigma^2 V_{ww})$. The proposed optimal strategy moves immediately to the boundary, so it does make $\mathcal{M}_t^b$ a local martingale in the third and final region. This completes the proof that $\mathcal{M}_t^b$ is a local martingale given the proposed optimal strategy and a local supermartingale given any feasible strategy. ∎

Next, we show that $\mathcal{M}_t^b$ is actually a supermartingale for all feasible strategies.

Lemma 3 *For all feasible strategies, $\mathcal{M}^b$ is a supermartingale.*

PROOF For any feasible strategy, Lemma 2 implies that $\mathcal{M}^b$ is a local supermartingale. Furthermore,

$$\text{(47)}\quad \begin{aligned} \mathcal{M}_t^b &\equiv \int_{s=0}^{t} e^{-\delta s} u(c_s)ds + e^{-\delta t} V(w_t, c_{t-}) \\ &\geq \int_{s=0}^{t} e^{-\delta s} u(c_{0-})ds + e^{-\delta t} u(c_{0-})/\delta \\ &= u(c_{0-})/\delta, \end{aligned}$$

because all feasible strategies must have $c_t \geq c_{0-}$ and because $u(c_{0-})/\delta$ is the smallest feasible value of $V(w_t, c_{t-})$ given c_{0-}, since $V(w_t, c_{t-}) \geq$

$V(c_{t-}/r, c_{t-}) \geq V(c_{0-}/r, c_{0-}) = u(c_{0-})/\delta$. This implies that $\mathcal{M}_t^b$ is a supermartingale for all feasible strategies because a local supermartingale that is bounded below is a supermartingale. ∎

The next logical step is to show that $\mathcal{M}_t^b$ is a martingale (and not just a local martingale) for the optimal strategy. First, it is useful to record some properties of the wealth process.

Lemma 4 *Fix $p \neq 0$. For any feasible strategy,*

$$(48)\quad w_t^p = w_0^p \exp\left(\int_{s=0}^t p(r + \frac{\alpha_s}{w_s}(\mu - r) - \frac{c_s}{w_s} + \frac{1}{2}(p-1)\left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2)ds\right)$$
$$\times \exp\left(\int_{s=0}^t p\frac{\alpha_s}{w_s}\sigma dZ_s - \frac{1}{2}\int_{s=0}^t p^2\left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2 ds\right),$$

where the second exponential is an exponential supermartingale. For the proposed optimal strategy, the exponential supermartingale is a martingale. Furthermore, for the optimal strategy there exists a constant b depending on the parameters and p such that

$$(49)\quad E\left[\frac{w_t^p}{p}\right] \leq \frac{w_0^p}{p}\exp(bt)$$

Proof The wealth change equation (6) can be rewritten as

$$(50)\quad \frac{dw_t}{w_t} = \left(r + \frac{\alpha_t}{w_t}(\mu - r) - \frac{c_t}{w_t}\right)dt + \frac{\alpha_t}{w_t}\sigma dZ_t.$$

Itô's Lemma tells us that $d\log(w^p) = pd\log(w) = pdw/w - (1/2)p(dw/w)^2$, since the logarithmic and power functions are smooth over the relevant domain since feasible strategies satisfy $w_t \geq C_{0-}/r$ by Corollary 1. Integrating the expression for $d\log(w^p)$ with (50) substituted in and then taking the exponential implies (48). The second exponential is clearly in the form of an exponential supermartingale.

For the proposed optimal strategy, α_s/w_s is in the compact set $[0,(\mu - r)/\sigma^2 R]$. This implies that the exponential supermartingale satisfies the Novikov condition (as given in Karatzas and Shreve [1988], Proposition 5.12) and the exponential supermartingale is therefore a martingale. If we let b be an upper bound of the integrand in the first exponential for $\alpha_s/w_s \in [0,(\mu - r)/\sigma^2 R]$ and $c_s/w_s \in [r^*, r]$, (49) follows. ∎

Lemma 5 *Given the proposed optimal strategy, $\mathcal{M}^b$ is a martingale.*

PROOF For the proposed optimal strategy we have that

$$(51)\quad \mathcal{M}_t^b = \int_{s=0}^t \alpha_s \sigma dZ_s.$$

The expected quadratic variation is given by

$$\begin{aligned}(52)\quad E[< M_t^b, M_t^b >] &= E[\int_{s=0}^t {\alpha_s}^2 \sigma^2 ds] \\ &= E[\int_{s=0}^t {w_s}^2 \frac{{\alpha_s}^2}{{w_s}^2} \sigma^2 ds] \\ &= \int_{s=0}^t E[{w_s}^2 \frac{{\alpha_s}^2}{{w_s}^2} \sigma^2] ds \\ &< \infty\end{aligned}$$

because σ^2 is constant, because α_s/w_s is uniformly bounded given the optimal strategy, and because the bound on $E[{w_s}^2]$ given by (49) of Lemma 4 is continuous and therefore integrable. (Positivity of the integrand implies that the integration and expectation operators commute.) Finite expected quadratic variation over all finite time intervals implies that the local martingale $\mathcal{M}^b$ is in fact a martingale. ∎

Now we are ready to address the asymptotic behavior of the residual term $E[e^{-\delta t} u(w_t)]$.

