Simulation-Based Pricing of Convertible Bonds

Manuel Ammann, Axel Kind, Christian Wilde *

University of St.Gallen / Goethe University Frankfurt

Abstract

We propose and empirically study a pricing model for convertible bonds based on Monte Carlo simulation. The method uses parametric representations of the early exercise decisions and consists of two stages. Pricing convertible bonds with the proposed Monte Carlo approach allows us to better capture both the dynamics of the underlying state variables and the rich set of real-world convertible bond specifications. Furthermore, using the simulation model proposed, we present an empirical pricing study of the US market, using 32 convertible bonds and 69 months of daily market prices. Our results do not confirm the evidence of previous studies that market prices of convertible bonds are on average lower than prices generated by a theoretical model. Similarly, our study is not supportive of a strong positive relationship between moneyness and mean pricing error, as argued in the literature.

JEL codes: G13, G14
Keywords: Convertible bonds, Pricing, American Options, Monte Carlo simulation

---

*This project is supported by the Swiss National Science Foundation. We are grateful for comments of Giovanni Barone-Adesi, Murray Carlson, Robert Engle, Stephen Figlewski, Fabio Mercurio, Suresh Sundaresan, Ingo Walter, Heinz Zimmermann, and participants of the Northern Finance Association Conference in Banff, Canada, the Quantitative Methods in Finance Conference in Cairns, Australia, the Australasian Finance and Banking Conference in Sydney, Australia, and of Research Seminars at WHU in Koblenz, Germany, University of Zürich, Switzerland, University of Konstanz, Germany, and Goethe University in Frankfurt, Germany. We thank Mace Advisers for providing access to the Mace Advisers convertible bond database. We acknowledge Zeno Dürr, Mark Evans, and Peter Hall for their assistance in obtaining the data and for helpful discussions. Manuel Ammann and Axel Kind are at the Swiss Institute of Banking and Finance, University of St.Gallen. Christian Wilde is at the Finance Department, Johann Wolfgang Goethe University Frankfurt. Email addresses: manuel.ammann@unisg.ch, axel.kind@unisg.ch, and wilde@finance.uni-frankfurt.de.

Simulation-Based Pricing of Convertible Bonds

Abstract

We propose and empirically study a pricing model for convertible bonds based on Monte Carlo simulation. The method uses parametric representations of the early exercise decisions and consists of two stages. Pricing convertible bonds with the proposed Monte Carlo approach allows us to better capture both the dynamics of the underlying state variables and the rich set of real-world convertible bond specifications. Furthermore, using the simulation model proposed, we present an empirical pricing study of the US market, using 32 convertible bonds and 69 months of daily market prices. Our results do not confirm the evidence of previous studies that market prices of convertible bonds are on average lower than prices generated by a theoretical model. Similarly, our study is not supportive of a strong positive relationship between moneyness and mean pricing error, as argued in the literature.

1. Introduction

To raise capital on financial markets, companies may choose among three major asset classes: equity, bonds, and hybrid instruments, such as convertible bonds. While issues arising from valuing equity and bonds are extensively studied by researchers in academia and industry, fewer articles focus on convertible bonds. This is surprising as convertible bonds cannot simply be considered as a combination of equity and bonds but present their own specific pricing challenges.

As hybrid instruments, convertible bonds are difficult to value because they depend on variables related to the underlying stock (price dynamics), the fixed income part (interest rates and credit risk), and the interaction between these components. Embedded options, such as conversion, call, and put provisions often are restricted to certain periods, may vary over time, and are subject to additional path-dependent features of the state variables. Sometimes, individual convertible bonds contain innovative, pricing-relevant specifications that require flexible valuation models. The purpose of this study is to present a pricing model based on Monte Carlo Simulation that can deal with these valuation challenges.

We implement this model and use it to perform the first simulation-based pricing study of the US convertible-bond market that accounts for early-exercise features.

Theoretical research on convertible bond pricing can be divided into three branches. The first pricing approach implies finding a closed-form solution to the valuation equation. It was initiated by Ingersoll (1977a), who applies the contingent claims approach to the valuation of convertible bonds. In this valuation model, the convertible-bond price depends on the firm value as the underlying state variable. More recently, Lewis (1991) develops a formula for convertible bonds that accounts for more complex capital structures, i.e. multiple issues. Bühler and Koziol (2002) focus on the possibility of non-block-constrained conversion and develop pricing formulas for simple convertible bonds. While very fast in computation, closed-form solutions are not suitable for empirical studies because they fail to account for a number of real-world specifications. Especially, dividends and coupon payments are often modeled continuously rather than discretely, early-exercise features are omitted, and path-dependent features are excluded.

The second pricing approach values convertible bonds numerically, using numerical partial differential equation approaches. Commercially available models for pricing convertible bonds, such as Bloomberg OVCV, Monis, and SunGard TrueCalcTM Convertible, belong to this category. The first theoretical model was introduced by Brennan and Schwartz (1977) who apply a firm-value-based approach and a finite-difference method for the pricing task. Brennan and Schwartz (1980) extend their pricing method by including stochastic interest rates. However, they conclude that the effect of a stochastic term structure on convertible-bond prices is so small that it can be neglected for empirical purposes. McConnell and Schwartz (1986) develop a pricing model based on a finite-difference method with the stock price as stochastic variable. To account for credit risk, they use an interest rate augmented by a constant credit spread. Since the credit risk of a convertible bond varies with respect to its moneyness, Bardhan et al. (1993) and Tsiveriotis and Fernandes (1998) propose an approach that splits the value of a convertible bond into a stock component and a straight bond component. Ammann et. al. (2003) extend this approach by accounting for call features with various trigger conditions. Also Hung and Wang (2002) propose a tree-based model that accounts for both stochastic interest rates and default probabilities but looses its recombining property. A further tree-based model is presented by Carayannopoulos and Kalimipalli (2003), who use a trinomial tree and incorporates the reduced-form

Duffie and Singleton (1999) credit-risk model. Similar credit-risk approaches are followed by Davis and Lishka (1999), Takahashi et al. (2001), and Ayache et al. (2003), who explicitly allow for non-zero recovery rates. To sum up, among numerical partial differential equation approaches, there are both binomial/trinomial trees (e.g. Takahashi et al., 2001, Ammann et al., 2003, and Carayannopoulos and Kalimipalli, 2003), finite difference (e.g. Brennan and Schwartz, 1980, Ayache et al., 2003, and Andersen and Buffum, 2004), and finite element methods (e.g. Barone-Adesi et al., 2003). Some of the proposed models provide sophisticated pricing and calibration solutions. Unfortunately, in the face of practical problems related to real convertible-bond specifications and limited data availability, the proposed approaches turn out to be practicable only in very few cases. For instance, Andersen and Buffum (2004) require for their calibration price series of several options and liquid straight bonds - a situation that is almost never given for typical convertible bond issuers. Finally, numerical partial differential equation approaches have to deal with some general challenges: computing time grows exponentially with the number of state variables, path dependencies cannot be incorporated easily, and the flexibility in modeling the underlying state variables is low.

The third class of convertible bond pricing methods uses Monte Carlo Simulation and may overcome many of the drawbacks of numerical partial differential equation approaches. Monte Carlo Simulation is very well suitable for modeling discrete coupon and dividend payments, for including more realistic dynamics of the underlying state variables, and for taking into account path-dependent call features. Typically, path dependencies arise from the fact that early redemption may only be allowed when the stock price exceeds a certain level for a pre-specified number of days in a pre-specified period, usually at least 20 out of the last 30 trading days. Finally, the relationship between the number of state variables and computing time is almost linear in our Monte Carlo framework and this can become advantageous when multiple state variables need to be modeled. Thus, the proposed model has a high degree of flexibility and is friendly with respect to future extensions. Despite all the natural advantages of the Monte Carlo approach, pricing American-style options such as those present in convertible bonds within a Monte Carlo pricing framework is a demanding task. In recent years, a considerable number of important articles have addressed the problem of pricing American-style options[1] by using a combination of Monte Carlo Simulation and dynamic programming. Bossaerts (1989), Li and Zhang

[1]In general, simulation techniques only allow for a finite number of early-exercise times and hence price *Bermudan options* rather than continuously exercisable American options. However, for a fairly large number of early-exercise dates, the Bermudan price may serve as an approximation for the price of the American option.

