Link-save trading

Katsiaryna KAVAL

Department of Statistics, University of Glasgow

Ilya MOLCHANOV[*][†]

Department of Mathematical Statistics and Actuarial Science, University of Berne

Abstract

Transaction costs involved while trading several assets may be described using bid-ask spread of the asset prices. We assume that the prices of several assets may be linked, so that transactions involving several assets have prices that are not necessarily equal to the sums of (bid or ask) prices of the individual assets. The family of possible price combinations forms a convex (random) set which changes in time and is called the set-valued price process. It is shown that the necessary and sufficient condition for no arbitrage is the existence of a martingale selection, i.e. a martingale that takes values in the set-valued price process. Examples and applications to option pricing are discussed.

Keywords: bid-ask spread; multiple assets; price process; set-valued process; transaction costs

[*]Corresponding author: I. Molchanov, Department of Mathematical Statistics and Actuarial Science, University of Berne, Sidlerstr. 5, CH-3012 Berne, Switzerland. Phone: 0041 31 6318801, Fax: 0041 31 6313870, e-mail: ilya@stat.unibe.ch

[†]Supported by Swiss National Science Foundation, Grant No 200021-100248

1 Introduction

It is well-known [3, 7] that no-arbitrage principle in economics corresponds to the fact that the price process forms a martingale under a certain measure equivalent to the basic probability measure $\mathbf{P}$. This fact is used to price derivatives by taking conditional expectations of the discounted payoff with respect to the martingale measure. Various reasons for market imperfections lead to deviations from the basic formulae of derivative pricing, see [10, 23]. These market imperfections may be caused by a number of reasons, for example short sale constraints, taxes and transaction costs. The foundations of derivative pricing under transaction costs were laid out in [5], see also the recent contributions in [13, 15, 21].

Here we follow an approach that suggests summarising effects caused by market imperfections (in particular, transaction costs) by assuming that every asset has two prices: one (lower) being the bid price and the other (higher) being the ask price. Then the price of an asset is represented by an interval, which may be random and usually also depends on time, as conventional asset prices do. This model for price process was considered by Jouini and Kallal [12], who proved that such an interval-valued price process admits no-arbitrage if and only if there exists a martingale (with respect to a probability measure equivalent to $\mathbf{P}$) that can be inscribed between the bid and ask prices. It was shown that in this case the derivative prices are given by intervals with end-points corresponding to the infimum and the supremum of the prices for the same derivative under all possible martingale measures. This approach was later cast into an axiomatic framework in [11] and extended to multiple assets. The axiomatic approach builds upon some admissibility axioms for the price of contingent claims. We refer the reader to [11] for an extensive survey of the literature related to pricing of contingent claims under transaction costs. The current paper is greatly inspired by the axiomatic approach developed in [11] and the tools presented in [21].

In difference to the prefect market where opposite transactions cancel each other, the presence of bid-ask spread means that opposite transactions do not simply close the position in the corresponding asset and so the long and short positions in the assets should be treated separately. In other words, a portfolio is represented by two vectors showing the amount of assets in long and short positions respectively and not simply the arithmetic difference of these two vectors (the net effect).

In the present paper we study contingent claim pricing for several assets in the presence of transaction costs represented as bid-ask spreads. The main novelty is to take into consideration the so-called link-save effect on related assets. It is sensible to assume that while trading related assets discounts on simultaneous purchase of several assets may be offered. This means that bid-ask spreads of several assets can no longer be treated separately of each other, i.e. the price ranges are described by general convex compact sets rather than parallelepipeds as it is the case in [11, 12]. Another effect is that combinations of several assets are no longer traded as the arithmetic sums of prices of their components (either as bid or ask prices depending on the type of the position). This novelty causes for generalising the concept of a portfolio, which now is described by a measure on the unit sphere in the d-dimensional space, where d is the number of traded assets.

We show that this framework naturally corresponds to the axiomatic developed in [11]. For instance, the fact that price functional is sublinear has now a clear geometric explanation by means of a convex set of all martingale selections for the price process. Note that our results are still applicable for various reasons of market imperfections as long as they can be described by means of a price set.

Following the technique developed in [11] and [21] we prove that no-arbitrage in our link-save model corresponds to existence of a martingale inscribed within the set-valued price process. The necessary mathematical techniques involve the concepts of a random set and a random measure [22].

The paper is organised as follows. Section 2 introduces the concept of a price set and explains its relationship with the sublinear property of the price functional. Section 3 shows that it is natural to describe the portfolios as measures in d-dimensional space. In the stochastic framework one works with the random measures. Various cones in the space of measures and the corresponding cones in the Euclidean space are described in Section 4. Section 5 introduces the time-dependent framework. The central concept here is the concept of a monotonicity of portfolios, which generalises the usual concept of coordinate-wise monotonicity for vector-valued portfolios. Section 6 shows how to use the framework of [21] in order to characterise the no-arbitrage property of a set-valued price process. Section 7 provides a characterisation of admissible price functionals of claims. The claims have the same interpretation as in the usual case, however, now the marketable claims have to be dominated by measure-valued portfolios, so that the net effect of the portfolio suffices to pay the claim. Section 8 treats the practical questions of determining the link-save functions that determine discounts and suggests several families of such functions in view of possible applications in practical trading. Section 9 describes several examples of set-valued price process and the corresponding contingent claims.

2 Bid-ask spreads and set-valued prices

Consider a single asset whose bid price is z' and ask price is z. To avoid instant arbitrage assume that the ask price is greater than or equal to the bid price, i.e. $z \geq z' \geq 0$. In other words it means that the asset price corresponds to the interval $[z', z]$.

For d different assets, their bid and ask prices are denoted respectively z'_i and z_i, where $z_i \geq z'_i \geq 0$ for all $i = 1, \dots, d$. They can be described geometrically as a parallelepiped in $\mathbb{R}^d$ with sides equal to the transaction costs (bid-ask spreads) of each stock. This parallelepiped is referred to as a *price set*.

A combined position in d assets (combination) is given by a vector $u = (u_1, \dots, u_d) \in \mathbb{R}^d$, where $u_i > 0$ means a long position and $u_i < 0$ means a short position in the correspondent asset. Thus a purchase of u_i units of the ith asset costs $u_i z_i$ and a sale of u_i units costs $-u_i z'_i$. The price of a combination u is denoted by $p(u)$. One of the typical conditions (axioms) imposed on the price functional is its sublinearity meaning that $p(u + u') \leq p(u) + p(u')$ and $p(\lambda u) = \lambda p(u)$ for every $\lambda > 0$, see [11]. The sublinearity means that the price of bought together assets is at most the sum of their individual asset prices.

It is known from convex analysis [20, p. 28] that any sublinear function corresponds to the support function of a certain convex set. Thus the price functional $p(u)$ can be represented as a support function of a convex set Z in $\mathbb{R}^d$, i.e.

$$p(u) = h(Z, u) = \sup\langle Z, u\rangle\,, \tag{2.1}$$

where $h(Z, u)$ is the support function of Z and

$$\langle Z, u\rangle = \{\langle z, u\rangle :\ z \in Z\}\,,$$

see Figure 2.1. Further we call Z a *price set*. It is always assumed that Z is a convex subset of $(0, \infty)^d$, which is guaranteed by requiring that $p(u) > 0$ and $p(-u) < 0$ for every $u = (u_1, \dots, u_d) \in \mathbb{R}^d_+ = [0, \infty)^d$ with at least one strictly positive coordinate.

The price set Z can be inscribed into a parallelepiped determined by bid-ask spreads of individual assets. The price set Z has a simple economical interpretation in terms of discounts that may be offered when trading related or linked assets. The sublinearity condition means that combinations of several assets can be bought at a lower price than the sum of ask prices of all individual assets and can be sold at a higher price than the sum of their bid prices. We call this effect *link-save*.

