RISK-FREE INTERNAL GAINS – BLACK AND SCHOLES RE-EXAMINED

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Abstract: In this paper we first show that if a not-necessarily-self-financing portfolio has instantaneously riskless internal gains, then on an infinitesimal time-interval, the increase in the internal gains on the portfolio is the same as the change in the price of that amount of bonds which has the same wealth as the portfolio has. Then, using this result, we re-examine the original derivation of the Black-Scholes formula, and conclude that contrary to common belief, the argument of Black and Scholes can be made completely rigorous, employing the same δ-hedge portfolio that they used and keeping all their mathematical formulas; but the explanations they gave to support their formulas must be replaced by others.

Keywords: mathematical finance, Black-Scholes formula, Wiener process, self-financing portfolio

1. Introduction

1.1. *Instantaneously risk-free internal gains and risk-free interest*

The Black-Scholes formula has come to occupy a fundamental position in the theory of contingent claims. At the same time, it is generally thought that the original derivation, by Black and Scholes, of the partial differential equation that leads to the formula was not correct. In this introduction, we summarize our approach to the issue as well as that of Black and Scholes; then, in Theorems 3 and 5, we supply some new results, with the help of which we finally show that in outline, the Black-Scholes derivation is correct.

It was not until having finished this paper that our attention was brought to Peter Carr's unpublished article [3], in which he comments on the same issue. Although his answer to the problem is essentially the same as ours, he gives no systematic, mathematical treatment of the notions involved and no proofs either, both of which we attempt to provide here.

We consider a complete, frictionless, continuous-time market model with a single bond of constant interest rate, where no arbitrage opportunities are allowed. (By an arbitrage opportunity, we mean the existence of a risk-free bond arbitrage, see Definition 1.) It is well-known, that if a self-financing portfolio is risk-free in such a model, then the wealth of the portfolio must appreciate at the bond's risk-free interest rate. We remind the reader that an adapted continuous process $(t,\omega) \mapsto X_t(\omega)$ is called *instantaneously risk-free*, if the increase in X_t from t_1 until t_2 can

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be expressed as an ordinary (non-stochastic) integral:

$$X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} Z_t \, dt \tag{1}$$

for some adapted process $(t, \omega) \mapsto Z_t(\omega)$, or which is the same,

$$dX_t = Z_t \, dt.$$

Our aim here is to study whether it is possible to relate instantaneous risk-freeness of a portfolio to the risk-free interest rate, even when the portfolio is not self-financing. The answer is "Yes," if it is not the total wealth of the portfolio which is riskless: If the process of *internal gains* on a portfolio is instantaneously riskless, then, instantaneously (that is, over an infinitesimal time-interval), the internal gain must be the same as the change in the price of the amount of bonds that has the same wealth as the portfolio. The crucial element in the proofs is the lack of arbitrage opportunities, for which the precise mathematical formulation is given in Definition 1.

The notion of *internal gains* and of the *influx of external funds* into the portfolio in case of a continuous-time model are investigated by Merton in [1]. Let V_t^{p} denote the price-process of the portfolio V_t; the change in the price over a time-interval, $V_{t_2}^{\mathrm{p}} - V_{t_1}^{\mathrm{p}}$, originates from two factors: the market prices of the instruments in the portfolio change, and money might flow in or out of the portfolio. The sum of these two parts is the total change in the wealth of the portfolio. The part that is due to the market, is called *internal gains*. The other part, the influx of external funds into the portfolio, is in a sense the "*cost*" of maintaining the portfolio over the time interval in question: it is the amount of money that we have to invest in the portfolio over this time interval. (We would like to emphasize that this "cost" has nothing to do with transaction costs in a market with friction.) Let us denote the internal gains over the time period $[t_1, t_2]$ by $\mathcal{G}_{t_1}^{t_2}(V.)$. The influx of external funds will be denoted by $\mathcal{C}_{t_1}^{t_2}(V.)$. It is not immediate how to identify these quantities for a portfolio in a continuous-time model. According to Merton in [1], they should be identified as in formulas (5) and (6) of the present paper. Merton arrives at these results through a discrete-time approximation; it seems not to be possible to derive the correct formulas purely by continuous-time considerations.