(1996), Grant et al. (1996), Andersen (2000), and García (2003) represent the early exercise rule via a finite number of parameters. The optimal exercise strategy and hence the price of the American-style option is obtained by maximizing the value of the option over the parameter space. Carrière (1996), Tsitsiklis and Van Roy (1999), Longstaff and Schwartz (2001), and Clément et al. (2002) apply standard backward induction and estimate the continuation value of the option by regressing future payoffs on a set of basis functions of the state variables. Tilley (1993), Barraquand and Martineau (1995), Raymar and Zwecher (1997) present methods based on backward induction that stratify the state space and find the optimal exercise decision for each subset of state variables. Broadie and Glasserman (1997a) and Broadie et al. (1997b) propose a method for calculating prices of American-style options with *simulated trees* that generate two estimates, one biased high and one biased low. Broadie et al. (1997a), Broadie and Glasserman (2004), Avramidis and Hyden (1999), and Boyle et al. (2000) develop stochastic-mesh methods with different choices for mesh weights. Finally, Broadie and Cao (2003), Haugh and Kogan (2004), and Rogers (2002) suggest a simulation method that uses a duality approach for pricing Bermudan options. A numerical comparison of different Monte Carlo approaches is provided by Fu et al. (2001).

Previous research to value convertible bonds by Monte Carlo Simulation is very limited. Buchan (1997, 1998) describes the application of the parametric optimization approach of Bossaerts (1989) to convertible bonds by employing the firm value as the underlying state variable and allowing for senior debt. However, in the empirical implementation, she assumes the conversion option to be European rather than American.

This paper contains a theoretical and an empirical contribution. First, we propose a stock-based pricing method for convertible bonds building on the enhanced Monte Carlo Simulation method by García (2003). This is a two-stage method designed to cope with the Monte Carlo bias that is inherent in one-stage methods. The two-step simulation method may be defined as a *parametric approach* because it uses a parametric representation for the early exercise decisions. The first step is an optimization, in which a set of Monte Carlo simulations is used to estimate parameter values representing strategies for early exercise and to generate an in-sample price. In a next valuation stage, the optimized parameter space is applied to a second set of simulated stock-price paths to determine an out-of-sample model price for the convertible bond. The actual point estimate is then obtained by averaging the in-sample and the out-of-sample estimates. The optimization method by García (2003) is preferred to other approaches (simulated trees, stratification algorithms, and stochastic meshes) because it is more parsimonious in allowing for multiple exercise opportunities. While the regression method by Longstaff and Schwartz (2001) is another suitable technique, the optimization-based approach by García (2003) has an attractive feature for empirical studies: the optimization algorithm can be terminated once a certain level of accuracy is reached. As outlined above, the simulation approach adopted in this paper has an inherent strength as it is flexible in incorporating the dynamics of the state variables. Furthermore, besides discrete coupon and dividend payments, the introduced method accounts for path-dependent call triggers as outlined in the offering circulars. Instead of using a firm-value model, the stock price is modeled directly, as proposed by McConnell and Schwartz (1986). Whereas the process parameters of a model based on the stock price can easily be estimated with standard methods, the fact that firm values are not observable makes firm-value models notoriously hard to calibrate. Since the presented method is cash-flow based, credit risk can easily be incorporated by discounting the payoffs subject to credit risk with the appropriate interest rate in the spirit of Tsiveriotis and Fernandes (1998).

The second contribution in this paper is an empirical analysis of the US convertible bond market. Despite the large size of international convertible bond markets, very little empirical research has been undertaken. Previous research in this area was performed by King (1986), who examines a sample of 103 American convertible bonds with a lattice-based method and the firm-value as stochastic variable. Using monthly price data and a convertible bond valuation model with Cox, Ingersoll, and Ross (1985) stochastic interest rates (CIR), Carayannopoulos (1996) empirically investigates 30 American convertible bonds for a one-year period beginning in the fourth quarter of 1989. Buchan (1997) uses a simulation-based approach to implement a firm-value model with a CIR term structure model for 35 Japanese convertible bonds. Buchan (1998) performs a pricing study for 37 US convertible bonds issued in 1994. However, the American property of convertible bonds is not accounted for in that study. Carayannopoulos and Kalimipalli (2003) investigate 25 US convertible bonds with a trinomial tree. Ammann et al. (2003) investigate on a daily basis 21 French convertible bonds with a binomial tree using the stock price as stochastic variable.

A drawback of many of those pricing studies is the small number of data points per convertible bond: Buchan (1997) tests her pricing model only for one calendar day (bonds priced per March

31, 1994), King (1986) for two days (bonds priced per March 31, 1977, and December 31, 1977), Carayannopoulos (1996) for twelve days (one year of monthly data), and Carayannopoulos and Kalimipalli (2003) for approximately two years of monthly data. In contrast, this study covers a larger sample using 69 months of daily price data, ranging from May 10, 1996, to February 12, 2002 and includes 32 convertible bonds in the US market. The US convertible bond market is chosen for its large size and the high number of rated issues.

A second drawback of the previous pricing studies is the simple modeling of the volatility of the underlying stock. This drawback is almost inherent to the lattice approaches adopted by King (1986), Carayannopoulos (1996), Carayannopoulos and Kalimipalli (2003), and Ammann et al. (2003). Although Buchan (1997) uses a simulation-based approach, her model does not fully exploit the potentials provided by Monte Carlo Simulation as a constant volatility is assumed for the stock dynamics. To take into account the clustering of stock volatility, we implement the model using a GARCH(1,1) specification.

The paper is organized as follows: First, we introduce the convertible bond pricing model that will be applied in the empirical investigation. Second, we describe the data set and present the specific characteristics of the convertible bonds examined. Third, we discuss the empirical methodology applied when implementing the model. Finally, the empirical study compares theoretical model prices with observed market prices and analyzes the results.

2. Pricing Convertible Bonds with Monte Carlo Simulation

2.1. The American Option Pricing Problem for Convertible Bonds

A standard, plain-vanilla convertible bond is a bond that additionally offers the investor the option to exchange it for a predetermined number of stocks during a certain, predefined period of time. The bond usually offers regular coupon payments and, in case it is kept alive, is redeemed at the time of maturity T with a pre-specified amount κN, where N is the face value of the convertible bond and κ is the final redemption ratio in percentage points of the face value. Although κ is equal to one for most convertibles, some issues are redeemed at premium with κ larger than one. Let us consider time discretely with daily frequency, i.e. that time t belongs to a finite set, $t \in [0,1,...T]$, where $t = 0$ indicates today, and $t = T$ the day of contractual maturity. In the case of conversion, the investor receives $n_t S_t$, where the conversion ratio n_t is the number of stocks the bond can be exchanged for, and S_t is the equity price (underlying) at time t. If the underlying stock differs from that of the issuing firm, the instrument is commonly called an *exchangeable*. Usually, convertible bonds additionally contain a call option, allowing the issuer to demand premature redemption in exchange for the call price K_t applicable at time t. The issuer is obliged to announce his intention to call a certain period in advance, referred to as the *call notice period*. If the convertible bond is called, the investor may want to exercise his conversion option at any time during the call notice period to receive the conversion value instead of the call price. Additionally, a putability feature is sometimes present. This entitles the investor to force the issuing firm to prematurely repurchase the convertible bonds for a certain predefined price P_t. All these embedded options may be restricted to certain periods of time or specific dates. To facilitate the formal exposition, we introduce three time sets, Ω_{conv}, Ω_{call} and Ω_{put}, that describe the dates at which the corresponding option is exercisable. Typically, the first possible conversion date precedes the first call opportunity and the last conversion opportunity is at maturity.

Thus, the payoff of a convertible bond depends on whether and when the investor and the issuer decide to exercise their options and trigger the termination of the convertible bond. Let τ^* be the optimal stopping time, i.e. the time at which it is optimal for either the issuer or the investor to terminate the convertible bond. Hereby, the investor maximizes the value of the convertible bond whereas the issuer acts in the opposite way. The resulting action may either be conversion, a call, forced conversion, or regular redemption when the bond matures. Formally, the optimal stopping time of the convertible bond is defined as $\tau^* = \min\{t : p(X_t, t) \neq 0\}$, where $p(X_t, t)$ is the payoff resulting from the convertible bond in state X_t at time t, given the optimal option-exercise behavior of both investor and issuer. The alternatives presented in Table 1 stand for all events that will cause the convertible bond to be terminated and reflect boundary conditions that impede arbitrage opportunities. Besides when reaching maturity, the convertible bond can be ended by a conversion into stock, by a call, or by a put. The optimal exercise decision critically depends on the value of continuation V_t', i.e. the value of the convertible bond if it is not exercised immediately. While the investor will convert (put) the bond as soon as $n_t S_t > V_t'$ ($P_t > V_t'$) for $t \in \Omega_{conv}$ ($t \in \Omega_{put}$), the issuer will call the convertible as soon as $V_t' > K_t$ for $t \in \Omega_{call}$. Thus, at each point in time, both investor and issuer decide whether they want to exercise their option or not

Table 1

Optimal exercise decision

This table presents the optimal option exercise behavior of both the issuer and the investor. The payoffs resulting from the optimal exercise decisions are listed in the first column of the table. The second column displays the conditions under which it is optimal to exercise. *Time restriction* indicates whether the embedded option can be exercised by the investor (issuer) at time t . Six outcomes are possible: voluntary conversion, put, call, forced conversion, redemption at maturity, or continuation of the convertible bond. V_t' is the conditional expected value of continuation, i.e. the value of holding the convertible bond for one more period instead of exercising immediately.

