# A Dynamic Analysis of Bid-Ask Spreads with Multiple Trade Sizes


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May 28, 2005


**Abstract**

This paper studies how the trade size and the historical sequence of trades affect bid-ask spreads, investors' trading strategies, and the market maker's learning process in a multi-period economy. First, we show that there is a nonzero cut-off size below which informed traders never buy or sell, and that larger trade sizes have positive bid-ask spreads, while smaller sizes do not. Then, we prove that the cut-off size decreases stochastically . We also derive the functional relationship between bid-ask spreads and trade sizes and show that bid-ask spreads are monotonically increasing in trade sizes. Moreover, we prove that when additional trade sizes are introduced to the market, the market maker's learning process can be impaired and the bid-ask spreads for the previously existing trade sizes can vanish under a mild condition. We prove that the smaller trade sizes that do not have a positive bid-ask spread result in zero price change, while for larger trade sizes the rate at which price change increases is a decreasing function of the trade size in all trading periods. Most of our results are broadly consistent with the empirical findings. Others provide testable hypotheses for future studies on the subject.



*We are grateful to Erzo Luttmer and Jan Werner for their helpful suggestions. Also, we would like to thank Hengjie Ai, Beth Allen, Marco Bassetto, Camelia Bejan, Florin Bidian, Borys Grochulski, John Rust, Raj Singh, and seminar participants at the University of Minnesota, the 2004 Villa Mondragone Workshop in Economic Theory and Econometrics in Rome for helpful comments. All errors are our own.


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# 1 Introduction

Trade sizes, bid-ask spreads, and price changes have been central to the analysis of market microstructure. Empirical market microstructure seeks to characterize the dynamics of prices, bid-ask spreads and trade sizes and has identified a strong link between trade sizes and the absolute value of price change for each trade size (see [14]). The motivation for this paper is to understand how trade sizes affect the dynamics of bid-ask spreads and price changes within the framework of theoretical market microstructure model. The market considered here has a market maker who trades a risky asset with traders and exposes bid and ask prices to traders. The presence of traders with superior information leads to a positive bid-ask spread. A bid-ask spread compensates a market maker for the risk of doing business with the informed traders.

Models of information based bid-ask spreads have been developed by Copeland and Galai [1] and Glosten and Milgrom [10]. In the Copeland and Galai model, the true value of the stock is public information after one trading round. In contrast, in the Glosten and Milgrom model, there are many trading rounds. Orders are assumed to be for one round lot[1], and there are two types of traders, either informed or uninformed. Glosten and Milgrom showed that the transaction prices are informative and spreads tend to decline with trade. They finally showed that bid-ask spread vanishes when the trading periods go to infinity.

One of important extensions of Glosten and Milgrom is Easley and O'Hara [6]. They extended Glosten and Milgrom by allowing two trade sizes. In an efficient market, the price of an asset should reflect the true value of the asset. However, empirically, we observe that block trades are made at worse prices than small trades, which is called "block trade premium", and that there are some asymmetric patterns in trade sizes and the direction of price changes. By allowing two trade sizes, Easley and O'Hara consider block trade premium and shows that under some condition, large trade size is transacted at worse price than small trade size. Then, they have tried to find how different an impact of each trade size on price is in two-period model. Easley and O'Hara consider only one and two-period models. Our model generalizes Glosten and Milgrom, and Easley and O'Hara.

The model that we have used is described as follows. As in Glosten and Milgrom [10] and Easley and O'Hara [6], there are three types of agents: a competitive single market maker, informed and uninformed traders. In the beginning of the whole game, Nature chooses the value of the risky asset

[1]According to [16], we can categorize all trades in stocks on the floors of exchanges into round lots and odd lots based on the unit of trading. A round lot is one for the unit of trading or some multiple thereof. An odd lot is one for less than the number of shares required for the unit of trading.

and informed traders observe the value. A market maker and uninformed traders do not observe the value. In each round of trades, a trader is randomly selected and comes to the market maker. She learns the market maker's price-quantity quotes and posts the order quantity and the price at which she will buy or sell the asset. If she is informed, she takes the profit-maximizing quotes. If she is uninformed, she posts the order for her liquidity needs. In this setting, we consider the market maker's regret-free pricing. Trades occurs sequentially and we analyze movement of the bid-ask spreads over time.

To understand why different trade sizes result in different price changes, consider the following situation. Suppose that there are many trade sizes available for traders and prices of all trade sizes are the same. Then, informed traders would rather trade larger trade sizes than smaller sizes if they know the future payoff of the asset. In this sense, the trade sizes that informed traders trade convey information. After large trades come into the market, the market maker can make an inference concerning the risky payoff of the asset. That is how trade sizes affect future price and impacts of trade sizes on price differ.

Notice that in this paper, we consider two different relationship. One is the relationship between trade sizes and price after each trade occurs. The other is the relationship between trade sizes and bid-ask spreads. As the reasons stated above, trade size has an impact on the market maker's expectation on the risky asset and then the consecutive price. At the same time, since bid-ask spreads also reflect the market maker's expectation on the risky asset, trade sizes also have a relationship with bid-ask spreads. In the paper, we derive these two functional relationships for empirical tests.

Our model follows the strand of Glosten and Milgrom and Easley and O'Hara, and we extend their models in two dimensions; time and number of trade sizes. The main difference between our model and their models is that we allow multiple (more than two types of) trade sizes in a multi-period economy. Incorporating multiple trade sizes more than two allows us to examine the relationship between more variety of trade sizes and price changes, and to derive this relationship as a function, which is called "price-impact functions". Moreover, a multi-period model allows us to study the dynamic change of trading behaviors and prices. Our model provides more enriched theoretical framework and testable hypotheses for future research on the subject.

The first contribution of the paper is a description of how trade size affects bid-ask spread, trading behavior and the market maker's learning process in a dynamic environment where multiple trade sizes are available. More specifically, we show that there is a cut-off size above which the informed traders sell or buy with strictly positive probabilities and larger trade sizes above the cut-off size have positive bid-ask spreads, while smaller sizes do not. Also, we show that the cut-off size decreases when both informed traders and liquidity traders trade in the same way. Interestingly, these results explain how smaller trade sizes start to have strictly positive bid-ask spreads and we show that it is more likely to happen. In Glosten and Milgrom, bid-ask spreads just decrease. However, our model shows that as trade goes on, smaller trade sizes more likely have strictly positive bid-ask spreads.

In relation with Glosten and Milgrom, we provide some interesting results. We prove that the market maker learns the true value of the asset almost surely when the number of the trading periods is infinity and thus bid-ask spreads eventually vanish almost surely. In addition, we prove that when additional trade sizes are introduced to the market, the market maker's learning process can be impaired and the bid-ask spreads for the previously existing trade sizes can vanish. In Glosten and Milgrom, the bid-ask spread vanishes as trades occur infinitely. The similar result holds in our model with multiple trade sizes. In addition to this result, we also prove that increasing the number of trade sizes complicates the market maker's learning process under a mild condition.

The second contribution of the paper is an explanation of the functional relationship between trade sizes and price changes. It becomes possible to describe this functional relationship as price-impact functions only within the model with multiple trade sizes more than two. We provide a theoretical framework for an empirical analysis of price-impact functions. We derive a theoretical price-impact function and prove that the smaller trade sizes that do not have a positive bid-ask spread result in zero price change, while for larger trade sizes the rate at which price change increases is a decreasing function of the trade size in all trading periods. That is, the function displays a concave form in the range of trade sizes which have strictly positive bid-ask spreads. We also show that as trading round goes, the price-impact function becomes flatter. Easley and O'Hara did not have this result because their model was two-period setting.

Our results about a price-impact function coincide with the empirical findings. Hasbrouck [11] and [12] measured a trade's information effect as the ultimate price impact of the trade innovation. The estimates for a sample of NYSE suggest that the impact is a positive and concave function of the trade size. We show that a theoretical price-impact function derived in our model also displays a concave form and also our result predicts how the price-impact function changes over time. This gives us an interesting testable hypothesis for a future research.

In the paper, we give an explanation of why price change is a concave function of trade sizes using the model with asymmetrically informed traders. Then, we show how curvature of the price-impact function changes along with the information structure. Another important result of the model is that price-impact function that we have derived become flatter as trade goes on. This implies that as more information spreads in a market, the impact of trade sizes on price becomes smaller.

The third contribution is a description of the functional relationship between trade sizes and bid-ask spreads. we derive the relationship between bid-ask spreads and the trade sizes. We show that the bid-ask spreads increases in the trade size. Also, the functional relationship between bid-ask spreads and the trade size shows the concave shape. Easley and O'Hara shows that under some condition, large size has a strictly positive bid-ask spread, while small size does not. We prove that the bid-ask spread increases monotonically in trade size by deriving the functional relationship between the bid-ask spread and trade size.

Most of our results are consistent with the empirical findings. Our results about information based bid-ask spreads coincide with the empirical findings of Glosten [8], which decomposes the bid-ask spread into two parts: one part due to asymmetric information and the other part due to other factors such as inventory costs and shows that the part of the spread due to asymmetric information and the part of the spread due to other factors affect the properties of the transaction-price process differently. Also, by using a maximum likelihood technique, Glosten and Harris [9] find that the asymmetric information component of the bid-ask spread is not economically significant for small trades, but increases with trade size. Our results about a cut-off size and the dynamics of the cut-off size are consistent with their results. Especially, our result shows that larger trade sizes than the cut-off size have strictly positive bid-ask spreads, while smaller sizes do not.

Overall, the paper links the theoretical market microstructure development with empirical work. Although the model still includes the stylized assumptions, the paper provides the following testable hypotheses:

- Information effect on price after small trade size is zero.
- Price change after large trade sizes are transacted has a concave relationship with trade sizes.
- Information component of bid-ask spread monotonically increases in trade sizes.
- Bid-ask spreads in large trade sizes have a concave relationship with trade sizes.

The structure of the paper is as follows. We introduce the model in the second section. In the third section, we provide the equilibrium analysis. In the fourth section, we construct a price-impact function and analyze it. Finally the fifth section concludes.

## 2 The Model of the Sequential Trades

We consider a model in which potential buyers and sellers trade assets with a competitive market maker. The market maker sets prices at which she will buy or sell any quantity of the traded asset. In our economy, one risky asset is traded for $T$ periods. There is also a risk-free asset with period-$T$ payoff normalized to $1$. The terminal payoff of the risky asset, $\tilde{V}$, is not known to all agents in the economy. In particular, $\tilde{V} \in \{\underline{V}, \overline{V}\}$ with the prior probabilities such that $0 < \Pr(\tilde{V} = \underline{V}) = \delta < 1$. We assume that $\underline{V} < \overline{V}$.

There are three types of agents in the economy:

1. *risk neutral informed traders*, who know the realization $V$ of the risky asset payoff $\tilde{V}$;

2. *liquidity traders*, whose trading motives are exogenously determined by a random variable (which will be specified later);

3. *a competitive risk neutral market maker*, who counter the trade orders made by informed traders and liquidity traders.

The order sizes (quantities) in which the risky asset can be traded are restricted. In particular, the traders can trade the risky asset only in the order sizes which are elements of the set

$$\Omega_n = \{-n, ..., -1, 0, 1, ..., n\}.$$

In our notation, $k$ and $-k$ represent purchase and sale of $k$ units of risky asset, respectively. $\Omega_n^+ := \{1, ..., n\}$ denotes the set of possible purchase order sizes while $\Omega_n^- := \{-n, ..., -1\}$ denotes the set of possible sales order sizes. The number $0$ represents no trade.

We also assume that the number of informed traders is large enough to make each act competitively. Since informed traders hold the information superiority about the asset value, they have a motivation for trades in order to maximize their expected profit. A market maker trades because there is some chance that she is dealing with an uninformed trader. In addition, bid-ask spreads can compensate her for doing business with informed traders.

Now we will describe how trades actually transpire. The $T$-period trading game unfolds according to the following timing structure:

1. In period $0$, nature chooses the realization $V$ of $\tilde{V}$, and all informed traders observe $V$.

2. In successive periods indexed by $t = 1, ..., T$, the events realize in the given order:

(a) Having observed the trades up to period $t-1$, the competitive market maker posts a price for each possible order size;

(b) A new trader (of either informed or liquidity type) arrives at the market and learns the price-quantity quotes of the market maker.

(c) If the trader is informed, she takes the profit-maximizing quote. If the trader is a liquidity trader, she trades in the order size determined by her liquidity needs.

3. In the end of period $T$, $V$ becomes public information.

Now we describe the information structure in the market. The type of the trader arriving in period $t$ is determined by the random variable $\tilde{\theta}_t$ which takes values in $\{i_v, l\}$. These letters, $i_v$ and $l$, denote informed type and liquidity type, respectively. The random variables $\tilde{\theta}_t$'s are i.i.d. across $t = 1, ..., T$, and

$$\Pr(\tilde{\theta}_t = i_v) = \mu, \quad t = 1, ..., T.$$

If period $t$ type is $l$, then the demand is determined by the random variable $\tilde{L}_t$ with

$$\Pr(\tilde{L}_t = q) = \gamma(q) > 0,$$

for $q \in \Omega_n$. The random variables $\{\tilde{L}_t\}_{t=1,...,T}$ are mutually independent. The random variables $\tilde{\theta}_t$, $\tilde{L}_t$, $\tilde{V}$ are mutually independent for all $t$, as well. We assume that an informed trader, who arrives at the market once, gets the chance to re-trade with probability $0$. Thus, an informed trader behaves myopically and ignores the effect of her trade on future periods. The structure of this trading game is common knowledge.

This is the game in which the market maker observes the realized demand at the end of each period. Let $q_t$ denote the order size that the market maker receives in period $t$. Thus, each $q_t$ is the action that is played in period $t$. A history $h_t = (q_1, ..., q_t)$ is the realized choices of actions at all periods before $t+1$. The space of all possible $t$-period histories, $t \geq 1$, is denoted by $\Omega_n^t := \prod_{\tau=1}^t \Omega_n$, and $h_t$ is taken to be the generic element of $\Omega_n^t$. Also, $h_t$, $t \geq 1$, is said to be *consistent* with $h_T = (q_1, ..., q_T) \in \Omega_n^T$ if $h_T = (h_t, q_{t+1}, ..., q_T)$. For notational convenience, we let $h_0 = \emptyset$. Let $\pi_t : \Omega_n^{t-1} \times \Omega_n \rightarrow I\!\!R$ be a price function. Then, $\pi_t(h_{t-1}, q)$ denotes the market maker's price-quantity quote of trade size $q$ given $h_{t-1}$.

Next we define trading strategies of informed traders. The market maker faces a different trader in each period. Since there are multiple trade sizes available and the market maker chooses a price for each quantity, an informed trader may possibly obtain the same profit from trading different sizes. In this case, the informed trader assigns positive probabilities to those trade sizes (that is, she will take the behavior strategy that assigns positive probabilities to these actions of trading these quantities in an information set).

First, we call as trading strategies a probability function $\psi : \Omega_n \to [0,1]$ such that $\sum_{q \in \Omega_n} \psi(q) = 1$. We also define support of $\psi$ as $\mathrm{supp}(\psi) := \{q \in \Omega_n | \psi(q) \neq 0\}$. We further let

$$\Delta(\Omega_n) := \left\{ \psi : \Omega_n \to [0,1] \middle| \sum_{q \in \Omega_n} \psi(q) = 1 \right\}$$

to be the set of trading strategies. Second, we define a trading strategy of informed traders in each period. In period $t$, observing the value of $V$ and history $h_{t-1}$, informed traders choose a trading strategy which assigns a probability distribution over trade sizes in $\Omega_n$. Formally, we can define a trading strategy of informed traders as follows:

**Definition 1** *A trading strategy of an informed trader in period t prescribes a probability distribution $\psi_t(V, h_{t-1}) \in \Delta(\Omega_n)$ over trade sizes in $\Omega_n$ for each $V \in \{\underline{V}, \overline{V}\}$ and history $h_{t-1}$.*

Notice that $\psi_t(V, h_{t-1})$ assigns a probability that an informed trader trades each quantity. We shall denote the assigned probability to trade a trade size $q \in \Omega_n$ by $\psi_t(q|V, h_{t-1})$. Game theoretically, a collection of $\psi_t(V, h_{t-1})$ is a behavior strategy over all trade sizes $q \in \Omega_n$.

Next, we consider optimal trading strategy of an informed trader in period $t$. Among trading strategies, the informed trader chooses the strategy which maximizes her expected profit given the price schedule given by the market maker. In other words, an informed trader which arrives at the market in period $t$ chooses the trading strategy assigning probability distribution which maximizes her profit.

**Definition 2** *An optimal trading strategy of an informed trader in period t with respect to $\pi_t$ is a probability distribution $\psi_t^*(V, h_{t-1}) \in \Delta(\Omega_n)$ over trade sizes in $\Omega_n$ for each $V \in \{\underline{V}, \overline{V}\}$ and history $h_{t-1}$ if for $\pi_t$,*

$$\psi_t^*(V, h_{t-1}) \in \arg \max_{\psi \in \Delta(\Omega_n)} \sum_{q \in \Omega_n} \psi(q) \left[ q \left( V - \pi_t(h_{t-1}, q) \right) \right] \tag{1}$$

Before we proceed, we consider the market maker's decision problem. She decides each period price schedule based on the history of trade sizes that she has ever received up to the current period and a Bayesian updated belief contingent on each quantity. Our important assumption is that the market maker earns zero expected profits on each purchase and each sale, and she faces no transaction costs. This is a central assumption about a market maker that Glosten and Milgrom [10] and Easley and

O'Hara [6] have used. Competition between market makers leads to such a description. Two market makers are enough to make competition and drive them to zero profit pricing.

To illustrate this, suppose that there are two market makers in this market. Suppose the first market maker put the ask price $A_1$ so that it is strictly greater than her expected value of the asset conditional on her information. Then, the second market maker will choose an ask price $A_2 < A_1$ which is greater than the conditional expected value of the asset to attract a trader. Notice that market makers are assumed to be uninformed of the value $V$ and risk neutral. In each period, an available information set is same for the market makers. Thus, the conditional expected value of the asset will be the same. In this way, the market makers set prices with zero expected profit. We do not explicitly assume the existence of multiple market makers because our focus in the paper is information based bid-ask spreads and as long as zero expected profit pricing is imposed, the result from having multiple market makers is equivalent with one from a single competitive market maker.

We consider how the market maker updates the belief according to each trade size. The market maker sets prices with zero expected profit by using the belief updated after each trade size. In order to move on to equilibrium analysis, we have to specify the market maker's pricing quotes and thus we need her updating formula. We assume that the market maker is a Bayesian. Now we specify her updating formula.

