How Equilibrium Prices Reveal Information in Time Series Models with Disparately Informed, Competitive Traders

Todd B. Walker [*]
University of Iowa

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Abstract

Accommodating asymmetric information in a dynamic asset pricing model is technically challenging due to the problems associated with higher-order expectations. That is, rational investors are forced into a situation where they must forecast the forecasts of other agents (i.e., form higher-order expectations). In a dynamic setting, this problem telescopes into the infinite future and the dimension of the relevant state space approaches infinity. By employing the frequency domain approach of Whiteman (1983) and Kasa (2000), this paper demonstrates how information structures previously believed to lead to disparate expectations in equilibrium (e.g., Singleton (1987)) converge to a symmetric equilibrium. The "revealing" aspect of the price process lies in the invertibility of the observed state space, which makes it possible for agents to infer the economically fundamental shocks, thus eliminating the need to forecast the forecasts of others.

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1 Introduction

In his seminal work, Keynes (1936) likened investment decisions to beauty contests in which competitors had to select the "prettiest faces" from a hundred photographs. Keynes argued that competitors will not pick the faces they find most attractive but will be forced into a situation where they must guess what average opinion expects the average opinion to be; that is, they must form higher-order expectations. This scenario accurately describes the behavior of differentially informed investors in a simple asset pricing model. Traders recognize that information disseminated through current and past prices reflects not only the information endowment of other investors, but also their attempts to forecast the forecasts of other traders. The main result of this paper is to model explicitly how information is disseminated through prices, and to show when this dissemination becomes too widespread to maintain asymmetric information in equilibrium. Models previously believed to impose asymmetric information in equilibrium [e.g., Singleton (1987)] are shown to degenerate into symmetric information, representative agent models.

The empirical failures of representative agent models have inspired a literature that introduces information heterogeneity and higher-order expectations in hopes of reconciling theory with data.[1] However, the computational problems associated with heterogeneous information have long proven to be a significant impediment to solving dynamic, intertemporal trading models with differentially informed investors. The crux of the problem is how to model higher-order expectations. Investors who anticipate what average opinion expects the average opinion to be form, what Keynes called, third-degree expectations. Due to the complexities of the fourth degree, Phelps (1983) quipped "one gets a vertiginous feeling, the eyes dull, and the face goes slack." Yet in a dynamic model, rational investors will fall into the trap of *infinite* regress where they must forecast what the average opinion expects the average opinion expects the average opinion..., *ad infinitum.* Agents behaving optimally must incorporate these infinite-order expectations into their beliefs, thus making the state space infinite dimensional. Moreover, as recently emphasized by Allen, Morris, and Shin (2003), average expectation operators fail to satisfy the law of iterated expectations. Therefore the price of an asset today in an economy with asymmetric information will *not* equal the representative agent's discounted expected value of the asset's payoff stream conditional on information available today, but the price will also encompass investors' higher-order beliefs.

To circumvent the problem of infinite regress, existing models typically make simplifying assumptions or cleverly construct the model to bypass infinite regress. Lucas (1975) was the first to capture the technical formalities of infinite regress in expectations, yet neatly sidestepped the problem by assuming traders got together each period and pooled their forecasts. Townsend (1983) did not make this simplifying assumption and resorted to a truncation strategy, where the state of the economy is revealed to all agents with a two-period lag.

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[1] Recent contributions include Allen, Morris, and Shin (2003), Bacchetta and van Wincoop (2004), and Woodford (2003).

Singleton (1987) employed the truncation solution of Townsend to study dynamic asset pricing when traders are forced into infinite regress.[2] In lieu of this truncation technique, which only offers an approximate solution, other approaches to elude the problem of infinite regress have included: constructing a hierarchical information structure with limited dispersion of information [e.g., Wang (1994), Wang (1993)]; assuming a continuum of investors and invoking the law of large numbers to remove all aggregate uncertainty [e.g., He and Wang (1995)]; assuming a finite or static environment in which the asset is liquidated at a certain date [e.g., Allen, Morris, and Shin (2003), Foster and Viswanathan (1996)].

Recently, there have been several advances in solving dynamic models with heterogeneous information. Sargent (1991) describes a method for computing equilibrium (vis-á-vis Townsend's partial revelation approach) in which the state of the economy is never revealed to the agents. By incorporating moving average components into agents' perceptions, he establishes a method to accommodate the infinite dimensional state space associated with dynamic asymmetric information models. In particular, Sargent appeals to the ability of low-order moving average (MA) representations, in certain special cases, to approximate spaces of infinite dimensional autoregressive processes. Using the techniques of Whiteman (1983), Kasa (2000) demonstrates how closed-form solutions that are difficult to obtain in the time domain (à la Sargent (1991)) are easily obtained in the frequency domain. More recently, Pearlman and Sargent (2005) show that the signal extraction problem of Townsend (1983) is not enough to preserve divergent beliefs when agents act rationally. By defining a new state variable (the agent's forecast error) and applying the recursive techniques of Pearlman, Currie, and Levine (1986), they show that every agent will make the same forecast in equilibrium, thus eliminating the need to forecast the forecasts of others.

These advancements have all been made in the context of Townsend (1983), which is a special case due to the unique structure of the model.[3] But it has long been suspected that this conclusion applies more generally to a broader class of models. Consider the concluding remarks of Singleton (1987),

> "Another interesting finding is that the equilibrium prices for the models with disparate information and partial, homogeneous information follow very similar time series properties. It remains to be seen whether this carries over to alternative parameterizations and information structures. Based on the findings to date, however, it appears that disparate information per se in a competitive market does not significantly effect the equilibrium price process."

The contribution of this paper is to show that a certain class of signal extraction problems, which includes Singleton (1987), is not general enough to generate and preserve divergent beliefs in a dynamic equilibrium where agents can learn from past forecast errors. While this has long been believed to be true, no formal

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[2] Bacchetta and van Wincoop (2004) also use Townsend's (1983) truncation technique to analyze exchange rate movements.

[3] The model of Townsend induces informational asymmetries by assuming that prices do not simultaneously clear markets due to production lags. Thus the resulting signal extraction problem is endemic to the economy [see Kasa (2000)] and is unlike the "typical" signal extraction problem (i.e., where there are more noises than signals).

analysis to date has proven this claim. I will use frequency domain techniques in both a linear model of Futia (1981) and a model that introduces risk aversion (i.e., negative-exponential utility maximizers) to show how endogenous variables reveal all privately held information. This, in turn, eliminates the need to forecast the forecasts of others and the asymmetric information equilibrium will be shown to degenerate into a symmetric information equilibrium. The next section introduces the model of Futia (1981) and describes the solution technique. Section 3 solves the model in a symmetric information setting and in an asymmetric information environment, and proves that these two information structures yield the same equilibrium. The final section of the paper introduces risk aversion and demonstrates how the results of the previous section continue to hold.

2 The Model

In this section of the paper, I briefly outline the model of Futia (1981). The basic features of the model include a competitive, Walrasian market structure with a single asset in fixed supply that is traded among speculative and nonspeculative or liquidity traders, who serve to break the no-trade theorem. Because of its simplicity, the model is conducive to examining the problems associated with heterogenous beliefs. Moreover, the model has a very general interpretation and thus has a broad appeal.

Supply of the asset is assumed to be in fixed quantity. The *net* supply of the asset s_t (total supply less nonspeculative demand at time t, less the mean difference) is assumed to be the sum of two stochastic components

$$s_t = A(L)\varepsilon_{1t} + B(L)\varepsilon_{2t} \tag{2.1}$$

where $A(L)$ and $B(L)$ are (possibly infinite-order) polynomials in nonnegative powers of the lag operator L with square-summable coefficients (i.e., $\sum_{j=0}^{\infty} A_j^2 < \infty$), and $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are mutually and serially uncorrelated. It is important to note that (2.1) places no restrictions on the serial correlation properties of $\{s_t\}$. Also, Futia's version of (2.1) was more special in that he assumed supply was a function of a single shock rather than the two used here. He restricted attention to a single shock because he assumed that traders see prices but not quantities. I will assume below that traders see both prices and quantities; avoidance of trivial stochastic singularities requires two independent shocks in (2.1).

The Wold Decomposition Theorem allows for such a general structure but in order to invoke the Wold Theorem, we must place more structure on the underlying uncertainty.[4] Following Futia (1981), a random variable will be deemed *admissible* if and only if it is a linear combination of independent, mean-zero Gaussian random variables with finite variance and square-summable coefficients. Therefore, I assume ε_{1t} and ε_{2t} are

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[4] See Sargent (1987) for details on the Wold Decomposition Theorem.

mutually independent for all t and distributed normally with mean zero and finite variances denoted by $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$. Let H denote the set of all admissible random variables; the completion in mean-square norm of these variables constitutes a well-known Hilbert space. Hilbert spaces generated by the random variables contained in the model will play an important role in determining the information content of the equilibrium price of the asset.

Suppose there exists a continuum of price-taking speculative traders indexed by $i \in [0, 1]$. The nonspeculative traders are assumed to have zero price elasticity of demand for the asset at all t, while each speculative trader has demand given by[5]

$$d_{it} = E^i_t p_{t+1} - \alpha p_t \tag{2.2}$$

where $\alpha = (1 + r) > 1$ is the same for all i and can be interpreted as the opportunity cost of investing in the asset. The important thing to note here is that expectations are indexed by i – that is, by trader. Heterogeneous information concerning market fundamentals (s_t) may generate divergent beliefs about next period's price of the asset across traders. To see how this entails forecasting the forecasts of others, note that market clearing for the asset requires

$$p_t = \alpha^{-1} \int_0^1 E^i_t p_{t+1} di - \alpha^{-1} s_t \tag{2.3}$$

Thus, following Singleton (1987), the equilibrium price of the asset at time t depends upon the market-wide average expectation of the asset at $t + 1$. However, each trader's forecast of p_{t+1} will depend upon the market-wide forecast of p_{t+2}, and so on, *ad infinitum.* If traders' information sets generate disparate expectations, the problem of infinite regress arises. When forecasting p_{t+1}, each trader must take into account every other traders' forecast of $p_{t+1}, p_{t+2}, \cdots$. If the equilibrium price does not reveal all privately held information, then the usual method of solving for the rational expectations equilibrium (e.g., Blanchard and Kahn (1980)) breaks down. The principal technical difficulty is that agents are extracting signals from *endogenous* variables. When endogenous variables convey information it becomes difficult to identify a tractable set of state variables because an agent's notion of the state of the economy would include other agents' forecasts of the asset price at indefinitely many future dates, thereby making the dimension of the state indefinitely large.