All that said, we have

$$V_{t_2}^{\mathrm{p}} - V_{t_1}^{\mathrm{p}} = \mathcal{G}_{t_1}^{t_2}(V.) + \mathcal{C}_{t_1}^{t_2}(V.).$$

Our aim is to show that if the process $\mathcal{G}_0^t(V.)$ is instantaneously risk-free, that is, if there is a continuous adapted process, Y_t, such that

$$d\mathcal{G}_0^t(V.) = Y_t \, dt,$$

then $Y_t = V_t^{\mathrm{p}} \cdot r$ almost everywhere for all t, hence (except for extreme pathologies),

$$d\mathcal{G}_0^t(V.) = V_t^{\mathrm{p}} \cdot r \, dt,$$

where r is the risk-free interest rate. This is our main result, and it is the conclusion of Theorem 3.

It is then natural to ask, what happens if instead of the gain, it is the inflow of external funds that is instantaneously risk-free. We state those results in Theorem 5, although we can say much less in that case.

As an application of Theorem 3, we shall revisit the original δ-hedge argument (sometimes called *the risk-free portfolio method*) of Black and Scholes, which they give to derive their famous formula. It is known that this original derivation has errors in it, and the common view is that Black and Scholes could produce their formula, only because two mathematical errors cancel each other, and that their argument cannot be repaired without significant changes. We shall show though that *the argument of Black and Scholes can be made completely rigorous, using the same δ-hedge portfolio that they used and keeping all their mathematical formulas, but we have to replace the explanations that they gave to support their formulas with others*. We begin by taking a detailed look at the major difficulty with their derivation.

1.2. *The notorious δ-hedge portfolio of Black and Scholes*

Black and Scholes start off by assuming that the price of the option, $w(x,t)$, depends only on the stock price x and the time t (their notation). In order to derive their formula, they consider the δ-hedge portfolio consisting of one stock held long and $1/w_1$ options held short, where w_1 denotes the derivative of w with respect to the first variable. The value of this hedging portfolio at each moment is then given by

$$x - \frac{1}{w_1} w, \tag{2}$$

(which is formula (2) in their paper as well). Of course, at this point they don't know the actual form of $w(x,t)$; further considerations will reveal what $w(x,t)$ must be. In order to maintain the $-1/w_1$ amount of options at each moment, continuous trading is necessary. A surprising claim then follows: the change in the value of the portfolio in a short interval Δt is

$$\Delta x - \frac{1}{w_1} \Delta w \tag{3}$$

(formula (3) of the paper). Although this is not the complete Itô differential, the authors continue without comment, leaving it to the reader to search for justification. The change of the portfolio's value would be given by (3) only if the neglected terms were 0, that is, if the portfolio were self-financing. However, as was pointed out in [4], for example (or in [5] on p129), if we use the $w(x,t)$ that Black and Scholes reach at the end of their argument, and check whether the the δ-hedge portfolio with this particular w is self-financing, it turns out not to be!

There has been some discomfort about a faulty argument being used to derive a correct formula. In the literature of the subject, some authors either ignore the problem (some use the original derivation in textbooks), or point out that in the derivation there is in fact another error that cancels the first (this other error being that later when Black and Scholes find that the price of their δ-hedge portfolio is instantaneously riskless, they conclude that is must be the same as that of the bond's, which is only true if the portfolio is self-financing, but it is not; see again [4] for more details), almost suggesting that it was pure luck that produced the right formula. In [5] pp. 127-130, Musiela and Rutkowski also note the problem, remarking only that the risk-free portfolio method works well in the discrete-time setting. A brief analysis of the problem can also be found in [6].

