Quasi-Real Indexing

The Pareto-Efficient Solution to Inflation Indexing

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August 1, 2005

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ABSTRACT

In a pure-exchange economy with one good, stochastic aggregate demand and supply, and consumers having the same relative-risk aversion, Pareto efficiency requires each individual's consumption to be proportional to aggregate supply. While neither nominal contracts nor pure inflation-indexed contracts provide this proportionality, quasi-real contracts do. Quasi-real contracts adjust for aggregate-demand-caused inflation but not for aggregate-supply-caused inflation, causing their real obligations to be proportional to aggregate supply. When consumers differ in their relative risk aversion, or experience stochastic utility or endowment shocks, they will need insurance and other risk-transfer contracts in addition to quasi-real contracts.

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1. Introduction

Like many of you, we are travelers to other universes. We leave our chaotic and complex universe to travel to simpler and more comprehendible universes with hopes of learning lessons that apply to our own universe. We have traveled to the well-known universe with certainty and perfectly competitive markets. We have marveled at how orderly is this universe; how this universe follows the first and second fundamental theorems of welfare economics. We have also traveled to the universe discovered by Arrow (1953) and Debreu (1959) where all markets are perfectly competitive, uncertainty does exist, but so do state-contingent securities. As in the previous universe, order exists, markets are complete, and the first and second fundamental theorems of welfare economics apply.

We have tried to travel to universes where markets are incomplete in an environment of uncertainty. These universes are less orderly than the two previously mentioned universes, and they do not necessarily follow the first and second fundamental theorems of welfare economics. We have also traveled to universes where money exists, only to find them either unsatisfactorily structured or too complex to easily understand what we need or want to learn to apply to our own universe. However, recently we have traveled to a universe where the "essence of money" permeates the economy even though no one holds money from one period to the next and where order prevails even without state-contingent securities. In this universe, they have tried nominal contracts, which they have since discarded. They have also tried contracts that were indexed to inflation, which they later also discarded. Now they use quasi-real contracts as discussed by

Eagle and Domian (1995), and they are as happy as they can be; no one can be made better off without making someone else worse off. We have learned lessons from our journey that apply to our own universe. The quasi-real indexing of that universe could be applied to our bonds, wages, and social security as well as many other contracts.

We invite you to join us in a journey back to this "essence-of-money" (EOM) universe. That journey will be a lot easier than most journeys to universes with incomplete markets or to universes where money exists. In fact, the journey will be almost as easy as traveling to an Arrow-Debreu (A-D) universe. Because of the close similarities of the EOM and A-D universes, we will first stop (in section 2) at an A-D universe that has a pure-exchange competitive equilibrium economy and where consumers have identical, constant-relative-risk-aversion (CRRA) utility functions. There we will learn about a characteristic of Pareto-efficient consumption allocations in that universe that many of us may not have previously noticed. In section 3, we will complete our journey to the EOM universe. We will study the history of contracts in that universe beginning with prepaid future contracts, then nominal contracts and inflation-indexed contracts, and then ending with their now Pareto-efficient quasi-real contracts. In section 4, we will graphically study how changes in nominal aggregate demand and in real aggregate supply affect the real payments on these contracts.

During our return trip, we will stop in section 5 at a slightly different A-D universe. While consumers in the universes of sections 2 and 3 have identical, CRRA utility functions, the consumers in section 5's universe have different utility functions with differing coefficients of relative risk aversion. From this universe we will learn when quasi-real contracts will not be perfectly efficient, but you still may leave that universe with great respect for that type of indexing.

After our journey is complete, we will sit back in section 6 and contemplate the lessons we have learned and ponder how quasi-real indexing could be applied within our own universe to bonds, wage contracts, and social security. We will also reminisce about our journey and contemplate undertaking other journeys to similar universes. The appendixes contain some mathematical development of some tools to make our trans-universe journey easier and more enjoyable. Enjoy your trip!

2. A Layover in the Complete-Markets Universe

Don't worry about buckling your seat belts; our means of transport is a safe one – our imaginations. We first have a brief stop in a simple A-D universe. Imagine this universe with its pure exchange, perfectly competitive economy with one consumption good. This universe consists of an infinite series of nonoverlapping generations.[1] Every consumer of any particular generation lives the same T+1 periods, although different generations can have different lives. Our indexing of time will refer to the time of life in any particular generation. For example, time t=0 represents the first period of life for a particular generation. The consumers receive endowments that differ among consumers and among periods. Within a generation, all consumers have the same identical, additively separable, constant-relative-risk-aversion (CRRA) utility function, although relative risk aversion can vary among the generations. For each state i and time t combination, a state-contingent security (also called an Arrow-Debreu security) exists at time 0. The buyer of the security pays the price Ω_{it} to the seller at time 0. The seller of the

[1] We imagined no overlapping of generations to preclude any possible Pareto improvement from implementing a pay-as-you-go social security system, which would detract from the main points of this paper. Also, while the general logic we present could have just applied to one generation, we imagined a series of generations so that there could be a past, present, and future both for generations and for markets.

security agrees to deliver one unit of the consumption good to the buyer of the security if and only if state i occurs in period t.

Let π_{it} be the probability of state i occurring at time t, n_t be the number of states of nature possible at time t, and β is the common time discount factor.[2] Each consumer's optimization problem is to choose his/her current consumption and holding of state-contingent securities to maximize $U(c_{j0}) + \sum_{t=1}^{T}\sum_{i=1}^{n_t}\pi_{it}\beta^t U(c_{jit})$ subject to $P_0 c_{j0} + \sum_{t=1}^{T}\sum_{i=1}^{n_t}\Omega_{it}x_{jit} = P_0 y_{j0}$ and $c_{jit} = y_{jit} + x_{jit}$ for all i and t where c_{jit}, and y_{jit}, respectively represent consumer j's consumption and endowment for state i at time t. The variable x_{jit} represents j's demand at time 0 for the state-contingent security that pays in state i at time t. (A negative value for x_{jit} indicates j is selling the security.)

For any particular generation in this universe, each consumer's utility takes the CRRA form, which is U(c) = $\frac{c^{1-\gamma}}{1-\gamma}$ when γ in (0, ∞) but not equal to one, or when γ equals one U(c)=ln(c) where ln is the natural logarithm. The parameter γ is the constant coefficient of relative risk aversion. Some necessary conditions for consumer j's optimization problem to be satisfied are for all j, i, and t, $\frac{U'(c_{j0})}{P_0} = \frac{\beta^t \pi_{it} U'(c_{jit})}{\Omega_{it}}$. Since U'(c) = $c^{-\gamma}$, this implies that:

$$c_{jit} = \left(\frac{\beta^t \pi_{it} P_0}{\Omega_{it}}\right)^{1/\gamma} c_{j0} \qquad \textbf{(1)}$$

[2] The states of nature at time t can be nodes in a "tree" of states that could expand over time.

Summing both sides over all consumers and then dividing both sides by Y_0 gives

$\frac{Y_{it}}{Y_0} = \left(\frac{\beta^t \pi_{it} P_0}{\Omega_{it}} \right)^{1/\gamma}$ where Y_0 is aggregate supply at time 0 and Y_{it} is aggregate supply at time t under state i. Substituting this into equation (1) and rearranging slightly gives:

$$c_{jit} = \frac{c_{j0}}{Y_0} Y_{it} \qquad \textbf{(2)}$$

Since c_{j0} and Y_0 are known at time 0, (2) implies c_{jit} is proportional to Y_{it}. For example, if aggregate supply drops by 10%, then all individuals reduce their consumption by 10%. Since the equilibrium in an A-D economy is Pareto efficient, (2) is a necessary condition for Pareto efficiency in this A-D universe.