Payoff $p(X_t,t)$	Condition	Time restriction	Action
n_tS_t	if $n_tS_t > V_t'$ and $P_t \leq n_tS_t$	for $t \in \Omega_{conv}$ for $t \in \Omega_{put} \cap \Omega_{conv}$	Voluntary conversion
P_t	if $P_t > V_t'$ and $n_tS_t < P_t$	for $t \in \Omega_{put}$ for $t \in \Omega_{conv} \cap \Omega_{put}$	Put
K_t	if $V_t' > K_t$ and $K_t \geq n_tS_t$	for $t \in \Omega_{call}$ for $t \in \Omega_{call} \cap \Omega_{conv}$	Call
n_tS_t	if $V_t' > K_t$ and $n_tS_t > K_t$	for $t \in \Omega_{call}$ for $t \in \Omega_{conv} \cap \Omega_{call}$	Forced conversion
κN	if $n_tS_t < \kappa N$	for $t = T \in \Omega_{conv}$	Redemption
0	otherwise		Continuation

and this decision is dependent on the continuation value. In the case of a call, the investor will convert the bond if the conversion value is above the call price (*forced conversion*), otherwise he will prefer to have it redeemed. The entries *Condition* and *Time restrictions* in Table 1 have to be read line by line, i.e. the condition in the second column of the table is checked only if the corresponding time restriction on the same line of the following (third) column is satisfied. Besides to certain predefined times, the possibility to call the convertible bond may be restricted by certain conditions to be satisfied, e.g. that the conversion value exceeds a pre-specified call trigger. The investor will make use of the option to put the convertible bond when the value of continuation falls below the put price. It follows that the convertible bond will be kept alive as long as $max\,(n_t S_t; P_t) \leq V_t' \leq K_t$, i.e. that neither the investor nor the issuer will execute their options and cause the convertible bond to terminate.

In addition to the payoff at the time of termination, the investor receives from his convertible bond investment all coupon payments that occurred prior to this date. Formally, the function $h\,(X_{\tau^*}, \tau^*)$ represents the payoff from a convertible bond with embedded call option in state X_{τ^*} and at time τ^*:

$$h\,(X_{\tau^*}, \tau^*) = p\,(X_{\tau^*}, \tau^*) + c\,(\tau^*) \tag{1}$$

where $p\,(X_{\tau^*}, \tau^*)$ is the payoff from the convertible bond at the optimal time of termination τ^* and $c\,(\tau^*)$ is the present value at time τ^* of all coupon payments accumulated during the existence of the bond, i.e. before τ^*. As will be seen later, whether $c\,(\tau^*)$ contains also accrued interest payments is an empirical matter that depends on the specification of the individual convertible bond.

The price of a convertible bond can be obtained by discounting all future cash flows under the risk-neutral measure. Thus, valuing convertible bonds implies determining

$$V_0 = E^{\mathbb{Q}}\left[e^{-\sum_{t=0}^{\tau^*-1} r(X_t,t)t} h\,(X_{\tau^*}, \tau^*)\right], \tag{2}$$

where V_0 is the current value of the convertible bond, τ^* is the optimal stopping time taking values in the finite set $\{0, 1, \ldots, T\}$, the function $h\,(X_{\tau^*}, \tau^*)$ represents the payoff from a convertible bond with embedded call option in state X_{τ^*} and at time τ^*, and the expectation $E^{\mathbb{Q}}\,[\cdot]$ is taken with respect to the equivalent Martingale measure $\mathbb{Q}$ defined using the riskless security as the numeraire. $r(X_t, t)$ is the interest rate between time t and $t+1$ in state X_t that is applicable for discounting cash flows from time $t+1$ to time t.

2.2. Characterizing the Optimal Exercise Decision

Before maturity, the optimal exercise strategy implies comparing the value of immediate exercise with the value from continuing, i.e. not exercising this period. The crucial step implies determining the conditional expected value of continuation V_t'. Formally, the value at a future time t of a convertible bond that is not exercised immediately, but held for one more period, is given by

$$V_t' = E^{\mathbb{Q}}\left[e^{-\sum_{t=0}^{\tau^*-1} r(X_t,t)t} h\left(X_{\tau^*},\tau^*\right) \mid \mathcal{F}_t\right] \quad (3)$$

where $\tau^* > t$ and $\mathcal{F}_t$ represents the information available at time t.

The continuation value V_t' can be expressed as a function of the state variables and time. In particular, for convertible bonds, there is a monotonous relation between V_t' and the state variables.[2] Hence, for obtaining a full description of any economically meaningful option-exercise behavior, it is sufficient to define for each embedded option only one *exercise boundary* Z_{conv}, Z_{call}, and Z_{put} for the conversion, call, and put option, respectively. For each option, the exercise boundary separates the exercise-region from the non-exercise region. The exercise boundaries describe the combined values of state variables for which investor and issuer are indifferent between exercising their options or not. For q state variables, the boundaries Z_{conv}, Z_{call}, and Z_{put} can be viewed as functions that associate to any date t and any values of $q-1$ state variables critical values for the remaining state variable q that trigger the exercise of the respective option. Z_t^{conv} denotes, for a specific date $t \in \Omega_{conv}$, the value of state variable q for which $V_t' = n_t S_t$. Similarly, Z_t^{call} denotes, for a specific date $t \in \Omega_{call}$, the value of state variable q for which $V_t' = K_t$ and Z_t^{put} denotes, for a specific date $t \in \Omega_{put}$, the value of state variable q for which $V_t' = P_t$.

In the case where the stock price S is the only state variable, it is optimal to exercise the options whenever $S_t > Z_t^{conv}$, $S_t > Z_t^{call}$, and $S_t < Z_t^{put}$, where Z_t^{conv}, Z_t^{call}, and Z_t^{put} are scalars. As described

[2] For example, V_t' is monotonically increasing in the stock price S, with $0 < dV_t'/dS < n$, given specific values for the other state variables. Therefore, for every embedded option, there is at most one S for which the continuation value is equal to the respective option payoff if exercised (K_t, $n_t S_t$, and P_t).

in Appendix A, for the numerical implementation, the exercise boundaries Z_{conv}, Z_{call}, and Z_{put} are approximated by parametric functions $G_{conv}(t;\theta_{conv})$, $G_{call}(t;\theta_{call})$, and $G_{put}(t;\theta_{put})$ with parameter sets θ_{conv}, θ_{call}, and θ_{put}.

2.3. Simulation Methodology

The pricing algorithm consists of two stages, an optimization stage and a valuation stage. In the first stage, the optimal exercise strategy of the investor and the issuer is estimated using a first set of simulated paths for the state variables. The parameter sets θ_{iss} and θ_{inv} govern the exercise behavior, or exercise strategy, of the issuer and investor, respectively. The exercise behavior of the issuer concerns solely the call option so that we can write without loss of generality $\theta_{iss} = \theta_{call}$. Since the investor's exercise behavior is related to both the conversion and the put option, we can conveniently write $\theta_{inv} = \left[\theta'_{conv} \;\; \theta'_{put}\right]'$. These exercise strategies determine the time of termination, or stopping time, τ, of the convertible bond. Hence, the value of the convertible bond given certain exercise strategies can be calculated by averaging the discounted payoffs of all simulation paths:

$$V\left(\theta_{inv}, \theta_{iss}\right) = \frac{1}{N}\sum_{i=1}^{N} e^{-\sum_{t=0}^{\tau^*-1} r(X_t,t)\tau_i(\theta_{inv},\theta_{iss})} h\left(X^i_{\tau_i(\theta_{inv},\theta_{iss})}, \tau_i\left(\theta_{inv}, \theta_{iss}\right)\right), \tag{4}$$

where X_t are realizations of the simulated state variables and N is the number of simulation paths.