In contrast, in representative agent models involving signal extraction from endogenous variables, the true state vector is latent from the optimizer and *beliefs* can serve as the defacto state variable (e.g. Muth (1960),

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[5] A more general demand will be studied in Section 3.

Lucas (1972)). Then the Kalman filter can be employed to estimate the hidden state. But when the state becomes infinite dimensional, as it does here, this method cannot be applied. The next section describes an alternative technique that preserves tractability.

2.1 Information and Solution Technique

Solutions to the model will be sought in the space spanned by square-summable linear combinations of the fundamental driving processes $\{\varepsilon_{1t}, \varepsilon_{2t}\}$, which implies that the conditional expectation of tomorrow's price $E_t^i(p_{t+1})$ will depend upon a collection of admissible random variables from H. The information set of agent i at time t, denoted by W_t^i, is a subset of H and represents the current and past values of the variables seen by trader i. Let J_t denote the set of information known by all traders at date t (i.e., $J_t \equiv \bigcup_i W_t^i$). There will be two types of information available to each agent – exogenous and endogenous. Exogenous information is by definition not affected by market forces and will be denoted by U_t^i. Endogenous information may be generated through market interactions of differentially informed agents. Assuming all agents behave rationally, the Projection Theorem (see Brockwell and Davis (1991)) implies that agent i's conditional expectation of p_{t+1} is the orthogonal projection of p_{t+1} on the smallest closed subspace of H which contains W_t^i. This subspace will include both exogenous and endogenous information, and since the collection of variables in H is jointly normal, conditional expectations will reduce to linear least-squares projections. Let $H_x(t)$ denote the space spanned by square-summable linear combinations of current and past values of x. Then trader i's expectation of p_{t+1} is given by

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|W_t^i] = \Pi[p_{t+1}|U_t^i \bigvee H_p(t)] \tag{2.4}$$

where Π denotes linear least-squares projection and $U_t^i \bigvee H_p(t)$ is standard notation for the "linear space spanned by U_t^i and $H_p(t)$." That is, agents use all exogenous information and information generated by current and past values of the equilibrium price in evaluating the expectation of tomorrow's price. The expectation of investor i is then simply the orthogonal projection of p_{t+1} onto the subspace generated by U_t^i and $H_p(t)$. The endogenous price process is then found by invoking market clearing. We are now ready to define a rational expectations equilibrium (REE).

Definition 1. *A rational expectations equilibrium is a stochastic process for* $\{p_t\}$, $p_t \in J_t$, *that satisfies market clearing* (2.3)*, where expectations are formed according to* (2.4)*.*

The requirement, $p_t \in J_t$, is what Futia (1981) dubbed the axiom of "no divine revelation." This condition prevents equilibrium prices at date t from conveying any more information than that which could be in principle be available to traders at date t.

The equilibrium of the model is computed as follows. First, each trader uses all available information at time t (W_t^i) to form beliefs about the current price process. These guesses will then be used to evaluate the conditional expectation of p_{t+1} via Wiener-Kolmogorov optimal prediction formulas. The appropriate form of equation (2.3) is then used to impose market clearing. In solving the subsequent fixed-point problem, I will appeal to the Riesz-Fischer Theorem and derive the solution in the frequency domain.[6] The benefit of working in the frequency domain is computational convenience. Working in the time domain, Sargent (1991) shows how to convert the infinite dimensional state space associated with the forecasting the forecasts of others problem of Townsend (1983) into a finite dimensional system. Rather than guessing an *infinite*-order autoregressive representation for beliefs (the state variable), Sargent models agents as forecasting by fitting low-order autoregressive, moving-average (ARMA) representations. By introducing moving-average components in agents' perceptions, Sargent utilizes the fact that low-order ARMA representations have infinite-order AR representations to generate an approximation of the infinite-dimensional solution. The drawback of this approach is that not only does one have to solve a fixed point problem in the coefficients of the ARMA process, but the *order* of the ARMA process must be matched as well. This led to the insight of Kasa (2000), who shows how this two-step process can be condensed into a single step by working in the frequency domain. As opposed to guessing a functional form for beliefs, applying the Kalman filter, and then attempting to match coefficients, the frequency domain allows one to work with a functional fixed point problem. Therefore coefficients and *order* of the ARMA process are matched simultaneously by using the theory of the residue calculus. However, this process will only generate a *candidate* equilibrium price process. Traders will surely condition on past prices, therefore in order for the information structure to preserve disparate expectations, the equilibrium price process must not reveal other agents' privately held information. In a sense, we must have an "informational fixed point" in order for the price process to be sustainable in equilibrium. If the candidate equilibrium price does reveal information to agents that they did not have in forming expectations, then the process just described will be repeated with the updated information sets of the traders until it does not.

In a symmetric equilibrium, agents' expectations coincide:

Definition 2. *We say a REE is symmetric if after observing the history of equilibrium prices $\{p_{t-j}\}_{j=0}^{\infty}$ all traders have identical information and make the same forecasts. That is*

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|U_t^i \bigvee H_p(t)] = E_t^j(p_{t+1}) = \Pi[p_{t+1}|U_t^j \bigvee H_p(t)]$$

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[6] Recall the Riesz-Fischer Theorem states there is an equivalence (i.e., an isometric isomorphism) between the space of square-summable sequences denoted by $\ell_2(-\infty,\infty)$ and the space of square integrable functions, $L^2[\pi,-\pi]$; the former is referred to as the time domain and the latter the frequency domain.

for all i and j.

3 Information and Equilibrium

3.1 Symmetric Information, Symmetric Equilibrium

As a baseline model, it is useful to first assume a symmetric information structure that avoids infinite regress.[7] To this end, suppose that every investor observes past prices and net supplies. That is, the common information set of every trader i is given by

$$W_t^i = \{p_{t-j}, s_{t-j} : j \geq 0\} \qquad \forall\ i.$$

Given this informational assumption, an investor's belief about the average equilibrium price will be represented by a linear combination of current and past values of p_t and s_t. In the absence of sunsopts, these stochastic processes will be driven by the underlying shocks $\{\varepsilon_{1t}, \varepsilon_{2t}\}$. For reasons outlined in Whiteman (1983) it is much simpler to calculate equilibrium prices and quantities if agents' expectations are computed under the assumption that they can see these underlying shocks. This "candidate" equilibrium will be realizable if and only if the space spanned by current and past values of the candidate $\{s_t, p_t\}$ process is identical to that spanned by current and past values of $\{\varepsilon_{1t}, \varepsilon_{2t}\}$. Thus we proceed by assuming agents see the shocks, compute equilibrium under that assumption and then check to verify that the equilibrium $\{s_t, p_t\}$ process would enable agents to uncover $\{\varepsilon_{1t}, \varepsilon_{2t}\}$. This candidate equilibrium is not a REE if this recovery is not possible. Thus we begin with the tentative assumption that every trader believes the average equilibrium price to be given by

$$p_t = \sum_{j=0}^{\infty} D_j\ \varepsilon_{1,t-j} + \sum_{j=0}^{\infty} F_j\ \varepsilon_{2,t-j} \equiv D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} \tag{3.1}$$

where $D(L)$ and $F(L)$ are infinite-order square summable polynomials in the lag operator L.[8] The Wiener-Kolmogorov optimal prediction formula gives the conditional expectation as

$$E_t^i(p_{t+1}) = L^{-1}[D(L) - D_0]\varepsilon_{1t} + L^{-1}[F(L) - F_0]\varepsilon_{2t}.$$

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[7] This case will not only serve as a benchmark but the next section shows how *ex ante* disparate information structures degenerate to this symmetric case.

[8] It is important to note that by working in the frequency domain one does not have to take a stance on the explicit functional form of beliefs, $D(L)$ and $F(L)$. This is especially convenient when beliefs contain moving average components because MA representations are difficult to handle in the time domain.

Imposing market clearing (2.3) and rearranging, one obtains

$$\alpha[D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}] = L^{-1}[D(L) - D_0]\varepsilon_{1t} + L^{-1}[F(L) - F_0]\varepsilon_{2t} - A(L)\varepsilon_{1t} - B(L)\varepsilon_{2t}.$$

Assuming that this expression holds for all realizations of $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$, the coefficients on ε_{1s} and ε_{2s} must match for every s. This implies the power series equalities: $\alpha D(z) = z^{-1}[D(z) - D_0] - A(z)$ and $\alpha F(z) = z^{-1}[F(z) - F_0] - B(z)$. Due to the symmetry of the problem, we can focus on solving the fixed-point problem for one process, say ε_{1t}. A little algebra gives

$$D(z)(1 - \alpha z) = D_0 + zA(z)$$

If the equilibrium price process p_t is to be in the set H, it must be the case that the coefficients D_j are square summable. The requirement of square-summability in the time domain corresponds to the requirement that $D(z)$ be analytic on the open unit disk $|z| < 1$ in the frequency domain. Whiteman (1983) shows that given $\alpha > 1$ this function will not be analytic unless the free parameter D_0 removes the singularity at $z = \alpha^{-1}$. This is achieved by setting the residue equal to zero and solving for D_0, which yields

$$\lim_{z \to \alpha^{-1}} D(z)(1 - \alpha z) = D_0 + \alpha^{-1}A(\alpha^{-1}) = 0$$
$$D_0 = -\alpha^{-1}A(\alpha^{-1}).$$

This implies $D(z)$ is *unique* and given by

$$D(z) = (1 - \alpha z)^{-1}(zA(z) - \alpha^{-1}A(\alpha^{-1}))$$

and thus the equilibrium price process has the form

$$p_t = \left[\frac{LA(L) - \alpha^{-1}A(\alpha^{-1})}{(1 - \alpha L)}\right]\varepsilon_{1t} + \left[\frac{LB(L) - \alpha^{-1}B(\alpha^{-1})}{(1 - \alpha L)}\right]\varepsilon_{2t} \tag{3.2}$$

which involves two instances of the prediction formula of Hansen and Sargent (1980). Equation (3.2) gives us the candidate equilibrium price process. We must now determine whether the *ex ante* informational assumptions support such a price process. To do so, we construct the "observer system" for agent i, which includes all exogenous information (U_t^i) and the endogenous information generated by p_t. Setting this system up for the symmetric case one obtains,

$$\begin{bmatrix} s_t \\ p_t \end{bmatrix} = \begin{bmatrix} A(L) & B(L) \\ \frac{LA(L)-\alpha^{-1}A(\alpha^{-1})}{(1-\alpha L)} & \frac{LB(L)-\alpha^{-1}B(\alpha^{-1})}{(1-\alpha L)} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

or more compactly

$$\mathbf{y}_t = M(L)\boldsymbol{\epsilon}_t \tag{3.3}$$

If this system is invertible, so that $\boldsymbol{\epsilon}_t$ may be obtained as a square-summable linear combination of current and past $\{\mathbf{y}_t\}$, then (3.3) is a REE. In order to proceed, we need to place more structure on the model; anticipating analysis below, we follow Singleton (1987) in the specification:[9]

Assumption 1. *The supply process is given by:*

$$A(L) = \frac{1}{(1-\rho L)}, \qquad B(L) = 1 + \varsigma L, \qquad 0 \leq |\rho|, |\varsigma| \leq 1$$

Note that under this assumption, supply is the sum of a first-order autoregression (AR(1)) and a first-order moving average (MA(1)). Depending on the relative sizes of $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$ and whether ρ and ς are nonzero, this is general enough to include the following special cases for the univariate representation of supply: white noise, MA(1), AR(1) (in the limit as $\sigma^2_{\varepsilon_2}/\sigma^2_{\varepsilon_1} \to \infty$), ARMA(1,1) and ARMA(2,1). It is now possible to determine conditions for the invertibility of (3.3).