Having learned about the proportionality between any Pareto-efficient consumption allocation[3] and aggregate supply when all consumers have identical CRRA utility functions, we are now ready to leave this A-D universe to complete our journey to the EOM universe.

3. The "Essence-of-Money" (EOM) Universe

To transport yourself to the EOM Universe, imagine the same as the A-D universe we just visited except for three changes: First, there are no state-contingent securities. Second, each consumer's endowment is proportional to real aggregate supply in each period, meaning that that proportion (k_{jt}) does not depend on the level of real aggregate supply.[4] Third, no money is held from one period to the next, but there is an exogenous stochastic nominal aggregate demand.

[3] There is not just one Pareto-efficient consumption allocation, but a continuum of such allocations with each allocation corresponding to a given level of endowments for each state. However, (2) applies to all of these Pareto-efficient allocations.

[4] In a pure exchange competitive equilibrium economy, the uncertainty can only manifest itself into the endowments or nominal aggregate demand since all other variables are endogenous. The assumption that endowments are proportional to aggregate supply means only aggregate supply and nominal aggregate demand are uncertain.

We asked one of the local consumers how there could be nominal aggregate demand without money being held from one period to the next. She told us that every period they wake up to see their new individual endowments of type-M goods and a collective endowment of type-C goods. Both type-M and type-C goods perish within the period and are infinitely divisible. Only type-C goods enter the consumers' utility functions. As a result, the residents of this universe use the term "consumption good" to refer to the type-C goods. The rule the inhabitants have always followed and plan to continue to always follow is that each consumer is allotted consumption goods based on his/her relative endowment of type-M goods.

Define m_{jit} to be the type-M good endowed unto consumer j in state i at time t and M_{it} to be the aggregate of the type-M good at time t (i.e., $M_{it} \equiv \sum_j m_{jit}$). This entitles the consumer to receive $\frac{m_{jit}}{M_{it}} Y_{it}$ of the consumption good in state i at time t. For example, assume the total of the type-M and type-C goods over all consumers for a particular period are 1000 units and 20,000 units respectively. If consumer j is endowed with 2.4 units of type-M goods, then consumer j will be entitled to 0.24% of the 20,000 type-C units or 48 units. In the EOM universe, $\frac{m_{jit}}{M_{it}} = k_{jt}$, a constant, which means that k_{jt} does not vary across states $i=1..n_t$.

The local consumers refer to the type-M goods as "money." Like our fiat money, their money has no intrinsic value; rather its value derives from the fact that it is used to obtain the consumption good. Unlike our money, their money perishes after one period, and consumers are unable to hold it from one period to another. Since they use their money once and then it disappears, we call their money "temporary money."

The history of this universe tells of a time when everyone just consumed their endowments, no more or no less. They quickly learned that they could make everyone better off by entering into contracts, but consumers in this universe only have the mental capacity to understand one type of contract existing. They, therefore, are unable to create enough state-contingent securities to complete their markets.

The first type of contract they tried was what they called an "F contract". This F contract is similar to many futures contracts in our own universe, except the payment for the contract is made in the period the contract is written. The F contract is essentially a prepaid futures contract just as in our own universe there are prepaid forward contracts (See McDonald, 2003, pp. 199-120). At time 0, there will be a total of T contracts outstanding, one each for times 1, 2, …, T. Under an F contract written at time 0 for time t, the buyer pays the seller the price of the contract in terms of the consumption good at time 0. In return, the seller agrees to pay the buyer one unit of the consumption good at time t.

The existence of this contract did help improve welfare. A consumer who had no endowment one period used an F contract to in essence "borrow" some consumption good in that period in exchange for giving up some of the consumption good in a later period. Similarly, a consumer who had some endowment in the earlier period but who expected no endowment in the later period was able to "lend" some consumption good in the earlier period in exchange for receiving some of the consumption good in the later period. Both the borrower and the lender were made better off by the F contract.

However, the F contract was not Pareto efficient. Remember that these consumers have the same utility functions as do the consumers in the A-D universe we just visited. Therefore, a necessary condition for Pareto efficiency is that each individual's consumption be proportional to the aggregate supply for that period. As explained earlier, the ratio of individual j's endowment of the consumption good to aggregate supply at time t was k_{jt},. Therefore, if the consumer j did not rebalance his/her holdings of unexpired F contracts, his/her consumption would have equaled $c_{jit} = k_{jt}Y_{it} + F_{jt}$ where F_{jt} is j's demand at time 0 for the F contract that delivers at time t. Dividing by Y_{it} gives $\frac{c_{jit}}{Y_{it}} = k_{jt} + \frac{F_{jt}}{Y_{it}}$, which clearly shows that j's consumption is not proportional to aggregate supply.[5]

In the EOM universe when only F contracts existed, the buyer of the F contract was entitled to one unit of the consumption good at time t regardless of aggregate supply. If aggregate supply was lower than expected, the seller of the F contract was still obligated to pay the one consumption unit to the buyer. To do so the seller had to reduce his/her consumption more than proportionately to the lower aggregate supply. Thus, the buyer consumed more, and the seller consumed less than required by Pareto-efficiency.

On the other hand, if aggregate supply was higher than expected, the seller of the F contract consumed more and the buyer consumed less than required by Pareto-efficiency. In essence, the seller was insuring the buyer against aggregate supply decreasing. However, given that both the buyer and seller had the same coefficients of relative risk aversion, the buyer really did not want to buy nor did the seller want to sell this insurance. That this unwanted insurance was part of the F contract was what caused the consumers in the EOM universe to abandon it.

To remove this insurance, the consumers devised the "B contract" to replace the F contract. The B contract was the same as the F contract except the contract was in terms of

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[5] This represents the consumption in the last period of the generation's life when consumers would have had no other contracts. However, earlier in life, a consumer would have been able to buy or sell unexpired F contracts to help spread the impact of any unexpected change in aggregate supply to consumption over the rest of one's life. The

money (the type-M good) instead of the consumption good. For a B contract written at time 0 for delivery at time t, the buyer at time 0 paid the seller the price of the contract in terms of money. The seller then used the money to obtain more consumption good at time 0. In return, the seller agreed to pay one unit of money to the buyer at time t. At time t, the buyer then used the money to obtain more consumption good.

We met with a local consumer who described the sequence of events that occurred each period when they used the B contract. First, they woke up to their individual money endowments and the collectively received endowment of the consumption good. All consumers then knew the aggregate money supply and the aggregate supply of the consumption good. Second, the consumers entered into the B contracts. Third, the consumers exchanged money to meet their B-contract obligations for both the expiring contracts and the contracts they entered into this period. Finally, the consumers presented their money to determine their allotment that period of the consumption good.

Initially when consumers used the B contract, the aggregate money supply was constant from period to period. During this time, the B contract worked very well. In fact, the B contract was Pareto efficient during this time. Under the B contract, j's consumption in state i at time t was $c_{jit} = \frac{m_{jit} + B_{jt}}{M_{it}} Y_{it}$, where B_{jt} was j's demand at time 0 for the B contract with delivery at time t. Since $k_{jt} = \frac{m_{jit}}{M_{it}}$, this can be rewritten as:

$$c_{jit} = \left(k_{jt} + \frac{B_{jt}}{M_{it}} \right) Y_{it} \tag{3}$$

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ability to rebalance indicates a sequential economy, that markets are open not only at time 0 but later periods as well. The need to rebalance is itself an indication of the Pareto inefficiency of the market.