To find the optimal conversion strategy, given a fixed call strategy, the initially chosen parameters encoding the put and the conversion strategy are altered until the algorithm finds a maximum for the convertible bond price:

$$\widehat{\theta}_{inv} = \underset{\theta_{inv}}{\arg\max}\, V\left(\theta_{inv}, \widehat{\theta}_{iss}\right), \tag{5}$$

where $\widehat{\theta}_{inv}$ indicates an estimate of the optimal exercise strategy of the investor. Subsequently, these parameters are applied to find a call strategy that minimizes the convertible-bond price:

$$\widehat{\theta}_{iss} = \underset{\theta_{iss}}{\arg\min}\, V\left(\widehat{\theta}_{inv}, \theta_{iss}\right). \tag{6}$$

To determine the final exercise strategy, this procedure is applied iteratively until the optimal parameters are obtained and a predefined accuracy is reached. The relevant stopping times $\tau_i^* \left(\widehat{\theta}_{inv}, \widehat{\theta}_{iss}\right)$ for each path i and the corresponding payoffs $h\left(X_{\tau_i^*}, \tau_i^*\right)$ for valuing a convertible bond are obtained by applying these optimized exercise rules to the simulated paths. Thus, as a result of this procedure, we obtain estimates of the optimal exercise strategies as well as an in-sample estimate of the price of the convertible bond.

In the second stage, the optimized exercise strategies from the first stage are applied on a second set of simulated paths of the state variables to determine the out-of-sample value of the convertible bond. The final point estimate is the average of the in- and out-of-sample estimates. While numerical experiments show that both the in-sample and the out-of-sample estimates converge to the true price of the convertible bond as the number of simulation paths increases, averaging the two results generates a more accurate point estimate.

Figure 1 presents a comparison of exercise boundaries obtained by a 6000-step binomial tree and the simulation-based model. For the sake of comparison the convertible bond only has only features that can be easily addressed within a standard binomial tree model. We investigate a simple case with the stock price as the only state variable and constant interest rates. As can be seen in the plots, the main features of the exercise boundaries are captured by the simulation model. The fact that the conversion boundary is lower in the simulation model can be easily explained. As long as the conversion boundary is higher than the call boundary, its exact position does not affect the price of the convertible bond. Thus, if during any step of the maximization procedure (cfr. equation (5)) the optimizer sets the conversion boundary in an arbitrary position above the call boundary, no change of the parameters $\theta conv$ will increase the price of the convertible and the current position of the conversion boundary will be the final one. The economic reason for the irrelevance of the exact position of the conversion boundary in the Monte Carlo algorithm is that, in this setting, the issuer will always call the convertible before voluntary conversion can become the optimal choice for the investor. Finally, the prices generated by the simulation model (106.4304 for the point estimate) are very close to those of the tree (106.405), supporting the convergence of the simulation model.[3]

[3]While Figure 1 is only one example, extensive numerical experiments obtained with several specifications confirm the convergence of the model. It is worth noting that in general the pricing results are sensitive to the level of the boundaries but not to their exact shape, i.e. that even simple shapes of the boundaries guarantee very accurate pricing results.

Figure 1. Exercise Boundaries

This graph shows exercise boundaries of a convertible bond obtained by a binomial tree (Panel A) and by the simulation-based model (Panel B). The tree generates a price of 106.405. The simulation model produces an in-sample price of 106.656 and an out of sample price of 106.2047 with standard deviations of 0.1128 and 0.1162, respectively, resulting in a point estimate of 106.4304. The parameter set for pricing the convertible bond is as follows. The volatility is equal to 40% ($\sigma = 0.4$), the initial stock price is 100 dollars ($S(0) = 100$), the dividend yield is continuously compounded ($\delta = 0.1$), the contractual time to maturity is two years ($T = 2$), the risk-free interest rate is 5% ($r = 0.05$), the call price is constant at 110 ($K = 110$), the put price is constant at 98 ($P = 98$), the nominal value is 100 ($F = 100$), the convertible bond pays no coupons ($c = 0.0$), the conversion ratio is one ($n = 1$), and 100 exercise opportunities are assumed. Pricing by simulation is performed with 4000 simulation paths ($N = 4000$) and 100 discretization steps which correspond to the 100 early-exercise opportunities. Both the call and the put boundary are obtained by interconnecting estimates at three different points in time with hermite polynomials. To make the results comparable, the binomial tree is obtained with 6000 steps but only 100 exercise opportunities.

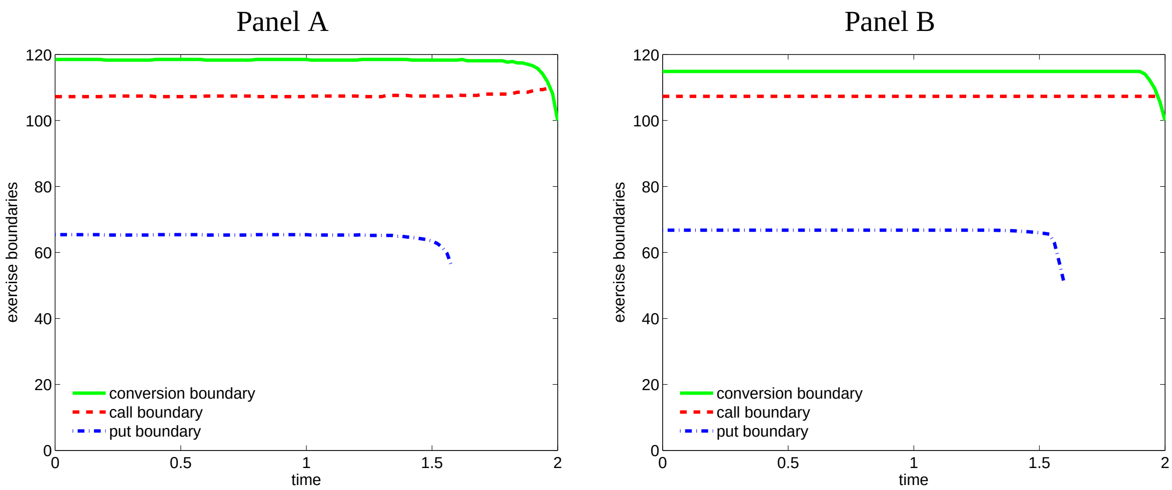


Table 2
Ratings of U.S. convertible bonds

Ratings according to Standard & Poor's Bond Guide of February 2002 of all U.S. convertible bonds as listed in the Mace Advisers' database. "+" and "-" denote rating differences within one class, for example AA+ or AA- for the AA class, as defined by Standard & Poor's.

Rating	"+"	"plain"	"-"	total	total (%)
AAA	-	3	0	3	1.60
AA	0	0	2	2	1.06
A	6	3	7	16	8.51
BBB	7	13	20	40	21.28
BB	12	1	5	18	9.57
B	16	23	27	66	35.11
CCC	14	12	7	33	17.55
CC	0	3	0	3	1.60
C	0	1	0	1	0.53
D	0	6	-	6	3.19
				188	100.00

3. Convertible Bond Data

We choose to investigate the U.S. domestic market because it is the largest market[4], it has a high ratio of rated issues[5], and we obtained accurate daily data from Mace Advisors. As can be seen from Table 2, all rating categories are represented in the U.S. market and 32.45% of the rated issues are investment-grade bonds. On February 12, 2002, the average maturity at issuance of an outstanding U.S. convertible bond was 11.5 years while the average time to maturity was 8.5 years. 25 out of the 588 convertible bonds in our data set have a maturity at issuance larger than 30 years. The bond issued with the shortest maturity is Coeur D'Alene with a time to maturity at issuance of 2.4 years and an extraordinarily high coupon of 13.375%. Only two convertibles out of 588 are not callable, 149 are putable, and in 92 cases callability is restricted by a trigger condition.

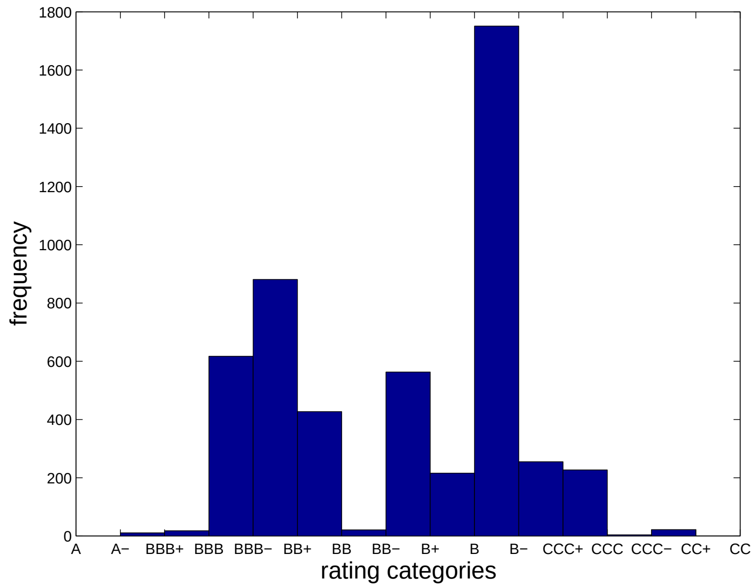
We consider for our analysis all domestic convertible bonds on the U.S. market outstanding as of February 12, 2002. Daily convertible-bond prices as well as corresponding synchronous stock prices

[4]On February 12, 2002, Mace Advisors had a data coverage of 588 convertible-bond issues with an average size of 379.6 million dollars.