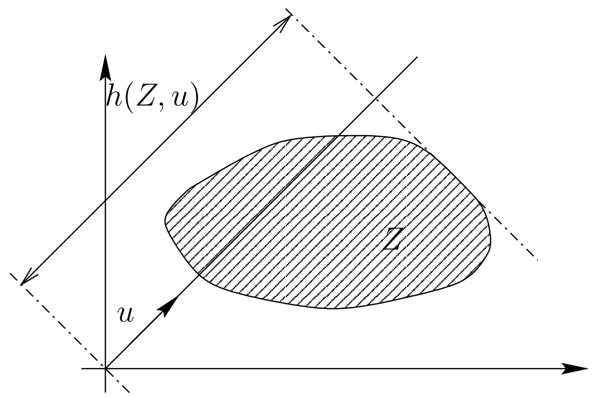


Figure 2.1: Support function of Z in direction u.

Example 2.1. If there is no link-save effect for two assets, then the price set is a rectangle, and the price of combination $u = (1, 1)$ is $p(1, 1) = z_1 + z_2$. In case of a possible link-save offers, the price set Z becomes a subset of the rectangle. Then $p(1, 1) = h(Z, (1, 1)) = z_1^B + z_2^B \le z_1 + z_2$, see Figure 2.2.

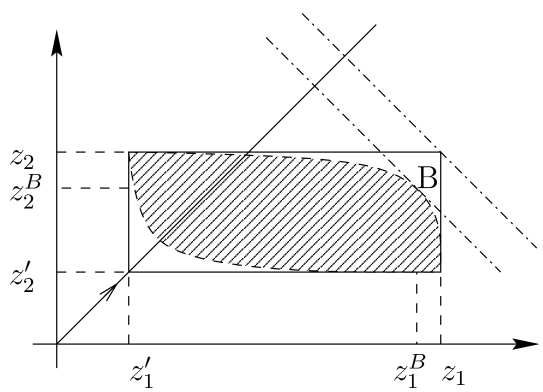


Figure 2.2: Price set for two assets in case of link-save discounts.

Example 2.2. If the price set Z is a ball in $\mathbb{R}^d$, see Figure 2.3 for $d = 2$, then the price of the combination u equals

$$p(u) = h(Z, u) = \langle z_0, u \rangle + R\|u\| ,$$

where z_0 is the centre of the ball. Therefore, the price $p(u)$ is the sum of the linear part and the term proportional to the transaction volume $\|u\|$.

Example 2.3. If the price set is a segment $[Z', Z'']$, see Figure 2.4, then most combinations are bought at the ask price for one asset and the bid

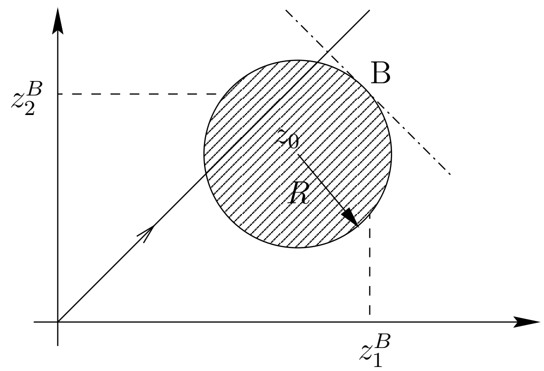


Figure 2.3: Price set Z is a ball, $d = 2$.

price for the other, i.e. for combination $\bar{u} = (\bar{u}_1, \bar{u}_2)$ the price is $p(\bar{u}) = h(Z, \bar{u}) = \bar{u}_1 z_1 + \bar{u}_2 z_2'$ and for combination $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ the price is $p(\tilde{u}) = h(Z, \tilde{u}) = \tilde{u}_1 z_1' + \tilde{u}_2 z_2$. If the vector $u = (u_1, u_2)$ is perpendicular to the segment, then the price of this combination stays unchanged when the first asset is bought at its bid price and the second at its ask price and vice versa, i.e. $p(u) = h(Z, u) = u_1 z_1' + u_2 z_2 = u_1 z_1 + u_2 z_2'$.

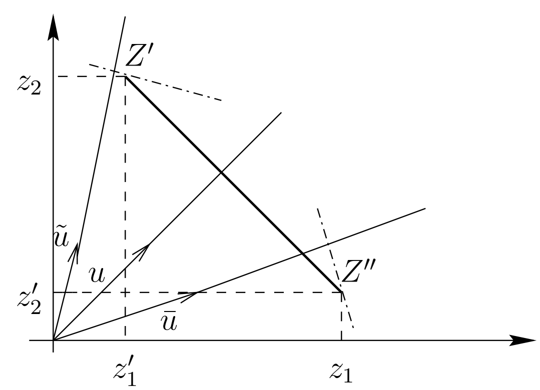


Figure 2.4: Price set is a segment.

Kabanov [14] and Schachermayer [21] described the transaction costs involved into exchange of d assets by a bid-ask matrix $\Pi = (\pi^{ij})_{ij=1}^d$, where π^{ij} is the number of units of asset i for which an agent can buy one unit of asset j. It is possible to represent their model using price sets if every exchange contact between two pairs of assets is considered separately and so the price set becomes a parallelepiped in $(d^2 - d)$-dimensional space with coordinate projections given by $[\frac{1}{\pi^{ji}}, \pi^{ij}]$ with $1 \leq i < j \leq d$.

In the other direction, it possible to represent the price set $Z = \{1\} \times [z'_2, z_2] \times \cdots \times [z'_d, z_d]$ using the bid-ask matrix

$$\pi^{ij} = \frac{z_i}{z'_j}\,,$$

where $z_1 = z'_1 = 1$. However, this matrix differs from those considered in [14, 21], since its diagonal entries π^{ii} are not necessarily equal to 1.

In the stochastic framework the prices are random and the price set Z becomes a *random convex compact set*. Fix the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. A random closed convex set is a random element $Z : \Omega \to \mathcal{C}$ defined on $(\Omega, \mathfrak{F}, \mathbf{P})$ with values in the family $\mathcal{C}$ of convex compact sets in $\mathbb{R}^d$, see [17, 22]. It is $\mathfrak{F}$-measurable in the sense that $\{Z \cap K = \emptyset\} \in \mathfrak{F}$ for each compact set K.

Definition 2.1. A random convex compact set Z is said to be integrably bounded if $\mathbf{E}\|Z\| < \infty$ and square integrable if $\mathbf{E}\|Z\|^2 < \infty$, where $\|Z\| = \sup\{\|x\| :\ x \in Z\}$ is the norm of Z.

3 Measure-valued portfolios

For several assets, a portfolio is defined to be a pair (θ, θ') of vectors $\theta = (\theta_1, \dots, \theta_d)$ and $\theta' = (\theta'_1, \dots, \theta'_d)$ that represent correspondingly the amounts of assets in long and short positions, see [11]. The following definition extends the concept of a portfolio for the case of link-save trading.

Definition 3.1. A portfolio θ is a measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Note that by a measure we always understand a non-negative measure and use the term signed measure otherwise.

Since the portfolio is a measure on $\mathbb{S}^{d-1}$ and the price of the combination u is the support function of the price set Z, the price of the portfolio can be expressed as

$$\langle Z, \theta\rangle = \int\limits_{\mathbb{S}^{d-1}} h(Z, u)\theta(du)\,. \tag{3.1}$$

Example 3.1. Let ask and bid prices of the assets be given by $z = (z_1, \dots, z_d)$ and $z' = (z'_1, \dots, z'_d)$ respectively. In the absence of link-save effect the vectors $\theta = (\theta_1, \dots, \theta_d)$ and $\theta' = (\theta'_1, \dots, \theta'_d)$ representing the portfolio correspond to an atomic measure on $\{\pm e_i :\ i = 1, \dots, d\}$ where $e_1, \dots, e_d$ are basis vectors in $\mathbb{R}^d$. The price of a portfolio is given by $\sum_{i=1}^d (\theta_i z_i - \theta'_i z'_i)$. Note that the corresponding atomic measure θ is non-negative, so that assets in short positions are designated by assuming that θ attaches non-negative weights to some set of points with negative coordinates. If Z is a rectangle that using a general measure θ as a portfolio is not different from representing the portfolio as two vectors θ and θ'.

