CONVEXITY ADJUSTMENT AND DELIVERY OPTION IN AUSTRALIAN DOLLAR 90 DAY BILLS FUTURES

MARC HENRARD

Abstract. Australian dollar bills futures are very particular, not only on the valuation at expiry but also for the maturity delivery option and the credit delivery option. This note consider only the interest rate part of the futures (marginning and maturity delivery option). An explicit formula for the convexity adjustment realted to the marginning in the HJM gaussian model is proposed. The delivery option is also studied but found to be (almost) worthless.

1. Introduction

This note origins in the need to price Australian Dollar 90 Day Bank Accepted Bills Futures and the lack of litterature on the subject. The AUD dollar futures are significantly different from the eurodollar futures in several caracteristics and a specific treatment is required. The first section details the description of those futures. All the details relevant to the pricing are collected.

In the second section the price and convexity adjustment are detailed in absence of delivery option. In practice the delivery option is (almost) worthless and the delivery will always be in the longest maturity.

Nevertheless in the third section the delivery option is priced. An explicit formula is obtain, even in presence of more than two deliverables. As intuitively expected in absence of *convertion factor* the delivery will (almost) always in the longuest maturity. But we show in the examples that if such a factor was introduced the delivery option could play a significant role.

Note that the settlement is done in bank bills. The swaps fix against an interbank average rate. There is no garantee that those two rates move in parallel. Consequently hedging swap with futures exposes to a basis risk.

2. Bills Futures description

The *settlement* of the AUD bills futures is *physical*. It means that at expiry of the future one (or several) Bank Accepted Bills will be delivered against payment of a settlement amount computed from the last price of the futures. Several maturities and issuers can be delivered by the future short side. It means that the futures combine an interest rate option and a credit option. The *credit part of the futures is not investigated* in this note and only one yield curve (without spread) is used to price all the instruments. The insterest rate option could be viewed as equivalent to the delivery option for US Treasury futures. The difference is that the bills are discount instruments (without coupon), no accrued interest is taken into account and there is no convertion factor. As the settlement is done on the price (not the yield) it is always advantageous for the short position to deliver the longest maturity possible. This is always the cheapest bill as long as the forward rates (with maturity on the longest maturity) are positive. Even if the model allows for negative interest rates (Gaussian HJM) in a first stage, to simplify the treatment, this (very low probability) event is disregard. A further analysis of the delivery option will prove that it is really worthless.

Date: First version: 9 September 2005; this version: 22 September 2005.

Key words and phrases. Australian dollar bills futures, convexity adjustment, delivery option, HJM one-factor model.

JEL classification: G13, E43.

AMS mathematics subject classification: 91B28, 91B24, 91B70.

The *dates* involved in the futures are the following. The *settlement date* of the futures is the second Friday of the delivery month. That date is denoted t_1 and also called *start date* of the bill. The *last trading*, which determine the price used for the settlement, is the previous business day. We denote that date by t_0. The maturity date of the delivered bill can be any bills with maturity between 85 and 95 days. As explain in the previous paragraph the longest maturity available will be used. As the settlement date is on a Friday the 95 days maturity will be on a Tuesday. That date will be used for the maturity except when it falls on a holiday in which case the previous business day will be used[1].

The *quoted price* of the futures is used only as a conventional number for quotation. It is not used directly for marginning or for settlement payment. The quoted price P is calculated from the *rate* (or *yield*) R as

$$P = 1 - R. \tag{1}$$

The rate is the ACT/365 money-market rate computed from the bills *value* V by

$$V = \frac{1}{1 + R\frac{90}{365}}. \tag{2}$$

Note that the number of days to compute the rate is conventionaly set to 90 even is the delivered paper as a different number of days to maturity.

The value is the settlement amount for the delivered bill. It means that on the last trading date it coincide with the settlement price of the bill.

$$V_{t_0} = \frac{P(t_0, t_2)}{P(t_0, t_1)} \tag{3}$$

This is the forward value for settlement at t_1 of the zero-coupon bond maturing in t_2 as viewed from the fixing date t_0.

The marginning of the futures is done on a daily basis on its value as computed from the quoted price through Equations (1) and (2).

Due to the frequent (daily) marginning and the short time between the future price capture (evening) and the marginning payment (next day morning), we use a continuous marginning model with immediate payment.

The delivery is physical in a bank negotiable certificate of deposit (NCD). We estimate that the last fixing will be equivalent to the mid price of the cheapest to deliver bill on expiry date. The reasoning behind the mid is a standard one. A market maker would be happy to reduce its exposure at a mid-market level. If the fixing price was above the mid level, a market maker long the underlying would be ready to sell at that level. He could do so by shorting the future and immediately delivering the Bill. This would bring the price down.