[5]188 issues out of the 588 convertible bonds in our data set have a rating in the Standard & Poor's Bond Guide of February, 2002. This is indicative of a rating coverage of more than 32%. In fact, the actual ratio of rated issues is likely to be higher than 32% because some convertibles may be rated by other rating agencies (e.g. Moody's) and not by Standard & Poor's.

Figure 2. Ratings of U.S. convertible bonds

This histogram splits the total number of pricing points of the sample into different classes according to the rating of the corresponding convertible bond at that time. The rating information is obtained from Standard & Poor's Bond Guide.



were made available by Mace Advisers. Convertible bonds with embedded cross-currency features are excluded. To estimate the parameters of the stock dynamics, only convertible bonds with a pre-sample stock history dating back at least until January 1, 1990, are included in the sample. Furthermore, we require for all convertible bonds in the sample that a rating be provided by Standard & Poor's Bond Guide, and - to be able to account for all relevant specifications for each convertible bond in detail - that the official and legally binding *offering circulars* be available. The latter proved to be necessary because some contractual provisions are so specific that they can hardly be collected in predefined data types, and electronic databases usually lack the needed flexibility to incorporate non-standard features. Rating changes for the single issues were followed up according to the monthly publications in Standard & Poor's Bond Guide. To account for a possible publication-lag and additional potential delays of rating adjustments by the rating agencies, we apply a filter that eliminates forty data points preceding rating changes that lead to a credit-spread change of at least 2 percentage points. As an additional liquidity requirement, we only consider data points with a bid-ask spread of less than 2 percentage points for both the convertible bond and the underlying stock.

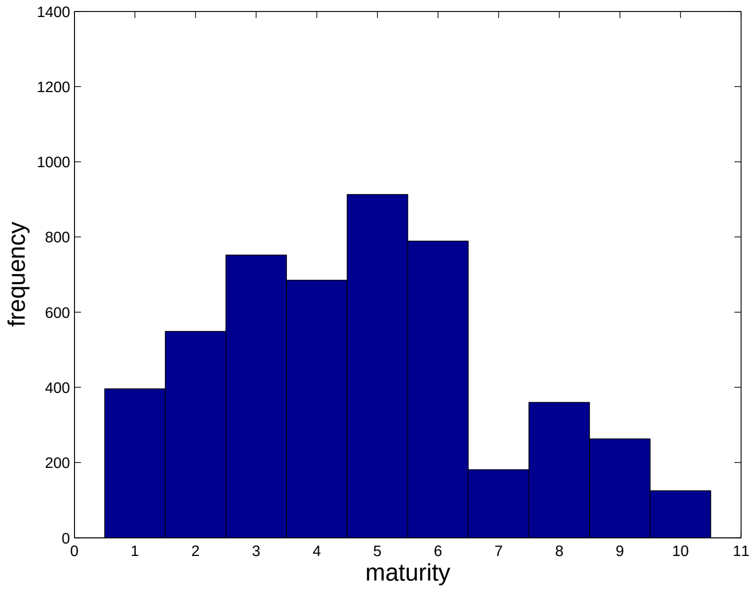
Table 3
Provisions of the convertible bonds in the sample

This table gives an overview of the analyzed convertible bonds with *convertible bond* referring to the name of the issuing firm, *date of issue*, *coupon* as percentage of the face value, and *maturity*. *Size* indicates the amount outstanding in million dollars as reported by Standard & Poor's Bond Guide. *Call* indicates whether the bond is redeemable at the option of the issuing company at any time prior to maturity during the period considered in this study. *Trigger* indicates the existence of an additional trigger condition to be satisfied in order to call the convertible. *Call notice period* indicates the number of days in advance the issuing company has to notify the investor before a call becomes effective. More often than not, the contractual provision specified in the legally binding offering circular states that upon call accrued interests are paid to the investor. *Rating* represents the Standard & Poor's Bond Guide rating as of February 2002.

Convertible bond	Date of issue	Coupon	Maturity	Size	Call	Trigger	Call Notice Period	Accrued interest paid at call	Rating
Adaptec	28-Jan-97	4.75%	01-Feb-04	230	Yes	No	15	Yes	B-
Alpharma	25-Mar-98	5.75%	01-Apr-05	125	Yes	No	30	Yes	B
Analog Dev.	26-Sep-00	4.75%	01-Oct-05	1200	Yes	No	30	Yes	BBB
Charming S.	17-Jul-96	7.50%	15-Jul-06	138	Yes	No	30	Yes	B
CKE Rest.	09-Mar-98	4.25%	15-Mar-04	159	Yes	No	30	Yes	CCC
Clear C. C. I	25-Mar-98	2.63%	01-Apr-03	575	Yes	No	15	Yes	BBB-
Clear C. C. II	17-Nov-99	1.50%	01-Dec-02	900	No	No	30	No	BBB-
Corning/Oak	20-Feb-98	4.88%	01-Mar-08	100	Yes	No	30	Yes	BBB-
Cypress S.	21-Jun-00	3.75%	01-Jul-05	250	Yes	No	20	Yes	B
Genesco	06-Apr-98	5.50%	15-Apr-05	104	Yes	No	30	Yes	B
Healthsouth	17-Mar-98	3.25%	01-Apr-03	443	Yes	No	30	Yes	BB+
Hexcel	18-Jul-96	7.00%	01-Ago-03	114	Yes	No	20	Yes	CCC+
Hilton H.	09-May-96	5.00%	15-May-06	494	Yes	No	30	Yes	BB+
Interpubl. G.	26-May-99	1.87%	01-Jun-06	361	Yes	No	30	No	BBB
Kerr McGee	21-Jan-00	5.25%	15-Feb-10	550	Yes	No	30	Yes	BBB-
Kulicke & S.	08-Dec-99	4.75%	15-Dec-06	175	Yes	No	30	Yes	B-
LAM R.	19-Aug-97	5.00%	01-Sep-02	310	Yes	Yes	20	No	B
LSI Logic	16-Mar-99	4.25%	15-Mar-04	345	Yes	No	30	Yes	B
NABI	02-Feb-96	6.50%	01-Feb-03	80.5	Yes	No	20	Yes	CCC-
Offshore L.	11-Dec-06	6.00%	15-Dec-03	80	Yes	No	30	Yes	B+
Omnicare	04-Dec-97	5.00%	01-Dec-07	345	Yes	No	30	Yes	BB+
Parker Drill.	21-Jul-97	5.50%	01-Aug-04	124	Yes	No	30	Yes	B-
Penn T. A.	20-Nov-96	6.25%	01-Dec-03	74.8	Yes	No	15	Yes	CC
Photronics	22-May-97	6.00%	01-Jun-04	103	Yes	No	20	Yes	B
Pogo Prod.	11-Jun-96	5.50%	15-Jun-06	115	Yes	No	30	Yes	BB
Providian F.	17-Aug-00	3.25%	15-Aug-05	402	Yes	No	30	Yes	B
Rite Aid	04-Sep-97	5.25%	15-Sep-02	650	Yes	No	30	Yes	CCC+
Safeguard S.	03-Jun-99	5.00%	15-Jun-06	200	Yes	No	20	Yes	CCC
Semtech	03-Feb-00	4.50%	01-Feb-07	400	Yes	No	30	Yes	CCC+
Service C.	18-Jun-01	6.75%	22-Jun-08	300	Yes	No	30	Yes	B
Silicon G.	07-Aug-97	5.25%	01-Sep-04	231	Yes	Yes	30	Yes	CCC-
St. Motor Pr.	20-Jul-99	6.75%	15-Jul-09	90	Yes	No	30	Yes	B+

Figure 3. Histogram of pricing points by maturity

This histogram splits the total number of pricing points of the sample into different classes according to the maturity of the convertible bond. Maturity (x-axis) is expressed in years and the frequency (y-axis) indicates the absolute number of pricing points for each maturity class. A maturity class of n covers pricing points with a time-to-maturity ranging from $n-0.5$ years to $n+0.5$ years.