Since k_{jt} and B_{jt} were both constant at time t, c_{jit} was proportional to Y_{it} regardless of state i as long as the aggregate money supply did not change with state i. When the aggregate money supply was constant, the B contract enabled consumers to in essence "borrow" or "lend" money to achieve the same Pareto-efficient allocations as they would have had in the A-D universe. The purchasing power of one B contract equaled aggregate supply over the aggregate money supply. Therefore, when the money supply was constant, the purchasing power of a B contract increased or decreased proportionately to changes in aggregate supply.

It amazed us to learn that the B contract by itself was able to do the same as a large number of state-contingent securities. The nominal nature of the B contract resulted in the purchasing power of the contract changing when aggregate supply changed just exactly as needed for the welfare of the EOM universe.

Just when the consumers in this universe started to think they were as well off as they could be, the aggregate money supply began changing. In some periods, it rose unexpectedly; in others it unexpectedly decreased. This caused the purchasing power of a B contract to change for reasons other than changes in aggregate supply of the consumption good. As seen in equation (3), an unexpected increase in the money supply caused the buyer of a B contract at time 0 to consume less than planned in period t and the seller to consume more than planned. On the other hand, an unexpected decrease in the money supply caused the buyer to consume more and the seller to consume less than planned. Since these consumers were risk averse, they disliked this increased uncertainty concerning their consumption.[6]

To try to learn how to improve the B contract, the EOM universe sent some researchers to our universe to learn from us. While most of our universe was too confusing for the visitors

[6] As with the F contract, rebalancing one's portfolio could have helped spread the impact of these unexpected money supply changes over the remainder of one's life rather than the consumer absorbing the whole impact in one period.

to understand, they did find our economic literature more comprehendible. From this literature, they learned about inflation-indexing and upon returning to the EOM universe, they designed the "R contract". They first defined the price level as $P_{it} \equiv M_{it}/Y_{it}$.[7] For an R contract written at time 0 for time t, the buyer paid the seller the price of the contract at time 0 in terms of money. In return, the seller promised to pay the buyer at time t money equal to the ratio P_{it}/P_0.

After trying the R contract out, the EOM universe found they had just completed a circle – they were back to having the equivalent of their F contract. Regardless what happened to aggregate supply in time t, the seller of an R contract was guaranteeing the buyer a specific amount of the consumption good.

The EOM universe sent their researchers back to our universe to scour our economic literature for some guidance. Finally, they found quasi-real indexing (Eagle and Domian, 1995), which only adjusts for inflation caused by aggregate-demand shocks and not for inflation caused by aggregate-supply shocks. The researchers returned to the EOM universe and designed a new contract which they called a "Q contract".

In a Q contract written at time 0 for time t, the seller promises to pay the buyer money equal to the ratio of the money supply at time t over the money supply at time 0.[8] Where Q_{jt} is consumer j's demand at time 0 for the Q contract for delivery at time t, j's consumption in state i at time t is $c_{jit} = \left(\dfrac{m_{jit} + Q_{jt} \dfrac{M_{it}}{M_0}}{M_{it}} \right) Y_{it}$. Remembering that $k_{jt} = m_{jit}/M_{it}$, the above simplifies to:

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[7] In the EOM universe, velocity always equals one because money is used only once before it perishes.

[8] Eagle and Domian (1995) present quasi-real indexing using the ratio of nominal GDP to the base year's nominal GDP. In the EOM universe, since velocity equals one, nominal GDP equals the money supply.

$$c_{jit} = \left(k_{jt} + \frac{Q_{jt}}{M_0} \right) Y_{it}.$$

Because k_{jt}, Q_{jt} and M_0 are known at time 0, individual j's consumption at time t will be proportional to aggregate supply regardless of the level of aggregate supply or the money supply.

As shown in the appendix, the Q contract has in fact resulted in Pareto efficiency being achieved in the EOM universe.[9]

4. A Graphical Understanding of the F, B, R, and Q Contracts

Before we leave the EOM universe, we will use graphs to help us better understand the differences amount these contracts. Figures 1 and 2 show how the real payments on these contracts vary *ceteris paribus* with respectively nominal aggregate demand and aggregate supply. Each curve is identified by combinations of the letters F, B, R, and Q to denote the contracts that apply to the curve.

The F contract was very much like our commodity futures contract except the buyer of an F contract paid the seller when the contract was written instead of at the delivery date. Essentially, the F contract was a bond where payments were made in terms of the consumption good both when the contract was written and at delivery. As shown in Figure 1, the real payment at expiration of the F contract was immune to changes in nominal aggregate demand.

The real payment on an F contract also was immune to changes in aggregate supply as shown in Figure 2. The seller guaranteed one consumption good regardless of aggregate supply. However, to meet that guarantee, the lender had to decrease his/her consumption more than proportionately to any decrease in aggregate supply. (On the other hand, if aggregate supply increased, the lender kept all the increase.) Since Pareto efficiency requires consumers with identical CRRA utility to proportionately reduce (increase) their own consumption with decreases (increases) in aggregate supply, this immunity of the real payment of the F contract to changes in aggregate supply caused the EOM universe to discard it.

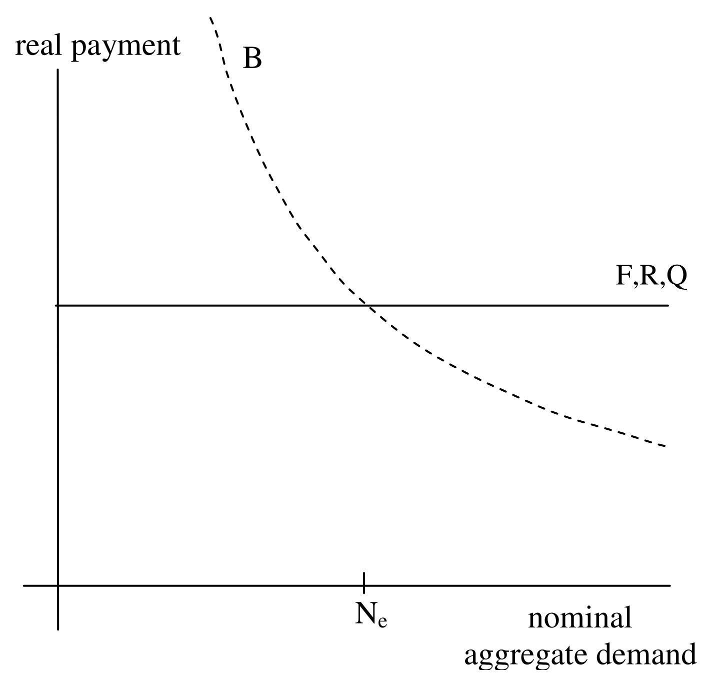

Figure 1: Real Payments on Contracts F, B, R, and Q as Functions of Nominal Aggregate Demand

The second contract the EOM universe tried was the B contract where the seller delivered one unit of money to the buyer at time t in exchange for the buyer paying the seller the contract's price at time 0. This B contract was equivalent to one of our nominal discount bonds or zero-coupon bonds. As shown in Figure 2, when nominal aggregate demand remains the same, the real payment on the nominal bond at time t varied proportionately with aggregate supply, enabling consumers to achieve a Pareto-efficient consumption allocation. For example, if aggregate supply decreased by 10%, the price level would rise by 10% causing the real payment on the nominal bond to decrease by 10%.