From a technical perspective the only number we need to model is the value. The value at expiry is known through Equation 3. Using the generic future price process theorem [**?**, Theorem 12.6] we obtain the price relatively easily.

3. Model, hypothesis, and preliminary results

We model interest rate products. The base assets are $P(t, u)$, the price in t of the zero-coupon bond paying 1 in u. We work in a Heath-Jarrow-Morton [**?**] one factor model framework (see for example the chapter *Dynamical term structure model* in [**?**]). The function P is positive and regular enough so that it can be written as

$$P(t, u) = \exp\left(-\int_t^u f(t, s)ds\right).$$

Let $A = \{(s, u) \in \mathbb{R}^2 : u \in [0, T] \text{ and } s \in [0, u]\}$. We work in a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W)_{0 \leq t \leq T}$.

[1] But no such event is visible in the calendar in the coming years.

H: There exists $\sigma : [0,T]^2 \to \mathbb{R}^+$ measurable and boundedwith $\sigma = 0$ on $[0,T]^2 \setminus A$ such that for some process $(r_s)_{0\leq t\leq T}$, $N_t = \exp(\int_0^t r_s ds)$ forms with some measure $\mathbb{N}$ a numeraire pair(with Brownian motion W_t),

$$\begin{aligned} df(t,u) &= \sigma(t,u)\int_t^u \sigma(t,s)ds\, dt - \sigma(t,u)dW_t \\ dP^N(t,u) &= P^N(t,u)\int_t^u \sigma(t,s)ds\, dW_t \end{aligned}$$

and $r_t = f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

In the case of the extended Vasicek or Hull-White model, the volatility function is given by $\nu(s,t) = \frac{\sigma}{a}(1-\exp(-a(t-s)))$. The parameter σ can be constant or time dependent.

The proof of the following technical lemmas can be found in [**?**]. Similar formula can be found in [**?**, (3.3),(3.4)] in the framework of coherent interest-rate models.

Lemma 1. *Let $0 \leq t \leq u \leq v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)}\exp\left(\int_t^u (\nu(s,v)-\nu(s,u))\,dW_s - \frac{1}{2}\int_t^u \left(\nu^2(s,v)-\nu^2(s,u)\right)ds\right).$$

4. Futures pricing

Theorem 1. *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the HJM one-factor model, the value of the AUD futures fixing on t_0 for the period t_1–t_2 is given by*

$$V_t = \frac{P(t,t_2)}{P(t,t_1)}\tilde{\gamma}(t)$$

where

$$\tilde{\gamma}(t) = \exp\left(\int_t^{t_0} \nu(s,t_1)(\nu(s,t_1)-\nu(s,t_2))ds\right).$$

Note that $\tilde{\gamma}$ (convexity adjustment) defined here is different from the one used for eurodollar futures. The role of t_1 and t_2 are inverted. This is due to the fact that in one case the marginning is done on the rate and in the other on the value.

Proof. Using the generic pricing future price process theorem [**?**, Theorem 12.6],

$$V_t = \mathrm{E}_{\mathbb{N}}\left[V_{t_0}|\,\mathcal{F}_t\right].$$

Using Lemma 1 twice, we obtain

$$\frac{P(t_0,t_2)}{P(t_0,t_1)} = \frac{P(t,t_2)}{P(t,t_1)}\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_2)-\nu^2(s,t_1)ds + \int_t^{t_0}\nu(s,t_2)-\nu(s,t_1)dW_s\right).$$

Only the second integral contains a stochastic part. This integral is normally distributed of variance $\int_t^{t_0}(\nu(s,t_2)-\nu(s,t_1))^2ds$. So the expected value of the ratio of discount factors is reduced to

$$\frac{P(t,t_2)}{P(t,t_1)}\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_2)-\nu^2(s,t_1)ds + \int_t^{t_0}(\nu(s,t_2)-\nu(s,t_1))^2ds\right)$$

and we have the announced result. □

In the case of the extended Vasicek model, the adjustment factor can be written explicitly

$$\ln\tilde{\gamma} = \frac{\sigma^2}{2a^3}(\exp(-at_2)-\exp(-at_1))(\exp(at_0)-\exp(at))(2-\exp(-a(t_1-t_0))-\exp(-a(t_1-t))).$$

5. CHEAPEST TO DELIVER

As indicated in the futures description there is a delivery option embedded in the future. Even if the option is probably (almost) worthless in this section its exact value is computed as a confirmation. The valuation is done in a more general set-up that include a *conversion factor* between the different deliverables.