Using the approach and results of the next section, we shall see that *the structure of the Black-Scholes derivation is correct*, although the comments and explanations that they provide to support their derivation are incorrect. As far as we can see, the mistakes are generated by not making a clear distinction between discrete and continuous time models. As we mentioned earlier, the risk-free portfolio method works well in the discrete-time setting. Black and Scholes apparently had discrete time approximations in mind when they were dealing with their continuous-time hedging portfolio. In the continuous limit though, they failed to give the right interpretations to the formulas they reached. For example, (3) in a continuous model, is not the total change of the price of the portfolio as they claimed, it is only the internal gain. The total change includes another part as well, the money inflow into the portfolio. This latter is not zero in their case, for the portfolio is not self-financing. The major difference between our approach to repair the derivation and that of other authors is the following: We do not suggest that instead of (3) they should have written the total change. We suggest exactly the opposite. (3) should be kept as it is, but *it should be called the internal gain on the portfolio.* Using our Theorem 3, we cast (3) and some of their formulas in new roles.

2. **Risk-Free Internal Gains and Risk-Free Interest**

Suppose that our complete, frictionless market has, besides a single, riskless bond (with price β_t) of constant interest rate r, n other market instruments with price processes $S_{1,t},\ S_{2,t},\dots,S_{n,t}$. A portfolio in such a market can be described by an $n+1$-tuple,

$$V_t = (L_t, M_{1,t}, M_{2,t}, \dots, M_{n,t}),$$

where $t \in [0,T]$ represents time, and $L_t,\ M_{1,t},\ \dots$ are adapted processes; L_t stands for the amount of bonds in the portfolio, $M_{1,t}$ the amount of the first instrument, $M_{2,t}$ the amount of the second, and so forth. The value (or price) of such a portfolio is of course

$$V_t^{\mathrm{p}} = L_t\beta_t + \sum_{j=1}^{n} M_{j,t}S_{j,t}.$$

Following Itô, the change in the price of this portfolio from time t_1 to t_2 can be written as

$$V_{t_2}^{\mathrm{p}} - V_{t_1}^{\mathrm{p}} = \mathcal{G}_{t_1}^{t_2}(V_\cdot) + \mathcal{C}_{t_1}^{t_2}(V_\cdot) \tag{4}$$

where

$$\mathcal{G}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2}\left[L_t\,d\beta_t + \sum_{j=1}^{n} M_{j,t}\,dS_{j,t}\right] \tag{5}$$

and

$$\mathcal{C}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2}\left[\beta_t\,dL_t + \sum_{j=1}^{n} S_{j,t}\,dM_{j,t}\right] + \int_{t_1}^{t_2}\left[\sum_{j=1}^{n} dS_{j,t}\,dM_{j,t}\right]. \tag{6}$$

As we mentioned in the introduction, according to Merton's analysis in [1], $\mathcal{G}_{t_1}^{t_2}(V_\cdot)$ should be interpreted as that part of the change in the price of the portfolio which arises from changes in the market prices only, the interpretation of $\mathcal{C}_{t_1}^{t_2}(V_\cdot)$ is the influx of external funds into our portfolio over this time interval (i.e. the "cost" of ensuring the right amount of instruments in the portfolio).

In this terminology, a portfolio V_t is self-financing if and only if

$$\mathcal{C}_{t_1}^{t_2}(V_\cdot) = 0 \tag{7}$$

almost everywhere for any time interval $[t_1, t_2] \subset [0,T]$.

It goes without saying that all the above stochastic processes are defined over and event space Ω with a filtering $\mathcal{F}_t$ and a probability measure P.

In order to avoid the pathologies of doubling portfolio-strategies, it is common to require that the losses on portfolios are bounded. We will not explicitly require this, which is only a matter of convenience from our part. Everything in the paper can be done so that this requirement is imposed on all portfolios, making the reasoning somewhat more cumbersome, but not at all more lucid.

We first recall a precise formulation of arbitrage opportunity from [6]:

Definition 1. *A self-financing portfolio W_t, is called a risk-free bond arbitrage on $[t_1, t_2]$ if there is a $\lambda \in \mathbf{R}$ such that $P(W^{\mathrm{p}}_{t_1}/\beta_{t_1} \leq \lambda) = 1$, $P(W^{\mathrm{p}}_{t_2}/\beta_{t_2} \geq \lambda) = 1$, and $P(W^{\mathrm{p}}_{t_2}/\beta_{t_2} > \lambda) > 0$.*

That is, the portfolio makes a profit with some nonzero probability, but it surely does not create a loss by the end of the interval.