Lemma 1. *The vector moving average representation* (3.3) *is invertible (making* (3.2) *a REE) provided*

$$\left| \frac{\alpha\varsigma\rho}{\alpha(\rho-\varsigma)+\varsigma\rho} \right| \leq 1 \tag{3.4}$$

Proof. A necessary and sufficient condition for (3.3) to be a fundamental representation for $[s_t, p_t]'$ is that the determinant of $M(z)$ be analytic and have no zeros inside the open unit disk. By direct calculation,

$$\begin{aligned} \det\, M(z) &= \frac{A(z)[zB(z)-\alpha^{-1}B(\alpha^{-1})]}{(1-\alpha z)} - \frac{B(z)[zA(z)-\alpha^{-1}A(\alpha^{-1})]}{(1-\alpha z)} = 0 \\ &= \frac{\alpha^{-2}[\alpha(\rho-\varsigma)+\varsigma\rho+\alpha\varsigma\rho z]}{(\alpha-\rho)(1-\rho z)}; \end{aligned} \tag{3.5}$$

the stated condition guarantees that det $M(z)$ does not contain any zeros inside the unit circle, and therefore (3.3) is invertible. □

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[9] Following Singleton (1987), the model was also solved with $A(L) = 1 + \phi L$. Adopting a different representation for $A(L)$ will alter the solution of the model slightly but the main results found here continue to hold given $|\phi| < 1$.

Invertibility makes (3.3) a fundamental (Wold) representation. This is important because it implies $M(L)$ has a one-sided inverse in nonnegative powers of L, so a corollary of the above lemma is that $H_t(s) \bigvee H_t(p) \equiv H_t(\varepsilon_1) \bigvee H_t(\varepsilon_2)$. Thus, the observables s_t and p_t span the same linear space as the underlying fundamental shocks ε_{1t} and ε_{2t}, and therefore the Hilbert spaces generated by $\{\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ...\}$ and $\{\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, ...\}$ must be identical (in mean-square sense). Thus, the equality

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|H_p(t) \bigvee H_s(t)] = \Pi[p_{t+1}|H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)] \tag{3.6}$$

holds for all i. By allowing traders to guess an equilibrium price that is a linear combination of the underlying shocks (3.1), equality (3.6) was implicitly assumed to hold. This suggests, and was subsequently proven, that by observing the combination of the history of net supplies and equilibrium prices agents would be able to infer the underlying shocks. Of course this relationship does not have to hold in equilibrium. The next section studies whether disparate expectations are preserved in this setup under Singleton's (1987) information structure.

3.2 Asymmetric Information, Symmetric Equilibrium

Adopting the information structure and setup of Singleton (1987), suppose there are two distinct groups of traders, in proportion k and $(1-k)$. Traders are not able to observe net supply directly, but *every* trader receives a private, noisy signal on $A(L)\varepsilon_{1t}$ and a public signal on $B(L)\varepsilon_{2t}$. Denote the two signals by

$$v_{it} = A(L)\varepsilon_{1t} + \eta_{it}, \qquad v_t = B(L)\varepsilon_{2t}$$

where η_{it} is assumed to be i.i.d, normally distributed with finite variance and is uncorrelated with all other shocks. Not only will each individual trader receive different realizations of η, but I will also assume that the two groups of traders differ in their qualities of information. Traders in group 1 (in proportion k) see private signals with smaller variance ($\sigma_{\eta 1}^2 < \sigma_{\eta 2}^2$). Of course agents will also be able to condition on current and past prices, and the information set of agent i at time t is given by

$$W_t^i = \{v_{t-j}, v_{it-j}, p_{t-j} : j \geq 0\} \tag{3.7}$$

The central question is, will this heterogenous information be enough to generate and preserve disparate expectations in equilibrium? The exogenous information structure is asymmetric; each investor receives a different realization of η_t and the two groups receive different qualities of information on average. However, in equilibrium, each trader will also extract information from current and past prices. If the equilibrium price provides a rich enough information structure to bridge the gap among traders, then the equilibrium will degenerate into the one studied in Section 3.1

The difference between this section and the previous section is that the agents must first solve the signal extraction problem, which relates the signals to the underlying shocks. After solving this problem, agents will then use this information to generate a guess of the equilibrium price process. Consider the following signal system for agent i,

$$\begin{bmatrix} v_{it} \\ v_t \end{bmatrix} = \begin{bmatrix} A(L) & 1 & 0 \\ 0 & 0 & B(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \eta_{it} \\ \varepsilon_{2t} \end{bmatrix}$$

or more compactly,

$$\mathbf{v}_t = G(L)\boldsymbol{\epsilon}_t. \tag{3.8}$$

Since there are more shocks than observables, agent i will not be able to "see" or infer both economically fundamental shocks $(\varepsilon_{1t}, \varepsilon_{2t})$. This implies, *a priori*, that a relationship analogous to (3.6) will not hold. That is,

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|H_{v_i}(t) \bigvee H_v(t)] \neq \Pi[p_{t+1}|H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)]. \tag{3.9}$$

So what will the agents be able to infer? To give a concrete example, I adopt Assumption 1 and seek a fundamental representation to replace (3.8). To find this moving average, we must first transform the private signal (v_{it}) into a fundamental, Wold representation – call it $C_1(L)\xi_{it}$. This is accomplished by factorization of the covariance generating function of the private signal. This covariance generating function is given by[10]

$$g_{v_i}(z) = \frac{\sigma^2_{\varepsilon_1}}{(1-\rho z)(1-\rho z^{-1})} + \sigma^2_{\eta_1}. \tag{3.10}$$

In establishing a fundamental representation, we seek a λ_1 and $\sigma^2_{\xi_1}$ such that

$$\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1}(1-\rho z)(1-\rho z^{-1}) = \sigma^2_{\xi_1}(1-\lambda_1 z)(1-\lambda_1 z^{-1}).$$

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[10] See Whittle (1983) or Sargent (1987) Chapter XI Section 18.

Setting λ_1 equal to the smaller root ensures $|\lambda_1| < 1$ and yields[11]

$$\lambda_1 = \frac{1}{2}\left[\left(\frac{\sigma^2_{\varepsilon_1}}{\sigma^2_{\eta_1}\rho}\right) + \left(\frac{1}{\rho} + \rho\right) - \left\{\left[\frac{\sigma^2_{\varepsilon_1}}{\sigma^2_{\eta_1}\rho} + \left(\frac{1}{\rho} + \rho\right)\right]^2 - 4\right\}^{1/2}\right].$$

The variance $\sigma^2_{\xi_1}$ is then found by the formula

$$\sigma^2_{\xi_1} = \frac{g_z(1)}{C_1(1)^2} = \frac{\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1}(1-\rho)^2}{(1-\lambda_1)^2}.$$

Note that Type 2 investors will have an analogous fundamental representation $C_2(L)\xi_{it}$ with λ_2 and $\sigma^2_{\xi_2}$ each a function of $\sigma^2_{\eta_2}$.

This factorization puts the signal in a form that the agents can use to predict next period's price and it also tells us the relationship between the signals and the fundamental shocks. To see this note that after the factorization, the signal system for agent i in group k becomes

$$\begin{bmatrix} v_{it} \\ v_t \end{bmatrix} = \begin{bmatrix} C_1(L) & 0 \\ 0 & B(L) \end{bmatrix} \begin{bmatrix} \xi_{it} \\ \varepsilon_{2t} \end{bmatrix}$$

where $C_1(L) = \frac{1-\lambda_1 L}{1-\rho L}$. By construction, this system is a fundamental (Wold) representation and can now be used to generate trader i's guess of the equilibrium price:

$$p_t = D_i(L)\xi_{it} + F(L)\varepsilon_{2t} \tag{3.11}$$

where ξ_{it} is related to the underlying shock ε_{1t} by the equation

$$\xi_{it} = C_1(L)^{-1}A(L)\varepsilon_{1t} + C_1(L)^{-1}\eta_{it} = \frac{\varepsilon_{1t}}{1-\lambda_1 L} + \frac{1-\rho L}{1-\lambda_1 L}\eta_{it}. \tag{3.12}$$

As the signal-to-noise ratio $(\sigma_{\varepsilon_1}/\sigma_{\eta_j})$ approaches infinity, λ_j approaches zero and the first component of (3.12) approaches ε_{1t}. Conversely as the signal-to-noise ratio approaches zero, λ_j approaches ρ. Therefore traders in group 1, who receive a more precise signal, will have a more accurate guess of the price process on average. Further, because the entire price sequence is observable in equilibrium, in order that agents have perceptions about the serial correlation properties of prices that are consistent with what is observed, the $D_i(L)$ functions

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[11] This also ensures ξ_{it} lies in the linear space spanned by current and lagged v_{it}'s. In other words, that ξ_{it} is the one step ahead prediction error of predicting v_{it} from its own past, $\xi_{it} = \Pi(v_{it}|v_{it-1}, v_{it-2}, ...)$; and therefore the Hilbert spaces generated by $\{v_{it}, v_{it-1}, ...\}$ and $\{\xi_{it}, \xi_{it-1}, ...\}$ are equivalent.