Take a function $\delta_t : \Omega_n^t \to [0, 1]$. Let $\delta_t(h_t)$ denote the probability of risky payoff equal to $\underline{V}$ given history $h_t$. From the initial distribution of $\tilde{V}$, we have $\delta_0 = \delta$. Using Bayesian updating, we now have for all $q \in \Omega_n$,

$$
\begin{aligned}
\delta_t(h_{t-1}, q) &:= \Pr(\tilde{V} = \underline{V} | h_{t-1}, q) \\
&= \frac{\Pr(\tilde{V} = \underline{V} | h_{t-1}) \Pr(q | h_{t-1}, \tilde{V} = \underline{V})}{\sum_{V \in \{\underline{V}, \overline{V}\}} \Pr(\tilde{V} = V | h_{t-1}) \Pr(q | h_{t-1}, \tilde{V} = V)} \qquad (2) \\
&= \frac{\Pr(\tilde{V} = \underline{V} | h_{t-1}) \left[\mu \, \psi_t(q | \underline{V}, h_{t-1}) + (1 - \mu) \, \gamma(q)\right]}{\sum_{V \in \{\underline{V}, \overline{V}\}} \Pr(\tilde{V} = V | h_{t-1}) \, \mu \, \psi_t(q | V, h_{t-1}) + (1 - \mu) \, \gamma(q)}. \qquad (3)
\end{aligned}
$$

Notice that $0 < \delta < 1$ and $\gamma(q) > 0$ for all $q \in \Omega_n$, for all $q$, the denominator of (3) is non-zero. Also, following the zero-profit pricing condition, the market maker sets the price equal to

$$
\pi_t(h_{t-1}, q) = \delta_t(h_{t-1}, q) \, \underline{V} + (1 - \delta_t(h_{t-1}, q)) \, \overline{V}, \quad \forall h_{t-1} \in \Omega_n^{t-1}, \forall q \in \Omega_n. \qquad (4)
$$

We say that a price function $\pi_t$ satisfies the zero-profit pricing condition if the equation (4) holds.

The updating process of the market maker's belief is summarized in the following (the letter MM means market maker in Figure 1.).

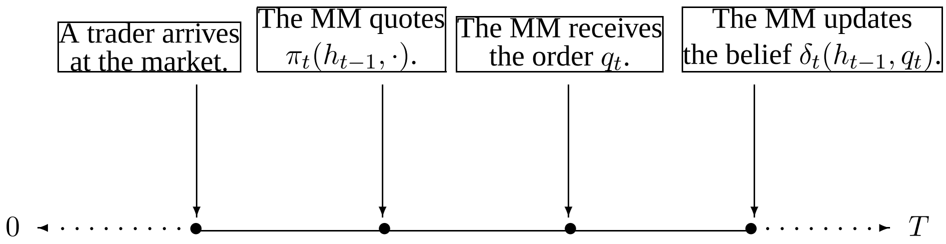


Figure 1. Updating the belief in period $t$ given the history $h_{t-1}$.

Finally, we define an equilibrium of this game. The equilibrium is defined as a perfect Bayesian equilibrium of the trading game and the zero-profit pricing condition.

**Definition 3** *An equilibrium consists of a sequence of price functions and optimal trading strategies of informed traders, $\{\pi_t^*, \psi_t^*\}_{t=1,2,\ldots,T}$, and posterior beliefs $\{\delta_t^*\}_{t=1,2,\ldots,T}$ such that for all $t \in \{1, \cdots, T\}$, all $q \in \Omega_n$, and all $h_t \in \Omega_n^t$,*

**(P1)** *$\pi_t^*$ satisfies the zero-profit pricing condition with respect to $\psi_t^*(V, h_{t-1})$,*

**(P2)** *for all $V \in \{\underline{V}, \overline{V}\}$, $\psi_t^*(V, h_{t-1})$ is an optimal trading strategy in every period $t$ with respect to $\pi_t^*$,*

**(B)** *for all $q \in \Omega_n$,*

$$\delta_t^*(h_{t-1}, q) = \frac{\Pr(\tilde{V} = \underline{V}|h_{t-1})\,[\mu\,\psi_t^*(q|\underline{V}, h_{t-1}) + (1-\mu)\,\gamma(q)]}{\sum_{V \in \{\underline{V}, \overline{V}\}} \Pr(\tilde{V} = V|h_{t-1})\,\mu\,\psi_t^*(q|V, h_{t-1}) + (1-\mu)\,\gamma(q)}. \tag{5}$$

The condition **(P1)** implies that the competitive market maker determines the price for each order size $q$ such that it is equal to the expected value of the risky asset payoff conditional on $q$ and the history of orders. The competition between potential market makers is not explicitly modelled here. Also, since in order to calculate the expected value of $\tilde{V}$ the market maker uses a updated belief contingent on each order size, the first condition explicitly implies that the belief has to be updated through a Bayesian rule.

The condition **(P2)** means that an informed trader chooses a trading strategy to maximize her profit. Also, note that the $T$-period history $h_T$ depends on the realization $\left(\{\theta_t\}_{t=1}^T, \{L_t\}_{t=1}^T\right)$ of the random vector $\left(\{\tilde{\theta}_t\}_{t=1}^T, \{\tilde{L}_t\}_{t=1}^T\right)$.

The condition **(B)** specifies the equilibrium belief. The market maker's belief in equilibrium follows the process defined in (5). For all $q \in \Omega_n$, $(1-\mu)\gamma(q)$ is non-zero and then the denominator of (5) is non-zero. Thus, the equilibrium belief path (5) is well-defined.

Notice that since a liquidity trader trades for her liquidity needs, she does not have strategic choices. Therefore, in an equilibrium there is no condition for her strategy to satisfy. Thus, the equilibrium definition above does not include any conditions for her. Actually, the trading game is played between a market maker and an informed trader. The market maker is uninformed and uses a Bayesian rule to update her belief. In this sense, the equilibrium definition has Bayesian game flavor. On the other hand, the market maker uses the zero profit condition for pricing assuming the competition between potential market makers, although this situation is not explicitly modelled. Thus the equilibrium notion that we are using here could be called a (Bayesian) Nash market equilibrium.

Finally, the bid-ask spread of quantity size $i$ is the difference between the ask price of quantity $q$ and the bid price of the same size. Thus, the period $t$ bid-ask spread for history $h_{t-1} \in \Omega_n^t$ and order size $q \in \{1, ..., n\}$ is formulated as

$$S_t(h_{t-1}, q) := \pi_t(h_{t-1}, q) - \pi_t(h_{t-1}, -q).$$

# 3 Equilibrium Analysis

## 3.1 Optimal Trading Strategies of the Informed Traders

Here we analyze the optimal informed trading strategies for given order histories. Our first result shows that if an optimal trading strategy assigns positive probability to a trade size $q$, all trade sizes larger than $q$ are also assigned positive probabilities. To motivate this result, let us consider an informed trader's size preference in a uniform-price setting: with a uniform price, the informed trader always prefers to buy the largest trade size if price is lower than the payoff and sells in the largest trade size if price is higher than the payoff. When there is differential pricing, as in our case, informed traders may avoid trading in the largest size with probability one fearing that such a strategy might lead the market maker to hike up the price for that size. As a result, trading that size with probability one would not be profitable for the informed traders.

First, we introduce one proposition about the basic properties of optimal trading strategies. The next proposition is helpful to define an equilibrium. For $\pi_t$ which satisfies (4), the following holds.

**Proposition 1** *For all possible $h_{t-1} \in \Omega_n^{t-1}$ and $t = 1, ..., T$, for zero-profit price functions $\pi_t^*$,*

1. $\psi_t^*(0|V,h_{t-1}) = 0\, \forall V \in \{\underline{V}, \overline{V}\}$.

2. $\psi_t^*(q|\overline{V},h_{t-1}) = 0 \quad \forall q \in \Omega_n^-$.

3. $\psi_t^*(q|\underline{V},h_{t-1}) = 0 \quad \forall q \in \Omega_n^+$.

Proposition 1 implies that an informed trader always trades in non-zero quantities. Also, the informed trader sells in case $V = \underline{V}$, and buys in case $V = \overline{V}$. Notice that a sale decreases the market maker's expectation of $\tilde{V}$ and a purchase increases it. Also, since $(1-\mu)\,\gamma(q)$ is always strictly positive in all non-zero trade sizes $q$, the denominator of (7) is non-zero for all non-zero $q$. Notice that there is no liquidity trader who trades $0$ and that the informed trader's trading zero quantity is of measure zero. So, when $q = 0$, the market maker can not obtain any information and so the belief stays the same. In other words, a null order does not affect the market maker's expectation of $\tilde{V}$. From these arguments, we can clarify the equilibrium belief as follows:

$$\delta_t^*(h_{t-1},q) = \begin{cases} \delta_{t-1}^*(h_{t-1}) & : \quad q = 0 \\ \frac{\delta_{t-1}^*(h_{t-1})[\mu\,\psi_t^*(q|\underline{V},h_{t-1})+(1-\mu)\,\gamma(q)]}{\delta_{t-1}^*(h_{t-1})\,\mu\,\psi_t^*(q|\underline{V},h_{t-1})+(1-\mu)\,\gamma(q)} & : \quad q \in \Omega_n^- \\ \frac{\delta_{t-1}^*(h_{t-1})\,(1-\mu)\,\gamma(q)}{(1-\delta_{t-1}^*(h_{t-1}))\mu\,\psi_t^*(q|\overline{V},h_{t-1})+(1-\mu)\,\gamma(q)} & : \quad q \in \Omega_n^+. \end{cases} \tag{6}$$

From (6), for all $q \in \Omega_n$,

$$0 < \delta_t(h_t) < 1, \quad \forall h_t \in \Omega_n^t, \forall t. \tag{7}$$

That is, the realization of $\tilde{V}$ is never fully revealed due to the non-degenerate possibility of liquidity trading at each non-zero order size.

$$\overline{k}_t(h_{t-1}) \;=\; \min\{k : (1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(i) + (1-\delta_t(h_{t-1},q_t))\mu > 0\}; \tag{8}$$

$$\underline{k}_t(h_{t-1}) \;=\; \min\{k : (1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(-i) + \delta_t(h_{t-1},q_t)\mu > 0\}. \tag{9}$$

The letters $\overline{k}_t(h_{t-1})$ or $\underline{k}_t(h_{t-1})$ are the minimal possible cut-off size above which the informed traders trade with strictly positive probabilities on the long side or the short side, respectively.

**Theorem 2** *For all $h_{t-1} \in \Omega_n^{t-1}$ and $t = 1,...,T$, there exists a cut-off size $\overline{k}_t(h_{t-1})$ and $\underline{k}_t(h_{t-1})$ with $\overline{k}_t(h_{t-1}) \geq 1$ and $\underline{k}_t(h_{t-1}) \geq 1$ such that* $\operatorname{supp}\{\psi_t^*(\overline{V},h_{t-1})\} = \{\overline{k}_t(h_{t-1}),\cdots,n\}$ *and* $\operatorname{supp}\{\psi_t^*(\underline{V},h_{t-1})\} = \{-n,\cdots,-\underline{k}_t(h_{t-1})\}$.

This simply means that an optimal trading strategy of informed traders is $k$ partially pooling for some $k$ with $1 \leq k \leq n$. By Theorem 2 we can classify the types of the optimal trading strategies.

DEFINITION. An optimal trading strategy of informed traders, $\psi_t^*$, is

- $k$ *partially pooling* on the long side if
  $0 < \psi_t^*(q|V, h_{t-1}) \leq 1$ for all $q \in \{k, k+1, ..., n\}$ and $\psi_t^*(q|V, h_{t-1}) = 0$ for all $q \in \{0, ..., k-1\}$.

- $k$ *partially pooling* on the short side if
  $0 < \psi_t^*(q|V, h_{t-1}) \leq 1$ for all $q \in \{-n, -n-1, ..., -k\}$ and $\psi_t^*(q|V, h_{t-1}) = 0$ for all $q \in \{0, ..., -k+1\}$.

Theorem 2 implies that all optimal trading strategies are $k$ partially pooling for some $k$, $1 \leq k \leq n$. For instance, $\psi_t(i|V, h_{t-1}) = 1$, for $i < n$, cannot occur for any $V$ and history $h_{t-1}$. As we will see in Theorem 2, there is a cut-off size beyond which the informed trader never trades with strictly positive probabilities. In this sense, the above classification does include all possible optimal trading strategies.

Notice that as special cases of $k$ partially pooling, we have the followings.

- $\psi_t^*$ is *separating* on the long side (short side) if $\psi_t^*$ is $n$ partially pooling on the long side (short side).

- $\psi_t^*$ is *completely pooling* on the long side (short side) if $\psi_t^*$ is 1 partially pooling on the long side (short side).

Now, we consider when an optimal trading strategy is separating or pooling. The following three propositions give us a necessary and sufficient condition for each kind of optimal trading strategies. Proposition 3 is about separating, Proposition 5 is about $k$ partially pooling, and Proposition 6 is about completely pooling.

**Proposition 3** *For all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$,*

1. *$\psi_t^*$ is separating on the long side if and only if $V = \overline{V}$ and*

$$\frac{n}{n-1} \geq 1 + \frac{(1 - \delta_{t-1}(h_{t-1}))\mu}{\gamma(n)(1-\mu)}; \tag{10}$$

2. *$\psi_t^*$ is separating on the short side if and only if $V = \underline{V}$ and*

$$\frac{n}{n-1} \geq 1 + \frac{\delta_{t-1}(h_{t-1})\,\mu}{\gamma(-n)(1-\mu)}. \tag{11}$$

This proposition says that as the probability of informed trading goes up, an optimal trading strategy is less likely to be separating. On the other hand, as the probability of informed trading goes down, an optimal trading strategy is more likely to be separating. One interpretation is that if the probability that informed traders trade the largest quantity is high, the market maker puts the very large bid-ask spread in the largest quantity. Then, the informed traders avoid trading the largest quantity and instead trade a different quantity size. In this sense, an optimal trading strategy is less likely to be separating as the probability of the informed trading increases. Reversely, as the probability of the informed trading decreases, an optimal trading strategy is more likely to be separating because the bid-ask spread in the largest quantity is not so large. As a direct corollary to Proposition 3, the following holds.

**Corollary 4** *In equilibrium, if $\mu$ is sufficiently close to $0$, $\psi_t^*$ is separating. On the other hand, if $\mu$ is sufficiently close to $1$, $V = \overline{V}$ and $q_{t-1} > 0$, $\psi_t^*$ is not separating in equilibrium. If $\mu$ is sufficiently close to $1$, $V = \underline{V}$ and $q_{t-1} < 0$, $\psi_t^*$ is not separating in equilibrium.*

**Proposition 5** *For all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$,*

1. *$\psi_t^*$ is $k_t(h_{t-1})$ partially pooling on the long side if and only if $V = \overline{V}$ and*

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}\left(1-\frac{i}{k_t(h_{t-1})}\right)\gamma(i) + (1-\delta_{t-1}(h_{t-1}))\mu \quad > \quad 0, \quad \textit{and} \tag{12}$$

$$(1-\mu)\sum_{i=k_t(h_{t-1})-1}^{n}\left(1-\frac{i}{k_t(h_{t-1})-1}\right)\gamma(i) + (1-\delta_{t-1}(h_{t-1}))\mu \quad \leq \quad 0; \tag{13}$$

2. *$\psi_t^*$ is $k_t(h_{t-1})$ partially pooling on the short side if and only if $V = \underline{V}$ and*

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}\left(1-\frac{i}{k_t(h_{t-1})}\right)\gamma(-i) + \delta_{t-1}(h_{t-1})\mu \quad > \quad 0, \quad \textit{and} \tag{14}$$

$$(1-\mu)\sum_{i=k_t(h_{t-1})-1}^{n}\left(1-\frac{i}{k_t(h_{t-1})-1}\right)\gamma(-i) + \delta_{t-1}(h_{t-1})\,\mu \quad \leq \quad 0. \tag{15}$$

We can see that the condition (12) for the set of trade sizes which the informed traders trade with the positive probabilities is complement of the condition (13) of the set of trade sizes which the informed traders trade with the probability of zero. The condition for the existence of the partially pooling equilibrium is that there must exist some $k$ above and beyond which each condition holds. Then, this $k$ will be a cut-off size above which the informed traders trade with strictly positive probabilities. As a special case of the above proposition, we obtain the following proposition for a completely pooling equilibrium.

**Proposition 6** *For all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$,*

1. *$\psi_t^*$ is completely pooling on the long side if and only if $V = \overline{V}$ and*
$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(i) + (1-\delta_{t-1}(h_{t-1}))\mu \quad > \quad 0; \tag{16}$$

2. *$\psi_t^*$ is completely pooling on the short side if and only if $V = \underline{V}$ and*
$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(-i) + \delta_{t-1}(h_{t-1})\,\mu \quad > \quad 0. \tag{17}$$

In the case of a completely pooling equilibrium, the informed traders trade all possible trade sizes with strictly positive probabilities. Therefore, in the above proposition we do not have condition like (13) in Proposition 5. The similar interpretation stated in Corollary 4 applies here. The first term of (16), $(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(i)$, is always negative. In order for (16) to hold, the second term, $(1-\delta_{t-1}(h_{t-1}))\mu$, has to be positive. Therefore, the probability of informed trading, $(1-\delta_{t-1}(h_{t-1}))\mu$, has to be positive. So, we can say that as the probability of informed trading is greater, an equilibrium is more likely to be completely pooling.

### 3.2 Existence and Uniqueness of Equilibrium

We have given the necessary and sufficient conditions for the existence of each kind of equilibrium. Before we move on to the analysis of dynamics of these equilibria, we consider whether those equilibria would always exist or different kinds of equilibrium exist simultaneously. The answer is no. The following proposition says that it is always the case that there exists the only optimal strategy of the informed traders in each period.

**Proposition 7** *For a history $h_{t-1} \in \Omega_n^{t-1}$, $1 \leq t \leq T$, an optimal trading strategy of informed traders $\psi_t^*$ is unique.*

From Proposition 5, inequalities (12) and (13) or inequalities (14) and (15), we can find a unique $k$. In other words, on the long side, a trade size that satisfies both (12) and (13) is a unique cut-off size. For example, when an equilibrium is $k$ partially pooling, $k+1$ makes the right hand sides of (12) and (13) greater than 0 and $k-1$ makes the right hand sides of (12) and (13) smaller than 0. Then, from Lemma 2, $\psi_t^*$ is determined uniquely (see appendix for details). Therefore, an optimal trading strategy of informed traders is uniquely determined.

Notice that Proposition 7 establishes that there can be no an optimal trading strategy $\psi_t^*$ with $\psi_t(n|V,h_{t-1}) = \psi_t(-n|V,h_{t-1}) = 0$ for any $V \in \{\underline{V},\overline{V}\}$. This means that the informed traders always trade the largest quantity with a strictly positive probability. As the direct corollary to the above proposition, we obtain the unique existence of an equilibrium.