Theorem 3 is our main result. It is the non-self-financing analogue of the well known fact that the price of an (instantaneously) risk-free self-financing portfolio must follow the price of an appropriate amount of the bond. The meaning of our result is that if the process of internal gains is instantaneously risk-free, then, the increase in the internal gains locally follows the increase in the price of that amount of the bond that initially has the same value as the value of the portfolio. We first prove the following lemma.

Lemma 2. *Suppose we have two portfolios V_t and $\tilde{V}_t$, such that for each $[t_1, t_2] \subset [0, T]$,*

$$\mathcal{G}^{t_2}_{t_1}(V_\cdot) = \int_{t_1}^{t_2} Y_t \, dt \tag{8}$$

and

$$\mathcal{G}^{t_2}_{t_1}(\tilde{V}_\cdot) = \int_{t_1}^{t_2} \tilde{Y}_t \, dt \tag{9}$$

hold, where Y_t and $\tilde{Y}_t$ are continuous adapted processes. If $V^{\mathrm{p}}_t = \tilde{V}^{\mathrm{p}}_t$ for all $t \in [0, T]$ and if the market model contains no risk-free bond arbitrage, then for each $t \in [0, T]$,

$$Y_t = \tilde{Y}_t \quad a.e. \ on\ \Omega. \tag{10}$$

Proof. The main idea is to reason from the lack of arbitrage opportunities in the following way: If at a certain moment, t_0, we recognize that $Y_{t_0}(\omega) > \tilde{Y}_{t_0}(\omega)$, then we start maintaining a long position in V_t, and a short position in $\tilde{V}_t$. The initial transaction at t_0, namely, creating the long position of V_{t_0} and the short position of $\tilde{V}_{t_0}$, has no cost, since we assumed that the two portfolios have equal prices. We maintain these positions until $Y_t(\omega)$ and $\tilde{Y}_t(\omega)$ become equal. Before that happens, the inflow of external funds into V_t is less than inflow into $\tilde{V}_t$ (since $Y_t(\omega) > \tilde{Y}_t(\omega)$), therefore, maintaining $V_t - \tilde{V}_t$ produces money surplus (see formula (21)), which we continually invest in the bond (the amount of the bond that piles up until time t this way will be denoted by L_t below, see (11)). $V_t - \tilde{V}_t$ together with these bonds constitute a portfolio W_t, which is a risk-free bond arbitrage: it is self-financing, it has zero wealth at the beginning, and as large a wealth at the end, as it is the value of the bonds that pile up this way. But we assumed that the market model accommodates no risk-free bond arbitrage, therefore $Y_{t_0}(\omega) \leq \tilde{Y}_{t_0}(\omega)$. $Y_{t_0}(\omega) \geq \tilde{Y}_{t_0}(\omega)$ is shown in a similar manner, so $Y_{t_0}(\omega) = \tilde{Y}_{t_0}(\omega)$. We now present the details of this argument.

Suppose there is a $t_0 \in [0,T]$ with $P(Y_{t_0} > \tilde{Y}_{t_0}) > 0$. For each event $\omega \in \Omega$, let

$$\tau(\omega) = \min\{t: \ t_0 \le t < T \text{ and } Y_t(\omega) \le \tilde{Y}_t(\omega), \text{ or } t = T\}.$$

The function τ is then a stopping time.

In what follows we use $a \vee b$ and $a \wedge b$, for two reals a and b, to denote the larger and the smaller of a and b, respectively.

We construct the following portfolios. Let

$$U_t(\omega) = \big(L_{(t_0 \vee t)\wedge \tau(\omega)}(\omega), 0, \ldots, 0\big) \tag{11}$$

where L_t is determined by $L_{t_0} \equiv 0$ and

$$dL_t = [Y_t - \tilde{Y}_t] \cdot \exp(-rt)/\beta_0 \, dt. \tag{12}$$

Let

$$W_t(\omega) = \begin{cases} U_t(\omega) \ \text{ for } \ t < t_0 \\ V_t(\omega) - \tilde{V}_t(\omega) + U_t(\omega) \ \text{ for } \ t \in [t_0, \tau(\omega)] \\ U_t(\omega) \ \text{ for } \ t > \tau(\omega). \end{cases} \tag{13}$$