must be identical within groups and proportional across groups:

$$\text{for } i \leq k: \quad D_i(L) = \chi_1 D(L), \qquad \text{for } k \leq i: \quad D_i(L) = \chi_2 D(L).$$

Still, this information setup implies disparate expectations for *every* i

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|H_{v_i}(t) \bigvee H_v(t)] = \Pi[p_{t+1}|H_{\xi_i}(t) \bigvee H_{\varepsilon_2}(t)].$$

In particular, for $i \leq k$,

$$E_t^i p_{t+1} = L^{-1}[D(L) - D_0]\chi_1 \xi_{it} + L^{-1}[F(L) - F_0]\varepsilon_{2t}.$$

Given that the η_{it}'s are i.i.d., I will assume that a version of the strong law of large numbers holds for each of the two groups so that the overall impact of the idiosyncratic shocks averages to zero. That is,

$$E_t^1 p_{t+1} \equiv \int_0^k E_t^i p_{t+1} di = kL^{-1}[D(L) - D_0]\xi_{1t} + kL^{-1}[F(L) - F_0]\varepsilon_{2t}$$

where

$$\xi_{1t} = \chi_1 \int_0^k \xi_{it} di = \frac{\chi_1 \varepsilon_{1t}}{(1 - \lambda_1 L)}$$

and analogously for $k \leq i \leq 1$. This assumption will *not* by itself lead to a symmetric REE because the traders have different qualities of information ($\sigma^2_{\eta_1} < \sigma^2_{\eta_2}$) which implies $\xi_1 \neq \xi_2$. It is important to note that $\chi_1 > \chi_2$ and $\lambda_1 < \lambda_2$ implies traders of group 1 have more accurate perceptions and therefore smaller one-step-ahead forecast errors. Thus heterogenous information and disparate expectations now exists across the two *groups.*

$$E_t^1(p_{t+1}) = \Pi[p_{t+1}|H_{\xi_1}(t) \bigvee H_{\varepsilon_2}(t)] \neq E_t^2(p_{t+1}) = \Pi[p_{t+1}|H_{\xi_2}(t) \bigvee H_{\varepsilon_2}(t)] \tag{3.13}$$

Because the idiosyncratic information integrates out, the equilibrium price will be a linear function of ε_{1t}. Thus, $p_t = D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}$, and χ_1 and χ_2 solve

$$\sigma^2_{\varepsilon_1} = \chi_1^2 \sigma^2_{\xi_1} = \chi_2^2 \sigma^2_{\xi_2}$$

i.e.,

$$\chi_1 = \left(\frac{\sigma_{\varepsilon_1}^2(1-\lambda_1)^2}{\sigma_{\varepsilon_1}^2 + \sigma_{\eta_1}^2(1-\rho)^2}\right)^{1/2} \quad \chi_2 = \left(\frac{\sigma_{\varepsilon_1}^2(1-\lambda_2)^2}{\sigma_{\varepsilon_1}^2 + \sigma_{\eta_2}^2(1-\rho)^2}\right)^{1/2}.$$

It is straightforward to verify that if p_t is of the form $D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}$, then the average forecast errors $p_{t+1} - E_t^1 p_{t+1}$ and $p_{t+1} - E_t^2 p_{t+1}$ are serially correlated. More troubling is the fact that individual forecast errors are also serially correlated. If this serial correlation can be exploited to improve predictions (as will be shown), perceptions (e.g., (3.11)) will not match reality, and we will not have a REE. The extent to which this additional information generated by the candidate price process can be exploited is the crux of the issue. Letting i and j denote the two groups of traders, the market clearing condition is

$$\alpha p_t = \int_0^k E_t^i p_{t+1} di + \int_k^1 E_t^j p_{t+1} dj - s_t \tag{3.14}$$

$$\alpha[D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}] = k\{L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_1}{1-\lambda_1 L} + L^{-1}[F(L) - F_0]\varepsilon_{2t}\}$$
$$+(1-k)\{L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_2}{1-\lambda_2 L} + L^{-1}[F(L) - F_0]\varepsilon_{2t}\} - A(L)\varepsilon_{1t} - B(L)\varepsilon_{2t}.$$

It is easy to see that the solution for $F(z)$ will be the same as the symmetric case; specifically,

$$F(z)^* = \frac{zB(z) - \alpha^{-1}B(\alpha^{-1})}{1-\alpha z}. \tag{3.15}$$

Equating coefficients on $\varepsilon_{1t}, \varepsilon_{1t-1}, ...$ yields the power series equality

$$D(z)[-\lambda_1\lambda_2\alpha z^3 + (\lambda_2+\lambda_1)\alpha z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)z + k\chi_1 + (1-k)\chi_2]$$
$$= D_0[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2] + z(1-\lambda_1 z)(1-\lambda_2 z)A(z).$$

The roots of the cubic equation

$$-\lambda_1\lambda_2\alpha z^3 + (\lambda_2+\lambda_1)\alpha z^2 - (\lambda_2\chi_1 k + (1-k)\lambda_1\chi_2 + \alpha)z + k\chi_1 + (1-k)\chi_2 \tag{3.16}$$

will determine the uniqueness of the candidate equilibrium encountered. The following proposition shows that for the supply process given by Assumption 1 and assuming the gross interest rate exceeds unity ($\alpha > 1$), there is a unique candidate price.

Proposition 3.1. *Given $\alpha > 1$ and Assumption 1, the price process generated by information* (3.7) *and the market clearing condition* (3.14) *is unique.*

Proof. See Appendix A.1 □

In order for the equilibrium price to be unique, we need *exactly* one root of (3.16) to lie inside the unit circle. If no root lies inside the unit circle, then the free parameter D_0 will not be pinned down and $D(z)$ will not be unique. If more than one root lies inside the unit circle, a square-summable $D(z)$ satisfying (3.16) does not exist. The proof found in Appendix A.1 shows that there exists exactly one root that lies inside the unit circle. Let θ denote this root. Then D_0 will be set to remove this singularity as before; in this case we have

$$D_0^* = -\frac{\theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta)}{(1-\lambda_2\theta)k\chi_1 + (1-\lambda_1\theta)(1-k)\chi_2}.$$

Substituting D_0^* into $D(z)$

$$\begin{aligned} &D(z)[-\lambda_1\lambda_2\alpha z^3 + (\lambda_2+\lambda_1)\alpha z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)z + k\chi_1 + (1-k)\chi_2] \\ &= \frac{-(\theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta))[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2]}{((1-\lambda_2\theta)k\chi_1 + (1-\lambda_1\theta)(1-k)\chi_2)} + z(1-\lambda_1 z)(1-\lambda_2 z)A(z) \end{aligned} \tag{3.17}$$

or more compactly

$$D(L)^*\varepsilon_{1t} = \left[\frac{D_0^*[(1-\lambda_2 L)k\chi_1 + (1-\lambda_1 L)(1-k)\chi_2] + L(1-\lambda_1 L)(1-\lambda_2 L)A(L)}{-\lambda_1\lambda_2\alpha L^3 + (\lambda_2+\lambda_1)\alpha L^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)L + k\chi_1 + (1-k)\chi_2}\right]\varepsilon_{1t}.$$

We now have our unique candidate equilibrium price process,

$$p_t = D(L)^*\varepsilon_{1t} + F(L)^*\varepsilon_{2t} \tag{3.18}$$

and we can construct the post-equilibrium observer system for a trader i:

$$\begin{bmatrix} v_{it} \\ v_t \\ p_t \end{bmatrix} = \begin{bmatrix} A(L) & 0 & 1 \\ 0 & B(L) & 0 \\ D(L)^* & F(L)^* & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \eta_{it} \end{bmatrix}$$

$$\mathbf{v}_t' = H(L)\boldsymbol{\epsilon}_t \tag{3.19}$$

Representation (3.19) corresponds to the information available to trader i at time t, W_t^i. If (3.19) is a fundamental (Wold) representation, then the Hilbert spaces spanned by $\{\mathbf{v}_t', \mathbf{v}_{t-1}', ...\}$ and $\{\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, ...\}$ are equivalent, and every trader will be able to infer the shocks ε_{1t} and ε_{2t} by observing the price sequence. In other words, even though we assumed (at (3.11)) that agents saw only a noisy signal on ε_{1t} when forming expectations, the market interactions of those agents injects enough information into the price to reveal the underlying $\{\varepsilon_{1t}\}$ process. Thus after conditioning on the current price, every investor will have identical forecasts in equilibrium. This leads to the following proposition.

Proposition 3.2. *Under Assumption 1, the candidate price process* (3.18) *reveals the fundamental shocks* $\{\varepsilon_{1t}\}$ *and* $\{\varepsilon_{2t}\}$ *to all traders.*

Proof. The proof is given in Appendix A.2. The intuition is as follows. □

Following Lemma 1, in order to prove that (3.19) is fundamental and reveals $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$, we must show that the determinant of $H(z)$ has no zeros inside the unit circle. Recall $B(z) = 1 + \varsigma z$ where $|\varsigma| < 1$. By direct calculation, the numerator of det $H(z)$ is

$$-(1+\varsigma z)(z-\theta)[Qz^2 - Rz + S]$$

where

$$\begin{gathered} Q = \lambda_1\lambda_2[k\chi_1(1-\lambda_2\theta) + \chi_2(1-k)(1-\lambda_1\theta)](1-\theta\rho) \\ R = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1\lambda_2 + (1-k)\chi_2\lambda_1] + (1-\theta\rho)[k\chi_1\lambda_1(1-\lambda_2\theta) + (1-k)\chi_2\lambda_2(1-\lambda_1\theta)] \\ S = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1 + (1-k)\chi_2]. \end{gathered}$$

The determinant of $H(z)$ has four zeros. The root $z = \theta$ is by construction (recall D_0^* was set to ensure such a numerator zero would cancel the like term in the denominator of $D(z)$) and the root $z = -\psi^{-1}$ lies outside the unit circle. Therefore, the roots of

$$Qz^2 + Rz + S \tag{3.20}$$

determine whether or not representation (3.19) is a fundamental (Wold) moving average. Appendix A.2 shows that for $\alpha > 1$ there exists no roots that lie inside the unit circle.