However, the downfall of these nominal bonds was in how they varied with nominal aggregate demand. As shown in Figure 1, when aggregate supply remained the same, the real

[9] Because the Q contract is Pareto efficient, markets need only be open at time 0 of each generation; Even if markets are open later in life, no one would want to make any trades.

payment on the bond decreased with nominal aggregate demand. When nominal aggregate demand exceeded its expected value (N_e), the price level unexpectedly increased, causing the real payment on the bond to decrease. This lower real payment made the borrower better off (i.e., higher real consumption) and the lender worse off. On the other hand, if nominal aggregate demand was less than expected, the real payment on the bonds increased, causing the borrower to be worse off and the lender better off. Since they are risk averse, both borrowers and lenders prefer a certain real payment over the uncertain real payments on B contracts caused by changes in nominal aggregate demand.

The next contract the EOM universe tried was the R contract, which was the equivalent of an inflation-indexed discount bond, which we sometimes refer to as a "real" bond. However, this real bond turned out to be essentially the same as the F contract. As shown in Figures 1 and 2, changes in neither nominal aggregate demand nor aggregate supply affected the real payment on a real bond. As with the F contract, the real bond failed to enable consumers to reach Pareto-efficient consumption allocations because the lender of a real bond guaranteed the real payment to the borrower even when aggregate supply changed.

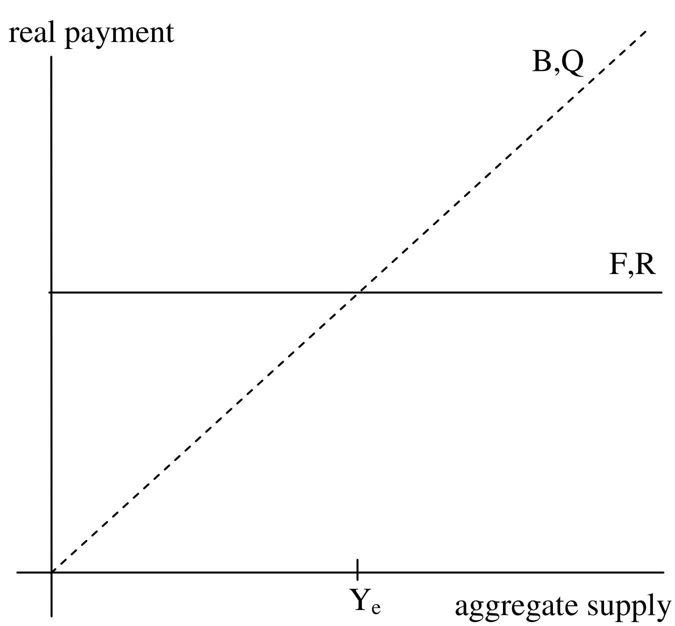

Figure 2: Real Payments of Contracts F, B, R, and Q as Functions of Aggregate Supply

The contract the EOM universe now uses is the Q contract, which essentially is a bond indexed for aggregate-demand-caused inflation or deflation but not for aggregate-supply-caused inflation or deflation. We will call this bond a "quasi-real bond" as did Eagle and Domian (1995). The real payment on a quasi-real bond is immune to changes in nominal aggregate demand as shown in Figure 1. On the other hand, the real payment on a quasi-real bond changes proportionately to changes in aggregate supply just as with a nominal bond (see Figure 2). Having real payments that are immune to changes in nominal aggregate demand, but proportional to aggregate supply is exactly what is needed to make the quasi-real bonds Pareto efficient when all consumers have identical CRRA utility functions.

In summary, Figures 1 and 2 show the following bond properties which do not depend on any utility assumptions:

1. The real payments on nominal bonds decrease with increases in nominal aggregate demand and increase proportionately with increases in aggregate supply.
2. The real payments on inflation-indexed bonds are immune to changes both in nominal aggregate demand and in aggregate supply.
3. The real payments on quasi-real bonds are immune to changes in nominal aggregate demand but increase proportionately with increases in aggregate supply.

5. A Layover in Another A-D Universe Without CRRA Utility

Having learned that in the EOM universe, problems existed with both nominal and inflation-indexed bonds, but that quasi-real bonds are Pareto efficient, we are now ready to leave this universe. As we leave, we wonder if the lessons we have learned here are applicable to our own universe. We wonder whether the Pareto efficiency of quasi-real bonds will persist beyond an economy with consumers having identical CRRA utility functions. We ponder whether quasi-real bonds in our universe will be able to duplicate the consumer allocation that would result from an economy with complete markets of state-contingent securities. To help answer that question, we now stop at a different A-D universe on our return trip and learn more about how a Pareto-efficient consumption allocation changes with aggregate supply when consumers differ in their risk aversion.

Imagine the A-D universe of section 2 with one change — consumers in this A-D universe have different utility functions. These utility functions are still all well behaved, i.e., they are risk averse, continuous, twice differentiable, and $\lim_{c \to 0} U(c) = -\infty$.[10] In this universe, bonds are not needed since state-contingent securities provide complete markets. However, this universe could help us think about how nominal, inflation-indexed, and quasi-real bonds would meet consumers' needs if these state-contingent securities did not exist.

Each consumer j in this universe has the following utility function:

$$U_{j0}(c_{j0}) + \sum_{t=1}^{T} \beta^t \sum_{i=1}^{n_t} \pi_{ji} U_{jt}(c_{jit}) \qquad \textbf{(4)}$$

where β is the time discount factor in common to all consumers. Since different consumers can have different utility functions at different times, the common time discount factor is not really a restrictive assumption.[11]

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[10] The latter assumption means that the utility function is unbounded from below as is the CRRA utility function. If we were dealing with continuous probability distribution functions, this would create a problem (see Arrow, 1965). However, throughout this paper, we assume discrete distributions. Nevertheless, some may criticize us including this latter assumption in the term "well behaved". We did so in order to preclude a corner solution for consumption in any period equaling zero.

[11] Suppose for complete generality the discount factor varies by consumer, by state, and by time; that instead of $\beta^t U_{jt}(c_{jit})$, consumer j's utility is $\beta^t_{jit} \tilde{U}_{jt}(c_{jit})$ where $\tilde{U}$ denotes the true utility function. If we set our beta equal to one and defined $U_{jt}(c_{jit}) \equiv \beta^t_{jit} \tilde{U}_{jt}(c_{jit})$, our formulation would take this situation into account. We chose to leave a constant time preference factor in our formulation to make it consistent with our including a time preference factor for the EOM universe.

Consumers in this A-D universe maximize (4) subject to the constraints that

$P_0 c_{j0} + \sum_{t=1}^{T}\sum_{i=1}^{n_t}\Omega_{it}x_{jit} = P_0 y_{j0}$ and $c_{jit} = y_{jit} + x_{jit}$ for all states i and future time periods j.