We want to show that W_t is a risk-free bond arbitrage. Observe, that for $t \in [0, t_0]$, $W_t = U_t = (0, 0, \ldots, 0)$, and for $t \in [\tau(\omega), T]$, $W_t(\omega) = (L_{\tau(\omega)}(\omega), 0, \ldots, 0)$. Note also that the assumption $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ implies that the price process

$$W_t^{\mathrm{p}} = U_t^{\mathrm{p}} = L_{(t_0 \vee t)\wedge\tau} \beta_t$$

is continuous.

We first show that W_t is self-financing. To this end, fix an arbitrary ω event, and then take an interval $[t_1, t_2] \subset [0, T]$ with $t_1 \le \tau(\omega)$ and $t_2 \ge t_0$. The definition of W_t shows that, on this ω,

$$\mathcal{C}_{t_1}^{t_2}(W_\cdot)(\omega) = \mathcal{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V_\cdot - \tilde{V}_\cdot)(\omega) + \mathcal{C}_{t_1}^{t_2}(U_\cdot)(\omega). \tag{14}$$

Since $U_t(\omega)$ is constant on $[t_1, t_0]$ and on $[\tau(\omega), t_2]$, there is no influx over these intervals:

$$\mathcal{C}_{t_1}^{t_2}(U_\cdot)(\omega) = \mathcal{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(U_\cdot)(\omega).$$

Then (6) and (12) show that

$$\begin{aligned} \mathcal{C}_{t_1}^{t_2}(U_\cdot)(\omega) &= \mathcal{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(U_\cdot)(\omega) && (15) \\ &= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} \beta_t \, dL_t(\omega) && (16) \\ &= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} \beta_0 \cdot \exp(rt) \cdot [Y_t(\omega) - \tilde{Y}_t(\omega)] \cdot \exp(-rt)/\beta_0 \, dt \\ &= \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} [Y_t(\omega) - \tilde{Y}_t(\omega)] \, dt. && (17) \end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{C}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(V_\cdot-\tilde V_\cdot)(\omega) &= \mathcal{C}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(V_\cdot)(\omega)-\mathcal{C}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(\tilde V_\cdot)(\omega) && (18)\\
&= V^{\mathrm{p}}_{t_2\wedge\tau(\omega)}-V^{\mathrm{p}}_{t_1\wedge t_0}-\tilde V^{\mathrm{p}}_{t_2\wedge\tau(\omega)}+\tilde V^{\mathrm{p}}_{t_1\wedge t_0}\\
&\quad -\mathcal{G}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(V_\cdot)+\mathcal{G}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(\tilde V_\cdot)(\omega) && (19)\\
&= -\mathcal{G}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(V_\cdot-\tilde V_\cdot)(\omega) && (20)\\
&= -\int_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}[Y_t(\omega)-\tilde Y_t(\omega)]\,dt, && (21)
\end{aligned}$$

where we used (4), (8), (9) and $V^{\mathrm{p}}_t=\tilde V^{\mathrm{p}}_t$. But then

$$\mathcal{C}_{t_1}^{t_2}(W_\cdot)(\omega)=\mathcal{C}_{t_1\vee t_0}^{t_2\wedge\tau(\omega)}(V_\cdot-\tilde V_\cdot)(\omega)+\mathcal{C}_{t_1}^{t_2}(U_\cdot)(\omega)=0.$$

So far we assumed that $t_1\le\tau(\omega)$ and $t_2\ge t_0$. If $t_2\le t_0$ or $t_1\ge\tau(\omega)$ then $\mathcal{C}_{t_1}^{t_2}(W_\cdot)(\omega)=0$ trivially by (13). Hence we get $\mathcal{C}_{t_1}^{t_2}(W_\cdot)(\omega)=0$ for any interval $[t_1,t_2]\subset[0,T]$. But ω was arbitrary, therefore W_t is self-financing.