Within the stochastic framework a portfolio becomes a *random measure*, see [22]. Note that a random measure θ is a random element whose values are measures. Its measurability is understood in the sense that $\theta(B)$ is a random variable for every Borel set B.

4 Cones associated with price sets

The following cones in the space of measures on $\mathbb{S}^{d-1}$ associated with the price set are defined similarly to [21], where these cones are defined in the Euclidean space.

Definition 4.1. For a given price set Z, define

- the solvency cone $K(Z)$, which is the family of measures θ such that $\langle Z, \theta \rangle \geq 0$;
- the cone $K'(Z)$ of portfolios available at price zero, which is the family of measures θ such that $\langle Z, \theta \rangle \leq 0$;
- the space of portfolios exchangeable to zero, $F(Z) = K(Z) \cap K'(Z)$.

Note that the cone $K'(Z)$ is not symmetric to $K(Z)$ in the usual sense, since all measures θ in $K(Z)$ are non-negative. However, every measure $\theta \in K'(Z)$ turns into a measure from $K(Z)$ by a symmetric transform of its arguments, i.e. subsets of the unit sphere. Note that the portfolios and the corresponding cones represent the physical quantities of the assets and their combinations.

Definition 4.2. The price system consistent with the price set Z is the cone $K^*(Z) \subset \mathbb{R}^d$ polar to $K'(Z)$, i.e.

$$K^*(Z) = \{w \in \mathbb{R}^d : \langle w, \theta \rangle \leq 0 \text{ for all } \theta \in K'(Z)\}. \tag{4.1}$$

Note that

$$\langle w, \theta \rangle = \int_{\mathbb{S}^{d-1}} \langle w, u \rangle \theta(du) = \langle w, \bar{\theta} \rangle$$

where

$$\bar{\theta} = \int_{\mathbb{S}^{d-1}} u\theta(du) \tag{4.2}$$

is the total holding of θ. Therefore, (4.1) can be written as follows

$$K^*(Z) = \{w \in \mathbb{R}^d : \langle w, \bar{\theta} \rangle \le 0 \text{ for all } \theta \in K'(Z)\}\,. \tag{4.3}$$

Proposition 4.1. *For any price set Z, one has*

$$K^*(Z) = \{cz : z \in Z,\ c \ge 0\}\,. \tag{4.4}$$

Proof. If $Z = \{z\}$ is the singleton, then $K'(Z)$ consists of all θ's such that $\langle z, \bar{\theta} \rangle \le 0$. Then $K^*(\{z\}) = \{cz : c \ge 0\}$ indeed is the ray passing through z.

The support function of $z \in Z$ is smaller than the support function of Z, whence $\langle z, \theta \rangle \le \langle Z, \theta \rangle$ and $K^*(Z)$ contains Z and so the right-hand side of (4.4).

If $v \in \mathbb{R}^d$ is not equal to cz for some $z \in Z$, then there exists a hyperplane passing through the origin that separates v and the set given by the right-hand side of (4.4). If θ is concentrated on the v side of the hemisphere generated by this plane, then $\langle v, \theta \rangle \ge 0$ whereas $\langle Z, \theta \rangle \le 0$. Therefore, such v does not belong to $K^*(Z)$. □

Proposition 4.2. *For any price set Z, the polar cone to $K^*(Z)$ is the family of $x \in \mathbb{R}^d$ such that $h(Z, x) \le 0$. This cone coincides with the set of all $\bar{\theta}$ for $\theta \in K'(Z)$.*

Proof. It follows from (4.4) that the polar cone to $K^*(Z)$ is $\{x : h(Z, x) \le 0\}$. With any such x we can associate the atomic measure θ with mass $\|x\|$ located at $x/\|x\|$ such that $\bar{\theta} = x$ and $\langle Z, \theta \rangle = h(Z, x) \le 0$, whence $\theta \in K'(Z)$.

Now consider an arbitrary measure $\theta \in K'(Z)$. By the sublinearity property of the support functions

$$h(Z, x) = h(Z, \int u\theta(du)) \le \int h(Z, u)\theta(du) \le 0\,,$$

where $x = \bar{\theta}$. □

Clearly, many convex sets Z share the same cone $K^*(Z)$. However, the correspondence is unique if we assume that the projection of Z onto one (say, the first) coordinate is $\{1\}$, that is

$$\langle Z, (1, 0, \ldots, 0) \rangle = \{1\}\,.$$

In the financial setting, this assumption is quite natural if the first coordinate of Z represents the bond.

5 Time-dependent trading

While the previous sections deal with the static case, the time and dynamics are crucial parts of financial conception. We consider a multiperiodic economy in which agents can trade a finite number of securities at discrete times $t = 0, 1, \dots, T$. For that reason, consider alongside with the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ a filtration, i.e. a nondecreasing right-continuous family $(\mathfrak{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathfrak{F}$. The random price set Z and the portfolio θ may also depend on time t.

Definition 5.1. A set-valued price process is a function $Z(t)$, $t \geq 0$, such that $Z(t)$ is $\mathfrak{F}_t$-measurable random convex compact subset of $(0,\infty)^d$ for each t.

Particularly important set-valued processes are set-valued martingales. The conditional expectation of $Z(t)$ given $\mathfrak{F}_s$, $s \leq t$, is defined as $\mathfrak{F}_s$-measurable random closed set $Y = \mathbf{E}(Z(t)|\mathfrak{F}_s)$ such that $\mathbf{E}(h(Z(t),u)|\mathfrak{F}_s) = h(Y,u)$ for every u from the unit sphere, see [9]. The process $Z(t)$ is called a *set-valued martingale* if $\mathbf{E}(Z(t)|\mathfrak{F}_s) = Z(s)$ whenever $s \leq t$. This is equivalent to the fact that $h(Z,u)$ is a martingale for every direction u.

The price of portfolio $\theta(t)$ for the price set $Z(t)$ can be calculated by (3.1). Since the short and long positions are treated separately for measure-valued portfolios, it is natural to assume that $\theta(t)$ represents the cumulative effect of trading up to time t, i.e. $\theta(\cdot)$ is monotonic in the sense that $\theta(t) - \theta(s)$ is a non-negative measure for all $s \leq t$. This corresponds to the conditions imposed on vector-valued portfolios in [11]. A *simple strategy* is a monotonic family $\theta(t)$, $t = 0, 1, \dots, T$, of measures on $\mathbb{S}^{d-1}$.

This monotonicity condition can be relaxed by requiring that the signed measure $\Delta\theta = \theta(t) - \theta(s)$ satisfies

$$\int h(K,u)\Delta\theta(du) \geq 0 \tag{5.1}$$

for every convex set K that contains the origin. Condition (5.1) is clearly satisfied if $\theta(t) - \theta(s)$ is a non-negative measure for all $s \leq t$. Note that for $Z(t)$ being a parallelepiped, (5.1) is equivalent to the coordinate-wise monotonicity of $(\theta_1(\cdot), \dots, \theta_d(\cdot))$ and $(\theta'_1(\cdot), \dots, \theta'_d(\cdot))$.

The following example recalls the definition of the self-financing strategy for a single asset with a bid-ask spread.

Example 5.1. Assume that there is only one asset with the ask price $z(t)$ and the bid price $z'(t)$ and the corresponding strategies $\theta(t)$ and $\theta'(t)$ that determine the amounts of this asset in long and short positions respectively. Following [11] the self-financing strategy is defined to satisfy

$$(\theta(t)-\theta(s))z(t) \leq (\theta'(t)-\theta'(s))z'(t)$$

for any $t \geq s$. After components displacement one gets

$$\theta(t)z(t)-\theta'(t)z'(t) \leq \theta(s)z(t)-\theta'(s)z'(t)\,, \tag{5.2}$$

where the left part is a cost of the portfolio at time t in prices at the same time moment, while the right part of the inequality represents the cost of the portfolio at time s but in prices of the present time t.

Writing down the left and the right parts of inequality (5.2) for general measure-valued portfolios justifies the following definition.