The upshot is simply that the assumption of asymmetric information is not sustainable in equilibrium. Traders will surely condition on past prices and update their forecasts accordingly. Showing (3.19) is a fundamental representation is tantamount to the argument that *all* traders will guess a price process of the form

$$p_t = D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}.$$

And therefore the idiosyncratic shock will not enter the average traders' perceptions. More importantly, the conditional expectation (3.9) will be replaced by

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)]. \tag{3.21}$$

All traders will share the same forecast! Hence there is no need to forecast the forecasts of others. This economy will then degenerate into the one studied in Section 3.1.

3.3 Relation to the Existing Literature

The results obtained here are analogous to the results found in Pearlman and Sargent (2005). Their results were achieved recursively in the time domain by defining a new state variable –the forecast error. This forecast error provided enough information to agents so as to reveal all privately held information in equilibrium. However the findings of Pearlman and Sargent (2005) apply only to the unique structure of Townsend (1983) (see Footnote 3). The previous section shows that asset prices also reveal private information in Singleton's (1987) setup.

The results derived here shed light on the consequences of the truncation approach of Townsend (1983) and Singleton (1987), who *assumed* the equilibrium would preserve disparate expectations and allowed traders to see the fundamental shocks with a two-period lag, that is $\{\varepsilon_{1,t-2}, \varepsilon_{1,t-3}, ...\}$ and $\{\varepsilon_{2,t-2}, \varepsilon_{2,t-3}, ...\}$. Therefore instead of matching the entire sequence for $D(z)$ and $F(z)$, the truncation approach requires that one match only a handful of coefficients, namely D_0, D_1, F_0 and F_1. This, however, only *approximates* the actual equilibrium. To see this more clearly, note that the privately held information of each trader under this informational assumption is given by

$$\begin{bmatrix} \varepsilon_{1,t-2} \\ \varepsilon_{2,t-2} \\ v_{it} \end{bmatrix} = \begin{bmatrix} L^2 & 0 & 0 \\ 0 & L^2 & 0 \\ A(L) & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \eta_{it} \end{bmatrix}$$

$$\mathbf{x}_t = \mathbf{M(L)}\boldsymbol{\epsilon}_t \tag{3.22}$$

The above representation is not a fundamental one due to the zero in the determinant of $\mathbf{M}(L)$ at $L = 0$; therefore traders will not be able to infer the economically fundamental shocks ε_{1t} and ε_{2t}. This signal extraction problem is slightly different from the one encountered in the previous section and is an artifact of the truncation approach. In order to obtain a fundamental representation, we must employ Blaschke matrices (see Hansen and Sargent

(1991)). Appendix B shows how Blaschke matrices are used to derive the fundamental representation

$$\mathbf{x}_t = \mathbf{M}^*(\mathbf{L})\boldsymbol{\epsilon}_t^* \tag{3.23}$$

where

$$\mathbf{M}^*(\mathbf{L}) = \hat{\mathbf{M}}(L)\mathbf{W}\mathbf{B}(\mathbf{L}), \qquad \boldsymbol{\epsilon}_t^* = \mathbf{B}(\mathbf{L^{-1}})\mathbf{W}'\boldsymbol{\epsilon}_t$$

$$\hat{\mathbf{M}}(\mathbf{L}) = \begin{bmatrix} L^2\sigma_{\varepsilon_1} & 0 & 0 \\ 0 & L^2\sigma_{\varepsilon_2} & 0 \\ A(L)\sigma_{\varepsilon_1} & 0 & \sigma_{\eta_j} \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} -\frac{\sigma_{\eta_j}}{\sqrt{A_0^2\sigma_{\varepsilon_1}^2+\sigma_{\eta_j}^2}} & 0 & \frac{\sigma_{\varepsilon_1}^2 A_0}{\sqrt{A_0^2\sigma_{\varepsilon_1}^2+\sigma_{\eta_j}^2}} \\ 0 & 1 & 0 \\ \frac{\sigma_{\varepsilon_1}^2 A_0}{\sqrt{A_0^2\sigma_{\varepsilon_1}^2+\sigma_{\eta_j}^2}} & 0 & \frac{\sigma_{\eta_j}}{\sqrt{A_0^2\sigma_{\varepsilon_1}^2+\sigma_{\eta_j}^2}} \end{bmatrix} \qquad \mathbf{B}(\mathbf{L}) = \begin{bmatrix} L^{-2} & 0 & 0 \\ 0 & L^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $j = 1, 2$ denoting the two distinct groups of traders.

Thus the fundamental innovations ($\boldsymbol{\epsilon}_t^*$) implied by this truncation approach are linear combinations of all the underlying shocks, including the idiosyncratic shock η_{it}. It is obvious, due to the law of large numbers, that the market-wide average of the equilibrium price will not include η_{it}. However the expectations will continue to be disparate across the two groups due to the presence of the variance of η in $\mathbf{W}$. Therefore when solving for the undetermined coefficients, the solution will slightly differ from the symmetric information equilibrium (3.2).

Comparing the informational assumption of (3.22) with that of Section 3.2, Equation (3.8) implies *less* information in that investors are not able to see ε_{1t} with a two-period lag.[12] In fact, the only information given to traders concerning ε_{1t} is that which can be extracted from the signal v_{it}. However, through the information contained in the candidate price process p_t, it was shown that investors actually had the information set of $\{\varepsilon_{1t}, \varepsilon_{1,t-1}, ...\}$ and $\{\varepsilon_{2t}, \varepsilon_{2,t-1}, ...\}$, the equilibrium derived in Section 3.1. Therefore assuming that investors could only condition on $\{\varepsilon_{1,t-2}, ...\}$ and $\{\varepsilon_{2,t-2}, ...\}$ only approximated the actual symmetric equilibrium. Indeed, when traders see $\{p_t\}$ in addition to (3.23), then all underlying uncertainty about the fundamental shocks is revealed. The next section extends this result to environments with risk aversion.

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[12] The public signal v_t could have been lagged by two periods with no change in the fundamental result. This is because all asymmetric information arises from the private signal.

4 Risk Aversion, Multiplicity and Price Revelation

Singleton (1987) examined a slightly different model than the one analyzed in the previous section by incorporating risk aversion via negative-exponential (CARA) preferences and by assuming the asset holder received a stochastic coupon payment (c_t) at the beginning of period t. Abstracting from constants, the coupon stream is given by

$$c_t = \sum_{j=0}^{\infty} c_j \varepsilon_{2,t-j} = C(L)\varepsilon_{2t}, \qquad \varepsilon_{2t} \sim N(0, \sigma^2_{\varepsilon_2}).$$

The assumptions concerning the underlying stochastic processes described in Section 2.1 continue to hold here. Investors may finance the purchase of these securities by borrowing at the constant rate r, which implies that trader i's wealth evolves according to

$$w_{it+1} = \nu_{it}(p_{t+1} + c_{t+1}) - (\nu_{it} p_t - w_{it})(1 + r)$$

where ν_{it} is the amount of the risky asset held at time t by agent i. Investors are assumed to be myopic in that they have a one-period investment horizon. Investment strategies are ranked according to the negative-exponential utility function

$$E^i_t - \exp[-\gamma_i w_{it+1}] \tag{4.1}$$

where γ_i is the coefficient of absolute risk aversion for agent i. It is well known that (4.1) can be calculated via the (conditional) moment generating function for the normal random variable $-\gamma_i w_{it+1}$, and that the corresponding demand for the risky asset of trader i is

$$d_{it} = (\gamma_i \delta_i)^{-1}[E^i_t p_{t+1} - \alpha p_t + E^i_t c_{t+1}] \tag{4.2}$$

where $\alpha = (1 + r) > 1$ and $\delta_i = \mathrm{Var}^i_t(p_{t+1} + c_{t+1})$. It is the conditional variance term δ_i that introduces nonlinearities into the model and leads to existence and multiplicity issues not encountered in the previous section. As in Section 3, the no-trade theorem is broken by assuming the supply of the risky asset is stochastic and given by[13]

$$s_t = A(L)\varepsilon_{1t} \qquad \varepsilon_{1t} \sim \mathrm{N}(0, \sigma^2_{\varepsilon_1}).$$

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[13] Adding an additional, orthogonal component (as in the previous section) would not alter the following results.

Consider the case of asymmetric information studied in Section 3.2, where each individual trader receives a private noisy signal of supply. That is,

$$v_{it} = A(L)\varepsilon_{1t} + \eta_{it}$$

As before, assume there are two groups of traders, in proportion k and $1-k$, that receive a different quality of signal ($\sigma^2_{\eta_1} < \sigma^2_{\eta_2}$). The observer system for agent i is the same as the previous section and given by

$$\begin{bmatrix} v_{it} \\ c_t \end{bmatrix} = \begin{bmatrix} C_1(L) & 0 \\ 0 & C(L) \end{bmatrix} \begin{bmatrix} \xi_{it} \\ \varepsilon_{2t} \end{bmatrix}$$

Invoking Assumption 1 with $C(L) = 1 + \varsigma L$ implies the derivations follow exactly as in the previous section, and therefore Equations (3.8) through (3.13) remain valid here. The only difference is the addition of the conditional variance term, δ. Given that agents initially only condition on the information extracted from signals and that they are able to match the serial correlation properties of the actual price process, the conditional variance terms will be identical across groups, i.e.,

$$\begin{aligned} \delta &= \mathrm{var}_t(p_{t+1}) + \mathrm{var}_t(c_{t+1}) + 2\mathrm{cov}_t(p_{t+1}, c_{t+1}) \\ &= D_0^2\sigma^2_{\varepsilon_1} + (F_0 + C_0)^2\sigma^2_{\varepsilon_2}. \end{aligned}$$

The corresponding market-clearing condition is given by

$$\begin{aligned} \alpha p_t &= \int_0^k E_t^i(p_{t+1} + c_{t+1})di + \int_k^1 E_t^j(p_{t+1} + c_{t+1})dj - \delta\gamma s_t \\ \alpha[D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}] &= k\{L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_1}{1-\lambda_1 L} + L^{-1}[F(L) - F_0]\varepsilon_{2t} + L^{-1}[C(L) - C_0]\varepsilon_{2t}\} \\ &+ (1-k)\{L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_1}{1-\lambda_1 L} + L^{-1}[F(L) - F_0]\varepsilon_{2t} + L^{-1}[C(L) - C_0]\varepsilon_{2t}\} - \delta\gamma A(L)\varepsilon_{1t} \end{aligned}$$