Now, if ω is such an event that $Y_{t_0}(\omega)>\tilde Y_{t_0}(\omega)$, then, for any $t\in(t_0,\tau(\omega))\neq\emptyset$, $Y_t(\omega)>\tilde Y_t(\omega)$, which implies $L_{\tau(\omega)}(\omega)>0$ since remember, (12) holds. Thus, along the initial hypotheses $P(Y_{t_0}>\tilde Y_{t_0})>0$, we arrive at

$$P(L_\tau>0)>0,$$

and therefore

$$P(W^{\mathrm{p}}_T/\beta_T>0)=P(L_\tau>0)>0.$$

But $W^{\mathrm{p}}_0/\beta_0=0/\beta_0=0$ and $W^{\mathrm{p}}_T/\beta_T=L_\tau\ge0$, that is, we have a risk-free arbitrage because the three requirements of Definition 1 are satisfied with $\lambda=0$. Risk free arbitrage is not allowed, so our initial hypotheses was wrong: $Y_t\le\tilde Y_t$ must hold almost everywhere. Proving $Y_t\ge\tilde Y_t$ a.e. is similar, hence $Y_t=\tilde Y_t$ almost everywhere.

□

Theorem 3. *Suppose we have a portfolio V_t and $\mathcal{G}_{t_1}^{t_2}(V_\cdot)$ has the representation*

$$\mathcal{G}_{t_1}^{t_2}(V_\cdot)=\int_{t_1}^{t_2}Y_t\,dt, \tag{22}$$

where Y_t a continuous adapted process. if the market model contains no risk-free bond arbitrage, then for all $t\in[0,T]$,

$$Y_t=V^{\mathrm{p}}_t\cdot r\quad \text{a.e. on }\Omega. \tag{23}$$

Proof. We choose a specific $\tilde V_t$ for the other portfolio in Lemma 2: one that consists of bonds only, and exactly V^{p}_t worth bonds: $\tilde V_t=(V^{\mathrm{p}}_t/\beta_t,0,\ldots,0)$. Then

$$\tilde V^{\mathrm{p}}_t=\frac{V^{\mathrm{p}}_t}{\beta_t}\beta_t=V^{\mathrm{p}}_t. \tag{24}$$

For this new portfolio,

$$\mathcal{G}_{t_1}^{t_2}(\tilde V_\cdot)=\int_{t_1}^{t_2}\frac{V^{\mathrm{p}}_t}{\beta_t}\,d\beta_t=\int_{t_1}^{t_2}V^{\mathrm{p}}_t\cdot r\,dt. \tag{25}$$

From Lemma 2, it follows that the integrands in the two market-based gains, (22) and (25), must agree:

$$Y_t = V_t^{\mathrm{p}} \cdot r$$

almost everywhere in Ω, for all $t \in [0, T]$.

□

Corollary 4. *Suppose we have two portfolios V_t and $\tilde{V}_t$, such that for each $[t_1, t_2] \subset [0, T]$,*

$$\mathcal{G}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2} Y_t \, dt \tag{26}$$

and

$$\mathcal{G}_{t_1}^{t_2}(\tilde{V}_\cdot) = \int_{t_1}^{t_2} \tilde{Y}_t \, dt \tag{27}$$

hold, where Y_t and $\tilde{Y}_t$ are continuous adapted processes. If the market model contains no risk-free bond arbitrage, then for all $t \in \Omega$,

$$V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}} \text{ a.e. on } \Omega \tag{28}$$

if and only if for all $t \in [0, T]$,

$$Y_t = \tilde{Y}_t \quad \text{a.e. on } \Omega. \tag{29}$$

Proof. According to Theorem 3, $Y_t = V_t^{\mathrm{p}} \cdot r$ a.e. and $\tilde{Y}_t = \tilde{V}_t^{\mathrm{p}} \cdot r$ a.e., so $Y_t = \tilde{Y}_t$ holds almost everywhere if and only if $V_t^{\mathrm{p}} \cdot r = \tilde{V}_t^{\mathrm{p}} \cdot r$ almost everywhere, which holds if and only if $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ a.e. (for all $t \in [0, T]$).