Definition 5.2. The *self-financing* portfolio process is a family of non-negative random measures (portfolios) $\theta(t)$ such that

(i) $\theta(t)$ is $\mathfrak{F}_t$-adapted, $t = 0,\dots,T$;

(ii) for all $0 \leq s \leq t \leq T$

$$\langle Z(t), \theta(t)\rangle \leq \langle Z(t), \theta(s)\rangle\,. \tag{5.3}$$

For $t = 0,\dots,T$ let $\Theta_t(x)$ be the convex cone in the family of random measures on $\mathbb{S}^{d-1}$ formed by $\theta(t)$ for all self-financing simple portfolio processes satisfying $\langle Z(0), \theta(0)\rangle \leq x$. Furthermore, let $A_t(x)$ be the (possibly random) cone in $\mathbb{R}^d$ that consists of all vectors v such that $v \leq \bar{\theta}$ coordinatewisely for some $\theta \in \Theta_t(x)$. Write A_t instead of $A_t(0)$.

The self-financing condition (5.3) means that for each $t = 1,\dots,T$ the increment $\theta(t)-\theta(t-1)$ belongs to the cone $K'(Z(t))$ of portfolios available at price zero.

Example 5.2. Let $Z(0) = \{1\}$ and $Z(1) = [0.5, 2]$. Then $\theta(0) = 3$, $\theta'(0) = 5$ and $\theta(1) = 2$, $\theta'(1) = 1$ is a non-monotonic sequence of portfolios that is however self-financing. Then $\langle Z(1), \theta(1)\rangle = 3.5$, while $\langle Z(0), \theta(0)\rangle = -2$, which means that it is possible to obtain a positive profit with a negative initial investment. It is easy to see that if $\theta(1)$ is coordinatewisely larger than $\theta(0)$, then this arbitrage opportunity disappears.

6 No-arbitrage and martingale selections

The following definition follows [21]. Note that $L^0(\Omega, \mathfrak{F}_t, \mathbf{P}; K)$ denotes the family of all d-dimensional random vectors with values in $K \subset \mathbb{R}^d$.

Definition 6.1. The price process $Z(t)$ satisfies no arbitrage property if

$$A_T \cap L^0(\Omega, \mathfrak{F}_T, \mathbf{P}; \mathbb{R}^d_+) = \{0\}\,. \tag{6.1}$$

The equation (6.1) asserts that if we have a self-financing portfolio v which is non-negative at time T almost surely then its value identically equals zero. This describes the impossibility to gain profit without investment.

Definition 6.2. The price set $\tilde{Z}$ is smaller than Z if $\tilde{Z} \subset Z$ and $h(\tilde{Z}, u) < h(Z, u)$ for all u such that the width

$$b(Z, u) = h(Z, u) - h(Z, -u)$$

is positive.

The set-valued price process $\tilde{Z}(t)$ is smaller then the set-valued process $Z(t)$ if $\tilde{Z}(t)$ is almost surely smaller then $Z(t)$ for all $t = 0, \dots, T$. The following definition describes the *robust no arbitrage* condition, first introduced in [21] for prices determined by cones.

Definition 6.3. The set-valued price process $Z(t)$ satisfies the *robust no arbitrage* condition if there exists a smaller set-valued price process $\tilde{Z}(t)$, such that $\tilde{Z}(t)$ satisfies the no arbitrage condition.

It is straightforward that in the perfect market conditions of no arbitrage and robust no arbitrage are equivalent, since single-valued price process $z(t)$ and the process $\tilde{z}(t)$ which is smaller than $z(t)$ coincide. The idea of the robust no-arbitrage assumes that in spite that some discounts for asset prices are possibly already offered there is still some opportunity for a broker to introduce greater discounts for prices of some trading assets without violating the no-arbitrage conditions.

A single-valued $\mathfrak{F}_t$-adapted process $z(t)$, such that $z(t) \in Z(t)$ (resp. $z(t)$ belongs to the relative interior of $Z(t)$) for all t, is called a (strict) *selection* and it is called a (strict) *martingale selection* if z itself is an $\mathfrak{F}_t$-martingale. The strict martingale selections appear in [21] under the name of *strictly consistent price process*. The following result shows that the existence of the strict martingale selection for the price process $Z(t)$ characterises the robust no-arbitrage condition. The key argument is that the robust no arbitrage condition implies that the cone A_T is closed in L^0 with respect to convergence in probability.

Theorem 6.1. *Assume that the first asset is a numéraire, meaning that the projection of $Z(t)$ onto the first coordinate is $\{1\}$, i.e. $\langle Z(t), (1, 0, \ldots, 0)\rangle = 1$ for all t. A set-valued price process $Z(t)$ satisfies the robust no arbitrage condition if and only if it admits a strict martingale selection $z(t)$.*

Proof. The fact that the set-valued price process $\tilde{Z}$ is smaller than Z is equivalent to the fact that $K^*(\tilde{Z})$ is contained in the relative interior of the cone $K^*(Z)$. This is essentially the robust no arbitrage condition from [21]. Theorem 1.7 of [21] establishes equivalence of this concept to the existence of a martingale z that takes values in the relative interior of $K^*(Z)$. Since the first coordinate of Z is $\{1\}$, the expectation of the first coordinate z_1 of z is 1, and we cane define a new probability measure $\tilde{P}$ that has z_1 as the Radon-Nikodym density with respect to $\mathbf{P}$. Then the process $\tilde{z} = z/z_1$ becomes a martingale with respect to $\tilde{\mathbf{P}}$, while by construction $\tilde{z}$ belongs to the relative interior of Z. ☐

7 Claims and price functionals

Let $L^0 = L^0(\Omega, \mathfrak{F}, \mathbf{P}; \mathbb{R}^d)$ be the family of d-dimensional random vectors defined on $(\Omega, \mathfrak{F}, \mathbf{P})$. For event $B \in \mathfrak{F}$, 1_B denotes the corresponding indicator random variable, i.e. 1_B equals 1 if B occurs and vanishes otherwise. A *claim* on several assets is a random vector $C = (C_1, \ldots, C_d) \in L^0$ whose elements determine the quantities of particular assets to be included in the claim. It is assumed that the claim can be exercised only at time T.

A portfolio θ is said to *superreplicate* (realise or dominate) the claim C if

$$\bar{\theta} = \int_{\mathbb{S}^{d-1}} u\theta(du) \geq C \tag{7.1}$$

coordinatewisely, i.e. the total holding of θ suffices to pay the claim C. For a single asset and portfolio (θ, θ') representing its amounts in long and short positions, inequality (7.1) turns into $\theta - \theta' \geq C$. Note that the cone $A_t(x)$ from Definition 5.2 consists of all claims that can be paid with a portfolio that satisfies the self-financing condition and with initial value at most x.

Define the price functional π for every contingent claim C by analogy with [11] as

$$\pi(C) = \inf \left\{ \langle Z(0), \theta(0)\rangle : \int u\theta(T)(du) \geq C \right\},$$

where the infimum is taken over all self-financing simple strategies $\theta(t)$, $t = 0, 1, \dots, T$, such that

$$\int u\theta(T)(du) \geq C\,. \tag{7.2}$$

In other words,

$$\pi(C) = \inf\{x :\ C \in A_T(x)\}\,.$$

The functional π represents the minimum cost necessary to replicate the contingent claim C at the final moment T, see (7.1). Recall that simple strategies are non-decreasing by definition.

The no-arbitrage condition in this framework can be formulated as $\pi(C) > 0$ for all $C \in L^0(\Omega, \mathfrak{F}, \mathbf{P}; \mathbb{R}^d_+)$. This would correspond to the fact that the closure of $A_T = A_T(0)$ intersects $\mathbb{R}^d_+$ only at the origin.

In the following we restrict attention to the square integrable case. Consider from now on the family $L^2 = L^2(\Omega, \mathfrak{F}, \mathbf{P})$ of square integrable claims. The price of a claim C is a functional on L^2 denoted by $p(C)$ with values in $[0, \infty]$. The *admissible* price functionals are described in [11] by the following properties.

(A1) The price functional is *sublinear*, i.e.

$$p(C + C') \leq p(C) + p(C'),$$
$$p(\lambda C) = \lambda p(C)$$

for all claims C and C' and $\lambda \geq 0$.