A necessary condition for this optimization problem to be satisfied for all consumers is that for all i, j, and t, $\frac{U'_{j0}(c_{j0})}{P_0} = \frac{\beta^t \pi_{it} U'_{jt}(c_{jit})}{\Omega_{it}}$, which implies

$$\frac{U'_{jt}(c_{jit})}{U'_{j0}(c_{j0})} = \frac{\Omega_{it}}{\beta^t \pi_{it} P_0} \text{ for all i, j, and t} \qquad \textbf{(5)}$$

Since j's Pareto-efficient consumption is a function solely of aggregate supply,[12] define the implicit function $\tilde{c}_{jt}(Y_t)$ to be how the Pareto-Efficient consumption by individual j at time t depends on aggregate supply.[13] It is extremely important to recognize $\tilde{c}_{jt}(Y_t)$ as a reduced form; it is not the structural consumption function. To help us avoid this confusion, we will refer to Y_t as aggregate supply at time t, not income.

Define $\tilde{a}_{jt}(Y_t) \equiv -\frac{U''_{jt}(\tilde{c}_{jt}(Y_t))}{U'_{jt}(\tilde{c}_{jt}(Y_t))}$, which is the function of how the coefficient of absolute risk aversion varies with aggregate supply. Define $\tilde{\rho}_{jt}(Y_t) \equiv \tilde{c}_{jt}(Y_t)\cdot \tilde{a}_{jt}(Y_t)$, which is the function of how the relative risk coefficient varies with aggregate supply. Also, define

[12] To see that j's Pareto-efficient consumption allocation is solely a function of aggregate supply, let state 1 and state 2 be any two states where aggregate supply are the same. State 2 could still differ from state 1 because of a different distribution of endowments or different probabilities. Set $\Omega_{2t} = \Omega_{1t}\pi_{2t}/\pi_{1t}$. If $c_{j2t} = c_{j1t}$ for all j, then if (5) holds for all j when i=1 then it also holds for all j when i=2 . Also, if $c_{j2t} = c_{j1t}$ for all j, then if markets clear for state 1 then they clear for state 2. Therefore, if $\tilde{c}_{jt}$ for j=1..m is the optimal consumption for one state, it is also the optimal consumption for another state with the same level of aggregate supply. Therefore, the competitive-equilibrium consumption allocation in an Arrow-Debreu economy is a function solely of aggregate supply.

[13] There is not just one Pareto-efficient consumption allocation, but rather a continuum of such allocations, each corresponding to a particular allocation of endowments across states. We can think about this Pareto-efficient consumption allocation as the one that corresponds to the existing allocation of endowments.

$\overline{\rho}_t(Y_t) \equiv \sum_{j=1}^{m}\left(\tilde{\rho}_{jt}(Y_t)\cdot\frac{d\tilde{c}_{jt}}{dY_t}\right)$, which is the weighted average of the relative risk coefficients using the derivatives of $\tilde{c}_{jt}(Y_t)$ as the weights. Finally, define $\tilde{\alpha}_j(Y_t) \equiv \frac{\tilde{\rho}_{jt}(Y_t)}{\overline{\rho}_t(Y_t)}$, which is how j's relative risk coefficient compares to the average relative risk coefficient.

For the general Arrow-Debreu pure-exchange economy, the appendix derives the following relationship:

$$\tilde{c}'_{jt}(Y_{it}) = \frac{1}{\tilde{\alpha}_{jt}(Y_{it})}\frac{\tilde{c}_{jt}(Y_{it})}{Y_{it}} \qquad \textbf{(6)}$$

If aggregate supply decreases by 1%, then (6) says that the Pareto-efficient consumption will decrease by half a percent for someone who has twice the average relative risk aversion, whereas it will decrease by 2% for someone having half the average relative risk aversion. By decreasing their consumption more than proportionately, the lower (relatively) risk-averse consumers are enabling the higher risk-averse consumers to reduce their consumption less than proportionately. In essence, the lower risk-averse consumers are providing insurance to the higher risk-averse consumers.[14]

This result will help answer some questions concerning the extent to which quasi-real bonds can help enable consumers to reach their Pareto-efficient consumption allocations when consumers differ in their relative risk aversion. However, it is time to return home from our journey.

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[14] The relationship in (6) is related to one derived by Viard (1993), although he assumed that all income was derived from past investments in risky assets whereas we assume all income comes from endowments.

6. Journey Reflections

Now let's open our eyes, be realistic, and return to our own universe. We see that money in our universe does not perish; we can hold it from one period to the next. We see many inflation-indexed contracts. We see governments issuing inflation-indexed bonds such as TIPS by the U.S. and inflation-linked gilts by the U.K. Governments index social security for inflation. Both governmental and business organizations often index pension funds and wages for inflation.

We do not, however, see quasi-real indexing in our universe. Quasi-real bonds with no other contracts enabled the EOM universe to achieve Pareto efficiency. Can quasi-real indexing help us improve economic efficiency?

Both risk-averse borrowers and risk-averse lenders will be better off if we eliminate any uncertainty caused by changes in nominal aggregate demand. Since the only difference between quasi-real bonds and nominal bonds is how their real payments are affected by nominal aggregate demand, quasi-real bonds are Pareto superior to nominal bonds. More generally, we conclude that quasi-real indexing is Pareto superior to no indexing for inflation.[15]

On the other hand, we cannot say that pure inflation-indexed bonds are Pareto superior to nominal bonds. While pure inflation-indexed bonds do adjust for aggregate-demand-caused inflation (or deflation), they also destroy the proportionality of the real payments of nominal bonds to aggregate supply which was so valuable in the EOM universe. Perhaps this is a reason that our economies have not embraced inflation indexing more than they have. As Adolph and Wolstetter (1991) state, "Roughly until the mid-1970s, most economists recommended full indexation as a cure to deal with the ills of unanticipated inflation. … But then came the oil-price shock of the early 1970s, and it became painfully clear that unanticipated inflation can be caused by unpredictable real as well as monetary distortions. Subsequently indexation was looked at with less enthusiasm."

The wage indexation literature founded by the work of Gray (1976) and Fischer (1977) followed the oil shocks of the 1970s. This literature recognized the difference between aggregate-demand-caused inflation and aggregate-supply-caused inflation. While this literature did not explicitly state that aggregate-demand-caused inflation is bad and aggregate-supply-caused inflation is good, it did conclude that (i) full inflation indexing should occur when only aggregate-demand shocks and no aggregate-supply shocks were possible, and (ii) no inflation indexing should occur if only aggregate-supply shocks and no aggregate-demand shocks were possible.[16] For more on this literature, see Dornbusch and Simonsen (1993), Adolph and Wolstetter (1991), Landerretche, Lefort, and Valdés (2002), and Heinemann (2004).

When both aggregate-demand and aggregate-supply shocks are possible, the wage indexation literature concluded that partial indexing is optimal. For example, assume that people expect that aggregate-demand or aggregate-supply shocks are equally likely. Then the optimum, according to the wage indexation literature, is for contracts to be half indexed. With such half indexing, a 10% inflation rate would contractually lead to a 5% increase in nominal wages. This 5% increase would result regardless whether the inflation was caused by an aggregate-demand or an aggregate-supply shock.

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[15] Some have expressed concerns that central banks may lose their incentive to keep inflation under control if there is inflation indexing, which equally applies to quasi-real indexing. However, if the central bank were committed to nominal income targeting aimed at zero or low inflation in the long run, this would not be an issue.