Lemma 6 *For all feasible strategies,* $\liminf_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] \geq 0$*. For the proposed optimal strategy,* $\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] = 0$*.*

Proof For all feasible strategies, $c_t \geq C_{0-}$ and $w_t \geq c_{t-}/r$ for all t. Therefore, $V(w_t, c_t) \geq V(c_t/r, c_t) \geq V(C_{0-}/r, C_{0-})$ for all strategies and consequently $\liminf_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] \geq \lim_{t\uparrow\infty}[e^{-\delta t}V(C_{0-}/r, C_{0-})] = 0$.

For the proposed optimal strategy, we consider separately the cases $R > 1$, $R = 1$, and $R < 1$. For the proposed optimal strategy and $R > 1$, V is everywhere negative and therefore $\liminf_{t\uparrow\infty} E[e^{-\delta t}u(w_t)] \geq 0$ implies $\lim_{t\uparrow\infty} E[e^{-\delta t}u(w_t)] = 0$. (Note: this argument for $R > 1$ is valid for all feasible strategies and not just the optimal strategy.)

For the proposed optimal strategy and $R = 1$, $V(w, c)$ is bounded below by

$$V(w, rw) = (\log(w) + \log(r))/\delta$$

and $V(w, c)$ is bounded above by

$$V(w, r^*w) = (\log(w) + \log(r^*))/\delta + \gamma(1/r^* - 1/r)^{1-R^*}.$$

Therefore, if $\lim_{t\uparrow\infty} E[e^{-\delta t}\log(w_t)] = 0$, then $\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] = 0$. Letting $p = 1$ in the expression (48) for w_t^p in the statement of Lemma 4,

$$(53)\quad \begin{aligned} E[\log(w_t)e^{-\delta t}] &= (\log w_0 + E[\int_{s=0}^{t}((r + \frac{\alpha_s}{w_s}(\mu - r) - \frac{c_s}{w_s} \\ &\quad - \frac{1}{2}\left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2)ds + \int_{s=0}^{t} \frac{\alpha_s}{w_s}\sigma dZ_s)])e^{-\delta t} \\ &\leq (\log w_0 + E[\int_{s=0}^{t}((r + \frac{(\mu - r)^2}{2\sigma^2})ds \\ &\quad + \int_{s=0}^{t} \frac{\alpha_s}{w_s}\sigma dZ_s)])e^{-\delta t} \end{aligned}$$

(where the quadratic form $(\mu - r)\alpha_s/w_s - \sigma^2(\alpha_s/w_s)^2/2$ in α_s/w_s was replaced by its largest possible value $(\mu - r)^2/2\sigma^2$ and c_s/w_s was replaced by its infimum 0 of feasible values)

$$= (\log w_0 + E[\int_{s=0}^{t}(r + \frac{(\mu - r)^2}{2\sigma^2})ds + \int_{s=0}^{t} \frac{\alpha_s}{w_s}\sigma dZ_s])e^{-\delta t}$$

$$
\begin{aligned}
&= (\log(w_0) + (r + \frac{(\mu - r)^2}{2\sigma^2})t)e^{-\delta t} \\
&\to 0 \quad \text{as } t \uparrow \infty,
\end{aligned}
$$

where α_s/w_s bounded implies that $E[\int_{s=0}^t ((\alpha_s/w_s)\sigma)^2 ds] < \infty$ and therefore that $E[\int_{s=0}^t (\alpha_s/w_s)\sigma dZ_s] = 0$.