(A2) The price functional is *lower-semicontinuous*, i.e. if a sequence $\{C_n, n \geq 1\}$ converges to C in L^2, i.e. $\mathbf{E}\|C_n - C\|^2 \to 0$, then $p(C) \leq \lim p(C_n)$.

(A3) The price functional p *induces no arbitrage*, i.e. $p(C) > 0$ if $C = (C_1, \dots, C_d) \in \mathbb{R}^d_+$ almost surely and with a positive probability at least one coordinate of C is positive.

(A4) $p(C) \leq \pi(C)$ for all $C \in L^2$.

The sublinearity property of the price functional implies that p is the support function of a compact convex set in L^2.

Consider a set-valued price process $Z(t)$, $t = 0, 1, \dots, T$, adapted to filtration $\mathfrak{F}_t$. The following result shows that the existence of the martingale selection for the price process $Z(t)$ is equivalent to the existence of an admissible price functional.

Theorem 7.1. *(**i**) There exists an admissible price functional p if and only if there exists a stochastic process $z^*(t)$ such that $z^*(t) \in Z(t)$ for all $t = 0, \dots, T$ and $z^*(t)$ is a $\mathbf{P}^*$-martingale for a measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that the Radon-Nikodym derivative $\rho = d\mathbf{P}^*/d\mathbf{P}$ is square-integrable and also $\mathbf{E}\|\rho z^*(T)\|^2 < \infty$.*
*(**ii**) If p satisfies conditions (A1)–(A4) then*

$$(7.3) \qquad p(C) \in \; [\inf \mathbf{E}^*\langle z^*(T), C\rangle, \sup \mathbf{E}^*\langle z^*(T), C\rangle] = [-p^*(-C), p^*(C)]$$

for all contingent claims C. The infimum and supremum in (7.3) are taken over all expectation operators $\mathbf{E}^*$ associated with probability measure $\mathbf{P}^*$ and all $\mathbf{P}^*$-martingales $z^*(t)$ such that $z^*(t) \in Z(t)$ for all $t = 0, \dots, T$. The functional

$$p^*(C) = \sup \mathbf{E}^*\langle z^*(T), C\rangle$$

is an admissible price functional.

Proof. (**i**) **Sufficiency.** Assume that there exists a martingale selection $z^*(t) \in Z(t)$, $t = 0, \dots, T$. Define the linear price functional by

$$p(C) = \mathbf{E}^*\langle z^*(T), C\rangle$$

for all C. This part of the proof aims to show that this price functional is admissible, i.e. it satisfies properties (A1)–(A4). It is easy to see that (A1) follows directly from the properties of expectation and scalar product. The lower semicontinuity of p required in (A2) follows from Fatou's lemma. Since $z^*(t)$ is a selection of $Z(t) \subset (0, \infty)^d$, we see that $p(C) > 0$ for a nonnegative C having at least one strictly positive coordinate on an event of a positive probability.

Consider a simple trading strategy θ with the trading dates satisfying $0 = t_0 \le t_1 \le \dots \le t_N = T$. Then

$$\mathbf{E}^*\Big[\langle z^*(t_n), \theta(t_n)\rangle - \langle z^*(t_n), \theta(t_{n-1})\rangle \,\Big|\, \mathfrak{F}_{t_{n-1}}\Big] = \mathbf{E}^*\Big[\langle z^*(t_n), \Delta\theta(t_n)\rangle \,\Big|\, \mathfrak{F}_{t_{n-1}}\Big]\,,$$

where $\Delta\theta(t_n) = \theta(t_n) - \theta(t_{n-1})$. By condition (ii) of Definition 5.2, $\theta(t_{n-1}) \preceq \theta(t_n)$. Since $0 \in Z(t_n) - z^*(t_n)$, the monotonicity condition yields

$$\mathbf{E}^*\Big[\langle Z(t_n) - z^*(t_n), \Delta\theta(t_n)\rangle \,\Big|\, \mathfrak{F}_{t_{n-1}}\Big] \ge 0,$$

By the self-financing assumption of Definition 5.2,

$$\mathbf{E}^*\Big[\langle Z(t_n), \Delta\theta(t_n)\rangle \,\Big|\, \mathfrak{F}_{t_{n-1}}\Big] \le 0\,.$$

Using the fact that z^* is a $\mathbf{P}^*$-martingale with respect to $\mathfrak{F}_t$ and the strategy $\theta(t)$ is self-financing one obtains

$$\begin{aligned}\mathbf{E}^*\Big[\langle z^*(t_n), \theta(t_n)\rangle \Big| \mathfrak{F}_{t_{n-1}}\Big] &\leq \mathbf{E}^*\Big[\langle z^*(t_n), \theta(t_{n-1})\rangle \Big| \mathfrak{F}_{t_{n-1}}\Big] \\ &\leq \langle z^*(t_{n-1}), \theta(t_{n-1})\rangle\,.\end{aligned}$$

By iterating and repeatedly using Definition 5.2, it is possible to deduce that

$$\mathbf{E}^*\langle z^*(T), \theta(T)\rangle \leq \langle z^*(0), \theta(0)\rangle \leq \langle Z(0), \theta(0)\rangle\,.$$

Thus

$$\begin{aligned}p(C) = \mathbf{E}^*\langle z^*(T), C\rangle &\leq \mathbf{E}^*\left[\int \langle z^*(T), u\rangle \theta(T)(du)\right] \\ &= \mathbf{E}^*\langle z^*(T), \theta(T)\rangle \leq \langle Z(0), \theta(0)\rangle\,.\end{aligned}$$

Taking infimum over all θ satisfying (7.2) we obtain that $p(C) \leq \pi(C)$. Therefore, $p(C)$ is an admissible price functional.

(**i**) **Necessity.** Assume that an admissible price functional exists. Define

$$\tilde{\pi}(C) = \inf\left\{\liminf_n \langle Z(0), \theta_n(0)\rangle : \int u\theta_n(T)(du) \geq C_n, C_n \to C\right\}. \tag{7.4}$$

Let Ψ be the set of positive linear forms on L^2. Consider the set M of all claims C such that $\tilde{\pi}(C)$ is finite. As shown in [12, Th. 2.1], there exists $\psi \in \Psi$ such that the restriction $\psi|_M$ of ψ onto M is dominated by $\tilde{\pi}$. According to the Riesz representation theorem applied to the space L^2, there exists $\rho = (\rho_1, \ldots, \rho_d) \in L^2$ such that

$$\psi(C) = \mathbf{E}\langle \rho, C\rangle$$

for all $C \in L^2$.

Let $\mathbf{P}^*(B) = \mathbf{E}[\rho_1 1_B]$ for B running over $\mathfrak{F}$. Since ψ is positive and linear it is easy to show that $\mathbf{P}^*$ is equivalent to $\mathbf{P}$. Furthermore, $\rho_1 \leq \|\rho\|$, whence $\rho_1 = d\mathbf{P}^*/d\mathbf{P}$ is square integrable. Since the first coordinate of Z is $\{1\}$, $\psi(1_\Omega, 0, \ldots, 0) \leq 1$ and $\psi(-1_\Omega, 0, \ldots, 0) \leq -1$, whence $\mathbf{P}^*(\Omega) = \mathbf{E}(\rho_1) = 1$.

Define $\mathbf{P}^*$-martingales $z_1^*, \ldots, z_k^*$ as the following conditional expectations

$$z_k^*(t) = \mathbf{E}^*[\rho_k/\rho_1 \big| \mathfrak{F}_t]\,, \quad k = 1, \ldots, d\,.$$

It remains to show that $z^*(t) = (z_1^*(t), \ldots, z_d^*(t)) \in Z(t)$ almost surely for every $t = 0, \ldots, T$.