[16] Gray's (1976) and Fischer's (1978) analyses found that full indexation reduced output fluctuations due to nominal shocks but increased output fluctuations due to real shocks. Their analyses differ from our analysis because rather than using Pareto-efficiency as the criterion for evaluating the welfare impacts of these shocks, they followed the standard in macroeconomic literature, which is to assume an ad-hoc objective function that minimizes a weighted average of the variance of inflation from its target (or the variance of the price level from its target) and the square

Our journey to the EOM universe in this paper has enabled us to see the Pareto implications of indexing. In the EOM universe, partial indexing would not have been Pareto efficient, but we saw that quasi-real indexing is Pareto efficient. While pure inflation indexing cannot distinguish *ex post* between aggregate-demand-caused and aggregate-supply-caused inflation, quasi-real indexing can. If a 10% inflation rate is solely caused by aggregate demand (which would show up as a 10% increase in nominal GDP), quasi-real indexing would increase the nominal payments on a contract by 10%. On the other hand, if a 10% inflation rate is solely due to a decrease in aggregate supply (meaning that nominal GDP does not change), then quasi-real indexing would not change the nominal contractual payments. If inflation increases 10% when NGDP increases 5% (meaning inflation is equally affected by aggregate-demand and aggregate-supply shifts), then quasi-real indexing would increase the nominal contract payment by 5%.

In the EOM universe, quasi-real indexing is Pareto efficient. However, the EOM universe has several simplifying characteristics. In the EOM universe, the variable k_{jt}, which represents the ratio of j's endowment to aggregate supply, is not stochastic meaning that k_{jt} does not vary across states of nature. Also, there are no utility shocks and all consumers have the same relative risk aversion in the EOM universe.

Eagle (2004) analyzes a general pure-exchange economy. His analysis allows stochastic endowment-to-real-aggregate-supply ratios, utility shocks, and differences in relative risk aversion among consumers. He finds that four types of contracts can approximately complete markets: (1) endowment-sharing contracts, (2) spending-sharing contracts, (3) real-aggregate-supply-risk-transfer (RASRT) contracts, and (4) "normal contracts." The first two types of

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of output from its target or potential (i.e. output gap). This differs from our analysis because we emphasize Pareto efficiency and we assume a pure-exchange economy where output is exogenous.

contracts are insurance contracts that deal with the unique or unsystematic risk associated with individuals' endowment-to-real-aggregate-supply ratios or their spending needs. The RASRT contracts are similar to Shiller's (1993) proposed GDP futures, which enable consumers with above average relative risk aversion to transfer some real-aggregate-supply risk to consumers having below average relative risk aversion. Eagle defines "normal contracts" as contracts whose real payments are proportional to real aggregate supply. Quasi-real contracts are normal contracts. Therefore, even in a more general economy, quasi-real contracts combined with endowment-sharing contracts, spending-sharing contracts, and RASRT contracts do lead to Pareto-efficiency.

Here on Earth, some contracts are purely inflation indexed, but none have yet to be quasi-real indexed. In order for pure inflation indexing to be Pareto superior to quasi-real indexing, the recipients of the contract's payments would need to have much more relative risk aversion than those who made the payments on these contracts. In essence with purely inflation-indexed contracts, there is a transfer of real-aggregate-supply risk from those receiving the payments to those making the payments. We believe that with most inflation-indexed contracts those who will be making the payments are unaware of the risk they are exposing themselves to as a result of the indexing.

For example, consider a government that issues inflation-indexed bonds and that inflation indexes many of its other obligations such as social security, pension funds for its workers, and its employees' wages. Suppose aggregate supply fell in half for some reason such as war, a terrorist attack, or a natural phenomenon like volcanic activity or a meteorite hitting the earth. Prices would then double if the central bank did not allow nominal aggregate demand to change. The government would then have to double its nominal payments on its inflation-indexed obligations. If the government neither increases its borrowing nor increases taxes (at a time when consumers are already struggling with less), then the government would have to cut its non-inflation-indexed obligations much more than proportionately in order to meet its inflation-indexed obligations. In fact, if half the government's budget was inflation indexed, this 50% reduction in aggregate supply and doubling of the price level would force the government to completely eliminate all of its non-inflation-indexed spending. Is this really what the government wants? Is this really what the taxpayers/beneficiaries want?[17]

Many have criticized governments for providing insurance. We believe that few realize the degree to which governments expose themselves to risk through their inflation-indexed obligations. If we take the insurance out of these inflation-indexed obligations, we would then have quasi-real indexing.

With pure inflation indexing, we multiply the obligation by the ratio of the current price level divided by the price level in the base year. Eagle and Domian (1995) propose achieving quasi-real indexing by multiplying the obligation by the ratio $\frac{N_t / N_0}{(1+g)^t}$ where N_t is nominal GDP in period t and g is the expected long-term growth rate in real GDP which would be stated in the quasi-real-indexed contract. This setup will result with adjustments to the obligation being made when nominal GDP increases at a rate greater than or less than g. Under rational expectations, the estimate of g is not important to the efficiency of the quasi-real bonds. If the market has a

[17] Some public finance literature does assume the government is risk neutral. However, if the taxpayers and beneficiaries are risk averse, and since those taxpayers and/or beneficiaries would have to absorb any undiversifiable risk that the government exposes itself to, a government acting in a risk neutral manner would be contrary to the preferences of taxpayers and/or beneficiaries. While the government may be able to diversify away some risk, one must only look at the corporate finance literature to realize that there is some risk (called systematic risk) that cannot be diversified away (See Brealey, Myers, and Marcus, 2005, p. 283). In particular, the risk related to the level of aggregate supply cannot be diversified away.

different estimate of g, the market will compensate for that difference through the quasi-real interest rate.

However, other practical issues concerning quasi-real indexing do need attention. For example, the timing of nominal GDP information, the lag of that information, the degree to which that information is subject to revision, and the reliability of that information all are important issues for us to consider as we try to develop quasi-real indexing for economies in our universe. Also, theoretically quasi-real indexing should be based on a measure of nominal aggregate demand rather than nominal income. The primary purpose of this paper is to discuss the theoretical desirability of quasi-real indexing; we will leave most of the issues related to the practical implementation of this indexing method to future research.

Many years ago, medical experts thought all cholesterol was bad; now they talk about good cholesterol and bad cholesterol. Most of the previous economic literature, with the exception of the wage indexation literature, has treated all inflation as bad. However, the EOM universe viewed inflation as either good or bad. We believe that the EOM universe's good-bad inflation view applies as much to our own universe as to theirs. Inflation caused by nominal aggregate demand shocks is bad and needs to be filtered out. However, inflation caused by aggregate supply shocks is good because it helps to efficiently reallocate consumption to changes in aggregate supply.

While this paper was able to generalize from identical CRRA utility to situations where consumers have different relative risk aversion, we did restrict our journey to universes with pure-exchange, closed economies. Future researchers should attempt to travel to similar universes that include production and open economies.

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APPENDIX

This appendix first derives the relationship (6) which shows how Pareto-Efficient consumption is related to relative risk aversion. Secondly, it shows that quasi-real bonds do lead to Pareto Efficiency in the EOM universe.

Relative Risk Aversion and Pareto-Efficient Consumption

Since equation (5) is true for all j,

$$\frac{U'_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} = \frac{U'_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} \qquad \textbf{(A1)}$$

for j=2..m. Totally differentiating (A1) with respect to Y_t gives $\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})}\frac{d\tilde{c}_{jt}}{dY_t} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})}\frac{d\tilde{c}_{1t}}{dY_t}$.