After filtering the sample with these criteria, we obtain a final sample of 32 convertible bonds, with price data ranging from May 10, 1996, to February 12, 2002. As shown in Table 3, most analyzed convertibles include a call option, allowing the issuer to repurchase the bond for a certain price K_t, called *call price* or *early redemption price*. When a convertible bond is called, the issuer has to notify the investor a certain period in advance about his intention to call the convertible. This provision bears some risk for the issuer in form of a failed forced conversion, in which case the issuer will have to redeem the bond in cash instead of shares. Thus, the issuer might want to avoid this eventuality by delaying the call. The call notice period in the US market is generally 30 days. However, it sometimes differs across the individual convertible bonds. Usually, the call price varies over time but is piecewise constant. For almost all examined convertibles, early redemption is restricted to a certain predetermined period. The period during which callability is not allowed is called the *call protection period*. An additional restriction to callability in form of a supplementary condition to be satisfied is given by the *call condition*. Callability is only allowed if the parity $n_t S_t$ exceeds a *call trigger* Ξ_t. The exact contractual specification of the call condition often states that the inequality $n_t S_t > \Xi_t$ must hold for a certain time (usually 20 out of the last 30 trading days) before the bond becomes callable. This “qualifying period” introduces a path dependent feature that can be accounted for better by a simulation-based convertible bond pricing method than by a lattice method. The call trigger is calculated as a percentage of either the early redemption price or the face value. If the trigger feature is present, the callability is called *provisional* or *soft*, if it is absent, the callability is called *absolute*, *unconditional*, or *hard*. Usually, the conversion ratio n_t is constant over time. It changes in case of an alteration of the nominal value of the shares (stock subdivisions or consolidations), extraordinary dividend payments and other financial operations that directly affect the stock price. Since stock splits are very common in the US market, the conversion ratio often changes over time and deviates quite substantially from the initial values stated in the offering circulars. To accommodate for this, we apply an equity correction factor and use the adjusted conversion ratio at any time. Conversion is possible within a certain period, called *conversion period*. For all issues in our sample, the end of the conversion period coincides with the maturity of the convertible bond. Some convertibles in the US market are *premium redemption* convertibles, i.e. the redemption at maturity is above par value. In this case, the final redemption is given by κN with the final redemption ratio κ larger than one. However, in our sample, all convertible bonds have a terminal redemption of 1000 dollars and κ is equal to one. Furthermore, while some convertible bonds in the

US market are traded ”dirty”, all bonds in our sample are traded ”clean”, i.e., the total purchase price is the quoted price plus accrued interest.

As depicted in Figure 2, the data points of convertible-bond prices in the sample cover all Standard and Poor’s rating categories ranging from A- to CCC-. The absence of higher investment grade convertible bonds and the presence of lower rated convertibles in our sample reflects the phenomenon that, in the US market, primarily small companies issue convertible bonds while more established companies rely on other means of financing. None of the convertible bonds in our sample actually defaulted during the examination period.

Figure 3 presents the frequency of single convertible bond prices for various maturity classes. While the convertible bonds in the US market have maturities of up to 30 years, the issues in our sample cover maturities ranging from half a year to slightly more than ten years, and have a mean maturity of approximately five years. Thus, convertible bonds belong to the class of derivative instruments with the longest maturities of all, largely surpassing even long term options which seldom reach up to three years.

4. Model Implementation

In this section we describe the model specification used in the empirical analysis and the estimation procedure of the parameters affecting the bond price. We describe the estimation of the underlying stock process, the interest rates, credit spreads, and dividends. All parameters are estimated out-of-sample.

4.1. Stock Dynamics

An important input parameter to be estimated is the volatility of the underlying stock price. This aspect becomes the more relevant the longer the maturity of the derivative to be valued. While research on stock volatility is plentiful, there is no consensus on which model should be applied for forecasting. A popular approach is the implied volatility concept. However, for two reasons, implied volatility is not suitable as input for the forecasting task in this study. First, most liquid options have maturities that are much shorter than maturities of convertible bonds. Second, issuers of convertible bonds are mostly small companies with no traded options outstanding. Third, several studies (e.g. Figlewski, 1997) suggest that implied volatility is not an unbiased estimator of realized volatility and thus should not be used for forecasting. Therefore, other alternatives have to be considered. We focus on possibilities to generate out-of-sample forecasts using volatility models that base on historical price movements.

For three reasons, we model the variance of the stock price with a GARCH(1,1) following Bollerslev (1986) and Duan (1995) instead of a continuous time process. First, the GARCH(1,1) model has proven capabilities of capturing the volatility patterns present in the data, in particular volatility clustering. Second, since simulation is intrinsically discrete, adopting a discrete time process makes discretization techniques redundant. Third, the estimation of the parameters is also naturally performed with discretely sampled data (daily frequency). The conditional variance of the GARCH(1,1) evolves as

$$\sigma_t^2 = w + a\varepsilon_{t-1}^2 + b\sigma_{t-1}^2, \tag{7}$$

where the ε_t are return shocks drawn from a normal distribution with a mean of zero and the respective conditional variance. Under the risk-neutral probability measure $\mathbb{Q}$ stock returns are assumed to depend on the conditional variance, and the dynamics of logarithmic stock returns:

$$ln(S_t/S_{t-1}) \equiv y_t = r - 0.5\sigma_t^2 + \sigma_t\varepsilon_t, \;\; \varepsilon_t \sim N(0,1), \tag{8}$$

where r is the risk-free interest rate.

For the empirical analysis, we calibrate the chosen volatility models with historical data. The parameters ψ are chosen to maximize the likelihood function

$$\ln L\left(\psi; \varepsilon_1, \varepsilon_2, ..., \varepsilon_T\right) = -0.5\ln\left(2\pi\right) - 0.5\sum_{t=1}^{T}\left(\ln\left(\sigma_t^2\right) + \left(\frac{\varepsilon_t^2}{\sigma_t^2}\right)\right). \tag{9}$$

The estimated parameters of the volatility models are presented in Table 4 for each convertible bond in the sample.

Table 4
Parameter estimates for GBM and GARCH(1,1)

σ represents the average of all annualized input volatilities that are used in the implementation of the GBM model and are estimated from as much pre-sample data as available to us (starting at least before 1990). The GARCH(1,1) equation is: $\sigma_t^2 = w + a\varepsilon_{t-1}^2 + b\sigma_{t-1}^2$. For comparison, the column denoted by GBM shows the parameter estimation for geometric Brownian motion.

Convertible bond	Parameters for the underlying process			
	GBM	GARCH(1,1)		
	σ	w	a	b
Adaptec	0.619601	3.07E-05	0.014910	0.965199
Alpharma	0.434762	0.000311	0.257896	0.377281
Analog Dev.	0.524330	5.51E-06	0.03007	0.964677
Charming S.	0.535953	1.05E-05	0.029823	0.960113
CKE Rest.	0.426572	0.00016	0.161979	0.615782
Clear C. C. I	0.357913	4.34E-05	0.097719	0.818231
Clear C. C. II	0.357913	4.34E-05	0.097719	0.818231
Corning/Oak	0.299630	2.61E-05	0.100687	0.829004
Cypress S.	0.538525	9.85E-05	0.077597	0.831327
Genesco	0.600186	0.000143	0.095389	0.805274
Healthsouth	0.429999	7.25E-05	0.089714	0.807925
Hexcel	0.471715	5.48E-06	0.008321	0.975708
Hilton H.	0.352844	1.71E-05	0.069696	0.898264
Interpubl. G.	0.313456	2.14E-06	0.038887	0.955268
Kerr McGee	0.289180	3.42E-06	0.04624	0.943652
Kulicke & S.	0.641227	1.37E-05	0.031839	0.960379
LAM R.	0.596124	0.000095	0.090067	0.841620
LSI Logic	0.542819	0.000164	0.072973	0.782621
NABI	0.964364	4.66E-05	0.070418	0.920182
Offshore L.	0.785796	0.000338	0.098008	0.757633
Omnicare	0.411482	4.04E-06	0.023983	0.970296
Parker Drill.	0.493291	8.76E-05	0.111824	0.798210
Penn T. A.	0.645585	2.31E-05	0.112127	0.880060
Photronics	0.713992	3.22E-05	0.054081	0.931388
Pogo Prod.	0.453902	8.27E-06	0.043163	0.947773
Providian F.	0.292776	6.09E-06	0.058140	0.924539
Rite Aid	0.385032	2.65E-06	0.026851	0.967054
Safeguard S.	0.675231	0.000115	0.007935	0.945687
Semtech	0.755131	9.13E-06	0.031023	0.965534
Service C.	0.394883	0.000933	0.025392	0.972872
Silicon G.	0.508525	0.000267	0.142702	0.599636
St. Motor Pr.	0.410191	1.85E-06	0.028817	0.968659

The stock price has to be adjusted for dividend payments. We accommodate for discrete dividends by subtracting them from the stock price at the appropriate dates. For each pricing, we assume that the dividend yield at the last ex-dividend date remains constant and applies until maturity.