To obtain an explicit formula we add a *separability condition* on the volatility factor similar to the one used in [**?**], [**?**] and [**?**].

(H): The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive functions g and h.

The computation linked to the minimum describing the delivery option will involve exponential fucntions. To clarify the proof of the main theorem we start to prove an calculs lemma on those functions.

Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$ be an increasing sequence of positive numbers and $(c_i)_{1\leq i\leq m}$ be a sequence of positive numbers. The exponential functions studied are $f_i(x) = c_i \exp(\alpha_i x)$.

The equations $f_i(x) = f_j(x)$ ($0 \leq j < i \leq m$) have a unique solution

$$x_{i,j} = \frac{1}{\alpha_j - \alpha_i} \ln\left(\frac{c_i}{c_j}\right).$$

To simplify the notation we define $x_{m+1,m} = -\infty$ and $x_{1,0} = +\infty$.

A sequence of those solution is created. Let $k_0 = m+1$ and $k_1 = m$. The indices in the sequence are defined recursively by

$$k_l \quad \text{is the index such that} \quad x_{k_l,k_{l-1}} = \min_{j<k_{l-1}} x_{k_{l-1},j}.$$

The sequence is constructed up to the moment where $k_n = 1$.

Lemma 2. *The sequence of solution $(x_{k_{i+1},k_i})_{0\leq i\leq n}$ constructed above is such that*

$$f_{k_i}(x) = \min_{j=1,\ldots,m} f_j(x) \quad \textit{for} \quad x \in (x_{k_{i-1},k_i}, x_{k_i,k_{i+1}}).$$

Theorem 2. *Let $0 \leq t \leq t_0 \leq t_1$ and $t_1 < s_1 < \cdots < s_m$. In the HJM one-factor model where the voaltility satisfy the condition (H), the price of a future fixing in t_0 for which the delivery is done in t_1 with the choice of zero-bond with maturities s_i and convertion factor b_i (i.e. the short position gives*

$$\min_{j=1,\ldots,m} b_i \frac{P(t_0,s_i)}{P(t_0,t_1)}$$

against receiving the last future price V_{t_0}) is given in t by

$$V_t = \sum_{i=1,\ldots,n} \frac{P(t,s_{k_i})}{P(t,t_1)} b_{k_i}\gamma_{k_i}\left(N(x_{k_i,k_{i+1}} - \alpha_{k_i}) - N(x_{k_{i-1},k_i} - \alpha_{k_i})\right)$$

where

$${\alpha_j}^2 = \int_t^{t_0} (\nu(s,s_j) - \nu(s,t_1))^2 ds$$

and

$$\gamma_j = \exp\left(-\frac{1}{2}\int_t^{t_0} \nu^2(s,s_j) - \nu^2(s,t_1) ds\right).$$

Proof. Using the generic future price process theorem [**?**, Theorem 12.6],

$$V_t = \mathrm{E}_{\mathbb{N}}\left[V_{t_0}\middle|\,\mathcal{F}_t\right]$$

where

$$V_{t_0} = \min_{j=1,\ldots,n} b_j \frac{P(t_0,s_j)}{P(t_0,t_1)}.$$

The notation $c_j = b_j \frac{P(t,s_j)}{P(t,t_1)} \exp(-\frac{1}{2}\int_t^{t_0} \nu^2(s,s_j) - \nu^2(s,t_1)ds)$ is used in the sequel. Like in the previous proof, using Lemma 1 twice, we obtain

$$b_j \frac{P(t_0,s_j)}{P(t_0,t_1)} = c_j \exp\left(\int_t^{t_0} \nu(s,s_j) - \nu(s,t_1)dW_s\right).$$

The volatility factor being deterministic, the stochastic integral is normally distributed with variance α_j^2 and the previous term can be written as

$$b_j \frac{P(t_0,s_j)}{P(t_0,t_1)} = c_j \exp(\alpha_j X)$$

with X a normal random variable. Note that the normal variable X is the same for all j. The condition (H) was used to ensure that the the variable is the same for all j.

Thanks to the previous lemma,

$$V_{t_0} = f_{k_i}(X) \text{ for } X \in (x_{k_{i-1},k_i}, x_{k_i,k_{i+1}}).$$

Spliting the integral of the expected value on the different intervals we obtain the announced result. □

6. Example

6.1. Convexity adjustment. We provide an example of pricing. The future is the September 07 one. The *fixing date* is Thursday 13 September 2007, the *settlement date* is Friday 14 September 2007 and the (longest) *maturity date* is Tuesday 18 December 2007.