If we divide the left and right sides of this by the left and rights sides of (A1) respectively, we get:

$$\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})}\frac{d\tilde{c}_{jt}}{dY_t} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})}\frac{d\tilde{c}_{1t}}{dY_t} \qquad \textbf{(A2)}$$

Since $\tilde{a}_{jt}(Y_t) \equiv -\frac{U''_{jt}(\tilde{c}_{jt}(Y_t))}{U'_{jt}(\tilde{c}_{jt}(Y_t))}$, multiplying both sides of (A2) by a minus sign and rearranging slightly gives:

$$\frac{\frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}} = \frac{\tilde{a}_{1t}}{\tilde{a}_{jt}} \qquad \textbf{(A3)}$$

By summing both sides of (A3) over all consumers, we get:

$$\frac{\sum_{j=1}^{m}\frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}} = \tilde{a}_{1t}\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}} \qquad \textbf{(A4)}$$

By equilibrium in the market for the consumption good at time t, $\sum_{j=1}^{m} c_{jt} = Y_t$, which also implies

that $\sum_{j=1}^{m}\frac{dc_{jt}}{dY_t} = 1$. Therefore, solving (A4) for $\frac{d\tilde{c}_{1t}}{dY_t}$ gives $\frac{d\tilde{c}_{1t}}{dY_t} = \frac{1/\tilde{a}_{1t}}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}}$. This and (A3) imply

that the following is true for all j.

$$\frac{d\tilde{c}_{jt}}{dY_t} = \frac{1/\tilde{a}_{jt}}{\sum_{s=1}^{m}\frac{1}{\tilde{a}_{st}}} \qquad \textbf{(A5)}$$

This result was first derived by Wilson (1968, see his theorem 5).

Next, we need to determine the value of $\bar{\rho}_t(Y_t)$. The following starts out with the definition of $\bar{\rho}_t(Y_t)$, then substitutes in the definition of $\tilde{\rho}_{jt}(Y_t)$ and the result in (A5):

$$\bar{\rho}_t(Y_t) \equiv \sum_{j=1}^{m}\left(\tilde{\rho}_{jt}(Y_t)\cdot\frac{dc_{jt}}{dY_t}\right) = \sum_{j=1}^{m}\left(\tilde{c}_{jt}\tilde{a}_{jt}\cdot\frac{\frac{1}{\tilde{a}_{jt}}}{\sum_{k=1}^{m}\frac{1}{\tilde{a}_{kt}}}\right) = \frac{\sum_{j=1}^{m}\tilde{c}_{jt}}{\sum_{k=1}^{m}\frac{1}{\tilde{a}_{kt}}}$$

However, the sum of consumption across all consumers in this pure exchange economy equals aggregate supply for that period. Therefore,

$$\bar{\rho}_t(Y_t) = \frac{Y_t}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}} \qquad \textbf{(A6)}$$

Now we are ready to derive (6). From the definition of $\tilde{\alpha}_j \equiv \frac{\tilde{\rho}_{jt}(Y_t)}{\overline{\rho}_t(Y_t)}$, we can

write $\tilde{\rho}_{jt} = \tilde{\alpha}_j \overline{\rho}_t$ and then replace $\tilde{\rho}_{jt}$ with $\tilde{c}_{jt}\tilde{a}_{jt}$ and $\overline{\rho}_t$ with (A6) to get $\tilde{c}_{jt}\tilde{a}_{jt} = \tilde{\alpha}_{jt} \frac{Y_t}{\sum_{j=1}^{m} \frac{1}{\tilde{a}_{jt}}}$.

Dividing both sides by $\tilde{a}_{jt} Y_t$ gives $\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_{jt} \frac{\frac{1}{\tilde{a}_{jt}}}{\sum_{j=1}^{m} \frac{1}{\tilde{a}_{jt}}}$. Using (A5), we can rewrite this as:

$\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_{jt} \frac{d\tilde{c}_{jt}}{dY_t}$. Dividing both sides by $\tilde{\alpha}_{jt}$ gives us (6).

Pareto Efficiency of Quasi-real Bonds in the EOM Economy

This section of the appendix shows that, when consumers have the same identical CRRA utility functions, the EOM economy with quasi-real bonds is Pareto efficient, that its consumption allocation is the same as the Arrow-Debreu economy with complete markets.

We will symbolize the expected value of a random variable X_{it} as $E[X_{*t}]$ where the "*" indicates that we are doing this expectation over all possible i=1..T. When consumers have the same identical CRRA utility functions, the consumption allocation for the Arrow-Debreu economy is:

$$c_{jo} = \frac{y_{j0} + Y_0^{\gamma} \sum_{t=1}^{T} \beta^t k_{jt} E\left[Y_{*t}^{1-\gamma}\right]}{1 + Y_0^{\gamma-1} \sum_{t=1}^{T} \beta^t E\left[Y_{*t}^{1-\gamma}\right]} \qquad \textbf{(A7)}$$

and for t=1..T and for i=1..n_t

$$c_{jit} = \frac{Y_{it}}{Y_0} c_{j0} \qquad \textbf{(A8)}$$

where the price of the state-contingent securities are

$$\Omega_{it} = P_0 \beta^t \pi_{it} \left(Y_0 \middle/ Y_{it} \right)^{\gamma} \tag{A9}$$

and the quantity demanded of each Arrow-Debreu security is

$$x_{jit} = c_{jit} - y_{jit} \tag{A10}$$

Below we briefly sketch a proof that equations (A7) through (A10) represent the Arrow-Debreu equilibrium. Without loss of generality, let $P_0 = 1$.

The next two lines use that $U'(c) = c^{-\gamma}$, (A8), and (A9) to show that the left and right sides of (5) are equal, which mean that the marginal conditions of each consumer's optimization problem are satisfied:

$$\frac{U'_{jt}(c_{jit})}{U'_{j0}(c_{j0})} = \left(\frac{c_{j0}}{c_{jit}} \right)^{\gamma} = \left(\frac{Y_0}{Y_{it}} \right)^{\gamma}$$

$$\frac{\Omega_{it}}{\beta^t \pi_{it} P_0} = \frac{P_0 \beta^t \pi_{it} \left(Y_0 \middle/ Y_{it} \right)^{\gamma}}{\beta^t \pi_{it} P_0} = \left(\frac{Y_0}{Y_{it}} \right)^{\gamma}$$

The constraints of the optimization problem can be reduced into the following single constraint:

$$P_0 c_{j0} + \sum_{t=1}^{T} \sum_{i=1}^{n_t} \Omega_{it} c_{jit} = W_0 = P_0 y_{j0} + \sum_{t=1}^{T} \sum_{i=1}^{n_t} \Omega_{it} y_{jit}$$

Substituting (A9) for Ω_{it}, (A8) for c_{jit}, and eventually (A7) for c_{j0} into the left side of the constraint gives:

$$P_0 c_{j0} + \sum_{t=1}^{T} \sum_{i=1}^{n_t} \Omega_{it} c_{jit} = P_0 c_{j0} + \sum_{t=1}^{T} \sum_{i=1}^{n_t} P_0 \beta^t \pi_{it} \left(Y_0 \middle/ Y_{it} \right)^{\gamma} \frac{Y_{it}}{Y_0} c_{j0} = P_0 c_{j0} \left(1 + Y_0^{\gamma - 1} \sum_{t=1}^{T} \beta^t \sum_{i=1}^{n_t} \pi_{it} Y_{it}^{1-\gamma} \right)$$

$$= P_0 c_{j0}\left(1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]\right) = P_0 \frac{y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t k_{jt} E\left[Y_{*t}^{1-\gamma}\right]}{1 + Y_0^{1-\gamma}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]}\left(1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]\right)$$

$$= P_0\left(y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t k_{jt} E\left[Y_{*t}^{1-\gamma}\right]\right)$$

Substituting (A9) for Ω_{it} and $y_{jit} = k_{jt}Y_{it}$ into the right side of the constraint gives:

$$P_0 y_{j0} + \sum_{t=1}^{T}\sum_{i=1}^{n_t}\Omega_{it}\, y_{jit} = P_0 y_{j0} + \sum_{t=1}^{T}\sum_{i=1}^{n_t} P_0\beta^t \pi_{it}\left(Y_0 / Y_{it}\right)^{\gamma} k_{jt} Y_{it} = P_0\left(y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t k_{jt}\sum_{i=1}^{n_t}\pi_{it} Y_{it}^{1-\gamma}\right)$$

$$= P_0\left(y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t E[Y_{it}^{1-\gamma}]\right)$$

which shows that the left and right sides of the constraint equal each other.