**Corollary 8** *There exists a unique equilibrium.*

The proof is done in the following way. From Proposition 3, Proposition 5 and Proposition 6, we prove that one of the three propositions must hold. Then, we can specify the cut-off size in each period. Thus, there is an optimal strategy. The existence of an equilibrium follows. Notice that the equilibrium consists of the optimal trading strategies and the price function. The price functions are given by the zero profit condition. Given the market maker's belief, we can determine the price-quantity quotes. As stated in Proposition 7, an optimal strategy is unique. Therefore, the equilibrium exists uniquely. Now we have given the necessary and sufficient conditions for each kind of equilibrium and established the unique existence of an equilibrium. From the point of a game theory, we have specified a behavioral strategy at each node. Also we have shown that a behavioral strategy exists uniquely at each node. The analysis so far is very static in the sense that we focus on period by period condition. Now we turn our attention to the dynamics of the bid-ask spreads in the next section.

### 3.3 Equilibrium Dynamics

In this section, we consider the market maker's learning process in the dynamics setting. In order to do so, we will extend the results that we have gotten in the previous section to the multiple periods environment. First, we consider the movement of each kind of equilibrium over time. The following proposition specifies the condition under which the set of trade sizes above which the informed traders trade with strictly positive probabilities expands. In other words, we have specified the condition under which a cut-off trade size decreases over time.

**Theorem 9** *Let $\{\pi_t^*,\psi_t^*,\delta_t^*\}_{t=1}^T$ be an equilibrium. Given $h_{t-1} \in \Omega_n^{t-1}$ and $h_t = (h_{t-1},q_t)$ for $q_t \in \Omega_n$,*

1. *if $\psi_t^*$ is $k_t(h_{t-1})$ partially pooling on the long side and $q_t > 0$, then $\psi_{t+1}^*$ is $k_{t+1}(h_{t-1},q_t)$ partially pooling on the long side with $k_{t+1}(h_{t-1},q_t) \le k_t(h_{t-1})$;*
2. *if $\psi_t^*$ is $k_t(h_{t-1})$ partially pooling on the short side and $q_t < 0$, then $\psi_{t+1}^*$ is $k_{t+1}(h_{t-1},q_t)$ partially pooling on the short side with $k_{t+1}(h_{t-1},q_t) \le k_t(h_{t-1})$.*

We have known that the cut-off size weakly decreases with a certain probability in each period. In other words, with that probability the cut-off size would not increase. In the next proposition, we consider the conditions under which the cut-off size decreases.

**Proposition 10** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium.*

1. *Suppose that for a history $h_{t-1} \in \Omega_n^{t-1}$ and $h_t = (h_{t-1}, q_t)$ for $q_t \in \Omega_n$, $\psi_t^*$ is $k_t(h_{t-1})$ partially pooling on the long side and $q_t > k_t(h_{t-1})$. Then $\psi_{t+1}^*$ is $k_{t+1}(h_{t-1}, q_t)$ partially pooling on the long side with $k_{t+1}(h_{t-1}, q_t) < k_t(h_{t-1})$ if*

$$\delta_t^*(h_t) \times \Big(\frac{1}{\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)} - \frac{1-\mu}{q_t \Pr_t(q_t \geq k_t(h_{t-1}))}\Big) \geq \frac{1-\mu}{\mu}\,\frac{1}{k_t(h_{t-1}) \times (k_t(h_{t-1}) - 1)}; \tag{18}$$

2. *Suppose that for given $h_{t-1} \in \Omega_n^{t-1}$ and $h_t = (h_{t-1}, q_t)$ for $q_t \in \Omega_n$, $\psi_t^*(V, h_{t-1})$ is $k_t(h_{t-1})$ partially pooling on the short side and $q_t < -k_t(h_{t-1})$. Then $\psi_{t+1}^*$ is $k_{t+1}(h_{t-1}, q_t)$ partially pooling on the short side with $k_{t+1}(h_{t-1}, q_t) < k_t(h_{t-1})$ if*

$$(1 - \delta_t^*(h_t)) \times \Big(\frac{1}{\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(-i)} - \frac{1-\mu}{|q_t| \Pr_t(q_t \leq -k_t(h_{t-1}))}\Big) \geq \frac{1-\mu}{\mu}\,\frac{1}{k_t(h_{t-1}) \times (k_t(h_{t-1}) - 1)}. \tag{19}$$

Notice that if the condition (18) or (19) does not hold, we will have $k_t(h_{t-1}) = k_{t+1}(h_{t-1}, q_t)$. One interpretation of the above proposition is that all trade sizes that the informed traders trade with strictly positive probabilities have bid-ask spreads and then in the next period the informed traders will trade the smaller quantity. As a direct corollary of Theorem 9, we have the following:

**Corollary 11** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium.*
*For a history $h_{t-1} \in \Omega_n^{t-1}$ and $h_t \in \{(h_{t-1}, q) : q \in \Omega_n\}$,*

1. *if $\psi_t^*$ is completely pooling on the long side (short side) and $q_t > 0 (q_t < 0)$, then $\psi_{t+1}^*$ is completely pooling on the long side (short side);*

2. *if $\psi_{t+1}^*$ is separating on the long side (short side) and $q_t > 0 (q_t < 0)$, then $\psi_t^*$ is separating on the long side (short side).*

Now we consider the dynamics of the market maker's equilibrium belief $\delta_t^*$. Proposition 3, Proposition 5 and Proposition 6 include endogenous variables, $\delta_t^*$. The following proposition gives the general formula of the market maker's belief and expresses the market maker's belief with only exogenous variables. We express the market maker's belief in a recursive way.

For a history $h_t = (q_1, ..., q_t) \in \Omega_n^t$, let

$$m^+(h_t) := \{\tau : \tau \le t, \delta_\tau(h_\tau) < \delta_{\tau-1}(h_{\tau-1}) \ h_{\tau-1} \text{ consistent with } h_t\},$$

and

$$m^-(h_t) := \{\tau : \tau \le t, \delta_\tau(h_\tau) > \delta_{\tau-1}(h_{\tau-1}) \ h_{\tau-1} \text{ consistent with } h_t\}.$$

Let $m(h_t) = m^+(h_t) \cup m^-(h_t)$. In words, $m^+(h_t)$ denotes the set of periods in which the market maker's belief is updated upward and $m^-(h_t)$ denotes the set of periods in which the market maker's belief is updated downward. Moreover, $m(h_t)$ denotes the set of periods in which the market maker's expectation of $\tilde{V}$ is non-trivially updated, i.e., it is the set of the periods $\tau$ which satisfies

$$\delta_\tau(h_\tau) \neq \delta_{\tau-1}(h_{\tau-1})$$

for $h_\tau$ and $h_{\tau-1}$ consistent with $h_t$.

**Proposition 12** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium. There exist functions $\delta_t^- : \Omega_n^t \times [0,1] \to [0,1]$ and $\delta_t^+ : \Omega_n^t \times [0,1] \to [0,1]$ such that for $\{h_t\}_{t=1}^T$ consistent with $h_T$, if $|m^-(h_t)| = \emptyset$, then $\delta_t^*(h_t) = \delta_t^+(h_t, \delta)$ and if $|m^+(h_t)| = \emptyset$, then $\delta_t^*(h_t) = \delta_t^-(h_t, \delta)$*[2].

An idea behind Proposition 12 is as follows. In order to derive the market maker's updated belief in period $t$, we first focus on the evolution of the market maker's belief associated with periods in $m^+(h_t)$ and then we consider periods in $m^-(h_t)$. In other words, $\delta_t^+$ gives us the market maker's belief that is updated downward given a history and the initial distribution of $\delta$. In the same way, $\delta_t^-$ gives us the market maker's updated belief that is updated upward given a history and the initial distribution of $\delta$. A history can be divided into two sets, $m^+(h_t)$ and $m^-(h_t)$. A function $\delta_t^+(h_t, \delta)$ gives us the market maker's updated belief associated with periods in $m^+(h_t)$ when $m^-(h_t)$ is an empty set and a function $\delta_t^-(h_t, \delta)$ gives us the market maker's updated belief associated with periods in $m^-(h_t)$ when $m^+(h_t)$ is an empty set. An equilibrium belief $\delta_t^*(h_t)$ is a combination of these two types of belief because in general both of $m^+(h_t)$ and $m^-(h_t)$ are non-empty.

Let us describe the relationship between $\delta_t^*(h_t)$, and $\delta_t^+(h_t, \delta)$ and $\delta_t^-(h_t, \delta)$. We consider the following situation: up to period $t_1$, $m^-(h_t)$ is empty and after $t_1$ up to $t_2$, $m^+(h_t)$ is empty. Then,

[2] The details are found in Appendix.

the market maker's equilibrium belief in period $t_1$ is given by $\delta^+_{t_1}(h_{t_1}, \delta)$, and then the market maker's equilibrium belief in period $t_2$ is given by $\delta^+_{t_1}(h_{t_1}, \delta^+_{t_1}(h_{t_1}, \delta))$. In a similar fashion, we repeat this procedure up to period $T$ and obtain a general formula of the market maker's belief $\delta^*_t(h_t)$.

Now that we know how to obtain the market maker's belief expressed by only exogenous variables, we combine it with propositions that we have seen in the previous section and see how the initial conditions influence on the optimal trading strategies of the informed traders. The most interesting result is the one applied to Proposition 3. Let $|m(h_t)|$ count the number of periods in the set $m(h_t)$. The following proposition describes when informed traders always trade the largest trade size, in other words when the optimal trading strategy in each period is separating. The idea is that since Proposition 3 specifies the necessary and sufficient conditions for an optimal trading strategy to be separating, by substituting $\delta^*_t$ derived in Proposition 12, we obtain the following proposition.

**Proposition 13** *Let $\{\pi^*_t, \psi^*_t, \delta^*_t\}_{t=1}^T$ be an equilibrium.*
*For a history $h_{t-1} \in \Omega_n^{t-1}$ and $h_t \in \{(h_{t-1}, q) : q \in \Omega_n\}$,*

1. *$\psi^*_t$ is separating on the long side if and only if $V = \overline{V}$ and*

$$\frac{n}{n-1} \geq 1 + \frac{\mu}{\gamma(n)\,(1-\mu) + \frac{\delta\,(\gamma(n)\,(1-\mu))^{|m(h_t)|}\,(\gamma(n)\,(1-\mu)+\mu)^{|m(h_t)|-1}}{1-\delta}},$$

2. *$\psi^*_t$ is separating on the short side if and only if $V = \underline{V}$ and*

$$\frac{n}{n-1} \geq 1 + \frac{\mu}{\gamma(-n)\,(1-\mu) + \frac{(1-\delta)(\gamma(-n)\,(1-\mu))^{|m(h_t)|}}{\delta(\gamma(-n)\,(1-\mu)+\mu)^{|m(h_t)|-1}}}.$$

An interesting thing about the above proposition is that if $\frac{n}{n-1} \geq 1 + \frac{\mu}{\gamma(n)\,(1-\mu)}$ or $\frac{n}{n-1} \geq 1 + \frac{\mu}{\gamma(-n)\,(1-\mu)}$, an equilibrium is separating even when the number of periods goes to infinity. The proposition says that if the probability of the informed trading is sufficiently small, an equilibrium is separating. In other words, if the probability of the liquidity trading is sufficiently large compared to the informed trading, an equilibrium is not separating.

## 3.4 Bid-Ask Spreads and The Market Maker's Learning Process

So far, we have given the general analysis of the dynamics of an equilibrium. Finally, in this section, we consider the bid-ask spreads. A bid-ask spread in each trade size is defined as the difference between bid-price and ask-price in each quantity. We will consider the behavior of the bid-ask spreads after the infinitely many transactions in this section.

Notice that:

$$S_t(h_{t-1}, q) := \left(\delta_t(h_{t-1}, -q) - \delta_t(h_{t-1}, q)\right)(\overline{V} - \underline{V}). \tag{20}$$

Bid-ask spread reflects the difference between the market maker's updated belief after setting the size $q$ and one after buying the size $q$. Let $\overline{K}_t(h_{t-1})$ and $\underline{K}_t(h_{t-1})$ denote the maximum and the minimum of $\underline{k_t}(h_{t-1})$ and $\overline{k_t}(h_{t-1})$, respectively. Also, let $S_t^*(h_{t-1}, q)$ denote the bid-ask spread in trade size $q$ associated with the equilibrium belief of $\delta_t^*(h_{t-1}, q)$ and $\delta_t^*(h_{t-1}, -q)$.

**Proposition 14** *$S_t^*(h_{t-1}, q) > 0$ for all $q \in \{\underline{K}_t(h_{t-1}), \cdots, n\}$ and $S_t^*(h_{t-1}, q) = 0$ for all $q \in \{1, \cdots, \underline{K}_t(h_{t-1}) - 1\}$.*

Proposition 14 implies that the larger trade sizes which informed traders trade with strictly positive probabilities have strictly positive bid-ask spreads, while the smaller sizes do not. An interpretation of Proposition 14 is straightforward. Since the market maker understands which trade sizes the informed traders trade, she assign strictly positive bid-ask spreads in those sizes in order to deal with the informed traders. Proposition 14 explains which sizes have strictly positive bid-ask spreads. Next, we consider when bid-ask spreads for each trade size, even larger sizes, vanish. In order to do so, we first calculate the bid-ask spreads. By substituting the optimal trading strategy derived in Lemma 2 (see Appendix) into (20), we obtain the following lemma:

**Lemma 1** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium. There exists a function $f^*(h_{t-1}, \cdot) : \Omega_n \to I\!R$ such that for a history $h_{t-1} \in \Omega_n^{t-1}$,*

$$S_t^*(h_{t-1}, q) = f^*(h_{t-1}, q)(\overline{V} - \underline{V}) \tag{21}$$

*for* $q = 1, ..., n$[3].

First, we consider how the bid-ask spread looks like as a function of trade sizes. As we can see from (20), bid-ask spread shows the difference of the market maker's beliefs after selling and buying the size. Thus we can say that we will consider how this difference changes over the trade sizes. In order to express it, we use the notion of "concavity." Before we proceed, we define concavity in our setting, because the concavity of functions are usually defined in a continuous domain and our domain is discrete.

[3]The details are found in Appendix.

**Definition 4** *A function $S_t(h_t, q)$ is defined to be concave in a discrete domain with respect to $q$ if for all $q^{'}$ and $q^{''}$ with $q^{'} < q^{''}$, $S_t(h_t, q^{'} + m) - S_t(h_t, q^{'}) > S_t(h_t, q^{''} + m) - S_t(h_t, q^{''})$ for all $m \in \mathcal{N}$.*

**Proposition 15** *In each interval of $[\underline{K}_t(h_{t-1}), \overline{K}_t(h_{t-1})]$ and $[\overline{K}_t(h_{t-1}), n]$, $S_t^*(h_t, q)$ is concave in a discrete domain with respect to $q$.*

The figure 1 is a continuous representation of the bid-ask spreads function $S_t^*(h_{t-1}, q)$ with respect to $q$. In the calculation, we have set the market maker's belief $\delta$ equal to $0.1$ and assumed that there are $10$ trade sizes available for traders. Also, in the first figure, liquidity traders are assumed to be distributed uniformly over all trade sizes and in the second figure, liquidity traders are distributed with poisson distribution of mean $5$, that is $\gamma(q) = \frac{5^q}{2q!} exp(-5)$. From the figure, we can see that when the probability of informed trading increases, bid-ask spreads of all quantity sizes become larger. This reflects the market maker's"risk hedging." Also, as Proposition 15 says, a continuous representation of $S_t^*(h_t, q)$ displays concave form in two intervals of $[\underline{K}_t(h_{t-1}), \overline{K}_t(h_{t-1})]$ and $[\overline{K}_t(h_{t-1}), n]$.

Informed traders are indifferent about trading those sizes in the intervals. Notice that there are two possible equilibrium, long side and short side. The market maker does not know which kind of equilibrium occurs. Therefore, the market maker considers both cases. In a case of long side equilibrium, informed traders are indifferent about trade sizes between $[\overline{k}_t(h_{t-1}), n]$ and in a case of short side, informed traders are indifferent between trade sizes in $[\underline{k}_t(h_{t-1}), n]$. It is often the case that these two cut off sizes are different. It is actually the case that as the market maker's belief changes, these trade sizes change. In this example of Figure 1, the market maker's belief $\delta$ is $0.1$. From our calculation, when liquidity traders are distributed uniformly, $\underline{k}_t(h_{t-1}) = 8$ for $\overline{k}_t(h_{t-1}) = 5$ in a case of $\mu = 0.2$. In other words, when $V = \overline{V}$, informed traders are indifferent about trading sizes above $5$ and when $V = \underline{V}$, informed traders are indifferent about trading sizes above $8$. The market maker does not know which case is realized. What the market maker knows is that the sizes between $5$ and $8$ are transacted only when $V = \overline{V}$ and the sizes above $8$ are transacted in either case. That is the reason why there is a kink in the function $S_t^*(h_t, q)$ defined in $q \in \{0, \cdots, 10\}$. This is true for the general case for the reasons explained above.

Notice that the bid-ask spread function for any distribution of liquidity traders displays the similar form. As we explained above, the concave form in the two intervals and the existence of the kink come from the condition that informed traders are indifferent between trading these sizes in each interval. When the market maker posts quotes, the market maker considers the distribution of liquidity traders and the informed traders' trading strategies. For example, as the extreme case, we consider the situation where almost all liquidity traders sell $9$ shares and very few traders trade at other sizes than $-9$. Informed traders are indifferent between selling the sizes larger than the cut-off size of $8$ on the short side of the above example. As presented in Lemma 2, (29), if $\gamma(-9) > \gamma(-10)$, then $\delta_t^*(-9|\underline{V}, h_{t-1}) > \delta_t^*(-10|\underline{V}, h_{t-1})$. However, since the informed traders are indifferent between selling both sizes, it is natural that the change of the market maker's belief after selling the size $10$ is as much as one after selling the size $9$.