□

It is now natural to ask what happens if not the internal gain, but the influx of external funds were represented by an ordinary integral. In this case we have the following theorem, which is analogous to Lemma 2; no proposition analogous to Theorem 3 or the corollary can be proven for this case.

Theorem 5. *Suppose we have two portfolios V_t and $\tilde{V}_t$, such that for each $[t_1, t_2] \subset [0, T]$,*

$$\mathcal{C}_{t_1}^{t_2}(V_\cdot) = \int_{t_1}^{t_2} Y_t \, dt \tag{30}$$

and

$$\mathcal{C}_{t_1}^{t_2}(\tilde{V}_\cdot) = \int_{t_1}^{t_2} \tilde{Y}_t \, dt \tag{31}$$

hold, where Y_t and $\tilde{Y}_t$ are continuous adapted processes. If $V_t^{\mathrm{p}} = \tilde{V}_t^{\mathrm{p}}$ for all $t \in [0, T]$ and if the market model contains no risk-free bond arbitrage, then for all $t \in \Omega$,

$$Y_t = \tilde{Y}_t \quad \text{a.e. on } \Omega. \tag{32}$$

Proof. The proof of this theorem follows line by line the proof of Lemma 2, except that W_t now should be defined as

$$W_t(\omega) = \begin{cases} U_t(\omega) \ \text{ for } \ t < t_0 \\ -[V_t(\omega) - \tilde{V}_t(\omega)] + U_t(\omega) \ \text{ for } \ t \in [t_0, \tau(\omega)] \\ U_t(\omega) \ \text{ for } \ t > \tau(\omega), \end{cases} \tag{33}$$

instead of equation (14), we have

$$\mathcal{C}_{t_1}^{t_2}(W.)(\omega) = -\mathcal{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) + \mathcal{C}_{t_1}^{t_2}(U.)(\omega) \tag{34}$$

here, and finally, equations (18) - (21) have to be replaced by

$$\mathcal{C}_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)}(V. - \tilde{V}.)(\omega) \quad = \quad \int_{t_1 \vee t_0}^{t_2 \wedge \tau(\omega)} [Y_t(\omega) - \tilde{Y}_t(\omega)]\, dt. \tag{35}$$

The rest is the same.

□

3. **Old-New Derivation of the Black-Scholes PDE**

3.1. *The derivation as an application of Theorem 3*

It is time now to turn our attention to the portfolio that Black and Scholes use to derive their formula. As we noted earlier, by keeping all formulas and equations of the original derivation, but providing them with new explanations, we present here a δ-hedge portfolio (or risk-free portfolio) argument that leads to the Black-Scholes PDE, and which is completely satisfactory both from the point of finance and of mathematics.

Let the random variable S_t denote the price of the stock in question at time $t \in [0, T]$; we assume that the price-process $(t, \omega) \mapsto S_t(\omega)$ satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \tag{36}$$

where μ and σ are constants and where B_t is the standard Brownian motion.

We then ask what happens, if there is a twice continuously differentiable function $f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ such that $(t, \omega) \mapsto f(t, S_t(\omega))$ gives the price-process of the European call option on the stock? As we shall see (as Black and Scholes saw), the δ-hedge portfolio argument bellow determines a PDE that such a function f must satisfy. Black and Scholes solved the PDE, and the set of solutions provided a unique twice-differentiable f for each possible call option on the given stock. This made the initial idea of searching for functions among the twice differentiable ones quite plausible, since it gave a unique solution for each situation.

Let $\partial_1 f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ denote the partial derivative with respect to the first (time) variable, whereas $\partial_2 f : [0, T] \times \mathbf{R}^+ \mapsto \mathbf{R}$ denotes the partial derivative with respect to the second variable. For simplicity, let $f(t, S_t)$ denote the random variable $\omega \mapsto f(t, S_t(\omega))$.

Let β_t mean the price of the bond again, which we accept to be governed by

$$d\beta_t = r\beta_t\, dt. \tag{37}$$

The original derivation of Black and Scholes requires a portfolio that contains linear combinations only of the bond, the stock and the European option on the stock. A portfolio like this is represented by a triple,

$$W_t = (L_t, M_{1,t}, M_{2,t}),$$

where $t \in [0, T]$ represents time, and L_t, $M_{1,t}$, $M_{2,t}$ are adapted processes; the first denotes the amount of bonds in the portfolio, the second the amount of stocks, the third stands for the quantity of options.