As before, both $D(z)$ and $F(z)$ must be analytic on the unit disk for all realizations of $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$, but the inclusion of the conditional variance has linked the two power series, which can be seen by rearranging and expanding the conditional variance term,

$$\begin{aligned} D(z)[-\lambda_1\lambda_2\alpha z^3 + (\lambda_2 + \lambda_1)\alpha z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)z + k\chi_1 + (1-k)\chi_2] \\ = D_0[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2] + z(1-\lambda_1 z)(1-\lambda_2 z)[\gamma D_0^2\sigma^2_{\varepsilon_1} + (F_0 + C_0)^2\sigma^2_{\varepsilon_2}]A(z) \end{aligned} \tag{4.3}$$

$$F(z)(1-\alpha z) = F_0 + C_0 - C(z) \tag{4.4}$$

There exists a potential pole at $z = |\alpha^{-1}| < 1$ in (4.4) unless the free parameter F_0 is set to remove the singularity. That is, $F_0 = C(\alpha^{-1}) - C_0$ and $F^*(z) = \frac{C(\alpha^{-1})-C(z)}{1-\alpha z}$. The root condition for uniqueness in (4.3) continues to be identical to the previous section (Equation (3.16)), and we know that there exists exactly one root (θ) that lies inside the unit circle. Then as before, D_0 must be set to remove the singularity at $z = \theta$. Thus we have

$$D_0[(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1] + [D_0^2\sigma^2_{\varepsilon_1} + C(\alpha^{-1})^2\sigma^2_{\varepsilon_2}]\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta) = 0 \tag{4.5}$$

Given that there are two roots that satisfy (4.5), there are exactly two equilibria. Moreover, the excess 'noise' associated with the stochastic coupon stream coupled with risk-averse investors severely diminishes the probability of finding equilibria that are real valued. In order for D_0 (and the corresponding price process) to be real valued, restrictions must be placed on the parameter values. Removing uncertainty in the coupon stream ($\sigma^2_{\varepsilon_2} = 0$) yields two equilibria with[14]

$$D_0^* = \begin{cases} -\frac{(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1}{\sigma^2_{\varepsilon_1}\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta)} \\ 0 \end{cases} \tag{4.6}$$

The second case implies $p_t = F^*(L)\varepsilon_{2t}$ for all t. However, if $\sigma^2_{\varepsilon_2} \neq 0$, then, from the quadratic formula, a necessary and sufficient condition for finding a real solution is given by the following restriction

$$\sigma^2_{\varepsilon_2} \leq \frac{[(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1]^2}{4\sigma^2_{\varepsilon_1}\gamma^2 A(\theta)^2\theta^2(1-\lambda_1\theta)^2(1-\lambda_2\theta)^2 C(\alpha^{-1})^2} \tag{4.7}$$

Assuming $\sigma^2_{\varepsilon_2}$ satisfies the above restriction then the number of roots in D_0 satisfying (4.5) will be the number of equilibria encountered. While the potential existence of multiple equilibria is disconcerting, we are more concerned here with the revelation properties of the price process(es). The candidate equilibrium price process(es) will be given by: $p_t = D(L)^*\varepsilon_{1t} + F(L)^*\varepsilon_{2t}$, where

$$D(L)^* = \frac{D_0^*[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2] + z(1-\lambda_1 z)(1-\lambda_2 z)\gamma(D_0^{*2}\sigma^2_{\varepsilon_1} + C(\alpha^{-1})^2\sigma^2_{\varepsilon_2})A(z)}{-\lambda_1\lambda_2\alpha z^3 + (\lambda_2+\lambda_1)\alpha z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)z + k\chi_1 + (1-k)\chi_2} \tag{4.8}$$

$$F(L)^* = \frac{C(\alpha)^{-1} - C(L)}{1-\alpha L}$$

The post-equilibrium observer system for trader i is then

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[14] Singleton(1987) found these equilibria numerically.

$$\begin{bmatrix} v_{it} \\ c_t \\ p_t \end{bmatrix} = \begin{bmatrix} A(L) & 0 & 1 \\ 0 & C(L) & 0 \\ D(L)^* & F(L)^* & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \eta_{it} \end{bmatrix}$$

$$\mathbf{x}_t = J(L)\boldsymbol{\epsilon}_t \tag{4.9}$$

As in the previous section, this observer system will reveal the underlying shocks $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ if (4.9) is a fundamental Wold representation; that is, if the determinant of $J(L)$ does not contain any zeros inside the unit circle. Assuming the minimum variance condition (4.7) is met with equality implies that there exists exactly one root that solves (4.5),

$$D_0^* = -\frac{(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1}{2\sigma_{\varepsilon_1}^2\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta)}.$$

The price process is therefore unique and given by

$$\begin{aligned} &D(z)^*[-\lambda_1\lambda_2\alpha z^3 + (\lambda_2+\lambda_1)\alpha z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha)z + k\chi_1 + (1-k)\chi_2] \\ &= \frac{(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1}{2\gamma\theta^2 A(\theta)(1-\lambda_1\theta)^2(1-\lambda_2\theta)^2}\Big[z(1-\lambda_1 z)(1-\lambda_2 z)A(z)[(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1] - \\ &\qquad \theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta)[(1-\lambda_1 z)(1-k)\chi_2 + (1-\lambda_2 z)k\chi_1]\Big] \end{aligned} \tag{4.10}$$

Comparing (3.17) with (4.10) it is easy to see how risk aversion enters the candidate equilibrium price process. More importantly, it is easy to see that the zeros of (3.17) and (4.10) will coincide. In other words, the zeros of the determinant of $J(L)$ are given by

$$\begin{aligned} \det J(L) &= -D(L)^* C(L) \\ &= -(1+\psi z)(z-\theta)[Qz^2 + Rz + S] \end{aligned}$$

The exact condition for price revelation encountered in Section 3.2! Appendix A.2 shows that there are no zeros that lie inside the unit circle and therefore the price process will reveal all privately held information. The equilibrium price of this economy is determined by a symmetric information structure. Moreover as $\sigma_{\varepsilon_2}^2$ falls below the minimum variance condition (4.7), the economy converges to the candidate equilibria with D_0^* given by (4.6). Using the techniques of this paper, it is easy to show that these candidate equilibria will also reveal and degenerate into symmetric information equilibria.

5 Conclusion

It seems doubtful that the bulk of fluctuations in asset markets is due primarily to differences in risk tolerance. A more likely alternative is that trade is generated by agents who are endowed with different sets of information. Unfortunately, tractable models of heterogenous information in a dynamic setting are difficult to construct and even more difficult to sustain in equilibrium. This paper has demonstrated explicitly how endogenous variables reveal the privately held information of other agents in a dynamic asset pricing model. It was shown, via frequency domain techniques, that the invertibility of the post-equilibrium observer system corresponds to the revelation of economically fundamental shocks. Traders then condition upon this "new" information and update their forecasts accordingly. Models and informational constructs previously believed to maintain asymmetric information (e.g., Singleton (1987)) were shown in fact to lead to revealing equilibria. The information content of equilibrium prices was too rich to sustain disparate information. This result is robust to various degrees of risk aversion. It was shown how introducing risk aversion leads to specific restrictions on parameter values. A minimum variance result was established that guaranteed a real equilibrium price process. Once these restrictions were satisfied, the equilibrium again revealed all privately held information and a symmetric equilibrium resulted.

An obvious extension of this research is to develop a dynamic rational expectations model that preserves asymmetric information. To that end, Kasa, Whiteman and Walker (2005) show how this can be achieved in the frequency domain by assuming a special information structure that gives rise to zeros in the post-equilibrium observer equations. These zeros prohibit equilibrium prices from revealing the economically fundamental shocks. Woodford (2003) achieves this goal in the time domain by following Sims (2001) in assuming that agents have limited capacity. That these efforts seem to require very special circumstances to preserve differential information in equilibrium when disparate expectations seem so widespread suggests that information differentials will constitute a fruitful area for research for some time to come.

Appendix A: Unique and Fundamental Representations

Appendix A.1

The following theorem will be used to show that two roots of (3.16) lie outside the unit circle with one real root inside the unit circle (see Barnett (1983)).

Theorem A (Jury)**.** *Given the real polynomial*

$$T(z) = \zeta_0 z^n + \zeta_1 z^{n-1} + \cdots + \zeta_{n-1} z + \zeta_n$$

construct the matrices Δ_1 *and* Δ_2

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & \Delta_1 \end{bmatrix} \qquad \Delta_1 = \begin{bmatrix} \zeta_0 & \zeta_1 & \cdots & \zeta_{n-1} \\ 0 & \zeta_0 & \cdots & \zeta_{n-2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_0 \end{bmatrix} \qquad \Delta_2 = \begin{bmatrix} 0 & \cdots & 0 & \zeta_n \\ 0 & \cdots & \zeta_n & \zeta_{n-1} \\ \vdots & . & & \vdots \\ \zeta_n & \cdots & \zeta_2 & \zeta_1 \end{bmatrix}$$

Let X *denote the principal submatrix of* Δ_1 *obtained by deleting the last row and last column; and let* Y *denote the submatrix of* Δ_2 *obtained by deleting the first row and last column. Define the two* $(n-1) \times (n-1)$ *matrices* $Z_1 = X - Y$ *and* $Z_2 = X + Y$*. If the inner matrices satisfy* $|Z_j^{(i)}| \neq 0$ *for all* i *and* $j = 1, 2$ *then define*

$$V \equiv \begin{cases} V(|Z_1^{(n-1)}|, |Z_1^{(n-3)}|, \ldots, |Z_1^{(1)}|, |Z_2^{(1)}|, \ldots, |Z_2^{(n-1)}|) & n \text{ even} \\ V(|Z_1^{(n-1)}|, |Z_1^{(n-3)}|, \ldots, |Z_1^{(2)}|, 1, |Z_2^{(2)}|, \ldots, |Z_2^{(n-1)}|) & n \text{ odd} \end{cases}$$

where $V(\cdot)$ *denotes the number of sign changes in the sequence. The number of roots of* $T(z)$ *inside and outside the unit circle are, respectively,* k *and* $n - k$*, where*

$$k = n - V - \tau$$

$$\tau = \begin{cases} 1, & (-1)^{n+V} T(1) T(-1) < 0 \\ 0, & (-1)^{n+V} T(1) T(-1) > 0 \end{cases}$$