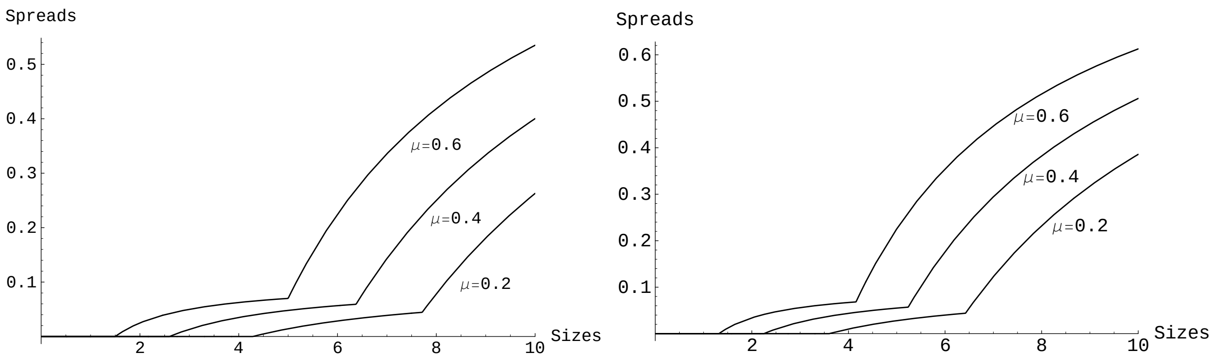


Figure 1: Bid-ask Spreads with Uniformly or Poisson Distributed Liquidity Traders

**Corollary 16** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium. Given $h_{t-1} \in \Omega_n^{t-1}$ and $h_t \in \{(h_{t-1}, q) : q \in \Omega_n\}$,*

1. *$S_t^*(h_{t-1}, n) > 0$.*

2. *if*

$$\frac{n}{n-1} \geq 1 + \max\left\{\frac{\mu}{\gamma(n)\,(1-\mu)}\,,\,\frac{\mu}{\gamma(-n)\,(1-\mu)}\right\},$$

*then*

$$S_t^*(h_{t-1}, q) \quad = \quad 0 \quad \textit{for } q \in \{1, ..., n-1\}.$$

The first part of the corollary says that there always exists a bid-ask spread in the largest quantity. Since the informed traders trade the largest quantity with a strictly positive probability, it is always the case that a bid-ask spread in the largest quantity is strictly positive. The condition in the second part is from the condition of Proposition 3. The second part says that if a separating equilibrium exists, a bid-ask spread in smaller quantities than the largest size is zero. This is straightforward because if there is no probability that the informed trade these smaller quantities, there is no need for the market maker to put the bid-ask spread. Next, we consider the behavior of the bid-ask spreads when we take the limit of the probability of liquidity trading.

**Corollary 17** *The followings hold.*

1. *As* $\gamma(n) + \gamma(-n) \to 0$*,* $S_t^*(h_{t-1}, n) \to c_1 > 0$*.*

2. *If*

$$\frac{n}{n-1} \leq 1 + \min\left\{ \frac{\mu}{\gamma(n)\,(1-\mu) + \frac{\delta}{(1-\delta)(\gamma(n)(1-\mu)+\mu)}}\,,\, \frac{\mu}{\gamma(-n)\,(1-\mu) + \frac{(1-\delta)}{\delta\,(\gamma(-n)(1-\mu)+\mu)}} \right\},$$

*then as* $\gamma(n-1) + \gamma(-n+1) \to p$ *for some positive* $p$*,* $S_t^*(h_{t-1}, n-1) \to c_2 > 0$*.*

The first part says that as the probability of informed trading in the largest quantity decreases to $0$, then the bid-ask spread vanishes. The second part says that as the probability of informed trading in the second largest size converges to some positive probability, the bid-ask spreads in that quantity do not vanish. An interpretation is that when the probability of liquidity trading in the largest trade size is sufficiently small, then the trading in that size will be most probably from the informed traders. Then, we would not have a separating equilibrium because the market maker will put a large bid-ask spread and the informed traders will trade different trade sizes. Therefore, the bid-ask spread in that quantity vanishes as the probability of liquidity trading decreases to zero.

An intuition of the second part is also similar to the first part. Here we have considered the second largest quantity. In a (completely or partially) pooling equilibrium, the informed traders trade the second largest size with a strictly positive probabilities. If there is some probability that a liquidity trading occurs in that quantity, it would be easier for the informed traders to 'hide' themselves in that quantity and the spreads in the second largest size do not vanish.

So far, we have considered bid-ask spreads across all trade sizes in a $T$-period game. Now, we turn our attention to bid-ask spreads in the case where $T$ become sufficiently large. In other words, we study the market maker's learning process when there are sufficiently large number of trading periods. Before we proceed, we introduce one proposition about when bid-ask spreads for all trade sizes vanish. From (21), the following proposition is straightforward.

**Proposition 18** *The following holds:* $S_t^*(h_{t-1}, q) = 0$ *if and only if* $\delta_t^*(h_{t-1}, q) = 0$ *or* $1$ *for* $q \in \Omega_n$*.*

An interpretation of Proposition 18 is as follows. When the market maker's belief is $0$ or $1$, the bid-ask spreads vanish. In other words, when the market maker learns whether or not the valuation of the asset is high, the bid-ask spreads for all trade sizes vanish. Otherwise, it does not. The result is natural in the sense that when the market maker learns the value of the asset, due to the zero profit pricing condition, she quote the bid and ask prices of each size at the exactly same level of the value of the asset. Therefore, the bid-ask spreads vanish.

An interesting is that even if the asset is of low value but if the market maker believes that the asset is of high value, the bid-ask spreads vanish. The question here is whether or not this could happen. Else, we would like to know whether or not the market maker learns the value of the asset.

Notice that: for a history $h_t$,

$$\begin{aligned} \mathrm{E}[\delta_{t+1}(h_t, q_{t+1})|h_t] &= \mathrm{E}[\Pr(\tilde{V} = \underline{V}|h_t, q_{t+1})|h_t] \\ &= \Pr(\tilde{V} = \underline{V}|h_t) \\ &= \delta_t(h_t). \end{aligned}$$

Therefore, the belief $\delta_t$ forms a martingale. An optimal strategy of informed traders prescribes probability distribution over all trade sizes in each period of $h_T$ and the market maker's belief assigns a probability of risky asset's being equal to the low value in each period given a history up to the period. We consider the market maker's equilibrium belief as a stochastic process.

**Theorem 19** *Let $T = \infty$. Then, $\lim_{t\to\infty} \delta_t(h_t) = \delta^*$ almost surely where $\delta^* = 0$ if $V = \overline{V}$ and $\delta^* = 1$ if $V = \underline{V}$.*

The sketch of the proof is as follows. Within the game of an infinitely many periods, we prove that the equilibrium belief converges to some random variable by Martingale convergence theorem. Finally, using the condition for the equilibrium belief, we can show that the random variable takes $0$ if $V = \overline{V}$ and $1$ if $V = \underline{V}$. Therefore, we can conclude that the market maker finally learns the true value of the asset almost surely. The important thing is that by Corollary 18, the bid-ask spreads vanish almost surely when the number of trading rounds goes to infinity.

### 3.5 The Number of Trade Sizes and Its Impact on Equilibrium

In this section, we investigate how the number of trade sizes affects bid-ask spreads and the market maker's learning process. To keep the notation simple, we previously ignored the fact that equilibrium is dependent on the largest trade size $n$ since $n$ has been fixed in our analysis till now. We introduce the following notation to carry out the comparative statics required for our investigation: when $n$ is the largest trade size, we let

- $\gamma_n$ denote the probability function for liquidity demand;

- $\{\pi^*_{t,n}, \psi^*_{t,n}, \delta^*_{t,n}\}_{t=1,...,T}$ denote the equilibrium;
- $\delta_{t,n}(h_t)$ denote the probability of risky payoff being equal to $\underline{\mathrm{V}}$ given the history $h_t$;
- $\mathrm{var}_n(V|h_t)$ denote the variance of risky payoff $V$ conditional on the history $h_t$;
- $S_{t,n}(h_{t-1}, q)$ denote the period $t$ bid-ask spread for the history $h_{t-1} \in \Omega^n_{t-1}$ and the trade size $q \in \{1, ..., n\}$.

If the market regulators add new trade sizes to the economy, how would this affect the bid-ask spreads for the previously existing trade sizes? This is the first question that we investigate in this section. Our analysis introduces new trade sizes by increasing the largest trade size $n$. We see that if sufficiently high number of new trade sizes are introduced into the economy and the probability of liquidity trading in these new sizes is high enough then the bid-ask spreads for the previously existing trade sizes vanish. Formally, we have the following:

**Proposition 20** *Let $\{\pi^*_{t,n}, \psi^*_{t,n}, \delta^*_{t,n}\}_{t=1,...,T}$ be the equilibrium, given the largest trade size $n \in \{1, 2, ...\}$. Suppose $\lim_{n\to\infty} \sum_{i=1}^n i\gamma_n(+i) = \lim_{n\to\infty} \sum_{i=1}^n i\gamma_n(-i) = \infty$. For any given trading period $t \geq 1$ and history $(h_{t-1}, q) \in \Omega^t_m$, there exists $N_{(h_{t-1},q)} \in \{m+1, m+2, ...\}$ such that for all $m' \geq N_{(h_{t-1},q)}$*

$$0 = S^*_{t,m'}(h_{t-1}, q) \leq S^*_{t,m}(h_{t-1}, q).$$

*The inequality above becomes strict if $\psi^*_{t,m}(V, h_{t-1}, q)$ is non-zero for some $V \in \{\underline{V}, \overline{V}\}$.*

The key to this result lies in the following observation. If pricing were uniform, informed traders would choose to trade in the largest available trade size since they can reap the highest profit by doing so. In anticipation of such behavior, the market maker puts a spread between the bid and ask prices of the large trade sizes, which compensates her for the risk of doing business with informed agents. Naturally, the spread for any given trade size decreases as the probability of liquidity trading in that size increases, because such an increase means lower risk of trading with informed traders. When the probability of liquidity trading in large trade sizes is high enough, spreads shrink and informed traders prefer to trade in these large quantities. Introducing new trade sizes by increasing the largest trade size $n$ essentially means adding larger trade sizes to the economy. If the number of these new and large trade sizes is sufficiently large and the probability of liquidity trading in these new sizes is sufficiently high, informed traders will trade in these new sizes and they will no more trade in the previously existing trade sizes. Since there is no more risk of doing business with informed traders in the previously existing trade sizes, these trade sizes will have no spreads between their bid and ask prices. Hence we have the result stated in Proposition 20.

Proposition 20 guarantees a high probability of liquidity trading in the newly introduced large trade sizes by imposing the condition

$$\lim_{n\to\infty}\sum_{i=1}^{n} i\gamma_n(+i) \;=\; \lim_{n\to\infty}\sum_{i=1}^{n} i\gamma_n(-i) \;=\; \infty.$$

This condition is satisfied, for instance, by the sequence of probability functions, $\{\gamma_n\}_{n=1}^{\infty}$, which uniformly distributes liquidity trading over the trade sizes in $\Omega_n$. This is the sequence, $\{\gamma_n\}_{n=1}^{\infty}$, with $\gamma_n : \Omega_n \to [0,1]$ and $\gamma_n(i) = \frac{1}{2n+1}, \;\; \forall i \in \Omega_n$.

Next we investigate how the introduction of new trade sizes affects the market maker's learning process. We measure the effect of a trade, in the size $q$ and period $t$, on the market maker's learning process by the precision of risky payoff $V$ conditional on her information and the knowledge that $q$ will be traded in that period, which is $\frac{1}{\mathrm{var}(V|h_{t-1},q)}$. So, the higher this conditional precision, the market maker's learning process improves, and the lower this conditional precision, the learning process gets impaired. The following proposition shows that introducing new trade sizes by increasing the largest trade size $n$ can actually impair the market maker's learning process for trades occurring in the previously existing trade sizes.

**Proposition 21** *Let $\{\pi^*_{t,n}, \psi^*_{t,n}\}_{t=1,\dots,T}$ be the equilibrium, given the largest trade size $n \in \{1,2,\dots\}$. Suppose $\lim_{n\to\infty}\sum_{i=1}^{n} i\gamma_n(+i) \;=\; \lim_{n\to\infty}\sum_{i=1}^{n} i\gamma_n(-i) \;=\; \infty$. For any given trading period $t \geq 1$ and history $(h_{t-1}, q) \in \Omega^t_m$, there exists $N_{(h_{t-1},q)} \in \{m+1, m+2, \dots\}$ such that for all $m' \geq N_{(h_{t-1},q)}$*

1. $\mathrm{var}_{m'}(V|h_{t-1},q) \geq \mathrm{var}_m(V|h_{t-1},q)$ *if* $\delta \leq \frac{1}{2}$, *and* $(h_{t-1},q) \in \left(\Omega^+_m \cup \{0\}\right)^t$,
2. $\mathrm{var}_{m'}(V|h_{t-1},q) \geq \mathrm{var}_m(V|h_{t-1},q)$ *if* $\delta \geq \frac{1}{2}$, *and* $(h_{t-1},q) \in \left(\Omega^-_m \cup \{0\}\right)^t$.

*The inequalities above become strict if $\psi^*_{t,m}(V, h_{t-1}, q)$ is non-zero for some $V \in \{\underline{V}, \overline{V}\}$, in addition to the conditions stated above.*

It is easy to check that all results in this paper hold when the space of trade sizes is taken as

$$\left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}.$$

Therefore, our analysis also allows us to investigate the implications of policy to introduce smaller trade unit in a stock market, that is, we can see the impacts of trading in smaller increments of stock numbers. The analysis above shows that this institutional change can cause bid-ask spreads to vanish and market maker's learning process to be impaired when orders are made in small trade size.

# 4 The Price-Impact Function

In this section, we consider the relationship between trade sizes and the expected price changes conditional on each trade size. This relationship is formed as a price-impact function. We define the function as:

$$R_t(h_{t-1}, q) = \pi_t(h_{t-1}, q) - \pi_{t-1}(h_{t-1}). \tag{22}$$

The function $R_t(h_{t-1}, q)$ shows the expected change of price conditional on trade size $q$ given the history $h_{t-1}$ in period $t$. We can rewrite (22) as:

$$R_t(h_{t-1}, q) = (\delta_{t-1}(h_{t-1}) - \delta_t(h_{t-1}, q))(\overline{V} - \underline{V}). \tag{23}$$

For a simplicity of calculation, we suppose $\overline{V} - \underline{V} = 1$. Notice that for $q > 0$, $R_t(h_{t-1}, q) \geq 0$ and for $q < 0$, $R_t(h_{t-1}, q) \leq 0$. We also denote the price-impact function in period $t$ after a history $h_{t-1}$ determined by the equilibrium variables by $R_t^*(h_{t-1}, q)$ for each $q \in \Omega_n$.

**Proposition 22** *Let $\{\pi_t^*, \psi_t^*, \delta_t^*\}_{t=1}^T$ be an equilibrium and suppose that we have $k_t$ partially pooling equilibrium on either the long side or the short side. For all $q \leq k_t$, $R_t^*(h_t, q) = 0$. For all $q > k_t$, $R_t^*(h_t, q)$ is concave in a discrete domain with respect to $q$.*

Proposition 22 says that among all trade sizes with strictly positive bid-ask spreads, in other words, larger than the cut-off size, the increment of price change decreases as trade sizes increase. Since trade sizes take on only integers, we do not explicitly use the words,“concave.” However, we can say that the approximated continuous function of the price-impact function displays a concave form.

Here, we have some numerical examples of the price-impact functions. Take $\mu = 0.5$ and $n = 10$. The Figure 1 shows the price-impact function in the case where $\gamma(q) = \frac{1}{2n}$. The first figure shows the function on the long side and the second one shows the function on the short side. Notice that trade sizes can take on only integers. That is, the liquidity traders are distributed uniformly. In this figures, we connected those points to approximate the functional form. The Figure 2 shows the price-impact function in case where $\gamma(q) = \frac{5^q}{2q!} exp(-5)$. That is, the liquidity traders are distributed with poisson distribution of mean $5$.

In the following figures, we plot $|R_t^*(h_{t-1}, q)|$ over all quantities. In both figures, points which intersects with $x$-axis are cut-off sizes. We can see that the cut-off size decreases as $\delta$ decreases on the long-side or as $\delta$ increases on the short-side. Also, we can see that after the cut-off size, the price impact takes on a concave functional form and that is why in a small quantity just after the cut-off size, change in impact is large.

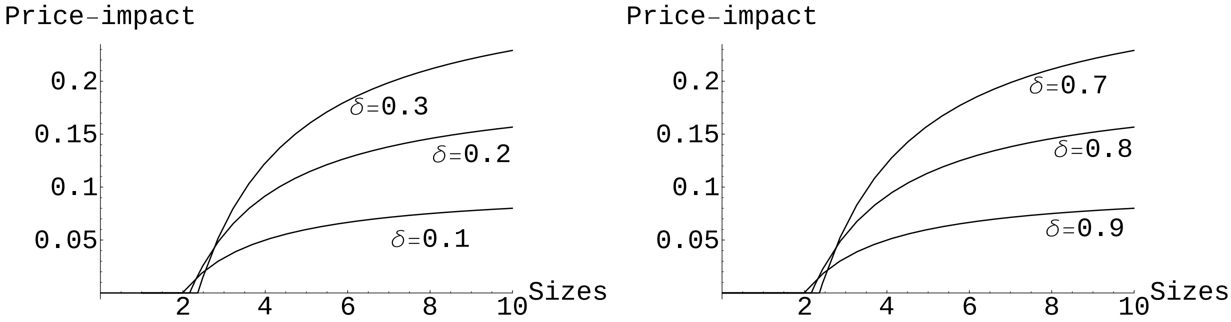


Figure 2: Price-impacts with uniform distributed liquidity traders

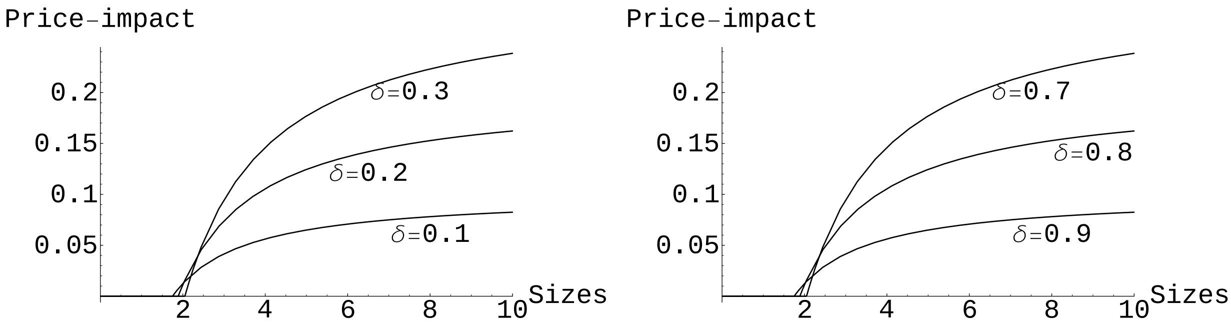


Figure 3: Price-impacts with poisson distributed liquidity traders

## 5 Concluding Remarks

The old adage of Wall street says that "it takes volume to move prices". The paper sheds light on the relationship between trade sizes and prices in a dynamic model. In a market with asymmetrically informed agents, trade sizes that informed traders trade convey information and therefore cause impacts on the security price. The impacts of trade sizes on price differ. It is important to understand what is the relationship between price change and trade size. For this purpose, this paper gives the enriched theoretical framework of the canonical model Glosten and Milgrom [10] and provide testable hypotheses for future studies on the subject.