More specifically, consider the portfolio X_t that includes exactly one stock and $-1/\partial_2 f(t,S_t)$ amount of options, i.e. $X_t = (0,1,-1/\partial_2 f(t,S_t))$ and

$$X_t^{\mathrm{p}} = S_t - f(t,S_t)/\partial_2 f(t,S_t). \tag{38}$$

Then, by (5), the market-based internal gain in our portfolio is

$$\mathcal{G}_{t_1}^{t_2}(X.) = \int_{t_1}^{t_2} \left[dS_t - \frac{1}{\partial_2 f(t,S_t)}\, d[f(t,S_t)] \right]. \tag{39}$$

By Itô, we see that

$$\begin{aligned} d[f(t,S_t)] &= \partial_1 f(t,S_t)\,dt + \partial_2 f(t,S_t)\,dS_t + \frac{1}{2}\partial_2\partial_2 f(t,S_t)\,dS_t^2 \\ &= \partial_2 f(t,S_t)\,dS_t + \left[\partial_1 f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2\partial_2 f(t,S_t)\right] dt, \end{aligned} \tag{40}$$

since according to (36), the box-calculus gives

$$dS_t^2 = \sigma^2 S_t^2\, dt. \tag{41}$$

Therefore,

$$dS_t - \frac{1}{\partial_2 f(t,S_t)}\, d[f(t,S_t)] = -\frac{1}{\partial_2 f(t,S_t)} \left[\partial_1 f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2\partial_2 f(t,S_t)\right] dt. \tag{42}$$

Hence

$$\mathcal{G}_{t_1}^{t_2}(X.) = \int_{t_1}^{t_2} \frac{-1}{\partial_2 f(t,S_t)} \left[\partial_1 f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2\partial_2 f(t,S_t)\right] dt. \tag{43}$$

The integral turns out to be just an ordinary one. Using Theorem 3 (in particular, equation (23)) and (38), we get that the integrand is given by

$$-\frac{1}{\partial_2 f(t,S_t)} \left[\partial_1 f(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_2\partial_2 f(t,S_t)\right] = \left[S_t - f(t,S_t)/\partial_2 f(t,S_t)\right] r \tag{44}$$

almost everywhere for all t. That is, if f is twice differentiable and gives the price of the option, it must certainly satisfy the equation above. Since S_t is a geometric Brownian motion, for any $t \in [0,T]$, and any $x \in \mathbf{R}^+$, $P[|S_t - x| \le 1/n] > 0$ whenever $n \in \mathbf{N}$, so there is an $\omega_n \in \Omega$ on which equation (44) is satisfied and $|S_t(\omega_n) - x| \le 1/n$. Since in equation (44), f and its derivatives are all continuous, we conclude that

$$-\frac{1}{\partial_2 f(t,x)} \left[\partial_1 f(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_2\partial_2 f(t,x)\right] = \left[x - f(t,x)/\partial_2 f(t,x)\right] r \tag{45}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^+$. After a bit of rearranging we receive the famous PDE:

$$\partial_1 f(t,x) = r f(t,x) - r x \partial_2 f(t,x) - \frac{1}{2}\sigma^2 x^2 \partial_2\partial_2 f(t,x). \tag{46}$$

3.2. *Connection with the original.*

Here, we briefly indicate that our derivation really follows the original line by line. To avoid confusion, we use double parentheses (( )) for referring to formulas of the paper of Black and Scholes, and parentheses ( ) for referring to ours.

Our equation (38) clearly corresponds to their formula ((2)). Their ((3)), which is not the increase in the value of the asset, but the increase that is due to the market only, appears as a stochastic integral on the right-hand side of (39). On their formulas ((4)) and ((5)), we have (40) and (43) to reflect. We arrive at the analogue of ((6)) by receiving equation (44) and (45) via a slightly different arbitrage argument (Theorem 3) then that of Black and Scholes. Finally, ((7)) and (46) are identical.

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