For the proposed optimal strategy and $R < 1$, $V(w,c)$ is bounded below by

$$
V(w, rw) = w^{1-R} r^{1-R}/\delta(1-R)
$$

and $V(w,c)$ is bounded above by

$$
V(w, r^* w) = w^{1-R}(r^{1-R}/\delta + \gamma(r^*)^{1-R}(1/r^* - 1/r)^{1-R^*}).
$$

Therefore, if $\lim_{t\uparrow\infty} E[e^{-\delta t} {w_t}^{1-R}] = 0$, then $\lim_{t\uparrow\infty} E[e^{-\delta t} V(w_t, c_t)] = 0$. Letting $p = 1 - R$ in the expression (48) for w_t^p in the statement of Lemma 4,

$$
\begin{aligned}
(54) \quad E[e^{-\delta t} {w_t}^{1-R}] &= w_0^{1-R} E\Bigg[\exp\Bigg(\int_{s=0}^t \Bigg((1-R)\Bigg(r + \frac{\alpha_s}{w_s}(\mu - r) - \frac{c_s}{w_s} \\
&\quad - \frac{1}{2} R \left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2\Bigg) - \delta\Bigg) ds\Bigg) \\
&\quad \times \exp\left(\int_{s=0}^t (1-R)\frac{\alpha_s}{w_s}\sigma dZ_s - \frac{1}{2}\int_{s=0}^t (1-R)^2 \left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2 ds\right)\Bigg] \\
&\leq w_0^{1-R} E\left[\exp\left(\int_{s=0}^t \left((1-R)\left(r + \frac{(\mu - r)^2}{2\sigma^2 R}\right) - \delta\right) ds\right)\right. \\
&\quad \left. \times \exp\left(\int_{s=0}^t (1-R)\frac{\alpha_s}{w_s}\sigma dZ_s - \frac{1}{2}\int_{s=0}^t (1-R)^2 \left(\frac{\alpha_s}{w_s}\right)^2 \sigma^2 ds\right)\right]
\end{aligned}
$$

(where the quadratic form $(\mu - r)\alpha_s/w_s - R\sigma^2(\alpha_s/w_s)^2/2$ in α_s/w_s was replaced by its largest possible value $(\mu - r)^2/2\sigma^2 R$ and c_s/w_s was replaced by the infimum 0 of feasible values)

$$
\leq u(w_0) \exp\left(\left((1-R)\left(r + \frac{\kappa}{R}\right) - \delta\right) t\right)
$$

(since $\kappa \equiv (\mu - r)^2/2\sigma^2$ and the second exponential in the previous expression is an exponential supermartingale by Lemma 4). But the coefficient of t in the exponent can be shown to be negative since $R > R^*$ and $\delta > 0$,[9] and we have verified the transversality condition $\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] = 0$ for the final case $R < 1$. ∎

Now we are ready to combine our results.

Proof of Theorem 1 Feasibility of the problem if and only if both $W_0 \geq C_{0-}/r$ and $r > 0$ was shown in Corollary 1.

For the proposed optimal strategy, $\mathcal{M}_t^b$ is a martingale by Lemma 5 and $\lim_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] = 0$ by Lemma 6. Therefore,

$$
\begin{aligned}
(55)\quad V(W_0, C_{0-}) &= \mathcal{M}_{0-}^b \\
&= \lim_{t\uparrow\infty} E[\mathcal{M}_t^b] \\
&= \lim_{t\uparrow\infty} E\left(\int_{s=0}^t e^{-\delta s}u(c_s)ds + e^{-\delta t}V(w_t, c_t)\right) \\
&= \lim_{t\uparrow\infty} E\left[\int_{s=0}^t e^{-\delta s}u(c_s)ds\right] + \lim_{t\uparrow\infty} E\left[e^{-\delta t}V(w_t, c_t)\right] \\
&= E\left[\int_{t=0}^\infty e^{-\delta s}u(c_s)ds\right],
\end{aligned}
$$

(where exchanging the expectation and the limit is justified by $u(c_s) \geq u(C_{0-})$), which confirms that $V(W_0, C_{0-})$ is the correct value function for the proposed optimum.

For any feasible strategy, $\mathcal{M}_t^b$ is a supermartingale by Lemma 3. Furthermore, we know that $\liminf_{t\uparrow\infty} E[e^{-\delta t}V(w_t, c_t)] \geq 0$ by Lemma 6. Therefore,

$$
\begin{aligned}
(56)\quad V(W_0, C_{0-}) &= \mathcal{M}_{0-}^b \\
&\geq \lim_{t\uparrow\infty} E[\mathcal{M}_t^b] \\
&= \lim_{t\uparrow\infty} E\left(\int_{s=0}^t e^{-\delta s}u(c_s)ds + e^{-\delta t}V(w_t, c_t)\right) \\
&= \lim_{t\uparrow\infty} E\left[\int_{s=0}^t e^{-\delta s}u(c_s)ds\right] + \lim_{t\uparrow\infty} E\left[e^{-\delta t}V(w_t, c_t)\right]
\end{aligned}
$$

[9]The coefficient is $1/R$ times $-rR^2 + (r - \kappa - \delta)R + \kappa$. This quadratic in R is negative for R larger than its single positive root $R = R^*$.

$$\geq \quad E\left[\int_{t=0}^{\infty} e^{-\delta s} u(c_s) ds\right],$$

(where exchanging the expectation and the limit is justified by $u(c_s) \geq u(C_{0-})$), which proves that no strategy can do better than the proposed optimum, which is therefore optimal.