Our main results are as follows. We have shown that there is a nonzero cut-off size above which informed traders possibly buy or sell, and that larger trade sizes have positive bid-ask spreads, while smaller sizes do not. Then, we have proved that the cut-off size decreases if both informed traders and liquidity traders trade in the same way. Moreover, we have proved that when additional trade sizes are introduced to the market, the market maker's learning process can be impaired and the bid-ask spreads for the previously existing trade sizes can vanish under a mild condition. Finally, we have proved that the smaller trade sizes that do not have a positive bid-ask spread result in zero price change, while for larger trade sizes the rate at which price change increases is a decreasing function of the trade size in all trading periods and then we have also derived the functional relationship between trade sizes and bid-ask spreads. The function for bid-ask spreads displays monotonically increasing and locally concave form. Most of our results are broadly consistent with the empirical findings.

Another important result of the model is that the price-impact function that we have derived more likely becomes flatter as trade goes on. As the market maker learns more about the risky asset in the sense that the market maker's belief becomes closer to the true probability, the information that each trade gives to the market maker decreases. This result shows how the information content that each trade has for the market maker changes over time. The dynamic setting of the model allows us to derive this result.

There have been many empirical works done on the analysis of price-impact functions. Gabaix et al. [5] and [4] empirically addressed the question of how stock prices respond to changes in demand. More specifically, using 116 most traded stock in NYSE, they consider the functional relationship between price changes and trade sizes transacted over fifteen minutes and found that a function of price change conditional on trade sizes displays a concave functional form that seems universal for all stocks. They consider fifteen-minute interval because large trade sizes tend to be split into several trades. Moreover, Gabaix et al. [3] proposes a theory of price-impact functions and shows that a price change is a concave function of the trade size. It uses a search model and claims that a price change is proportional to the square root of trade size. In their model, traders consider a cost of waiting and price-impact that their trade makes. If they sell too soon, they can shorten waiting time but they have to accept a large impact on price after their large trade. Large traders are willing to wait for a longer time to moderate their price impact.

In our model, traders can trade only once and cannot choose when to trade. So, by assumption, we eliminate the possibility that informed traders trade dynamically and manipulate the market. One important extension of our model could be to relax this assumption of myopic informed traders and to see how dynamic behavior of traders affects the price process. Within such an environment, traders can buy many shares and influence the price. After the price increases, they can re-sell the shares. In other words, there could be an arbitrage opportunity depending on the price-impact function. Investigating the conditions that prevents the arbitrate opportunity would be very interesting.

Another stylized assumption in the paper is that there are only two possible values of the risky payoff and informed traders know the exact value of the risky asset. Considering more complicated information structure will be interesting extension. For example, we can consider when we allow more variety of information signals, how this affects a current shape of price impact functions. Investigating an empirical price-impact function and bridging our theoretical results with empirical findings of price-impact functions will be promising for the future research.

## Notation

$$\begin{aligned} V_t^*(h_{t-1}, i) &= \delta_t^*(h_{t-1}, i)\,\underline{V} + (1 - \delta_t^*(h_{t-1}, i))\,\overline{V}. \\ t_p &= |m^+(h_t)|. \\ t_m &= |m^-(h_t)|. \end{aligned}$$

## Appendix A

***Proof of Lemma 1.***

The proof is straightforward from (6).

Let $q > 0$. For $q \in (0, \overline{k}_t(h_{t-1}))$, $\delta_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1})$. Moreover, by using the formula, we can calculate: for $q > \overline{k}_t(h_{t-1})$,

$$\delta_t^*(h_{t-1}, q) = \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n i\gamma(i)}{q[(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n \gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu]}. \tag{24}$$

For $-q \in (-\underline{k}_t(h_{t-1}), 0)$, $\delta_t^*(h_{t-1}, -q) = \delta_{t-1}^*(h_{t-1})$. Moreover, by using the formula, we can calculate:

$$\delta_t^*(h_{t-1}, -q) = 1 - \frac{(1 - \delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n i\gamma(-i)}{q[(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n \gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu]}. \tag{25}$$

Let $\overline{K}_t$ denote $\max\{\underline{k}_t(h_{t-1}), \overline{k}_t(h_{t-1})\}$. Then, for $q > \overline{K}_t$, we have:

$$\begin{aligned} \delta_t^*(h_{t-1}, -q) - \delta_t^*(h_{t-1}, q) &= 1 - \frac{(1 - \delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n i\gamma(-i)}{q[(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n \gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu]} \\ &\quad - \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n i\gamma(i)}{q[(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n \gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu]}. \end{aligned}$$

Next, we consider the sizes $q < \overline{K}_t(h_{t-1})$. In doing so, we consider the following two cases: Case (I) $\underline{k}_t(h_{t-1}) \leq \overline{k}_t(h_{t-1})$ and Case (II) $\underline{k}_t(h_{t-1}) > \overline{k}_t(h_{t-1})$.

Case (I) $\underline{k}_t(h_{t-1}) \leq \overline{k}_t(h_{t-1})$

For $q \in [\underline{k}_t(h_{t-1}), \overline{k}_t(h_{t-1})]$,

$$\delta_t^*(h_{t-1}, -q) - \delta_t^*(h_{t-1}, q) = 1 - \frac{(1 - \delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n i\gamma(-i)}{q[(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n \gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu]} - \delta_{t-1}^*(h_{t-1}). \tag{26}$$

Case (II) $\underline{k}_t(h_{t-1}) > \overline{k}_t(h_{t-1})$

For $q \in [\overline{k}_t(h_{t-1}), \underline{k}_t(h_{t-1})]$,

$$\delta_t^*(h_{t-1}, -q) - \delta_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1}) - \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n i\gamma(i)}{q[(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n \gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu]}. \tag{27}$$

Moreover, in both cases, for $q < \underline{K}_t(h_{t-1})$, $\delta_t^*(h_{t-1}, -q) - \delta_t^*(h_{t-1}, q) = 0$.

Therefore, we can define the function $f_t^*(h_{t-1}, q)$ as follows:

$$f_t^*(h_{t-1}, q) = \begin{cases} 1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n i\gamma(-i)}{q[(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n \gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu]} \\ \qquad -\frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n i\gamma(i)}{q[(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n \gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu]} & \text{for } q > \overline{K}_t \\ 1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n i\gamma(-i)}{q[(1-\mu)\sum_{i=\underline{k}_t(h_{t-1})}^n \gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu]} - \delta_{t-1}^*(h_{t-1}) & \text{for } q \in [\underline{k}_t(h_{t-1}), \overline{k}_t(h_{t-1})] \\ & \text{ if } \underline{k}_t(h_{t-1}) \le \overline{k}_t(h_{t-1}) \\ \delta_{t-1}^*(h_{t-1}) - \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n i\gamma(i)}{q[(1-\mu)\sum_{i=\overline{k}_t(h_{t-1})}^n \gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu]} & \text{for } q \in [\overline{k}_t(h_{t-1}), \underline{k}_t(h_{t-1})] \\ & \text{ if } \overline{k}_t(h_{t-1}) < \underline{k}_t(h_{t-1}) \\ 0 & \text{for } q < \underline{k}_t(h_{t-1}). \end{cases}$$

■

**Lemma 2** *In a $k$ pooling equilibrium on the long side, $\psi_t(\cdot|\overline{V}, h_{t-1})$ is of the form:*

$$\psi_t^*(j|\overline{V}, h_{t-1}) = \begin{cases} \dfrac{(1-\mu)\displaystyle\sum_{i=k}^n (1-\frac{i}{j})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu}{(1-\delta_{t-1}(h_{t-1}))\mu \displaystyle\sum_{i=k}^n \frac{i\gamma(+i)}{j\gamma(+j)}}, & : \quad j \in \{k, \cdots, n\} \\ 0 & : \quad j \in \{1, \cdots, k-1\}. \end{cases} \tag{28}$$

*In a $k$ pooling equilibrium on the short side, $\psi_t^*(\cdot|\underline{V}, h_{t-1})$ is of the form:*

$$\psi_t^*(-j|\underline{V}, h_{t-1}) = \begin{cases} \dfrac{(1-\mu)\displaystyle\sum_{i=k}^n (1-\frac{i}{j})\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu}{\delta_{t-1}^*(h_{t-1})\mu \displaystyle\sum_{i=k}^n \frac{i\gamma(-i)}{j\gamma(-j)}} & : \quad j \in \{k, \cdots, n\} \\ 0 & : \quad j \in \{1, \cdots, k-1\}. \end{cases} \tag{29}$$

***Proof of Lemma 2.***

We consider the optimal trading strategy $\psi_t^*$ in period $t$. Suppose that an informed trader strictly prefers trading in $\{k, \cdots, n\}$ to trading in $\{k-1, \cdots, q_1\}$. Fix an arbitrary $j \ge k$. Then, we have: for all $i \ge k$ with $i \ne j$,

$$j[\overline{V} - \pi_t^*(h_{t-1}, j)] = i[\overline{V} - \pi_t^*(h_{t-1}, i)], \tag{30}$$

and for all $i < k$, we have:

$$j[\overline{V} - \pi_t^*(h_{t-1}, j)] \ge i[\overline{V} - \pi_t^*(h_{t-1}, i)]. \tag{31}$$

Thus, we obtain: for all $i \in \{k, \cdots, n\}$,

$$\frac{i}{j} = \frac{\delta_t^*(h_{t-1}, j)}{\delta_t^*(h_{t-1}, i)}.$$

Therefore, we obtain: for all $i \in \{k, \cdots, n\}$,

$$(1 - \delta_{t-1}^*(h_{t-1}))\mu[\psi_t^*(j|\overline{V}, h_{t-1})\frac{i\gamma(+i)}{j\gamma(+j)} - \psi_t^*(i|\overline{V}, h_{t-1})] = (1 - \mu)(1 - \frac{i}{j})\gamma(+i).$$

The above equation must hold in general for any $i \in \{k, \cdots, n\}\backslash\{j\}$. Summing them up over $i \in \{k, \cdots, n\}\backslash\{j\}$, we obtain:

$$(1-\delta_{t-1}^*(h_{t-1}))\mu[\psi_t^*(j|\overline{V}, h_{t-1}) \sum_{i=k, i\neq j}^{n} \frac{i\gamma(+i)}{j\gamma(+j)} - \sum_{i=k, i\neq j}^{n} \psi_t^*(i|\overline{V}, h_{t-1})] = (1-\mu) \sum_{i=k, i\neq j}^{n} (1-\frac{i}{j})\gamma(+i).$$

Since $\sum_{i=k, i\neq j}^{n} \psi_t^*(i|\overline{V}, h_{t-1}) = 1 - \psi_t^*(j|\overline{V}, h_{t-1})$, we obtain:

$$\psi_t^*(j|\overline{V}, h_{t-1}) = \frac{(1 - \mu) \sum_{i=k}^{n} (1 - \frac{i}{j})\gamma(+i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu}{(1 - \delta_{t-1}^*(h_{t-1}))\mu \sum_{i=k}^{n} \frac{i\gamma(+i)}{j\gamma(+j)}}. \tag{32}$$

Since $0 < \psi_t(j) \leq 1$, the following two conditions must hold:

$$(1 - \mu) \sum_{i=k}^{n} (1 - \frac{i}{j})\gamma(+i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu \leq (1 - \delta_{t-1}^*(h_{t-1}))\mu \sum_{i=k}^{n} \frac{i\gamma(+i)}{j\gamma(+j)}, \tag{33}$$

and

$$(1 - \mu) \sum_{i=k}^{n} (1 - \frac{i}{j})\gamma(+i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu > 0. \tag{34}$$

Note that we always have:

$$(1 - \mu) \sum_{i=k}^{n} (1 - \frac{i}{j})\gamma(+i) \leq (1 - \delta_{t-1}^*(h_{t-1}))\mu \sum_{i=k}^{n} \frac{i\gamma(+i)}{j\gamma(+j)}$$

because the left hand side is always negative and the right hand side is always positive. Therefore, the first condition is always satisfied. So, if the second condition is satisfied, we will have $\psi_t(i)$ with $0 < \psi_t(i) \leq 1$ which satisfies the zero-profit condition and equal profit condition for an informed trader between trading in 1 and other quantities. On the other hand, if there exists a long side completely pooling equilibrium, we must have both conditions satisfied. Finally we conclude that the equilibrium $\psi_t^*$ is of the form (28). The proof of a short side completely pooling equilibrium will be done similarly with a long side case. ■

## Appendix B

***Proof of Theorems, Propositions, and Corollaries***

***Proof of Proposition 1.*** Let $\pi_t^*$ satisfy the zero-profit condition.

1. Suppose for some $t$ and $h_{t-1} = (q_1, ..., q_{t-1})$, $\psi_t^*(0|V, h_{t-1}) > 0$. This implies that

$$\begin{aligned} & 0 \geq q\,(\,V - \pi_t^*(h_{t-1}, q)) \quad \forall q \in \Omega_n \\ \implies \quad & 0 \geq q\left(V - \delta_t^*(h_{t-1}, q)\,\underline{V} - (1 - \delta_t^*(h_{t-1}, q))\,\overline{V}\right) \quad \forall q \in \Omega_n. \end{aligned} \tag{35}$$

From (7), we have $0 < \delta_t^*(h_{t-1}, q) < 1$ for all $t$. Thus,

$$\begin{aligned} & q\left(\underline{V} - \delta_t^*(h_{t-1}, q)\,\underline{V} - (1 - \delta_t^*(h_{t-1}, q))\,\overline{V}\right) > 0 \quad \forall q \in \Omega_n^-, \text{ and} \\ & q\left(\overline{V} - \delta_t^*(h_{t-1}, q)\,\underline{V} - (1 - \delta_t^*(h_{t-1}, q))\,\overline{V}\right) > 0 \quad \forall q \in \Omega_n^+. \end{aligned}$$

This contradicts with (35).

2. Let $V = \overline{V}$. Suppose there exist $t$ and $h_{t-1} = (q_1, ..., q_{t-1})$ such that $\psi_t^*(-i|\overline{V}, h_{t-1}) > 0$ for some $i \in \{1, ..., n\}$. Then

$$-i\left(\overline{V} - \pi_t^*(h_{t-1}, -i)\right) \geq q\left(\overline{V} - \pi_t^*(h_{t-1}, q)\right) \quad \forall q \in \Omega_n. \tag{36}$$

Since $0 < \delta_t^*(h_{t-1}, q) < 1$ for all $t$ and for all $q \in \Omega_n$,

$$\overline{V} > \pi_t^*(h_{t-1}, q) := \delta_t^*(h_{t-1}, q)\,\underline{V} + (1 - \delta_t^*(h_{t-1}, q))\,\overline{V}, \quad \forall q \in \Omega_n.$$

Then

$$q\left(\overline{V} - \pi_t^*(h_{t-1}, q)\right) > 0 > -i\left(\overline{V} - \pi_t^*(h_{t-1}, -i)\right), \quad \forall q \in \Omega_n^+,$$

and this contradicts with (36).

3. The proof is similar to 2.

■

***Proof of Theorem 2.*** The proof will be done by contradiction. On the contrary, suppose that there is an equilibrium such that an informed trader chooses: $\psi_t^*(j|\overline{V}, h_{t-1}) > 0$ but $\psi_t^*(i|\overline{V}, h_{t-1}) = 0$ for some $j < i$ (thus, $j < i$). Then by zero profit condition, in an equilibrium, $\pi_{t-1}^*(h_{t-1}, i) = V_{t-1}^*(h_{t-1})$. Moreover, since the informed trader must prefer trading in $j$ to $i$ , the following must hold:

$$j[\overline{V} - \pi_{t-1}^*(h_{t-1}, j)] \geq i[\overline{V} - \pi_{t-1}^*(h_{t-1}, i)]. \tag{37}$$

Note that

$$\overline{V} - \pi^*_{t-1}(h_{t-2}, j) = \frac{\delta^*_{t-1}(h_{t-2}, j)(1-\mu)\gamma(+j)}{(1-\delta^*_{t-1}(h_{t-2}, j))\psi^*_t(j|V, h_{t-1})\mu + (1-\mu)\gamma(+j)}(\overline{V} - \underline{V}), \tag{38}$$

and

$$\begin{aligned} \overline{V} - \pi^*_{t-1}(h_{t-1}, i) &= \overline{V} - V^*_{t-1}(h_{t-1}) \\ &= \delta^*_{t-1}(h_{t-2}, i)(\overline{V} - \underline{V}). \end{aligned} \tag{39}$$

Thus, we must have:

$$\begin{aligned} \frac{j}{i} &\geq \frac{\overline{V} - \pi^*_{t-1}(h_{t-2}, i)}{\overline{V} - \pi^*_{t-1}(h_{t-2}, j)} \\ &= 1 + \frac{(1-\delta^*_{t-1}(h_{t-2}, j))\psi^*_t(j|V, h_{t-1})\mu}{(1-\mu)\gamma(+j)}. \end{aligned} \tag{40}$$

Since $\frac{j}{i} < 1$ and $\psi^*_t(j|\overline{V}, h_{t-1}) > 0$ by assumption, this is impossible. So, we would not have this kind of equilibrium. The proof of the second part will be done similarly with the previous lemma. In this case, note that

$$\pi^*_{t-1}(h_{t-2}, -j) - \underline{V} = \frac{(1-\delta^*_{t-1}(h_{t-2}, j))(1-\mu)\gamma(-j)}{\delta^*_{t-1}(h_{t-2}, j)\psi^*_t(j|V, h_{t-1})\mu + (1-\mu)\gamma(-j)}(\overline{V} - \underline{V}),$$

and

$$V^*_{t-1}(h_{t-1}) - \underline{V} = (1-\delta^*_{t-1}(h_{t-2}, j))(\overline{V} - \underline{V}).$$

Thus, at the final step in order to get a contradiction, we have:

$$\frac{j}{i} \geq 1 + \frac{\delta^*_{t-1}(h_{t-2}, j)\psi^*_t(j|V, h_{t-1})\mu}{(1-\mu)\gamma(-j)}.$$

This is impossible. So, we can not have this kind of equilibrium. ■

***Proof of Proposition 3.***

First we consider the situation of $V = \underline{V}$. In this case, we have a short side separating equilibrium if an informed trader prefers to sell the largest quantity of the asset. Hence, for all $i < n$, we must have:

$$n[\pi^*_t(h_{t-1}, -n) - \underline{V}] \geq i[\pi^*_t(h_{t-1}, -i) - \underline{V}]. \tag{41}$$

Then,

$$\begin{aligned} \pi^*_t(h_{t-1}, -n) - \underline{V} &= (1-\delta^*_{t-1}(h_{t-1}))(\overline{V} - \underline{V}) - \frac{\mu\delta^*_{t-1}(h_{t-1})(1-\delta^*_{t-1}(h_{t-1}))(\overline{V} - \underline{V})}{\gamma(-n)(1-\mu) + \mu\delta^*_{t-1}(h_{t-1})} \\ &= \frac{(1-\delta^*_{t-1}(h_{t-1}))(1-\mu)}{\gamma(-n)\delta^*_{t-1}(h_{t-1})\mu + (1-\mu)\gamma(-n)}(\overline{V} - \underline{V}). \end{aligned} \tag{42}$$

Notice that since the informed prefer to trade in the largest quantity, any other quantities do not change the market maker's evaluation of the asset. Therefore, we have: for all $i < n$,

$$V_t^*(h_{t-1}, -i) = V_{t-1}^*(h_{t-1}).$$

Therefore, for all $i < n$,

$$\begin{aligned}\pi_t^*(h_{t-1}, -i) - \underline{V} &= V_t^*(h_{t-1}, -i) - \underline{V} \\ &= V_{t-1}^*(h_{t-1}) - \underline{V} \\ &= (1 - \delta_{t-1}^*(h_{t-1}))(\overline{V} - \underline{V}). \end{aligned} \tag{43}$$

Thus, by substituting (42) and (43) into (41) we have a short side separating equilibrium if for all $i < n$,

$$\begin{aligned}\frac{n}{i} &\geq \frac{\pi_t^*(h_{t-1}, -i) - \underline{V}}{\pi_t^*(h_{t-1}, -n) - \underline{V}} \\ &= 1 + \frac{\delta_{t-1}^*(h_{t-1})\mu}{\gamma(-n)(1-\mu)}. \end{aligned} \tag{44}$$

Note that for all $i \leq n-1$, we have

$$\frac{n}{i} \leq \frac{n}{n-1}.$$

Therefore, we have a short side separating equilibrium if

$$\frac{n}{n-1} \geq 1 + \frac{\delta_{t-1}^*(h_{t-1})\mu}{\gamma(-n)(1-\mu)}.$$

On the other hand, if the above equation holds, we have:

$$n[\pi_t^*(h_{t-1}, -n) - \underline{V}] \geq i[\pi_t^*(h_{t-1}, -i) - \underline{V}].$$

So, an informed trader chooses: $\psi_t^*(-n) = 1$ and $\psi_t^*(-q) = 0$ for $q \in \{1, \cdots, n-1\}$. Hence, this gives us the sufficient condition for a short side separating equilibrium. Next we consider the situation of $V = \overline{V}$. In this case, we have a long side separating equilibrium if for all $i < n$,

$$n[\overline{V} - \pi_t^*(h_{t-1}, n)] \geq i[\overline{V} - \pi_t^*(h_{t-1}, i)].$$

Similarly with the case of $V = \underline{V}$, we obtain the first part of the lemma. ■

***Proof of Corollary 4.***

Take a sequence $\{\mu_k\}$ which converges to $0$. Then, there exists a sequence $\{\delta_{t-1,k}^*\}$ such that for each $\mu_k$, the Baysian rule (6) holds. Notice that even when $\mu = 0$, the Baysian rule (6) holds. By the continuity of $\frac{[1-\delta^*_{t-1,k}(h_{t-1})]\mu_k}{\gamma(-n)(1-\mu_k)}$, the condition (3) holds. Therefore, we can conclude that if $\mu$ is sufficiently close to 0, $\psi^*_t$ is separating. This completes our proof of the first part.