Consider any $u \in \mathbb{S}^{d-1}$ and the claim $C = (C_1, \dots, C_d) = (-h(Z(t), u)1_B + u_1 1_B, u_2 1_B, \dots, u_d 1_B)$ for some $B \in \mathfrak{F}_t$. This contingent claim is duplicable, i.e. the combination $u = (u_1, \dots, u_d)$ is bought at the price $h(Z(t), u) - u_1$ and is paid with the corresponding amount of the bond at time t if $\omega \in B$. This strategy costs nothing, whence

$$\begin{aligned}
\mathbf{E}^* \left[\Big(- h(Z(t), u) + u_1 + \sum_{i=2}^{d} u_i \frac{\rho_i}{\rho_1} \Big) 1_B \right] &= \mathbf{E} \left[\Big(- \rho_1 h(Z(t), u) + \sum_{i=1}^{d} u_i \rho_i \Big) 1_B \right] \\
&= \mathbf{E}\langle \rho, C \rangle \\
&= \psi(C) \le \tilde{\pi}(C) \le \pi(C) \le 0 \,.
\end{aligned}$$

Therefore,

$$\mathbf{E}^* \left[\left(-h(Z(t), u) + \sum_{i=1}^{d} u_i \frac{\rho_i}{\rho_1} \right) 1_B \right] \le 0 \,,$$

whence

$$\mathbf{E}^* \left[\sum_{i=1}^{d} u_i \frac{\rho_i}{\rho_1} 1_B \right] \le \mathbf{E}^*[h(Z(t), u) 1_B]$$

for all $B \in \mathfrak{F}_t$ and $t = 0, \dots, T$. This yields

$$h(z^*(t), u) = \langle z^*(t), u \rangle \le h(Z(t), u)$$

for all unit vectors u, i.e. $z^*(t) \in Z(t)$ almost surely. Since, $Z(T)$ is a square integrable random set, $\mathbf{E}\|z^*(T)\|^2 \le \mathbf{E}\|Z(T)\|^2 < \infty$ and also $\mathbf{E}\|\rho_0 z^*(T)\|^2 < \infty$.

(**ii**) Theorem 2.2 in [12] implies that $\tilde{\pi}(C)$ equals the supremum of $\psi(C)$ over all $\psi \in \Psi$ satisfying $\psi|_M \le \tilde{\pi}$. The functional ψ is defined as $\psi(C) = \mathbf{E}\langle \rho, C \rangle$, whence

$$\psi(C) = \mathbf{E}^* \left[\sum_{i=1}^{d} \frac{\rho_i}{\rho_1} C_i \right] = \mathbf{E}^* \langle z^*(T), C \rangle \,.$$

Since p is an admissible price functional, $p \le \tilde{\pi}$. Applying this result to C and $-C$, we deduce that

$$p(C) \in [\inf \mathbf{E}^* \langle z^*(T), C \rangle, \sup \mathbf{E}^* \langle z^*(T), C \rangle] \,,$$

where the infimum and supremum are taken over all expectation operators $\mathbf{E}^*$ associated with a probability measure $\mathbf{P}^*$ and all $\mathbf{P}^*$-martingales $z^*(t)$ from part (**i**). The proof is finished by noticing that p^* satisfies (A1)–(A4). $\square$

Remark 1. The sufficiency in part **(i)** of Theorem 7.1 can be proved under relaxed assumptions similarly to the proof carried over in [6] for a single asset case. In particular, it suffices to assume that z^* is a supermartingale and the square integrability may be replaced by the absolute integrability condition.

It has been noticed in Section 3 that any admissible price functional corresponds to a convex set in L^2. The following result interprets this convex set. It follows directly from Theorem 7.1.

Corollary 7.2. *Under conditions of Theorem 7.1, any admissible price functional associated with a set-valued price process $Z(t)$ is dominated by the support function of the set $Z^*(T)$ of all martingale selections $z^* \in Z$ at time T.*

This result means that the family of martingale selections of a set-valued process plays a crucial role in defining admissible price functionals. The set of martingale selections itself forms a set-valued martingale $Z^*(t) \subset Z(t)$. In a view of this it is important to characterise the largest set-valued martingale contained in the price process.

8 Link-save functions

In practice it is important to have convenient and relatively simple ways of describing a price set. Naturally, it is impossible to provide (say, newspaper) quotes for all possible combinations of linked assets. Here we suggest describing prices of various combinations of assets by families of functions that are determined by bid and ask prices of individual assets and additional parameters that determine the amount of discount.

Following Section 2 the price set should be convex to avoid the possibility of arbitrage, hence the functions should determine a convex set inscribed into the parallelepiped given by the bid-ask prices of individual assets. For simplicity, we consider only the case of two related linked assets. We assume that if assets are kept in opposed positions then they are traded at their original bid and ask prices, while discounts are available if the short or long position is taken simultaneously in both assets. Below we suggest several templates of functions that may be used to determine the amount of link-save.

We assume that the discounts are symmetric for short and long positions in the both assets, so that if the upper bound of the price set is given by a function F_1, then the lower bound is determined by

$$F_2(s) = 1 - F_1(1 - s)\,.$$

In general, the lower (short link-save) function F_2 may be obtained using other parameters than the upper (long link-save) function F_1.

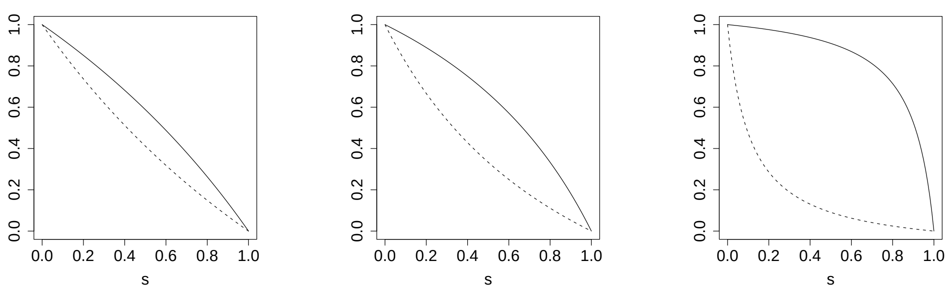
These functions are defined on the unit square and then can be adjusted to the general bid-ask prices by translations and rescaling. The rescaled link-save functions is given by

$$\bar{F}_i(y) = (z_2 - z_2')F_i((y - z_1')/(z_1 - z_1')) + z_2', \quad y \in [z_1', z_1], \; i = 1, 2 .$$

Example 8.1. Consider a family of functions based on the Hamacher parametric family described in [16] and given by

$$F_1(a, s) = \frac{1 - s}{1 - as},$$

where $a \in [0, 1]$ is a parameter that determines the shape of the function. Figure 8.1 shows how the shape of the price set depends on the parameter a. Smaller values of a correspond to larger discounts, while if parameter a tends to 1 links-save effect disappears.



(a) $a = 0.3$ (b) $a = 0.5$ (c) $a = 0.9$

Figure 8.1: Modified Hamacher family of functions: F_1 is a solid line and F_2 is a dashed line.

The price of a combination $u = (u_1, u_2)$ under the introduced link-save discount function is given by the support function of the price set. If $u_1, u_2 \geq 0$, then the point $z \in Z$ which maximises the scalar product $\langle z, u \rangle$ lies on the boundary formed by the upper link-save function, i.e. z has the coordinates $(y, \bar{F}_1(y))$. The value of y that maximises the scalar product $\langle z, u \rangle = u_1 y + u_2 \bar{F}_1(y)$ is

$$\tilde{y} = \frac{z_1 - z_1' + a z_1'}{a} - \frac{\sqrt{u_1 u_2 (1 - a)(z_1 - z_1')(z_2 - z_2')}}{a u_1}.$$

Thus, the support function is

$$h(Z,u) = \begin{cases} \langle \tilde{z}, u \rangle, & \tilde{y} \in [z_1', z_1]\,, \\ \max(u_1 z_1 + u_2 z_2', u_1 z_1' + u_2 z_2), & \text{otherwise}\,, \end{cases}$$