The goods market for time 0 clears as shown below where we substitute (A7) for c_{j0}:

$$\sum_{j=1}^{m} c_{j0} = \sum_{j=1}^{m}\frac{y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t k_{jt} E\left[Y_{*t}^{1-\gamma}\right]}{1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]} = \frac{\sum_{j=1}^{m} y_{j0} + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t\left(\sum_{j=1}^{m} k_{jt}\right) E\left[Y_{*t}^{1-\gamma}\right]}{1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]}$$

$$= \frac{Y_0 + Y_0^{\gamma}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]}{1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]} = \frac{Y_0\left(1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]\right)}{1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E\left[Y_{*t}^{1-\gamma}\right]} = Y_0$$

The market for the good at each time t clears for each state i as shown below:

$$\sum_{j=1}^{m} c_{jit} = \sum_{j=1}^{m}\frac{Y_{it}}{Y_0} c_{j0} = \frac{Y_{it}}{Y_0}\sum_{j=1}^{m} c_{j0} = \frac{Y_{it}}{Y_0} Y_0 = Y_{it}$$

Summing (A10) across all consumers j gives

$$\sum_{j=1}^{m} x_{jit} = \sum_{j=1}^{m} c_{jit} - \sum_{j=1}^{m} y_{jit} = Y_{it} - Y_{it} = 0$$

which means that the markets for the state-contingent securities clear. Since each consumer's optimization problem is satisfied and since all markets clear, (A7) through (A10) does represent the equilibrium to the A-D economy with identical CRRA utility.

The equilibrium for the EOM economy with quasi-real bonds will be the same consumption allocation expressed in equations (A7) and (A8). In addition, for t=1..T and i=1..n_t,

$$V_t = \beta^t \frac{E\left[Y_{*t}^{1-\gamma}\right]}{Y_0^{1-\gamma}} \tag{A11}$$

$$Q_{jt} = M_0 \left(\frac{c_{j0}}{Y_0} - k_{jt} \right) \tag{A12}$$

Because goods markets clear for the consumption allocations in (A7) and (A8) for the Arrow-Debreu economy, they clear for the EOM economy. Remaining to prove are (i) that consumers' optimization problems are satisfied, and (ii) that the market for quasi-real bonds clears.

When only quasi-real bonds exist, the consumer j's optimization problem is to maximize

$U(c_{j0}) + \sum_{t=1}^{n_t} \beta^t \sum_{i=1}^{n_t} \pi_{it} U(c_{jit})$ subject to

$$P_0 c_{j0} + \sum_{t=1}^{T} V_t Q_{jt} = P_0 y_{j0} \tag{A13}$$

and for t=1..T and i=1..n_t,

$$c_{jit} = y_{jit} + \frac{Q_{jt}}{M_0} Y_{it} \tag{A14}$$

The necessary and sufficient conditions for this problem to be satisfied are that the constraints (A13) and (A14) are met and:

$$\frac{U'(c_{j0})}{P_0} = \frac{\beta^t}{V_t}\sum_{i=1}^{n_t}\frac{\pi_{it}U'(c_{jit})Y_{it}}{M_0} \qquad \textbf{(A15)}$$

Since consumers in the EOM universe have identical CRRA utility functions, (A15) becomes

$$\frac{1}{P_0(c_{j0})^\gamma} = \frac{\beta^t}{V_t}\sum_{i=1}^{n_t}\frac{\pi_{it}Y_{it}}{M_0(c_{jit})^\gamma}$$

To show that this is satisfied, into the right side of the above we replace M_0 with P_0Y_0, c_{jit} with (A8), and V_t with (A11):

$$\frac{\beta^t}{\beta^t\,\frac{E[Y_{*t}^{1-\gamma}]}{Y_0^{1-\gamma}}}\sum_{i=1}^{n_t}\frac{\pi_{it}Y_{it}}{P_0Y_0(c_{j0})^\gamma\left(Y_{it}/Y_0\right)^\gamma} = \frac{Y_0^{1-\gamma}}{E[Y_{*t}^{1-\gamma}]}\frac{Y_0^\gamma}{Y_0}\frac{\sum_{i=1}^{n_t}\pi_{it}Y_{it}^{1-\gamma}}{P_0(c_{j0})^\gamma} = \frac{1}{P_0(c_{j0})^\gamma}$$

To show that the constraint (A13) is satisfied, into the left side of (A13) we substitute (A11) for V_t, (A12) for Q_{jt}, and eventually (A7) for c_{j0}:

$$P_0c_{j0} + \sum_{t=1}^{T}\beta^t\,\frac{E[Y_{*t}^{1-\gamma}]}{Y_0^{1-\gamma}}M_0\left(\frac{c_{j0}}{Y_0} - k_{jt}\right) = P_0c_{j0} + \sum_{t=1}^{T}\beta^t\,\frac{E[Y_{*t}^{1-\gamma}]}{Y_0^{1-\gamma}}\left(P_0c_{j0} - k_{jt}P_0Y_0\right) =$$

$$P_0\frac{y_{j0} + Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}]}{1 + Y_0^{\gamma-1}\sum_{t=1}^{T}\beta^t E[Y_{*t}^{1-\gamma}]}\left(1 + \sum_{t=1}^{T}\beta^t\,\frac{E[Y_{*t}^{1-\gamma}]}{Y_0^{1-\gamma}}\right) - P_0Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}] =$$

$$P_0\left(y_{j0} + Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}]\right) - P_0Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}] =$$

$$P_0y_{j0} + P_0Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}] - P_0Y_0^\gamma\sum_{t=1}^{T}\beta^t k_{jt}E[Y_{*t}^{1-\gamma}] = P_0y_{j0}$$

The reader also can easily show that the constraint (A14) is satisfied by replacing c_{jit} with (A8), Q_{jt} in (A14) with (A12) and noting that $y_{jit} = k_{jt}Y_{it}$.

Because the goods market at time 0 clears, each of the markets for quasi-real bonds also clears as is shown below:

$$\sum_{j=1}^{m} Q_{jt} = \sum_{j=1}^{m} M_0 \left(\frac{c_{j0}}{Y_0} - k_{jt} \right) = M_0 \left(\frac{\sum_{j=1}^{m} c_{j0}}{Y_0} - \sum_{j=1}^{m} k_{jt} \right) = M_0 \left(\frac{Y_0}{Y_0} - 1 \right) = 0$$

Since all markets clear and each consumer's optimization problem is satisfied, equations (A7), (A8), (A11), and (A12) represent the equilibrium for the EOM economy. Since the consumption allocation is the same as for the Arrow-Debreu economy with complete markets, the consumption allocation is Pareto efficient.