Suppose that $V = \overline{V}$. Then, take a sequence $\{\mu_k\}$ which converges to 1. Suppose that $q_{t-1} > 0$. Then, by the Baysian rule ( 6), there exists a sequence $\{\delta^*_{t-1,k}\}$ such that for each $\mu_k$,

$$\delta^*_{t-1,k}(h_{t-2}, q_{t-1}) = \frac{\delta^*_{t-2}(h_{t-2})\,(1-\mu_k)\,\gamma(q_{t-1})}{(1-\delta^*_{t-2}(h_{t-2}))\mu_k\,\psi^*_{t-1}(q_{t-1}|\overline{V}, h_{t-2}) + (1-\mu_k)\,\gamma(q_{t-1})}. \tag{45}$$

By the continuity of the right hand side of (45), $\delta^*_{t-1,k}(h_{t-2}, q_{t-1})$ converges to 0 as $\mu_k$ converges to 1. Notice that for all $\mu \in (0,1)$, $\delta^*_{t-2}(h_{t-2}) \in (0,1)$. Hence, the condition (3) does hold. Then, our desired results follow by Proposition 3.

When $V = \underline{V}$, we can apply the similar argument by using $\delta^*_{t-1}$ instead of $1 - \delta^*_{t-1}$ in the above argument. This completes our proof of the second part.■

***Proof of Proposition 5.***

First, we consider a long side partially pooling equilibrium in which an informed trader strictly prefers trading in $\{k, \cdots, n\}$ to trading in $\{k-1, \cdots, 1\}$. By Theorem 2, we can focus on this setting without loss of generality. In this equilibrium, an informed trader is indifferent between trading in $\{k, \cdots, n\}$. Therefore, we have: for all $i \geq k$

$$n[\overline{V} - \pi^*_t(h_{t-1}, n)] = i[\overline{V} - \pi^*_t(h_{t-1}, i)], \tag{46}$$

and for all $i < k$, we have:

$$k[\overline{V} - \pi^*_t(h_{t-1}, k)] \geq i[\overline{V} - \pi^*_t(h_{t-1}, i)] \tag{47}$$

It must be possible to choose a function $\psi^*_t$ with $0 < \psi^*_t(q\overline{V}, h_{t-1}) \leq 1$ for all $q \in \{k, \cdots, n\}$ and simultaneously satisfy the equal profit condition for any informed traders which is given above and the zero-profit condition for the market maker. Similarly with the proof of Lemma 2, we obtain:

$$(1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(-i) + (1-\delta^*_{t-1}(h_{t-1}))\mu > 0. \tag{48}$$

Now, we consider the condition for the trade sizes in which the informed never trade. This condition is expressed as (31). Then, we must have: for all $i < k$,

$$\frac{k}{i} \geq 1 + \frac{(1-\delta^*_{t-1}(h_{t-1}))\psi^*_t(k|\overline{V}, h_{t-1})\mu}{(1-\mu)\gamma(+k)}. \tag{49}$$

Since $\frac{k}{i} \geq \frac{k}{k-1}$ for all $i \leq k-1$, we obtain:

$$\frac{k}{k-1} \geq 1 + \frac{(1-\delta_{t-1}(h_{t-1}))\psi_t^*(k|\overline{V}, h_{t-1})\mu}{(1-\mu)\gamma(+k)}. \tag{50}$$

By Lemma 2 and (50), we obtain:

$$\frac{k}{k-1} \geq 1 + \frac{(1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu}{(1-\mu)\gamma(+k)\sum_{i=k}^{n}\frac{i\gamma(+i)}{k\gamma(+k)}}. \tag{51}$$

By multiplying both sides by $(1-\mu)\gamma(+k)\sum_{i=k}^{n}\frac{i\gamma(+i)}{k\gamma(+k)}$, we obtain:

$$0 \geq (1-\frac{k}{k-1})(1-\mu)\sum_{i=k}^{n}\frac{i\gamma(+i)}{k} + (1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu. \tag{52}$$

Notice that:

$$\begin{aligned}
&(1-\frac{k}{k-1})(1-\mu)\sum_{i=k}^{n}\frac{i\gamma(+i)}{k} + (1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(+i) \\
&\qquad = (1-\mu)\sum_{i=k}^{n}(\frac{i\gamma(+i)}{k} - \frac{i\gamma(+i)}{k-1} + \gamma(+i) - \frac{i\gamma(+i)}{k}) \\
&\qquad = (1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(+i) \\
&\qquad = (1-\mu)\sum_{i=k-1}^{n}(1-\frac{i}{k})\gamma(+i).
\end{aligned}$$

Therefore, by (52), we obtain:

$$0 \geq (1-\mu)\sum_{i=k-1}^{n}(1-\frac{i}{k})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu.$$

Hence, we have obtained the desired condition.

The proof of a short side $k$ pooling equilibrium will be done similarly with a long side case. Next, we will prove if part of the lemma. Suppose that there exists a $k \in \{1, ..., n\}$ such that

$$(1-\mu)\sum_{i=k}^{n}(1-\frac{i}{k})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu > 0, \tag{53}$$

and

$$(1-\mu)\sum_{i=k-1}^{n}(1-\frac{i}{k-1})\gamma(+i) + (1-\delta_{t-1}^*(h_{t-1}))\mu \leq 0. \tag{54}$$

Then, we will prove that $\psi_t$ defined in Lemma 2 maximizes the informed's profit and the followings hold: for all $i \geq k$

$$k[\overline{V} - \pi_t^*(h_{t-1}, k)] = i[\overline{V} - \pi_t^*(h_{t-1}, i)], \tag{55}$$

and for all $i < k$, we have:

$$k[\overline{V} - \pi_t^*(h_{t-1}, k)] \geq i[\overline{V} - \pi_t^*(h_{t-1}, i)]. \tag{56}$$

First, we will prove the equation (46) holds. It holds if and only if the following holds:

$$\frac{k}{i} = \frac{[\overline{V} - \pi_t^*(h_{t-1}, i)]}{[\overline{V} - \pi_t^*(h_{t-1}, k)]}. \tag{57}$$

By Lemma 4, we have known that:

$$\pi_t^*(h_{t-1}, q) = \delta_t^*(h_{t-1}, q)\underline{V} + (1 - \delta_t^*(h_{t-1}, q))\overline{V} \quad \text{for} \quad q \in \Omega_n. \tag{58}$$

Therefore,

$$\overline{V} - \pi_t^*(h_{t-1}, i) = \delta_t^*(h_{t-1}, i)(\overline{V} - \underline{V}). \tag{59}$$

Since

$$\delta_t^*(h_{t-1}, k) = \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\gamma(+k)}{(1 - \delta_{t-1}^*(h_{t-1}))\psi_t^*(k|V, h_{t-1})\mu + (1-\mu)\gamma(+k)}, \tag{60}$$

The equation (46) holds if and only if

$$\frac{k\gamma(+k)}{i\gamma(+i)} = \frac{(1 - \delta_{t-1}^*(h_{t-1}))\psi_t^*(k|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+k)}{(1 - \delta_{t-1}^*(h_{t-1}))\psi_t^*(j|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+j)}. \tag{61}$$

By plugging $\psi_t(k|\overline{V}, h_{t-1})$ and $\psi_t(i|\overline{V}, h_{t-1})$ from Lemma 4 into the above, a simple calculation verifies that the above equation holds. Therefore, we can conclude that with $\psi_t(i|\overline{V}, h_{t-1})$ defined in Lemma 4 the profits of trading in these quantity sizes from $k$ to $n$ are same. Next, we will prove that for all $i < k$, we have:

$$k[\overline{V} - \pi_t^*(h_{t-1}, k)] \geq i[\overline{V} - \pi_t^*(h_{t-1}, i)]. \tag{62}$$

Similarly with the above, this holds if and only if

$$\frac{k\gamma(+k)}{i\gamma(+i)} \geq \frac{(1 - \delta_{t-1}(h_{t-1}))\psi_t^*(k|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+k)}{(1 - \delta_{t-1}^*(h_{t-1}))\psi_t^*(j|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+k)}. \tag{63}$$

By plugging $\psi_t^*(k|\overline{V}, h_{t-1})$ and $\psi_t^*(i|\overline{V}, h_{t-1})$ from Lemma 4 into the above, a simple calculation verifies that the above inequality holds. Therefore, we can conclude that with $\psi_t^*(i|\overline{V}, h_{t-1})$ defined in Lemma 4 the profit of trading in $k$ is higher than one of trading in smaller sizes. Finally, we will prove that there does not exist any other $\psi_t^{'}$ that yields higher profits than $\psi_t^*$. ■

***Proof of Proposition 6.***

The proof is similar to Proposition 5. The existence of a completely pooling equilibrium is proved by the existence of the strictly positive $\psi_t^*(i|\overline{V}, h_{t-1})$ for all $i$. Then, we can derive the condition similarly to Proposition 5. ■

***Proof of Proposition 7.***

The proof for the short side is similar to one for the long side. Therefore, we will prove only the short side. Suppose that $V = \underline{V}$. Moreover, suppose that there exists a size $k$ with $2 < k < n$ such that

$$\begin{aligned} (1-\mu)\sum_{i=k}^{n}\left(1-\frac{i}{k}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu &> 0, \quad \text{and} \\ (1-\mu)\sum_{i=k-1}^{n}\left(1-\frac{i}{k-1}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\,\mu &\leq 0. \end{aligned}$$

Then, by Proposition 5, $\psi_t^*$ is $k$ partially pooling. By Lemma 2, we can uniquely determine the informed strategy $\psi_t^*$.

Now, suppose that there does not exist such a size $k$. Notice that the following holds for any $k$ with $k > 2$:

$$(1-\mu)\sum_{i=k}^{n}\left(1-\frac{i}{k}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu > (1-\mu)\sum_{i=k-1}^{n}\left(1-\frac{i}{k-1}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\,\mu.$$

Hence, we have:

$$(1-\mu)\sum_{i=n-1}^{n}\left(1-\frac{i}{n-1}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu > \cdots > (1-\mu)\sum_{i=1}^{n}\left(1-\frac{i}{1}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu.$$

Therefore, we must have:

$$(1-\mu)\sum_{i=1}^{n}\left(1-\frac{i}{1}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu > 0, \tag{64}$$

or

$$(1-\mu)\sum_{i=n-1}^{n}\left(1-\frac{i}{n-1}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu \leq 0. \tag{65}$$

When the inequality (64) holds, by Proposition 6, $\psi_t$ must be $k$ completely pooling. By Lemma 2, we can uniquely determine the informed strategy $\psi_t$.

On the other hand, suppose that the inequality (65) holds. Then, we have:

$$(1-\mu)\left(1-\frac{n}{n-1}\right)\gamma(-n) + \delta_{t-1}^*(h_{t-1})\mu \leq 0. \tag{66}$$

By arranging the order of the inequality (66), we obtain:

$$\frac{n}{n-1} \geq 1 + \frac{\delta^*_{t-1}(h_{t-1})\,\mu}{\gamma(-n)(1-\mu)}.$$

Therefore, by Propositionprop:separating, $\psi^*_t$ is separating. Hence, we can specify the informed strategy by $\psi^*_t(i|V, h_{t-1}) = 1$ for $i = n$ and $\psi^*_t(i|V, h_{t-1}) = 0$ otherwise. So, in this case we can also uniquely determine the informed strategy $\psi^*_t$. This completes our proof. ■

***Proof of Corollary 8.***

The proof is straightforward from Proposition 7. ■

***Proof of Theorem 9.***

Since proofs for the long side and the short side are similar, we prove the long side equilibrium. Suppose that in period $t$, we have a long side partially pooling equilibrium, in which an informed trader would buy quantities $i \geq k_t(h_{t-1})$. Then, by Proposition 5, we have:

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}(1-\frac{i}{k_t(h_{t-1})})\gamma(+i) + (1-\delta^*_t(h_t))\mu > 0. \tag{67}$$

Since an informed trader would trade in only $i$ such that $i \geq k_t(h_{t-1})$, by the market maker's updating formula we must have $\delta^*_{t+1}(h_t, i) \leq \delta^*_t(h_t)$. Therefore, we have:

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}(1-\frac{i}{k_t(h_{t-1})})\gamma(+i) + (1-\delta^*_{t+1}(h_t, i))\mu > 0. \tag{68}$$

By Proposition 5, we can conclude that in period $t$ we will have a (partially) pooling equilibrium, in which an informed trader would buy quantities $i \geq k_t(h_{t-1})$. This gives us the desired result of $k_t(h_{t-1}) \geq k_{t+1}(h_{t-1}, q_t)$. The above argument completes the first part of the proposition. ■

***Proof of Proposition 10.***

Since proofs for the long side and the short side are similar, we prove the long side equilibrium. Suppose that in period $t$, we have a long side partially pooling equilibrium, in which an informed trader would buy quantities $i \geq k_t(h_{t-1})$. Then, by Proposition 5, we have:

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}(1-\frac{i}{k_t(h_{t-1})})\gamma(+i) + (1-\delta^*_t(h_t))\mu > 0. \tag{69}$$

We will have $k_t(h_{t-1}) > k_{t+1}(h_{t-1}, q_t)$ if and only if in period $t+1$ the following holds:

$$(1-\mu)\sum_{i=k_t(h_{t-1})-1}^{n}(1-\frac{i}{k_t(h_{t-1})-1})\gamma(+i) + (1-\delta^*_{t+1}(h_t, q_t))\mu > 0. \tag{70}$$

The inequality (70) holds if the following inequality holds:

$$(1-\mu)\sum_{i=k_t(h_{t-1})-1}^{n}(1-\frac{i}{k_t(h_{t-1})-1})\gamma(+i) - (1-\mu)\sum_{i=k_t(h_{t-1})}^{n}(1-\frac{i}{k_t(h_{t-1})})\gamma(+i) \\ \geq (\delta^*_{t+1}(h_t, q_t)) - \delta^*_t(h_t))\mu.$$

The inequality (71) holds if and only if the following holds:

$$(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}(\frac{i}{k_t(h_{t-1})-1} - \frac{i}{k_t(h_{t-1})})\gamma(+i) \leq (\delta^*_t(h_t) - \delta^*_{t+1}(h_t, q_t))\mu. \tag{71}$$

Notice that:

$$(\delta^*_t(h_t) - \delta^*_{t+1}(h_t, q_t)) = \frac{\delta^*_t(h_t)(1-\delta^*_t(h_t))\psi^*_t(q_t|V, h_{t-1})\mu}{(1-\delta^*_t(h_t))\psi^*_t(q_t|V, h_{t-1})\mu + (1-\mu)\gamma(q_t)}. \tag{72}$$

By substituting $\psi^*_t(q_t|V, h_{t-1})$ and re-arranging terms, we conclude that we will have $k_t(h_{t-1}) > k_{t+1}(h_{t-1}, q_t)$ under the condition:

$$\delta^*_t(h_t) - \frac{\delta^*_t(h_t)\ \sum_{i=k_t(h_{t-1})}^{n}\frac{i\,\gamma(i)}{q_t}}{\sum_{i=k_t(h_{t-1})}^{n}\gamma(i) + (1-\delta^*_t(h_t))\frac{\mu}{1-\mu}} \geq \frac{1-\mu}{\mu}\,\frac{1}{k_t(h_{t-1})\times(k_t(h_{t-1})-1)}\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i).$$

Notice that

$$\delta^*_t(h_t) - \frac{\delta^*_t(h_t)\ \sum_{i=k_t(h_{t-1})}^{n}\frac{i\,\gamma(i)}{q_t}}{\sum_{i=k_t(h_{t-1})}^{n}\gamma(i) + (1-\delta^*_t(h_t))\frac{\mu}{1-\mu}} \\ = \delta^*_t(h_t)\,(1 - \frac{\frac{1}{q_t}}{\frac{\sum_{i=k_t(h_{t-1})}^{n}\gamma(i)}{\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)} + (1-\delta^*_t(h_t))\frac{\mu}{(1-\mu)\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)}}).$$

Therefore, we have $k_t(h_{t-1}) > k_{t+1}(h_{t-1}, q_t)$ if and only if

$$\delta^*_t(h_t)(1 - \frac{\frac{1}{q_t}}{\frac{\sum_{i=k_t(h_{t-1})}^{n}\gamma(i)}{\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)} + (1-\delta^*_t(h_t))\frac{\mu}{(1-\mu)\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)}}) \\ \geq \frac{1-\mu}{\mu}\,\frac{1}{k_t(h_{t-1})\times(k_t(h_{t-1})-1)}\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i). \tag{73}$$

Dividing both sides of the inequality (73) by $\sum_{i=k_t(h_{t-1})}^{n} i\,\gamma(i)$, we obtain the desired condition. ■

***Proof of Corollary 11.***

Omitted. ■

***Proof of Proposition 12.***

We will construct the market maker's belief by using initial exogenous variables. Our plan is first to focus on the periods in $m^+(h_t)$ and derive a recursive formulation for these periods and then to repeat a similar method to the periods in $m^-(h_t)$.