4.2. Interest Rates

All interest rate data employed in this study is obtained from the Federal Reserve. The time series of the risk-free interest-rates were extracted from T-Bill and T-Note prices and cover maturities from 3 months to 30 years on a daily basis. We obtain through interpolation the complete term structure of spot rates at any time.

Since the inclusion of stochastic interest rates is associated with additional computational costs, a term structure model is only appropriate if the gain in pricing precision is significant. To investigate the real valuation effects of stochastic interest rates, prices of convertible bonds generated by a pricing model under the assumption of constant interest rates are compared with prices obtained by a model that incorporates a CIR term-structure model. This comparison is performed in Table 5 for several initial stock prices and for different correlation values between the stochastic processes of the two state variables: stock price and interest rate. To keep the example sparse and realistic, both a European-style convertible bond and a convertible bond with embedded call and put options is priced under both a constant and stochastic interest rate. A maturity of two years is chosen for the valuation. In order to use a truly reliable specification of the interest-rate process, we adopt the term-structure parameters in Ait-Sahalia (1996). The parameters are estimated via GMM using seven-day Eurodollar deposit rates with daily frequency from June 1, 1973 to February 25, 1995. The inclusion of stochastic interest rates does not generate important deviations, with percentage price changes always smaller than half a percentage point in absolute terms. In general, the difference between prices obtained with and without an interest rate model is higher for convertible bonds that are at-the-money and where the correlation between stock-price innovations and interest-rate is different from zero. As can be seen from a comparison of Panel A and B, the presence of early-exercisable options further reduces the impact of stochastic interest rates. For correlations close to zero the effect of stochastic interest rates is remarkably low with an impact in the range of a couple of cents. While the results presented in Table 5 are clearly dependent on the specific convertible bonds and interest rate parameters assumed, they confirm results obtained by Brennan and Schwartz (1980) and Buchan (1997). Simulation experiments based on a broader set of convertible bonds confirm the qualitative results presented in this table.

Since the impact of the correlation between the innovations of stock-returns and interest rates is strong, it seems crucial to empirically investigate this parameter. Table 6 provides both daily and weekly empirical correlations (with confidence intervals) for all the issues in our sample. While no daily correlation exceeds 0.2, in four cases, the correlation is estimated to be slightly larger than 0.1. Since these low correlation values may depend on the daily frequency, a look at monthly data can be useful. While for some issues monthly correlations have much higher values (for instance Pogo Prod. with 0.268), no correlation value is statistically different from zero at the 1% confidence level. Even at the 10% confidence level, only 7 out of 32 issues have a correlation that is statistically different from zero. Given the high parameter uncertainty and the low pricing impact these estimated correlations would generate, it is questionable whether using stochastic interest rates is beneficial for our empirical analysis. In fact, also Brennan and Schwartz (1980) argue that, for empirical pricing purposes, stochastic interest rates can be neglected without important losses in accuracy. Hence, although our model would easily incorporate stochastic interest rates, the overall pricing benefit would be very limited and would not justify the additional computational costs. For these reasons we perform the pricing study without stochastic interest rates.

4.3. Credit Risk

We account for credit risk in the spirit of Tsiveriotis and Fernandes (1998) and discount the cash flows subject to credit risk with the appropriate interest rate. This can easily be done since the simulation approach presented in this paper is cash-flow based. Thus, coupon payments[6], the final redemption payment, and the call price in the event of a call are subject to credit risk. The stock price, on the other hand, is not and should therefore be discounted with the risk-free interest rate. In this approach, credit spread can be implemented as constant or as following a process correlated with other state variables.

[6]Most convertible bonds in the US market provide coupon payments. The most popular payment frequency is semiannual. We accommodate for discrete coupon payments at the appropriate dates and with the appropriate frequencies.

Table 5

Pricing impact of stochastic interest rates

This table shows the percentage price impact of a term-structure model on prices of European-style (Panel A) as well as callable and putable convertible bonds (Panel B) for different initial stock prices and for different values of the correlation between stock and interest rate. Different initial stock prices imply different moneyness values for the convertible bonds. Moneyness ranges from 0.24 to 2.37 with corresponding stock prices ranging from S=20 to S=200. The number of paths in each simulation run is 5000, with the same random-number series for each pricing. 'std' refers to the standard deviation of the Monte Carlo estimate. All convertible bonds have a face value $F = 100$, maturity $T = 2$ years, conversion ratio $\gamma = 1.0$, and coupon $c = 0$. The issuing firm pays continuously compounded dividends, $\delta = 0.1$, and is not entitled to call back the convertible bond at any time apart from maturity. The stock price follows a geometric Brownian motion, $\frac{dS_t}{S_t} = (r_t - \delta)dt + \sigma_S dW_{S,t}$, with volatility σ_S=0.4, and the instantaneous interest rate follows a one-factor CIR interest-rate process, $dr_t = \kappa_r(\theta_r - r_t)\,dt + \sigma_r\sqrt{r_t}dW_{r,t}$, with an initial short rate r=0.06, and parameters as estimated via GMM by Ait-Sahalia (1996): θ_r=0.090495, κ_r=0.89218, σ_r=0.180948. The correlations $\rho_{S,r}$ between dW_S and dW_r range from $\rho_{S,r}$=-0.5 to $\rho_{S,r}$=+0.5.

stock price	20	60	80	85	100	120	200
moneyness	0.24	0.71	0.95	1.01	1.19	1.42	2.37
Panel A: European-Style Convertible Bond							
	constant interest rates						
price	84.26	87.37	93.00	94.87	101.39	112.10	167.60
std	0.00	0.19	0.35	0.39	0.52	0.70	1.35
	stochastic interest rates (changes in %)						
-0.5	0.01	-0.08	-0.19	-0.25	-0.28	-0.32	-0.47
-0.2	0.00	-0.01	-0.05	-0.09	-0.10	-0.13	-0.20
0	0.00	0.00	0.00	0.00	0.01	0.01	0.00
0.2	0.01	0.02	0.02	0.01	0.09	0.16	0.23
0.5	0.01	0.15	0.16	0.15	0.28	0.45	0.60
Panel B: Callable and Putable Convertible Bond							
	constant interest rates						
price	98.00	98.00	98.95	100.03	105.51	120.00	200.00
std	0.00	0.00	0.07	0.08	0.09	0.00	0.00
	stochastic interest rates (changes in %)						
-0.5	0.00	0.00	-0.06	-0.12	-0.03	0.00	0.00
-0.2	0.00	0.00	-0.07	-0.07	-0.06	0.00	0.00
0	0.00	0.00	0.00	-0.03	-0.03	0.00	0.00
0.2	0.00	0.00	0.01	-0.03	-0.08	0.00	0.00
0.5	0.00	0.00	0.01	0.03	0.03	0.00	0.00

Table 6
Empirical correlation between stock returns and interest rates

This table reports for each issue in the sample the daily and monthly correlation between stock returns and changes in the interest rate. The table reports point estimates for the correlations as well as the lower (LCI) and higher (HCI) 10% confidence intervals. For obtaining these quantiles we first transform the statistical correlation ρ in the following way: $\widehat{\rho} = \frac{1}{2} ln\left(\frac{1+\rho}{1-\rho}\right)$. $\widehat{\rho}$ is an asymptotically normal correlation measure bounded between $[-\infty, +\infty]$. Second, the quantiles of $\widehat{\rho}$ are calculated as $\widehat{\rho}_{LCI} = \widehat{\rho} - z\left(\frac{\alpha}{2}\right)\frac{1}{\sqrt{n-3}}$ and $\widehat{\rho}_{HCI} = \widehat{\rho} + z\left(\frac{\alpha}{2}\right)\frac{1}{\sqrt{n-3}}$. Third, we re-transform the interval boundaries to ρ_{LCI} and ρ_{HCI}, respectively. For each issue, calculations are performed with all data used in the empirical analysis.