Recall the cubic equation (3.16) is given by

$$T(z) = -\lambda_1 \lambda_2 \alpha z^3 + (\lambda_2 + \lambda_1) \alpha z^2 - (\lambda_2 \chi_1 k + (1-k) \lambda_1 \chi_2 + \alpha) z + k \chi_1 + (1-k) \chi_2$$

Notice first that $T(\cdot) > 0$ for any $z \leq 0$ and

$$\begin{aligned} T(1) &= -\lambda_1\lambda_2\alpha + \lambda_2 + \lambda_1)\alpha - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \alpha + k\chi_1 + (1-k)\chi_2 \\ &= -\alpha(1-\lambda_1)(1-\lambda_2) + k\chi_1(1-\lambda_2) + (1-k)\chi_2(1-\lambda_2) \\ &= (1-\lambda_1)(1-\lambda_2)\left[\frac{k\sigma_\varepsilon}{\sqrt{\sigma_\varepsilon + \sigma_{\eta_1}^2(1-\rho)^2}} + \frac{(1-k)\sigma_\varepsilon}{\sqrt{\sigma_\varepsilon^2 + \sigma_{\eta_2}^2(1-\rho)^2}} - \alpha\right] < 0 \end{aligned}$$

Therefore there exists at least one real root inside the unit circle. Forming $|Z_1|$ and $|Z_2|$,

$$Z_1 = \begin{bmatrix} -\alpha\lambda_1\lambda_2 & \alpha(\lambda_1+\lambda_2) - k\chi_1 - (1-k)\chi_2 \\ -k\chi_1 - (1-k)\chi_2 & -\alpha\lambda_1\lambda_2 + (1-k)\lambda_1\chi_+ k\lambda_2\chi_1 + \alpha \end{bmatrix},$$

$$|Z_1| = \alpha^2[(\lambda_1\lambda_2)^2 - \lambda_1\lambda_2] + \alpha[k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2) + (1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2)] - (k\chi_1 + (1-k)\chi_2)^2$$

$$Z_2 = \begin{bmatrix} -\alpha\lambda_1\lambda_2 & \alpha(\lambda_1+\lambda_2) + k\chi_1 + (1-k)\chi_2 \\ k\chi_1 + (1-k)\chi_2 & -\alpha\lambda_1\lambda_2 - (1-k)\lambda_1\chi_2 - k\lambda_2\chi_1 - \alpha \end{bmatrix},$$

$$|Z_2| = \alpha^2[(\lambda_1\lambda_2)^2 + \lambda_1\lambda_2] - \alpha[k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2) + (1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2)] - (k\chi_1 + (1-k)\chi_2)^2$$

Given $n = 3$, we have $V \equiv V(|Z_1|, 1, |Z_2|)$ and $k = 3 - V - \tau$. Note that $T(1)T(-1) < 0$ and therefore $\tau = 1$ given $V = 1$, and $\tau = 0$ given $V = 0$ or $V = 2$. In order to show that $k = 1$, we must rule out the case that $V = 0$. Therefore, we must show that $|Z_1| > 0$ implies that $|Z_2| < 0$. To that end, we must first establish a few inequalities. First, define

$$\vartheta_1 = \frac{\sigma_\varepsilon}{\sqrt{\sigma_\varepsilon + \sigma_{\eta_1}^2(1-\rho)^2}}, \qquad \vartheta_2 = \frac{\sigma_\varepsilon}{\sqrt{\sigma_\varepsilon + \sigma_{\eta_2}^2(1-\rho)^2}}, \qquad \chi_1 = \vartheta_1(1-\lambda_1), \qquad \chi_2 = \vartheta_2(1-\lambda_2)$$

and suppose

$$\begin{gathered} k\chi_1 + (1-k)\chi_2 = k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2) + (1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2) \\ k\chi_1(1-\lambda_1-\lambda_2+\lambda_1\lambda_2^2) + (1-k)\chi_2(1-\lambda_1-\lambda_2+\lambda_1^2\lambda_2) = 0 \\ (1-\lambda_1)(1-\lambda_2)[k\vartheta_1(1-\lambda_1-\lambda_2\lambda_1) + (1-k)\vartheta_2(1-\lambda_2-\lambda_1\lambda_2)] = 0 \end{gathered}$$

which gives[15]

$$k^* = \frac{\vartheta_2(1-\lambda_2) - \vartheta_2\lambda_1\lambda_2}{\lambda_1\lambda_2(\vartheta_1 - \vartheta_2) - \vartheta_1(1-\lambda_1) + \vartheta_2(1-\lambda_2)}$$

Assuming $k^* \in (0,1)$ and noting

$$\frac{\chi_2\vartheta_1 - \chi_1\vartheta_2}{\chi_2 - \chi_1 + \lambda_1\lambda_2(\vartheta_1 - \vartheta_2)} = k^*\vartheta_1 + (1-k^*)\vartheta_2 < \alpha$$

gives the result

$$\begin{aligned} k^*(1-\lambda_1)\vartheta_1 + (1-k^*)(1-\lambda_2)\vartheta_2 &= k^*\chi_1(\lambda_1 + \lambda_2 - \lambda_1\lambda_2^2) + (1-k^*)\chi_2(\lambda_1 + \lambda_2 - \lambda_1^2\lambda_2) \\ &= \frac{(\chi_2\vartheta_1 - \chi_1\vartheta_2)\lambda_1\lambda_2}{\chi_2 - \chi_1 + \lambda_1\lambda_2(\vartheta_1 - \vartheta_2)} < \alpha\lambda_1\lambda_2 \qquad \text{(A.1.1)} \end{aligned}$$

Moreover, given

$$k < \frac{\vartheta_2(1-\lambda_2) - \vartheta_2\lambda_1\lambda_2}{\lambda_1\lambda_2(\vartheta_1 - \vartheta_2) - \vartheta_1(1-\lambda_1) + \vartheta_2(1-\lambda_2)}$$

yields

$$k\chi_1 + (1-k)\chi_2 < k\chi_1(\lambda_1 + \lambda_2 - \lambda_1\lambda_2^2) + (1-k)\chi_2(\lambda_1 + \lambda_2 - \lambda_1^2\lambda_2) < \alpha\lambda_1\lambda_2 \qquad \text{(A.1.2)}$$

where the last inequality follows due to $\chi_2 < \chi_1$.

Solving the equality

$$\alpha\lambda_1\lambda_2 = k\chi_1(\lambda_1 + \lambda_2 - \lambda_1\lambda_2^2) + (1-k)\chi_2(\lambda_1 + \lambda_2 - \lambda_1^2\lambda_2)$$

for k gives

$$k^{**} = \frac{\alpha\lambda_1\lambda_2 - \chi_2(\lambda_1 + \lambda_2 - \lambda_1^2\lambda_2)}{\chi_1(\lambda_1 + \lambda_2 - \lambda_1\lambda_2^2) - \chi_2(\lambda_1 + \lambda_2 - \lambda_1^2\lambda_2)}$$

Plugging k^{**} into $k\chi_1 + (1-k)\chi_2$ yields

$$\alpha\lambda_1\lambda_2 < \lambda_1\lambda_2[\alpha(\chi_1 - \chi_2) + \chi_2\chi_1(\lambda_2 - \lambda_1)] = k^{**}\chi_1 + (1-k^{**})\chi_2 \qquad \text{(A.1.3)}$$

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[15] It is interesting to note in passing that given $\lambda_1 = \lambda_2$ the above inequalities can be established via the unique properties associated with the golden ratio (i.e., the positive number that solves the equation $\phi + \phi^{-1} = 1$).

where the inequality can be established by noting

$$\vartheta_1(1-\lambda_1)[1-\vartheta_2(1-\lambda_2)\lambda_1]-\vartheta_2(1-\lambda_2)[1+\vartheta_1(1-\lambda_1)\lambda_2]<1<\alpha$$

Moreover, as $k > k^{**}$

$$\alpha\lambda_1\lambda_2 < k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2) < k\chi_1+(1-k)\chi_2 \tag{A.1.4}$$

The last inequality follows because $k^* < k^{**}$.

Inequalities (A.1.1) through (A.1.4) can be summarized as follows

$$k \in \begin{cases} (0,k^*) & k\chi_1+(1-k)\chi_2 < k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2) < \alpha\lambda_1\lambda_2 \\ (k^*,k^{**}) & k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2) < k\chi_1+(1-k)\chi_2 \\ (k^{**},1) & \alpha\lambda_1\lambda_2 < k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2) < k\chi_1+(1-k)\chi_2 \end{cases}$$

Now given $|Z_2| = \varepsilon$ where $\varepsilon > 0$ and small, then $|Z_1|$ is positive if and only if

$$\begin{gathered} 2\alpha[k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2)]-2\alpha^2\lambda_1\lambda_2+\varepsilon>0 \\ [k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2)] > \alpha\lambda_1\lambda_2-\frac{\varepsilon}{2\alpha} \end{gathered}$$

However, in order for $|Z_2|$ to remain positive, the above inequality implies

$$\alpha\lambda_1\lambda_2 > k\chi_1+(1-k)\chi_2$$

Notice that as ε approaches 0 we have the ordering[16]

$$k\chi_1+(1-k)\chi_2 < \alpha\lambda_1\lambda_2 < k\chi_1(\lambda_1+\lambda_2-\lambda_1\lambda_2^2)+(1-k)\chi_2(\lambda_1+\lambda_2-\lambda_1^2\lambda_2)$$

which is prohibited by (A.1.1)-(A.1.4).

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[16] This approach appeals to the continuity of roots; that is, $T(z)+\varepsilon$ will have the same number of roots inside the unit circle as $T(z)$ given ε sufficiently small (see Jury (1982)).