First, we consider all periods of $m^+(h_t)$ that the market maker's belief is changed downward. We first consider the situation where $m^-(h_t) = \emptyset$. By the updating formula, we can obtain the following:

$$\begin{aligned}\delta_t^*(h_{t-1}, j) &= 1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(\psi_t^*(j|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+j))}{(1-\delta_{t-1}^*(h_{t-1}))\psi_t^*(j|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+j)} \\ &= \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\gamma(+j)}{(1-\delta_{t-1}^*(h_{t-1}))\psi_t^*(j|\overline{V}, h_{t-1})\mu + (1-\mu)\gamma(+j)} \qquad (74)\end{aligned}$$

For $j$ with $\psi_t^*(j|\overline{V}, h_{t-1}) > 0$, by Lemma 2 we obtain:

$$\delta_t^*(h_{t-1}, j) = \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\gamma(+j)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{j\gamma(+j)}}{(1-\delta_{t-1}^*(h_{t-1}))\mu + (1-\mu)\gamma(+j)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{j\gamma(+j)} + (1-\mu)\sum_{i=k_t(h_{t-1})}^n (1-\frac{i}{j})\gamma(+i)}.$$

When the realized demand in period $t$, $q_t$, is $j$, we can rewrite the above as follows:

$$\delta_t^*(h_{t-1}, q_t) = \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\gamma(q_t)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{q_t\gamma(q_t)}}{(1-\delta_{t-1}^*(h_{t-1}))\mu + (1-\mu)\gamma(q_t)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{q_t\gamma(q_t)} + (1-\mu)\sum_{i=k_t(h_{t-1})}^n (1-\frac{i}{q_t})\gamma(+i)}.$$

Let $a_t = \frac{1}{\delta_t^*(h_{t-1},q_t)}$. For the simplicity of notation, we ignore $h_t$ for a temporary purpose. Then, we obtain:

$$a_t = a_{t-1}\cdot(1 + \frac{\mu + (1-\mu)\sum_{i=k_t(h_{t-1})}^n (1-\frac{i}{q_t})\gamma(+i)}{(1-\mu)\gamma(q_t)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{q_t\gamma(q_t)}}) - \frac{\mu}{(1-\mu)\gamma(q_t)\sum_{i=k_t(h_{t-1})}^n \frac{i\gamma(+i)}{q_t\gamma(q_t)}}. \qquad (75)$$

This is linear in $a_{t-1}$ and thus we can boil this form down to the expression of $a_0$. Therefore, we obtain the following:

$$\begin{aligned}a_t &= a_0 \cdot \Pi_{\tau=1}^{t_p}(1 + \frac{(1-\mu)\sum_{i=k_\tau(h_{\tau-1})}^n (1-\frac{i}{q_\tau})\cdot\gamma(+i) + \mu}{(1-\mu)\gamma(q_\tau)\cdot\sum_{i=k_\tau(h_{\tau-1})}^n \frac{i\gamma(+i)}{q_\tau\gamma(q_\tau)}}) \\ &- \sum_{\tau=1}^{(t-1)_p}[\Pi_{s=\tau+1}^{t_p}(1 + \frac{(1-\mu)\sum_{i=k_s(h_{s-1})}^n (1-\frac{i}{q_s})\cdot\gamma(+i) + \mu}{(1-\mu)\gamma(q_s)\cdot\sum_{i=k_s(h_{s-1})}^n \frac{i\gamma(+i)}{q_s\gamma(q_s)}})]\cdot\frac{\mu}{(1-\mu)\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^n \frac{i\gamma(+i)}{q_\tau\gamma(q_\tau)}} \\ &- \frac{\mu}{(1-\mu)\gamma(q_{t_p})\sum_{i=k_{t_p}(h_{t_p-1})}^n \frac{i\gamma(+i)}{q_{t_p}\gamma(q_{t_p})}}.\end{aligned}$$

Finally, we obtain the following:

$$\begin{aligned}
\frac{1}{\delta_t^*(h_t)} &= \frac{1}{\delta} \cdot \Pi_{\tau=1}^{t_p}(1 + \frac{(1-\mu)\sum_{i=k_\tau(h_{\tau-1})}^{n}(1-\frac{i}{q_\tau}) \cdot \gamma(+i) + \mu}{(1-\mu)\gamma(q_\tau) \cdot \sum_{i=k_\tau(h_{\tau-1})}^{n} \frac{i\gamma(+i)}{q_\tau \gamma(q_\tau)}}) \\
&- \sum_{\tau=1}^{(t-1)_p} [\Pi_{s=\tau+1}^{t_p}(1 + \frac{(1-\mu)\sum_{i=k_s(h_{s-1})}^{n}(1-\frac{i}{q_s}) \cdot \gamma(+i) + \mu}{(1-\mu)\gamma(q_s) \cdot \sum_{i=k_s(h_{s-1})}^{n} \frac{i\gamma(+i)}{q_s\gamma(q_s)}})] \cdot \frac{\mu}{(1-\mu)\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n} \frac{i\gamma(+i)}{q_\tau\gamma(q_\tau)}} \\
&- \frac{\mu}{(1-\mu)\gamma(q_{t_p})\sum_{i=k_{t_p}(h_{t_p-1})}^{n} \frac{i\gamma(+i)}{q_{t_p}\gamma(q_{t_p})}}.
\end{aligned}$$

Simplifying each term, we can obtain $\delta_t^+(h_t, \delta)$.

Next, we consider all periods $m^-(h_t)$ that the market maker's belief is changed upward. Similarly with the above, we will construct $\delta_t^+(h_t, \delta)$. We start with the period $0$ and the market maker's belief is $\delta$. In the period $t+1$, $\psi_{t+1}^*(\underline{V}, h_{t-1})$ is a $k_{t+1}(h_{t-1}, q_t)$ pooling on the short side if and only if: there exists a $k_{t+1}(h_{t-1}, q_t)$ such that

$$(1-\mu)\sum_{i=k_{t+1}(h_{t-1},q_t)}^{n}(1-\frac{i}{q_{k_{t+1}(h_{t-1},q_t)}})\gamma(-i) + \delta_t^*(h_t)\mu > 0, \tag{76}$$

and

$$(1-\mu)\sum_{i=k_{t+1}(h_{t-1},q_t)-1}^{n}(1-\frac{i}{q_{k_{t+1}(h_{t-1},q_t)-1}})\gamma(-i) + \delta_t^*(h_t)\mu \leq 0, \tag{77}$$

where

$$\begin{aligned}
\frac{1}{1-\delta_t^*(h_t)} &= (1-\delta)\Pi_{\tau=1}^{t_m}(1 + \frac{(1-\mu)\sum_{i=k_\tau(h_{\tau-1})}^{n}(1-\frac{i}{q_{k_\tau(h_{\tau-1})}}) \cdot \gamma(-i) + \mu}{(1-\mu)\gamma(q_\tau) \cdot \sum_{i=k_\tau(h_{\tau-1})}^{n} \frac{i\gamma(-i)}{q_\tau\gamma(q_\tau)}}) \\
&- \sum_{\tau=1}^{(t-1)_m} [\Pi_{s=\tau+1}^{t}(1 + \frac{(1-\mu)\sum_{i=k_s(h_{s-1})}^{n}(1-\frac{i}{q_s}) \cdot \gamma(-i) + \mu}{(1-\mu)\gamma(q_s) \cdot \sum_{i=k_s(h_{s-1})}^{n} \frac{i\gamma(-i)}{q_s\gamma(q_s)}})] \cdot \frac{\mu}{(1-\mu)\gamma(-q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n} \frac{i\gamma(i)}{q_\tau\gamma(q_\tau)}} \\
&- \frac{\mu}{(1-\mu)\gamma(q_{t_m})\sum_{i=k_{t_m}(h_{t_m-1})}^{n} \frac{i\gamma(-i)}{q_{t_m}\gamma(q_{t_m})}}.
\end{aligned}$$

Simplifying each term, we can obtain $\delta_t^-(h_t, \delta)$.

In summary, $\delta_t^*(h_t)$ will be given by the followings.

Define $\delta_t^+(h_t, \cdot)$ and $\delta_t^+(h_t, \cdot)$ as follows: $\delta_t^+(h_t, \delta) = \delta$ if $t_p = 0$; otherwise,

$$\delta_t^+(h_t,\delta) = \frac{1}{\frac{1}{\delta}\Pi_{\tau=1}^{t_p}(1+A_\tau) - \sum_{\tau=1}^{(t-1)_p}\left[\frac{\mu}{(1-\mu)\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n}\frac{i\gamma(i)}{q_\tau\gamma(q_\tau)}}\Pi_{s=\tau+1}^{t_p}(1+A_s)\right] - \frac{\mu}{(1-\mu)\gamma(q_t)\sum_{i=k_t(h_{t-1})}^{n}\frac{i\gamma(i)}{q_t\gamma(q_t)}}},$$

where

$$A_\tau = \frac{(1-\mu)\sum_{i=k_\tau(h_{\tau-1})}^{n}\left(1-\frac{i}{q_\tau}\right)\gamma(i) + \mu}{(1-\mu)\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n}\frac{i\gamma(i)}{q_\tau\gamma(q_\tau)}}, \quad \tau = 1, ..., t;$$

and $\delta_t^-(h_t, \delta) = \delta$ if $t_m = 0$; otherwise,

$$\delta_t^-(h_t,\delta)=1-\frac{1}{\frac{1}{(1-\delta)}\prod_{\tau=1}^{t_m}(1+B_\tau)-\sum_{\tau=1}^{(t-1)m}\left[\frac{\mu}{(1-\mu)\,\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n}\frac{i\gamma(-i)}{q_\tau\gamma(q_\tau)}}\prod_{s=\tau+1}^{t_m}(1+B_s)\right]-\frac{\mu}{(1-\mu)\gamma(q_t)\sum_{i=k_t(h_{\tau-1})}^{n}\frac{i\gamma(-i)}{q_t\gamma(q_t)}}},$$

where

$$B_\tau = \frac{(1-\mu)\sum_{i=k_\tau(h_{\tau-1})}^{n}(1-\frac{i}{q_\tau})\gamma(-i)+\mu}{(1-\mu)\gamma(q_\tau)\sum_{i=k_\tau(h_{\tau-1})}^{n}\frac{i\gamma(-i)}{q_\tau\gamma(q_\tau)}}, \tau = 1, ..., T.$$

■

***Proof of Proposition 13***

First, we consider the long side. By Proposition 12, we have:

$$\begin{aligned}\frac{1}{\delta_t^*(h_t)} &= \frac{1}{\delta}\cdot(1+\frac{\mu}{(1-\mu)\gamma(+n)})^{t_m} \\ &\quad -\sum_{\tau=1}^{t_m-1}(1+\frac{\mu}{(1-\mu)\gamma(+n)})^{\tau}\cdot\frac{\mu}{(1-\mu)\gamma(+n)}-\frac{\mu}{(1-\mu)\gamma(+n)}.\end{aligned}$$

Therefore, we obtain:

$$\delta_t^*(h_t) = \frac{\delta}{(1+\frac{\mu}{(1-\mu)\gamma(+n)})^{t_m}-\delta((1+\frac{\mu}{(1-\mu)\gamma(+n)})^{t_m}-1)}. \quad (78)$$

By re-arranging terms, we obtain:

$$\delta_t^*(h_t) = \frac{\delta[(1-\mu)\gamma(+n)]^{t_m}}{(1-\delta)[(1-\mu)\gamma(+n)+\mu]^{t_m}+\delta[(1-\mu)\gamma(+n)]^{t_m}}. \quad (79)$$

This gives us the form. With Proposition 3, we have proved the first part of the proposition. The second part about the short side is similar with the first part.■

***Proof of Proposition 14.***

From the updating rule (6), if $q < \underline{k}_t(h_{t-1})$, $\delta_t^*(h_t, q) = \delta_{t-1}^*(h_{t-1})$. On the other hand, if $q < \overline{k}_t(h_{t-1})$, $\delta_t^*(h_t, q) = \delta_{t-1}(h_{t-1})$. Therefore, if $q < \underline{K}_t(h_{t-1})$, $S_t^*(h_{t-1}, q) = 0$.

For $q > \max\{\underline{k}_t(h_{t-1}), \overline{k}_t(h_{t-1})\}$, $S_t^*(h_{t-1}, q) > 0$ because $\delta_t^*(h_{t-1}, -q) > \delta_{t-1}^*(h_{t-1}) > \delta_t^*(h_{t-1}, q)$.

Suppose $\underline{k}_t(h_{t-1}) \geq \overline{k}_t(h_{t-1})$. Then, for $q \in [\underline{k}_t(h_{t-1}), \overline{k}_t(h_{t-1})]$, $\delta_t^*(h_{t-1}, -q) > \delta_{t-1}^*(h_{t-1}) = \delta_t^*(h_{t-1}, q)$. Therefore, we obtain $S_t^*(h_{t-1}, q) > 0$. On the other hand, suppose $\underline{k}_t(h_{t-1}) > \overline{k}_t(h_{t-1})$.

Then, for $q \in [\overline{k}_t(h_{t-1}), \underline{k}_t(h_{t-1})]$, $\delta_t^*(h_{t-1}, -q) = \delta_{t-1}^*(h_{t-1}) > \delta_t^*(h_{t-1}, q)$. Therefore, we obtain $S_t^*(h_{t-1}, q) > 0$. Finally, we can conclude that if $q < \underline{K}_t(h_{t-1})$, $S_t^*(h_{t-1}, q) = 0$ and $q \geq \underline{K}_t(h_{t-1})$, $S_t^*(h_{t-1}, q) > 0$. ■

***Proof of Proposition 18.***

Omitted. ■

***Proof of Theorem 19.***

An optimal strategy of informed traders prescribes probability distribution over all trade sizes in each period of $h_T$ and the market maker's belief assigns a probability of risky asset's being equal to the low value in each period given a history up to the period. We consider the market maker's equilibrium belief as a stochastic process. Let $\mathcal{H}_t$ denote a sigma field generated by all the possible histories $h_t$. By Theorem 3 in Page 432 of Fristedt and Gray [2], there uniquely exists a conditional distribution over $\delta_t$ given $(\delta_0, \cdots, \delta_{t-1})$. Thus, we can take a probability space which contain the space of $(\{\underline{V}, \overline{V}\} \times \{\Omega_n\}^\infty, \{\underline{V}, \overline{V}\} \times \mathcal{H}_\infty, \mathcal{P})$.

First, we will prove that $\delta_T$ converges almost surely to a random variable $\delta^*$. By the martingale convergence theorem, $\delta_T$ converges almost surely to a random variable $\delta'$. That is, for sufficiently large $t$ and arbitrarily small $\epsilon$,

$$\Pr(h_t : |\delta_t(h_t) - \delta'| > \epsilon) = 0. \tag{80}$$

Second, we will prove that $\delta^*$ is 1 if $V = \underline{V}$ and 0 if $V = \overline{V}$, respectively. The proof will be done by contradiction. Suppose that $V = \overline{V}$. On the contrary, suppose that there exists a set of histories $\{h_t\}$ with positive probability which $\delta_t$ converges to an interval $(p-\epsilon, p+\epsilon)$ for some $p \in (0,1)$ and arbitrary small $\epsilon$. Then, by the Bayesian updating rule (6), if an informed trader or a liquidity trader who trades higher volume than $k_t(h_t - 1)$ arrives at the market, the belief $\delta_t$ deviates from the interval of $(p-\epsilon, p+\epsilon)$. In other words, the belief deviates from the interval with probability $\mu + \sum_{i=k_t(h_t-1)}^{n} \gamma(i)$. This is a contradiction with the "almost surely" convergence of $\delta_t$.