We use a flat yield curve on 15 September 2005 with 0/N, T/N, 1m, 3m, 6m, 9m, 12m, 2y and 3y equal to 6%. Up to 12m the rate are money market, ACT/365; the 2y and 3y are swap ACT/365 with quarterly payments; and no holidays are used. The curve constructed is suppose to be the appropriate one to value the bills futures.

The convexity adjustment using a mean reversion parameter $a = 0.01$ and a volatility $\sigma = 0.015$ is $\tilde{\gamma} = 0.999745$. The value is 98.4362%, which give a rate of 6.4429% and a quoted price of 93.5571%. For comparison the forward price is 98.4613% (a 10.5 bps difference).

6.2. Delivery option. When there are only two deliverable bills and they have maturity 10 December 2007 and 18 December 2007, the difference in price between the futures inclusing the delivery option and the one on the longest maturity bill is -0.016 bps. The risk neutral probability of the shortest maturity to be delivered is 0.99%.

6.3. Convertion factor. A artificial example with convertion factors is created to show the potential impact of the delivery option. The base for this example is the September 2007 future. The deliverable are seven deposits with maturities spaced by one week starting on 18 December 2007. The convesation factor for the first deposit is 1. For the other one are constructed with an adjustment $d_i = -1.75 - (i-1) * 0.25$ $(1 \leq i \leq 6)$ and

$$b_i = b_{i-1} \exp(d_{i-1}(\alpha_{i-1} - \alpha_i)).$$

Those numbers were chosen to have a *nice graph* (Figure 1) with the intersection $x_{i,j}$ nicely spaced between -0.65 and 0.65.

In Table 1 the figures related to the example are repoted. The second row contains the figures $N(x_{k_i,k_{i+1}} - \alpha_{k_i}) - N(x_{k_{i-1},k_i} - \alpha_{k_i})$. Their sum is in the last column. Note that the sum is slightly below 1. The side long the future will receive the worst of the bills, and shoudl pay less than the price for any of them. The convertion factor is reported in the third row. The factors $P(t,s_{k_i})/P(t,t_0)\gamma_i$ are in the last row. The futures value with the delivery option is 0.983107.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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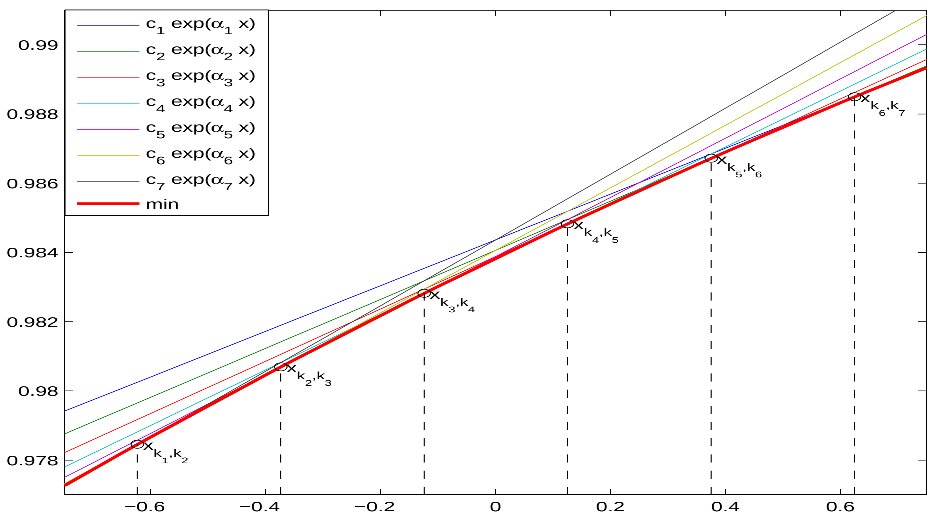

FIGURE 1. Delivery option: minimum f_j function in relation to normally distributed X.

Result	1	2	3	4	5	6	7	
N	0.263399	0.087353	0.096075	0.099296	0.096437	0.088012	0.268357	0.998929
Conv. fact.	1.000000	1.000856	1.001836	1.002938	1.004165	1.005515	1.006990	
Coef.	0.984362	0.984061	0.983881	0.983820	0.983880	0.984060	0.984360	

TABLE 1. Delivery option with convertion factor example.

DERIVATIVES GROUP, BANKING DEPARTMENT, BANK FOR INTERNATIONAL SETTLEMENTS, CH-4002 BASEL (SWITZERLAND)

E-mail address: Marc.Henrard@bis.org