Appendix A.2

This section shows that the quadratic given by (3.20) has both roots outside of the unit circle. This implies that the representation (3.19) is fundamental, hence eliminating the need to forecast the forecasts of others. Recall the quadratic is given by

$$Y(z) = Qz^2 - Rz + S \tag{3.14}$$

$$Q = \lambda_1\lambda_2[k\chi_1(1-\lambda_2\theta) + (1-k)\chi_2(1-\lambda_1\theta)](1-\theta\rho)$$

$$R = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1\lambda_2 + (1-k)\chi_2\lambda_1] + (1-\theta\rho)[k\chi_1\lambda_1(1-\lambda_2\theta) + (1-k)\chi_2\lambda_2(1-\lambda_1\theta)]$$

$$S = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1 + (1-k)\chi_2]$$

First notice that Q, R and S are all strictly positive, implying that $Y(0) > 0$ and all the corresponding roots must be strictly positive. Moreover $Y(1) > 0$; ignoring $(1-k)\chi_2$ terms (due to symmetry) yields

$$\begin{aligned} Y(1) &= k\chi_1[\lambda_1\lambda_2(1-\lambda_2\theta)(1-\theta\rho) - (1-\lambda_1\theta)(1-\lambda_2\theta)\lambda_2 - (1-\theta\rho)\lambda_1(1-\lambda_2\theta) + (1-\lambda_1\theta)(1-\lambda_2\theta)] \\ &= k\chi_1(1-\lambda_2\theta)(1-\lambda_2)[(1-\lambda_1\theta) - \lambda_1(1-\theta\rho)] \end{aligned}$$

which is clearly positive because $(1-\lambda_1\theta) > \lambda_1(1-\theta\rho)$.

We now need to rule out the case that both roots lie inside the unit circle. We do this by showing that the minimum lies outside the unit circle.

$$z^* = \text{argmin } Y(z) = \frac{R}{2Q} > 1$$

Again focusing on $k\chi_1$, we need to show that

$$\begin{aligned} R - 2Q &= k\chi_1\lambda_2(1-\lambda_1\theta)(1-\lambda_2\theta) + (1-\theta\rho)k\chi_1\lambda_1(1-\lambda_2\theta) - 2[\lambda_1\lambda_2 k\chi_1(1-\lambda_2\theta)(1-\theta\rho)] > 0 \\ &= k\chi_1(1-\lambda_2\theta)\{\lambda_2[(1-\lambda_1\theta) - \lambda_1(1-\rho\theta)] + (1-\rho\theta)\lambda_1(1-\lambda_2)\} \end{aligned}$$

which is clearly positive.

Appendix B: Deriving a Fundamental Representation

Recall from (3.22) that the privately held information is given by the following system

$$\begin{bmatrix} \varepsilon_{1,t-2} \\ \varepsilon_{2,t-2} \\ v_{it} \end{bmatrix} = \begin{bmatrix} L^2 & 0 & 0 \\ 0 & L^2 & 0 \\ A(L) & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \eta_{it} \end{bmatrix}$$

$$\mathbf{x}_t = \mathbf{M(L)}\boldsymbol{\epsilon}_t \tag{A.1}$$

The covariance generating function of the above representation is given by

$$G_{\mathbf{x}}(z) = \mathbf{M(z)\Sigma M(z^{-1})}, \qquad \mathbf{\Sigma} = \begin{bmatrix} \sigma^2_{\varepsilon_1} & 0 & 0 \\ 0 & \sigma^2_{\varepsilon_2} & 0 \\ 0 & 0 & \sigma^2_{\eta_j} \end{bmatrix}$$

where $j = 1, 2$ contingent upon the group of investor. Following Rozanov (1967), we can find an equivalent (same covariance generating function) representation through the use of orthogonal matrices $\mathbf{B}(L)$ and $\mathbf{W}$. The corresponding fundamental representation is given by

$$\mathbf{x}_t = \mathbf{M}^*(\mathbf{L})\boldsymbol{\epsilon}^*_t \tag{A.2}$$

where

$$\mathbf{M}^*(\mathbf{L}) = \hat{\mathbf{M}}(L)\mathbf{W}\mathbf{B}(\mathbf{L}), \qquad \boldsymbol{\epsilon}^*_t = \mathbf{B(L^{-1})W'}\boldsymbol{\epsilon}_t$$

$$\hat{\mathbf{M}}(\mathbf{L}) = \begin{bmatrix} L^2\sigma_{\varepsilon_1} & 0 & 0 \\ 0 & L^2\sigma_{\varepsilon_2} & 0 \\ A(L)\sigma_{\varepsilon_1} & 0 & \sigma_{\eta_j} \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} -\frac{\sigma_{\eta_j}}{\sqrt{A_0^2\sigma^2_{\varepsilon_1}+\sigma^2_{\eta_j}}} & 0 & \frac{\sigma^2_{\varepsilon_1}A_0}{\sqrt{A_0^2\sigma^2_{\varepsilon_1}+\sigma^2_{\eta_j}}} \\ 0 & 1 & 0 \\ \frac{\sigma^2_{\varepsilon_1}A_0}{\sqrt{A_0^2\sigma^2_{\varepsilon_1}+\sigma^2_{\eta_j}}} & 0 & \frac{\sigma_{\eta_j}}{\sqrt{A_0^2\sigma^2_{\varepsilon_1}+\sigma^2_{\eta_j}}} \end{bmatrix} \quad \mathbf{B}(\mathbf{L}) = \begin{bmatrix} L^{-2} & 0 & 0 \\ 0 & L^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $\mathbf{B(L)}$ is a Blaschke matrix that removes the root at zero by "flipping" the root outside the unit circle, while the matrix $\mathbf{W}$ serves to ensure that the Blaschke matrix does not introduce any unwanted poles into the fundamental representation.[17] It is important to note that the orthogonal properties of the matrices $\mathbf{W}$ and $\mathbf{B(L)}$ (i.e.,$\mathbf{WW'} = I$ and $\mathbf{B(z)B(z^{-1})} = I$) ensure that the fundamental representation continues to emit the same covariance generating function. This representation is unique up to an orthogonal constant and implies that investors will have knowledge of a linear combination of the innovations ($\boldsymbol{\epsilon}^*_t$).

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[17] See Lippi and Reichlin (1994) for a good introduction to Blaschke matrices and fundamental representations.

References

Allen, F., S. Morris, and H. S. Shin (2003): "Beauty Contests, Bubbles and Iterated Expectations in Asset Markets," Cowles Foundation Discussion Paper No. 1406.

Amato, J. D., and H. S. Shin (2004): "Imperfect Common Knowledge and the Information Value of Prices," Working Paper, Bank for International Settlements.

Bacchetta, P., and E. van Wincoop (2004): "Can Information Heterogeneity Explain the Exchange Rate Puzzle?," Working Paper, Studienzentrum Gerzensee.

Barnett, S. (1983): *Polynomials and Linear Control Systems.* Marcel Dekker, New York.

Blanchard, O. J., and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305–1312.

Brockwell, P. J., and R. A. Davis (1991): *Time Series: Theory and Methods.* Springer Verlag, New York, second edn.

Foster, F. D., and S. Viswanathan (1996): "Strategic Trading when Agents Forecast the Forecasts of Others," *Journal of Finance*, 51(4), 1437–1478.

Futia, C. A. (1981): "Rational Expectations in Stationary Linear Models," *Econometrica*, 49(1), 171–192.

Hansen, L. P., and T. J. Sargent (1980): "Formulating and Estimating Dynamic Linear Rational Expectations Models," *Journal of Economic Dynamics and Control*, 2, 7–46.

——— (1991): "Two Difficulties in Interpreting Vector Autoregressions," in *Rational Expectations Econometrics*, ed. by L. P. Hansen, and T. J. Sargent. Westview Press.

He, H., and J. Wang (1995): "Differential Information and Dynamic Behavior of Stock Trading Volume," *Review of Financial Studies*, 8, 919–972.

Jury, E. I. (1982): *Theory and Application of the z-Transform Method.* John Wiley and Sons, Inc., New York.

Kasa, K. (2000): "Forecasting the Forecasts of Others in the Frequency Domain," *Review of Economic Dynamics*, 3, 726–756.

——— (2004): "Asset Pricing with Heterogeneous Beliefs: A Frequency-Domain Approach," mimeo, Simon Fraser University.

Keynes, J. M. (1936): *The General Theory of Employment, Interest and Money.* Macmillan, London.

Lippi, M., and L. Reichlin (1994): "VAR Analysis, Nonfundamental Representations, Blaschke Matrices," *Journal of Econometrics*, 63, 307–325.

Lucas, Jr., R. E. (1972): "Expectations and the Neutrality of Money," *Journal of Economic Theory*, 4, 103–124.

——— (1975): "An Equilibrium Model of the Business Cycle," *Journal of Political Economy*, 83, 1113–1144.

Muth, J. F. (1960): "Optimal Properties of Exponentially Weighted Forecasts," *Journal of American Statistical Association*, 55, 299–306.

Pearlman, J. G., D. Currie, and P. Levine (1986): "Rational Expectations Models with Private Information," *Economic Modelling*, 3(2), 90–105.

Pearlman, J. G., and T. J. Sargent (2005): "Knowing the Forecasts of Others," *Review of Economic Dynamics*, 8(2), 480–497.

Phelps, Edmund, S. (1983): "The Trouble with 'Rational Expectations' and the Problem of Inflation Stabilization," in *Individual Forecasting and Aggregate Outcomes*, ed. by R. Frydman, and E. Phelps, pp. 31–45. Cambridge University Press, Cambridge.

Rozanov, Y. A. (1967): *Stationary Random Processes.* Holden-Day, San Francisco.

Sargent, T. J. (1987): *Macroeconomic Theory.* Academic Press, 2 edn.

——— (1991): "Equilibrium with Signal Extraction from Endogenous Variables," *Journal of Economic Dynamics and Control*, 15, 245–273.

Sims, C. A. (2001): "Implications of Rational Inattention," Unpublished Paper, Princeton University.

Singleton, Kenneth, J. (1987): "Asset Prices in a Time Series Model with Disparately Informed, Competitive Traders," in *New Approaches to Monetary Economics*, ed. by W. Barnett, and K. Singleton. Cambridge University Press, Cambridge.

Townsend, R. M. (1983): "Forecasting the Forecasts of Others," *Journal of Political Economy*, 91, 546–588.

Wang, J. (1993): "A Model of Asset Prices under Asymmetric Information," *Review of Economic Studies*, 60(2), 249–282.

——— (1994): "A Model of Competitive Stock Trading Volume," *Journal of Political Economy*, 102, 127–168.

Whiteman, C. (1983): *Linear Rational Expectations Models: A User's Guide*. University of Minnesota Press, Minneapolis.

Whittle, P. (1983): *Prediction and Regulation by Linear Least-Square Methods*. University of Minnesota Press, Minneapolis.

Woodford, M. (2003): "Imperfect Common Knowledge and the Effects of Monetary Policy," in *Knowledge, Information, and Expectations in Modern Macroeconomics*, ed. by P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford. Princeton University Press, Princeton, N.J.