It remains to show that that $\lim_{t\to\infty} \delta_t(h_t) = 0$. On the contrary, suppose that it converges to 1. In other words, for any given $\epsilon'$ there exists a $t_1$ such that for all $t > t_1$, $|1 - \delta_t(h_t)| < \epsilon'$. Similarly with the above, if an informed trader or a liquidity trader who trades higher volume than $k_t(h_t - 1)$, the belief $\delta_t(h_t)$ deviates from the interval of $(1-\epsilon', 1)$ for sufficiently small $\epsilon'$. Therefore, this is a contradiction. This gives us the desired result. ■

***Proof of Corollary 16.***

*1.* As we have seen in (7),

$$0 < \delta_t(h_t) < 1, \quad \forall h_t \in \Omega_n^t, \forall t.$$

By Lemma 1, since

$$\frac{(1-\mu)\left(\psi_t^*(i|\overline{V}, h_{t-1})\gamma(-i) + \psi_t^*(-i|\overline{V}, h_{t-1})\gamma(i)\right) + \mu\, \psi_t^*(i|\overline{V}, h_{t-1})\, \psi_t^*(-i)}{\left(\delta_t^*(h_{t-1})\, \mu\, \psi_t^*(i|\overline{V}, h_{t-1}) + (1-\mu)\,\gamma(i)\right)\left((1-\delta_t^*(h_{t-1}))\, \mu\, \psi_t^*(-i|\overline{V}, h_{t-1}) + (1-\mu)\,\gamma(-i)\right)}\,(\overline{V} - \underline{V}) > 0,$$

we can obtain the desired result. ■

*2.* Suppose that

$$\frac{n}{n-1} \geq 1 + \max\left\{\frac{\mu}{\gamma(n)\,(1-\mu)}\,,\, \frac{\mu}{\gamma(-n)\,(1-\mu)}\right\}.$$

Then, we have:

$$\frac{n}{n-1} \geq \max\left\{1 + \frac{\mu}{\gamma(n)\,(1-\mu) + \frac{\delta\,(\gamma(n)\,(1-\mu))^{m(h_t)}\,(\gamma(n)\,(1-\mu)+\mu)^{m(h_t)-1}}{1-\delta}}\,, 1 + \frac{\mu}{\gamma(-n)\,(1-\mu) + \frac{(1-\delta)(\gamma(-n)\,(1-\mu))^{m(h_t)}}{\delta(\gamma(-n)\,(1-\mu)+\mu)^{m(h_t)-1}}}\right\}.$$

Then by Proposition 3, $R(\Omega_n)_t$ is separating. Therefore, $\psi_t(i) = 0$ for all $i \in \{-n+1, ..., 0, ..., n-1\}$. Thus, from Lemma 1, we can obtain the desired result. ■

***Proof of Corollary 17.***

*1.*

As $\gamma(n) + \gamma(-n)$ goes to zero, $\psi_t^*(n|\overline{V}, h_{t-1})\gamma(-n) + \psi_t^*(-n|\overline{V}, h_{t-1})\gamma(n)$ goes to zero. Notice that since the informed traders always choose the largest quantity with strictly positive probabilities, we have: $\psi_t(n|\overline{V}, h_{t-1}) > 0$ and $\psi_t(-n|\overline{V}, h_{t-1}) > 0$. Therefore, we have: $\psi_t^*(n|\overline{V}, h_{t-1})\, \psi_t^*(-n|\overline{V}, h_{t-1}) > 0$. From Lemma 1, we can obtain the desired result. ■

*2.* Suppose that

$$\frac{n}{n-1} \leq 1 + \min\left\{\frac{\mu}{\gamma(n)\,(1-\mu) + \frac{\delta}{(1-\delta)(\gamma(n)(1-\mu)+\mu)}}\,,\, \frac{\mu}{\gamma(-n)\,(1-\mu) + \frac{(1-\delta)}{\delta\,(\gamma(-n)(1-\mu)+\mu)}}\right\}.$$

Then, we have

$$\frac{n}{n-1} \leq \min\left\{ 1+\frac{\mu}{\gamma(n)\,(1-\mu)+\frac{\delta\,(\gamma(n)\,(1-\mu))^{m(h_t)}\,(\gamma(n)\,(1-\mu)+\mu)^{m(h_t)-1}}{1-\delta}}\,,1+\frac{\mu}{\gamma(-n)\,(1-\mu)+\frac{(1-\delta)(\gamma(-n)\,(1-\mu))^{m(h_t)}}{\delta(\gamma(-n)\,(1-\mu)+\mu)^{m(h_t)-1}}} \right\}.$$

Then, by Proposition 3, $\psi_t^*$ is pooling. We consider the spreads in the second largest size, $S_t^*(h_{t-1}, n-1)$. Since in a pooling equilibrium the informed traders trade the second largest size with a strictly positive probability, we have $\psi_t^*(n-1|\overline{V}, h_{t-1}) \neq 0$ or $\psi_t^*(-n+1|\overline{V}, h_{t-1}) \neq 0$. Then, as $\gamma(n-1) + \gamma(-n+1)$ goes to some positive number $p$, $\psi_t^*(n-1|\overline{V}, h_{t-1})\gamma(-n+1) + \psi_t^*(-n+1|\overline{V}, h_{t-1})\gamma(n-1)$ also goes to some positive number. Remember that as in (7),

$$0 < \delta_t(h_t) < 1, \quad \forall h_t \in \Omega_n^t, \forall t.$$

Therefore, from Lemma 1, we can conclude that the spreads, $S_t^*(h_{t-1}, n-1)$, converges to some positive number $c$. ■

***Proof of Proposition 20.***

It is easy to verify that

$$\begin{aligned} S_{t,n}^*(h_{t-1}, q) &= \delta_{t,n}^*(h_{t-1}, q)\,(1 - \delta_{t,n}^*(h_{t-1}, q))\,\mu \\ &\times \frac{(1-\mu)\Big(\psi_{t,n}^*(h_{t-1},q)\gamma_n(-i)+\psi_{t,n}^*(h_{t-1},-q)\gamma_n(i)\Big)+\mu\,\psi_{t,n}^*(h_{t-1},q)\,\psi_{t,n}^*(h_{t-1},-q)}{\Big(\delta_{t-1,n}^*(h_{t-1})\,\mu\,\psi_{t,n}^*(h_{t-1},q)+(1-\mu)\,\gamma_n(q)\Big)\Big((1-\delta_{t-1,n}(h_{t-1}))\,\mu\,\psi_{t,n}^{h_{t-1}}(-q)+(1-\mu)\,\gamma_n(-q)\Big)}\,(\overline{V} - \underline{V}), \end{aligned} \tag{81}$$

for any $n \geq 1$ and $(h_{t-1}, q) \in \Omega_n^t$. For sufficiently large $N_{(h_{t-1},q)} \in \mathbb{Z}_{++}$, $\forall m' \geq N_{(h_{t-1},q)}$ and $\forall k \leq q$

$$(1-\mu)\sum_{i=k}^{m'}\left(1 - \frac{i}{q}\right)\gamma_{m'}(+i) + (1 - \delta_{\tau-1}^*(h_{\tau-1}))\mu \quad < \quad 0,$$

because

$$\begin{aligned} &\lim_{m'\to\infty}\left((1-\mu)\sum_{i=k}^{m'}\left(1 - \frac{i}{q}\right)\gamma_{m'}(+i) + (1 - \delta_{\tau-1}^*(h_{\tau-1}))\mu\right) \;=\; \\ &= \lim_{m'\to\infty}\left(-\frac{1-\mu}{q}\sum_{i=k}^{m'} i\gamma_{m'}(+i) + \underbrace{(1-\mu)\sum_{i=k}^{m'}\gamma_{m'}(+i) + (1 - \delta_{\tau-1}^*(h_{\tau-1}))\mu}_{\leq 1}\right) \\ &= -\infty. \end{aligned}$$

Consequently, from Lemma 2, $\psi^*_{\tau,m'}(q|\overline{V}, h_{t-1}) = 0$ and $\psi^*_{\tau,m'}(-q|\underline{V}, h_{t-1}) = 0$ $\forall m' \geq N_{(h_{t-1},q)}$. Following (81), $\forall m' \geq N_{(h_{t-1},q)}$

$$0 \quad = \quad S^*_{t,m'}(h_{t-1}, q) \quad \leq \quad S^*_{t,m}(h_{t-1}, q). \qquad \blacksquare$$

***Proof of Proposition 21.***

First, note that, given the largest trade size $n$ and history $(h_{t-1}, q)$,

$$\text{var}_n(V|h_{t-1}, q) \quad = \quad \delta^*_{t,n}(h_{t-1}, q)\left(1 - \delta^*_{t,n}(h_{t-1}, q)\right)(\overline{V} - \underline{\text{V}})^2. \tag{82}$$

Also, following (6), for any given largest trade size $n$ and trading period $t$,

$$\begin{aligned} &\delta^*_{t,n}(q_1, ..., q_{t-1}, q_t) \leq \delta^*_{t-1,n}(q_1, ..., q_{t-1}) \leq ... \leq \delta^*_{1,n}(q_1) \leq \delta, \quad \forall (q_1, ..., q_t) \in (\Omega^+_n \cup \{0\})^t, \\ &\delta^*_{t,n}(q_1, ..., q_{t-1}, q_t) \geq \delta^*_{t-1,n}(q_1, ..., q_{t-1}) \geq ... \geq \delta_{1,n}(q_1) \geq \delta, \quad \forall (q_1, ..., q_t) \in (\Omega^-_n \cup \{0\})^t. \end{aligned} \tag{83}$$

We now prove part (1) of Proposition 21 by induction on trading period $t$. Suppose

$$\lim_{n\to\infty} \sum_{i=1}^{n} i\gamma_n(+i) = \lim_{n\to\infty} \sum_{i=1}^{n} i\gamma_n(-i) = \infty,$$

and let $\delta \leq \frac{1}{2}$.

- <u>$t = 1$:</u> We shall show that $\forall q \in \Omega^+_m \cup \{0\}$ $\exists N_q \in \{m + 1, m + 2, ...\}$ such that $\forall m' \geq N_q$

  $$\text{var}_{m'}(V|q) \geq \text{var}_m(V|q).$$

  If $q = 0$, then following (6)

  $$\delta^*_{1,n}(q) = \delta, \ \forall n \in \mathbb{Z}_{++} \quad \overset{by\,(82)}{\Longrightarrow} \quad \text{var}_{m'}(V|q) = \text{var}_m(V|q), \ \forall m', m \in \mathbb{Z}_{++},$$

  and we are done.

  Suppose $q \neq 0$. Then, for sufficiently large $N_q \in \mathbb{Z}_{++}$, $\forall m' \geq N_q$ and $\forall k \leq q$

  $$(1 - \mu) \sum_{i=k}^{m'} \left(1 - \frac{i}{q}\right) \gamma_{m'}(+i) + (1 - \delta)\mu \quad < \quad 0,$$

because

$$\lim_{m'\to\infty}\left((1-\mu)\sum_{i=k}^{m'}\left(1-\frac{i}{q}\right)\gamma_{m'}(+i)+(1-\delta)\mu\right) =$$

$$= \lim_{m'\to\infty}\left(-\frac{1-\mu}{q}\sum_{i=k}^{m'} i\gamma_{m'}(+i)+\underbrace{(1-\mu)\sum_{i=k}^{m'}\gamma_{m'}(+i)+(1-\delta)\mu}_{\leq 1}\right)$$

$$= -\infty.$$

Then from Lemma 2, $\psi_{1,m'}(q)=0\ \forall m'\geq N_q$ since $\psi_{1,m'}$ always takes non-negative values as a probability function. Thus, following (6), $\forall m'\geq N_q$

$$\delta^*_{1,m'}(q)=\delta \quad\Longrightarrow\quad \mathrm{var}_{m'}(V|q)=\delta(1-\delta)(\overline{V}-\underline{\mathrm{V}})^2. \tag{84}$$

On the other hand, following (6),

$$\delta_{1,m}(q)\leq\delta \quad\overset{\text{since }\delta\leq\frac{1}{2}}{\Longrightarrow}\quad \mathrm{var}_m(V|q)\leq\delta(1-\delta)(\overline{V}-\underline{\mathrm{V}})^2. \tag{85}$$

(84) and (85) together prove that $\forall q\in\Omega_m^+\ \ \exists N_q\in\{m+1,m+2,...\}$ such that $\forall m'\geq N_q$

$$\mathrm{var}_{m'}(V|q)\geq\mathrm{var}_m(V|q).$$

- Induction hypothesis for $t=\tau-1$: $\forall\,(h_{\tau-2},q)\in(\Omega_m^+\cup\{0\})^{\tau-1}\quad\exists N_{(h_{\tau-2},q)}\in\{m+1,m+2,...\}$ such that $\forall m'\geq N_{(h_{\tau-2},q)}$

$$\mathrm{var}_{m'}(V|h_{\tau-2},q)\geq\mathrm{var}_m(V|h_{\tau-2},q).$$

- $t=\tau$: Given the induction hypothesis for $t=\tau-1$, let us show that $\forall\,(h_{\tau-1},q)\in(\Omega_m^+\cup\{0\})^{\tau}$ $\exists N_{(h_{\tau-1},q)}\in\{m+1,m+2,...\}$ such that $\forall m'\geq N_{(h_{\tau-1},q)}$

$$\mathrm{var}_{m'}(V|h_{\tau-1},q)\geq\mathrm{var}_m(V|h_{\tau-1},q).$$

If $q=0$, then $\delta^*_{\tau,n}(h_{\tau-1},q)=\delta^*_{\tau-1,n}(h_{\tau-1}),\ \forall n\in\mathbb{Z}_{++}$ following (6). Thus, from (82) and the induction hypothesis for $t=\tau-1$, $\forall m'\geq N_{(h_{\tau-1})}$

$$\mathrm{var}_{m'}(V|h_{\tau-1},q)=\mathrm{var}_{m'}(V|h_{\tau-1})\geq\mathrm{var}_m(V|h_{\tau-1})=\mathrm{var}_m(V|h_{\tau-1},q)$$

and we are done by letting $N_{(h_{\tau-1},q)} = N_{(h_{\tau-1})}$. Suppose $q \neq 0$. For sufficiently large $N \in \mathbb{Z}_{++}$, $\forall m' \geq N$ and $\forall k \leq q$

$$(1-\mu)\sum_{i=k}^{m'}\left(1-\frac{i}{q}\right)\gamma_{m'}(+i) + (1-\delta^*_{\tau-1}(h_{\tau-1}))\mu \quad < \quad 0,$$

because

$$\begin{aligned}
&\lim_{m'\to\infty}\left((1-\mu)\sum_{i=k}^{m'}\left(1-\frac{i}{q}\right)\gamma_{m'}(+i) + (1-\delta^*_{\tau-1}(h_{\tau-1}))\mu\right) \quad = \\
&= \lim_{m'\to\infty}\left(-\frac{1-\mu}{q}\sum_{i=k}^{m'} i\gamma_{m'}(+i) + \underbrace{(1-\mu)\sum_{i=k}^{m'}\gamma_{m'}(+i) + (1-\delta^*_{\tau-1}(h_{\tau-1}))\mu}_{\leq 1}\right) \\
&= -\infty.
\end{aligned}$$

Hence, from Lemma 2, $\psi^*_{\tau,m'}(q|V,h_{\tau-1}) = 0 \ \forall m' \geq N$ because $\psi^{h_{\tau-1}}_{\tau,m'}$ takes only non-negative values as a probability function. Now from (6), $\forall m' \geq N$

$$\delta^*_{\tau,m'}(h_{\tau-1},q) = \delta^*_{\tau-1,m'}(h_{\tau-1}),$$

which implies

$$\mathrm{var}_{m'}(V|h_{\tau-1},q) = \delta_{\tau-1,m'}(h_{\tau-1})(1-\delta^*_{\tau-1,m'}(h_{\tau-1}))(\overline{V}-\underline{\mathrm{V}})^2 = \mathrm{var}_{m'}(V|h_{\tau-1}). \quad (86)$$

Also, using (83), we get

$$\delta^*_{\tau,m}(h_{\tau-1},q) \leq \delta^*_{\tau-1,m}(h_{\tau-1}) \leq \frac{1}{2}$$

since $(h_{\tau-1},q) \in (\Omega^+ \cup \{0\})^\tau$ and $\delta \leq \frac{1}{2}$. Thus, we have

$$\mathrm{var}_m(V|h_{\tau-1},q) \leq \delta^*_{\tau-1,m}(h_{\tau-1})(1-\delta^*_{\tau-1,m}(h_{\tau-1}))(\overline{V}-\underline{\mathrm{V}})^2 = \mathrm{var}_m(V|h_{\tau-1}). \quad (87)$$

Using (86), (87) and the induction hypothesis for $t = \tau-1$, we prove that $\forall(h_{\tau-1},q) \in (\Omega^+_m \cup \{0\})^\tau \ \ \exists N_{(h_{\tau-1},q)} \in \{m+1, m+2, ...\}$ such that $\forall m' \geq N_{(h_{\tau-1},q)}$

$$\mathrm{var}_{m'}(V|h_{\tau-1},q) \geq \mathrm{var}_m(V|h_{\tau-1},q)$$

by letting $N_{(h_{\tau-1},q)} = \max\{N, N_{(h_{\tau-1})}\}$.

Part (2) can be proved using an induction argument similar to the one above. It is also easy to check that the inequalities in the statement of the proposition are strict when $\psi_{t,m}(h_{t-1}, q)$ is non-zero. ■

***Proof of Proposition 22.***

Suppose that we have $k_t(h_{t-1})$ partially pooling equilibrium. First, we suppose $q > 0$. For $q \in (0, k_t(h_{t-1}))$, $R_t^*(h_{t-1}, q) = 0$. Moreover, by using the formula, we can calculate: for $q > k_t(h_{t-1})$,

$$R_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1}) - \frac{\delta_{t-1}^*(h_{t-1})(1-\mu)\sum_{i=k_t(h_{t-1})}^{n} i\gamma(i)}{q[(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu]}. \tag{88}$$

Second, we suppose $q < 0$. For $q \in (-k_t(h_{t-1}), 0)$, $R_t^*(h_{t-1}, q) = 0$. Moreover, by using the formula, we can calculate:

$$R_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1}) - 1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)\sum_{i=k_t(h_{t-1})}^{n} i\gamma(-i)}{q[(1-\mu)\sum_{i=k_t(h_{t-1})}^{n}\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu]}. \tag{89}$$

In both cases, since $\delta_{t-1}^*(h_{t-1})$, $\mu$, and $\gamma(i)'$s are exogenous, $R_t^*(h_{t-1}, q)$ is a concave function of $q$ in $(0, k_t(h_{t-1}))$ or $(-k_t(h_{t-1}), 0)$. ■

## References

[1] COPELAND, T. AND D. GALAI (1983), “Information effect on the bid-ask spread”, *Journal of Finance*, **38**, 1457-1469

[2] FRISTEDT, B. AND L. GRAY (1996), ‘‘A Modern Approach to Probability Theory”, Birkhäuser, Boston

[3] X. GABAIX, P. GOPIKRISHNAN, V. PLEROU AND H. STANLEY (2004), “A theory of large fluctuations in stock market activity”, mimeo

[4] X. GABAIX, P. GOPIKRISHNAN, V. PLEROU AND H. STANLEY (2003), “A theory of power-law distributions in financial market fluctuations ”, *Nature*, **423**, 267-270

[5] X. GABAIX, P. GOPIKRISHNAN, V. PLEROU AND H. STANLEY (2002), “Quantifying stock-price response to demand fluctuations”, *Physical Review E*, **66**, 027104:1-027104:4

[6] D. EASLEY AND M. O’HARA (1987), “Price, trade size, and information in securities markets”, *Journal of Financial Economics*, **19**, 69-90

[7] D. EASLEY, N. KIEFER AND M. O’HARA (1997), “One day in the life of a very common stock”, *The Review of Financial Studies*, **10**, 805-835

[8] GLOSTEN, L. (1987), “Components of the bid-ask spreads and the statistical properties of transaction prices”, *Journal of Finance*, **5**, 1293-1307

[9] GLOSTEN, L. AND L. HARRIS (1988), “Estimating the components of the bid-ask spread”, *Journal of Financial Economics*, **14**, 21-42

[10] GLOSTEN, L. AND P.R. MILGROM (1985), “Bid, ask and transaction prices in a specialist market with heterogenously informed traders”, *Journal of Financial Economics*, **14**, 71-100

[11] J. HASBROUCK(1991), “Measuring the information content of stock trades”, *The Journal of Finance*, **XLVI**, 179-207

[12] J. HASBROUCK(1988), “Trades, quotes, inventories, and information”, *Journal of Financial Economics*, **22**, 229-252

[13] C. JONES, G. KAUL AND M. LIPSON(1994), “Transactions, volume, and volatility”, *The Review of Financial Studies*, **7**, 631-651

[14] KARPOFF, J. (1987), "The relation between price change and trading volume: A Survey", *Journal of Financial and quantitative Analysis*, **22**

[15] TAKAYAMA, S. (2004), "Impact of trading sizes on price and information", *mimeo*

[16] TEWELES, E.S., R.J. BRADLEY AND T.M. TEWELES (1992), *The Stock Market*, John Wiley & Sons, Inc., New York, NY