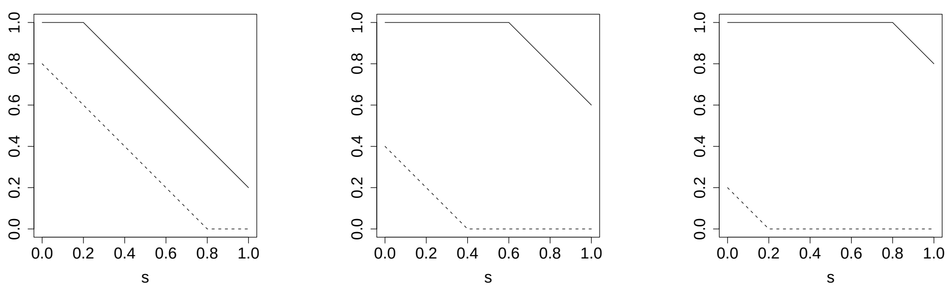
where

$$\langle \tilde{z}, u \rangle = u_1 \tilde{y} + u_2 \bar{F}_1(\tilde{y}) = u_1 \frac{z_1 - z_1' + a z_1'}{a} + u_2 \frac{z_2 - z_2' + a z_2'}{a} - \frac{2}{a}\sqrt{u_1 u_2 (1-a)(z_1 - z_1')(z_2 - z_2')}\,.$$

Example 8.2. The modified Lukasiewicz family of functions, see [16], depends on a single parameter $a \in [0,1]$ and is given by

$$F_1(a,s) = \min(1,\, 1 - s + a)\,.$$

Then $F_1(0,s) = 1 - s$ and $F_1(1,s) = 1$. Smaller values of a correspond to more substantial discounts being offered, see Figure 8.2.



(a) $a = 0.2$ (b) $a = 0.6$ (c) $a = 0.8$

Figure 8.2: Modified Lukasiewicz family of functions.

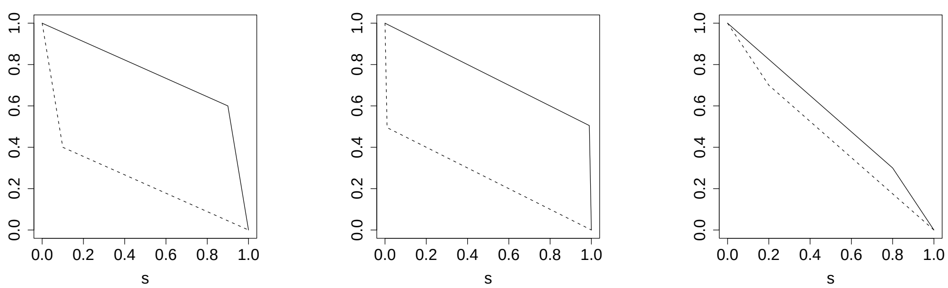
The value of support function of Z in the direction $u = (u_1, u_2)$, $u_1, u_2 \geq 0$ is

$$h(Z,u) = \begin{cases} u_1 z_1 + u_2(z_2' + a(z_2 - z_2')), & u_1/u_2 > (z_2 - z_2')/(z_1 - z_1')\,, \\ u_1(z_1' + a(z_1 - z_1')) + u_2 z_2, & \text{otherwise}\,. \end{cases}$$

Example 8.3. Another possible link-save function is given by

$$F_1(a,b,s) = \begin{cases} 1 - bs\,, & a = 0\,, \\ \min(1 - s\frac{b}{1-a}, \frac{1-b}{a} - s\frac{1-b}{a})\,, & a \in (0,1)\,, \\ 1 - s\,, & a = 1\,, \end{cases}$$

where a and b are two parameters from $[0,1]$ chosen to satisfy $a + b \leq 1$. Figure 8.3 shows several examples of functions from this parametric family. If both parameters tend to zero, the discounts disappear. If the sum of a and b approaches 1, the discounts become more substantial.



(a) $a = 0.1$, $b = 0.4$ (b) $a = 0$, $b = 0.5$ (c) $a = 0.2$, $b = 0.7$

Figure 8.3: A family of functions with two parameters.

The price process $Z(t)$ can be determined by means of given above link-save functions changing with time. It suffices to specify bid and ask price processes and the way the function's parameters change.

For more than two assets link-save functions can be defined using a template convex body K that is rescaled and translated to fit the parallelepiped formed by bid and ask prices of individual assets. An alternative way uses multivariate copulas, i.e. joint distribution functions with uniform marginals, see [19].

9 Examples

Consider the price set $Z(t)$ which is a set-valued martingale itself. In this case it is possible to find a dense set of martingale selections $z^*(t) \in Z(t)$, i.e. $Z(t)$

at any given t is the closure of the set of martingale selections, see [8, 9]. Since martingale selections are dense, the infimum and supremum of $\mathbf{E}^*\langle Z^*(T), C\rangle$ over all martingale selections z^* coincides with the expected values of the support function of $Z(T)$ with respect to the martingale measure, i.e.

$$p(C) \in [-\mathbf{E}^* h(Z(T), -C), \mathbf{E}^* h(Z(T), C)]\,.$$

The right-hand side can be expressed as $[h(\mathbf{E}^*(Z(T)), -C), h(\mathbf{E}^*(Z(T)), C)]$, where $\mathbf{E}^*(Z(T))$ is the selection (set-valued) expectation of the random compact set $Z(T)$ with respect to the martingale measure, see Aumann [2] and Artstein and Vitale [1].

Example 9.1. Let the price set $Z(t)$ be a Cartesian product of $\{1\}$ and the ball of radius $R(t)$ with centre at $(z_1(t), z_2(t))$. Assume that both $R(t)$ and $z(t) = (1, z_1(t), z_2(t))$ are martingales. Then

$$p(C) \in \Big[\langle \mathbf{E}^*[z_0(T)], C\rangle - \mathbf{E}^*[R(T)]\|C\|, \langle \mathbf{E}^*[z_0(T)], C\rangle + \mathbf{E}^*[R(T)]\|C\|\Big]\,.$$

For instance, the formula above is applicable for a call option on the weighted sum of several assets with payoff

$$\max(q_1S_1 + q_2S_2 - K, 0) = (q_1S_1 + q_2S_2 - K)_+\,,$$

where K is a strike price of the option, S_1, S_2 are spot prices of underlying assets and q_1, q_2 are constants. The claim in this case is

$$C = \Big(-K\mathbf{1}_{q_1S_1+q_2S_2>K}, q_1\mathbf{1}_{q_1S_1+q_2S_2>K}, q_2\mathbf{1}_{q_1S_1+q_2S_2>K}\Big)\,. \tag{9.1}$$

If the radius $R(t)$ is an integrable submartingale, then the Doob decomposition can be applied to represent $R(t)$ as a sum of a martingale $R_0(t)$ and a predictable increasing process $A(t)$, such that $A(0) = 0$. Then the above formulae can be applied to the set-valued process $Z^*(t)$ being the ball with the same centre as the ball $Z(t)$ but radius $R_0(t)$. Then Z^* is the largest set-valued martingale included in Z.

Example 9.2. Let $z_\sigma(t) = s_0 \exp\{\sigma W_t - \sigma^2 t/2\}$ be the geometric Brownian motion with volatility parameter σ. Assume that σ is unknown and the only information available is that σ belongs to an interval $[\sigma_1, \sigma_2]$. This situation has been described, for instance, in [4, 18]. In our framework, consider the set-valued price process

$$Z(t) = \{z_\sigma(t) :\ \sigma \in [\sigma_1, \sigma_2]\}$$

which is an interval $Z(t) = [z'(t), z(t)]$ formed by all possible values of z_σ over all admissible σ. Because the upper bound of $Z(t)$ is the supremum of martingales and the lower bound is the infimum of martingales, it is easily seen that $z(t)$ is a submartingale and $z'(t)$ is a supermartingale. One says that $Z(t)$ is a set-valued submartingale, since $\mathbf{E}[Z(t)|\mathfrak{F}_s] \supseteq Z(s)$ wherever $s \le t$.

It is possible to apply the Doob decomposition to $z'(t)$ and $z(t)$ separately. That is, $z(t) = M_t + A_t$ where M_t is a martingale and A_t is a predictable increasing process. Furthermore, $Z(t) = [M'_t, M_t] + [-A'_t, A_t]$, where $[M'_t, M_t]$ is a set-valued martingale and A'_t, A_t are predictable increasing processes.