Finally, existence of a solution given $\delta > 0$ and $R > R^*$ follows by construction from the proven solution. Existence of a solution for $\delta > 0$ and $C_{0-} = rW_0$ follows from the uniqueness of the feasible strategy as discussed at the start of the proof of Lemma 2 and the the finiteness of its value function $\int_0^\infty e^{-\delta t} u(C_{0-}) dt = u(C_{0-})/\delta$.

Conversely, nonexistence for $\delta \leq 0$, $R \leq R^*$ and $C_{0-} = rW_0$ follows from divergence of the value for the unique feasible strategy (since $R \leq R^*$ implies $u(C_{0-}) > 0$). Nonexistence for $\delta > 0$, $R \leq R^*$ and $C_{0-} < rW_0$ follows from from getting unbounded value from a sequence of strategies indexed by T, wherein $c_t \equiv C_{0-}$ and $\alpha = (W_t - C_{0-}/r)(\mu - r)/2\sigma^2 R$ for $t < T$ and $c_t \equiv rw_T$ and $\alpha = 0$ for $t \geq T$. The calculation showing unboundedness is virtually identical to that in (54) and the earlier footnote 3. ∎

4 Conclusion

We have introduced the investment and consumption withdrawal problem for an agent with intolerance for any decline in standard of living. The problem is solved in closed form for conditionally time separable constant relative risk aversion preferences. The solution has a portfolio strategy similar to the constant proportions portfolio insurance strategy marketed by Fischer Black at Goldman Sachs, with an additional ratcheting feature that parallels ratcheting in consumption. Three generalizations are also solved. One generalization replaces the assumption of no decline with an exogenously specified maximal rate of decline. This generalization can be transformed into the original problem for which we have a complete solution. The second generalization has an investment problem with two goods, one of which has consumption that cannot be decreased and the other of which is unconstrained. The third generalization caps the proportion of wealth that can be placed in the risky asset. While the third generalization is not solved in closed form, the solution is characterized closely enough to admit simple numerical solution.

It is hoped that these models will be of use in understanding security markets in general and in particular to universities and individuals who face rigidities in their expenditures.

References

Arrow, Kenneth J., 1965, *Aspects of the Theory of Risk-Bearing* (Yrjo Jahnsson Lectures) Helsinki: Yrjo Jahnssonin Säätiö.

Clark, J., 1970, "The representation of functionals of Brownian motion by stochastic integrals, *Annals of Mathematical Statistics*, **41**, 1282–1295.

Constantinides, George M., 1990, "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy* **98**, 519–543.

Davis, M. H. A., and A. R. Norman, Portfolio selection with transaction costs, *Mathematics of Operations Research* **15** 676–713.

Detemple, Jerome B., and Fernando Zapatero, 1991, "Asset Prices in an Exchange Economy with Habit Formation," Columbia University Graduate School of Business working paper.

Dixit, Avinash, 1992, "Investment and Hysteresis" *Journal of Economic Perspectives* **6**, 107–132.

Dybvig, Philip H. and Chi-fu Huang, 1988, "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Streams" *Review of Financial Studies* **1**, 377–401.

Fleming, W. H., and R. W. Rishel, 1975, *Deterministic and Stochastic Optimal Control* Springer-Verlag: New York.

Grossman, Sanford, and Zhongquan "Joe" Zhou, 1993, "Optimal Investment Strategies for Controlling Drawdowns," University of Pennsylvania Wharton School working paper.

J. Michael Harrison, 1985, *Brownian Motion and Stochastic Flow Systems* Wiley: New York.

Ingersoll, Jonathan E., Jr., 1992, "Optimal Consumption and Portfolio Rules With Intertemporally Dependent Utility of Consumption" *Journal of Economic Dynamics and Control* **16**, 681–712.

Karatzas, Ioannis, and Steven Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag.

McDonald, Robert, and Daniel Siegel, 1985, "Investment and the Valuation of Firms when There Is an Option to Shut Down," *International Economic Review* **26**, 331–349.

McDonald, Robert, and Daniel Siegel, 1986, "The Value of Waiting to Invest" *Quarterly Journal of Economics* **101**, 707–727.

Merton, Robert C., 1971, "Optimal Consumption and Portfolio Rules in a Continuous-time Model," *Journal of Economic Theory* **3**, 373–413.

Merton, Robert C., 1973, "Theory of rational option pricing," *Bell Journal of Economics and Management Science* **4**, 141–183.

Pratt, John W., 1964, "Risk-Aversion in the Small and in the Large," *Econometrica* **32**, 122–136.