In numerical evaluations, it is easy to obtain expressions for z' and z as minimum and maximum of z_σ over all possible σ. Then one can use a discretisation to calculate the ingredients of the Doob decomposition using sums of conditional expectations. For instance, if $t_k = n\Delta t$, $k = 0, \dots, n$, then

$$A_{t_n} = \sum_{k=1}^{n} \left(\mathbf{E}[z(t_k)|\mathfrak{F}_{t_{k-1}}] - z(t_{k-1})\right) .$$

Example 9.3. Consider two assets with bid ask prices $z'_i = y_i(1-\lambda_i)$ and $z_i = y_i(1+\lambda_i)$, $i = 1, 2$, and the link-save discount determined by the modified Hamacher family of functions $F(s)$ with parameter a, see Example 8.1. Assume that (y_1, y_2) is a martingale. Let $u_1, u_2 \ge 0$. To find the price of a combination $u = (u_1, u_2)$ we have to maximise $u_1 y_1(1+s) + u_2 F(y_1(1+s))$ for $s \in [-\lambda_1, \lambda_1]$. It is easily seen that the maximum is achieved at $s = s^*$, where

$$s^* = \begin{cases} -\lambda_1, & \tilde{s} \le -\lambda_1, \\ \tilde{s}, & -\lambda_1 < \tilde{s} < \lambda_1, \\ \lambda_2, & \tilde{s} \ge \lambda_1 , \end{cases}$$

and

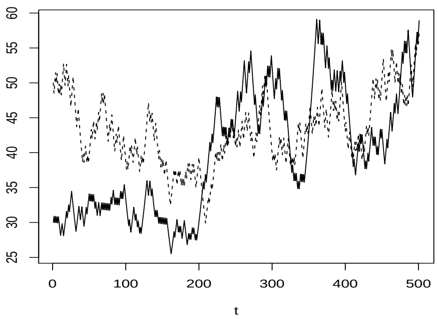
$$\tilde{s} = \frac{1}{a}\left[(2-a)\lambda_1 - 2\sqrt{\lambda_1\lambda_2 \frac{u_2}{u_1}\frac{y_2}{y_1}(1-a)}\right] .$$

Then

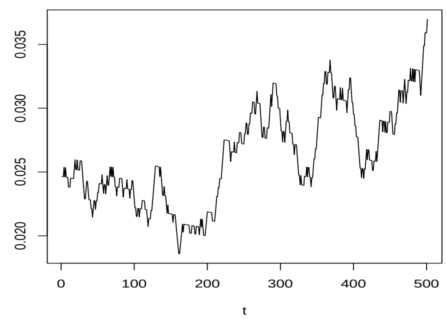
$$h(Z, u) = u_1 y_1 + u_2 y_2 + u_1 y_1 s^* + \lambda_2 u_2 y_2 \left[2\frac{\lambda_1 - s^*}{2\lambda_1 - a s^* - a\lambda_1} - 1\right] .$$

Since this price is obtained by maximisation of martingales, $h(Z(t), u)$ is a submartingale that can be decomposed into the sum of a martingale M_t^u and a predictable increasing process A_t^u, see Figure 9.1. The corresponding set-valued martingale is then given by

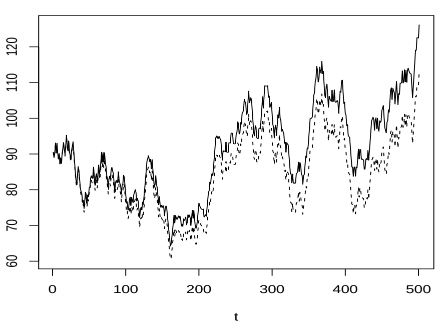
$$Z^*(t) = \{z : \langle z, u\rangle \le M_t^u,\ \|u\| = 1\} .$$



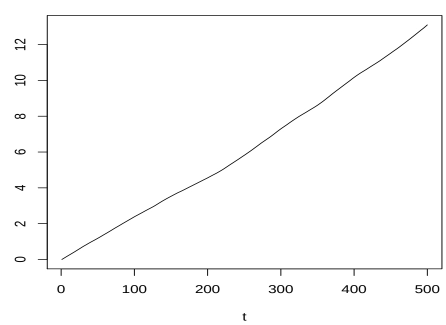
(a) Martingales y_1 (solid line) and y_2 (dashed line)



(b) $\mathbf{E}(h(Z(t_k),u)|\mathfrak{F}_{t_{k-1}}) - h(Z(t_{k-1}),u)$



(c) $h(Z(t),u)$ (solid line) and the martingale M_t^u (dashed line)



(d) The increasing process A_t^u

Figure 9.1: An example of the price process $h(Z(t),u)$ and the corresponding Doob decomposition for $u = (1,1)$, $\lambda_1 = \lambda_2 = 0.5$, $t = 0, \ldots, 500$ and $a = 0.5$.

References

[1] Artstein, Z., Vitale, R. A., 1975, A strong law of large numbers for random compact sets, Annals of Probability 3, 879–882.

[2] Aumann, R. J., 1965, Integrals of set-valued functions, Journal of Mathematical Analysis and Applications 12, 1–12.

[3] Avellaneda, M., Lawrence, P., 2000, Quantitative Modeling of Derivative Securities, Chapman & Hall/CRC, Boca Raton, FL.

[4] Avellaneda, M., Levy, A., Parás, A., 1995, Pricing and hedging derivative securities in markets with uncertain volatilities, Applied Mathematical Finance 2, 73–88.

[5] Bensaid, B., Lesne, J.-P., Pagès, H., Scheinkman, J., 1992, Derivative asset pricing with transaction costs, Mathematical Finance 2, 63–86.

[6] Bosi, G., Zuanon, M. E., 2001, Existence of price functionals with bid-ask spreads on the space of all integrable contingent claims, Unpublished manuscript.

[7] Harrison, M., Kreps, D., 1979, Martingales and arbitrage in multiperiod security markets, Journal of Economic Theory 20, 381–408.

[8] Hess, C., 1991, On multivalued martingales whose values may be unbounded: martingale selectors and Mosco convergence, Journal of Multivariate Analysis 39, 175–201.

[9] Hess, C., 1999, Conditional expectation and martingales of random sets, Pattern Recognition 32, 1543–1567.

[10] Hull, J. C., 2000, Options, Futures, & Other Derivatives, Prentice-Hall International Inc., Upper Sadle River, NJ.

[11] Jouini, E., 2000, Price functionals with bid-ask spreads: an axiomatic approach, Journal of Mathematical Economics 34, 547–558.

[12] Jouini, E., Kallal, H., 1995, Martingales and arbitrage in securities markets with transaction costs, Journal of Economic Theory 66, 178–197.

[13] Kabanov, Y., Rásonyi, M., Stricker, C., 2002, No-arbitrage criteria for financial markets with efficient friction, Finance and Stochastics 6, 371–382.

[14] Kabanov, Y. M., 1999, Hedging and liquidation under transaction costs in currency markets, Finance and Stochastics 3, 237–248.

[15] Kabanov, Y. M., Stricker, C., 2001, The Harrison-Pliska arbitrage pricing theorem under transaction costs, Journal of Mathematical Economics 35, 185–196.

[16] Klement, E. P., Mesiar, R., Pap, E., 2000, Triangular Norms, Kluwer Acad. Publ., Dordrecht.

[17] Matheron, G., 1975, Random Sets and Integral Geometry, Wiley, New York etc.

[18] Mykland, P. A., 2000, Conservative delta hedging, Annals of Applied Probability 10, 664–683.

[19] Nelsen, R. B., 1999, An Introduction to Copulas, vol. 139 of *Lecture Notes in Statistics*, Springer-Verlag, New York.

[20] Rockafellar, R. T., 1970, Convex Analysis, Princeton University Press, Princeton, NJ.

[21] Schachermayer, W., 2004, The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time, Mathematical Finance 14, 19–48.

[22] Stoyan, D., Kendall, W. S., Mecke, J., 1995, Stochastic Geometry and its Applications, Wiley, Chichester, 2nd edn.

[23] Wilmott, P., 1998, Derivatives, Wiley, Chichester.