Convertible	daily point estimate	daily LCI	daily HCI	monthly point estimate	monthly LCI	monthly HCI
Adaptec	0.033	-0.009	0.076	0.044	-0.052	0.138
Alpharma	0.048	0.005	0.090	0.061	-0.035	0.155
Analog Dev.	0.052	0.009	0.094	0.020	-0.075	0.115
Charming S.	0.004	-0.038	0.047	0.041	-0.055	0.135
CKE Rest.	0.069	0.021	0.116	0.071	-0.036	0.177
Clear C. C. I	0.086	0.038	0.133	0.069	-0.038	0.175
Clear C. C. II	0.037	-0.006	0.079	-0.016	-0.111	0.079
Corning/Oak	0.061	0.019	0.103	0.024	-0.071	0.119
Cypress S.	0.051	0.009	0.093	0.025	-0.071	0.120
Genesco	0.025	-0.017	0.067	0.076	-0.019	0.170
Healthsouth	0.021	-0.022	0.063	-0.047	-0.142	0.048
Hexcel	0.035	-0.008	0.077	0.026	-0.069	0.121
Hilton H.	0.019	-0.024	0.061	0.029	-0.067	0.124
Interpubl. G.	0.011	-0.031	0.053	-0.020	-0.115	0.076
Kerr McGee	0.025	-0.018	0.067	0.009	-0.086	0.104
Kulicke & S.	0.012	-0.031	0.054	0.027	-0.068	0.122
LAM R.	0.130	0.088	0.172	0.143	0.048	0.235
LSI Logic	0.085	0.033	0.136	-0.009	-0.126	0.107
NABI	0.040	-0.003	0.082	0.094	-0.001	0.187
Offshore L.	0.106	0.064	0.148	0.139	0.044	0.231
Omnicare	0.042	-0.001	0.084	0.106	0.011	0.199
Parker Drill.	0.074	0.032	0.117	0.092	-0.003	0.185
Penn T. A.	0.024	-0.040	0.087	0.017	-0.127	0.159
Photronics	0.024	-0.018	0.066	0.009	-0.086	0.104
Pogo Prod.	0.152	0.083	0.218	0.268	0.118	0.407
Providian F.	0.025	-0.017	0.068	0.119	0.024	0.212
Rite Aid	0.041	-0.001	0.083	0.093	-0.002	0.187
Safeguard S.	0.101	0.030	0.172	0.103	-0.060	0.261
Semtech	0.100	0.058	0.142	0.123	0.028	0.215
Service C.	0.021	-0.026	0.069	-0.019	-0.124	0.087
Silicon G.	0.091	0.049	0.133	0.109	0.014	0.202
St. Motor Pr.	-0.009	-0.051	0.033	-0.008	-0.103	0.087

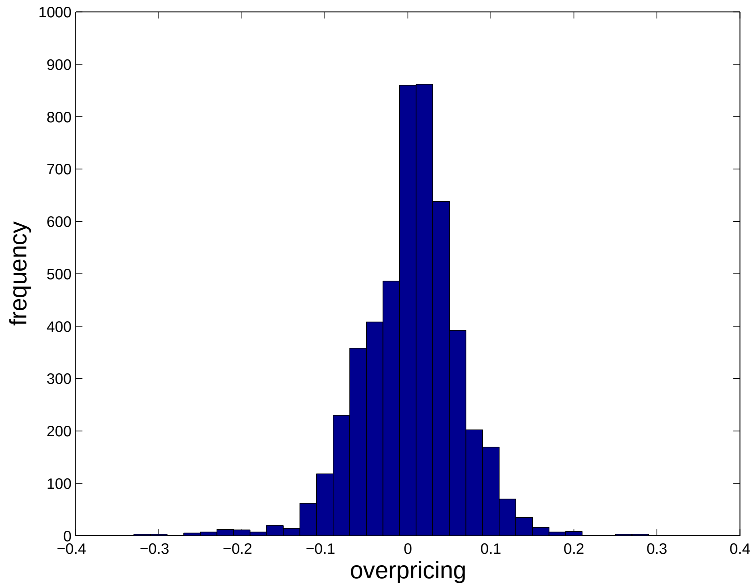
Unfortunately, for most convertible bonds in the sample, there are no straight bonds outstanding - let alone with a maturity corresponding to that of the convertible bond - that could be used to extract the appropriate issue-specific credit spreads for our implementation. In addition, such a procedure to obtain the credit spreads has the drawback that issue-specific characteristics of the convertible bonds, such as seniority, are not accounted for. Thus, to obtain credit spreads, we extract from the Yield Book database monthly time series of credit spreads for several rating categories according to Standard and Poor's Bond Guide. For all investment grade rating categories, we further obtain monthly credit-spread time series covering four maturity classes (1-3, 3-7, 7-10, and over 10 years). While this procedure allows to account for issue-specific convertible bond characteristics through applying the rating, it has several obstacles that potentially could influence the pricing results. First, the credit spreads represent averages of bonds outstanding within the same rating category. Second, ratings change over time. The publication we refer to only has a monthly updating frequency. Additionally, this procedure does not account for potential lags and, more importantly, differences in market valuations and the rating assessment by Standard and Poor's. The resulting estimation error of the credit spreads is potentially very relevant in our sample since it primarily consists of lower rated bonds with higher credit spreads.

5. Empirical Analysis of the US Convertible Bond Market

In this section, convertible-bond prices observed in the US market are compared with theoretical prices obtained using the proposed simulation-based model. Figure 4 presents the distribution of percentage deviations between model prices and empirical prices. On average, market prices are 0.36% higher than model prices, with a standard deviation of 6.17%. This result stands in contrast to some previous studies that use different pricing approaches and smaller data samples. In those studies, model prices are higher than market prices on average. Moreover, those studies have in common a mean price deviation between model and market prices that is substantially larger than 0.36%. King (1986) investigates a sample of 103 American convertible bonds and finds that market prices are 3.75% below model prices on average. Carayannopoulos (1996) obtains for 30 US convertible bonds and one year of monthly price data an even larger price deviation, with market prices lower than model prices by 12.9% on

Figure 4. Distribution of Pricing Deviation

This histogram splits the total number of pricing points of our sample into different classes. Overpricing (x-axis) denotes the relative pricing error (market price / model price -1). Frequency (y-axis) indicates the absolute number of pricing points in each class.



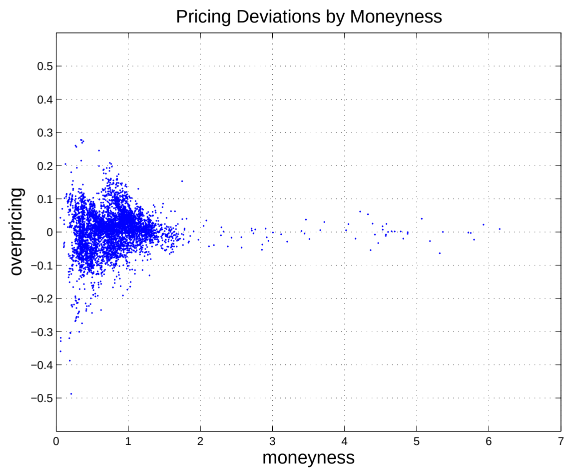
average. Ammann et al. (2003) investigate 21 French convertible bonds and report that market prices are on average 3.24% lower than model prices.

Table 7 and Figure 5 show the percentage price deviation between each daily observed market price and the theoretical fair values as generated by our model in relation to two important characteristics of the convertible bond: moneyness and credit rating. Panel A of Figure 5 plots these daily price deviations with respect to the moneyness of the convertible bond, calculated as the ratio between conversion value and investment value. The investment value is defined as the value of the convertible bond under the hypothetical assumption that the conversion option does not exist and that the credit spread is zero. The latter proves to be useful because the credit spread is potentially subject to an estimation error, as we do not observe issue-specific credit spreads but infer them from issues with the same rating. Thus, taking into account credit risk when calculating the investment value would lead to incorrect moneyness values. However, since disregarding credit risk leads to moneyness values that are slightly downward-biased, we should imagine at-the-money convertibles to have a moneyness of less than one in Panel A of Figure 5. The results in Table 7 suggest that the error dispersion decreases with the moneyness. This

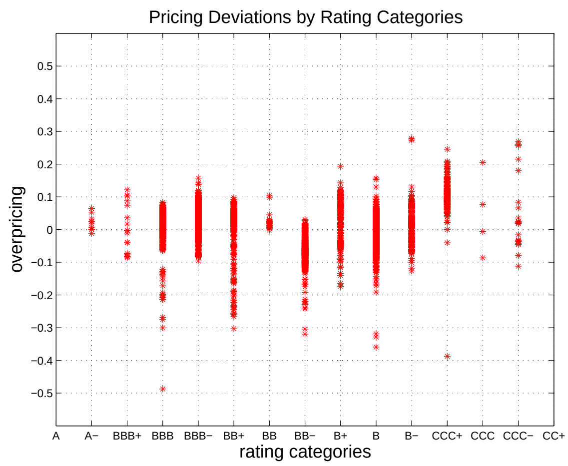
Figure 5. Pricing deviation by moneyness and rating classes

This graph shows the percentage price deviation between each daily observed market price in the sample and its corresponding theoretical fair value as generated by the simulation-based method plotted against the moneyness level of the convertible bond. Moneyness is calculated by dividing the conversion value by the investment value. The conversion value is the value of shares that can be obtained by converting the bond. The investment value denotes the value of the convertible bond under the hypothetical assumptions that the conversion option does not exist and default never occurs. The rating is attributed to each convertible bond according to Standard & Poor's Bond Guide. The data in the sample cover rating categories (x-axis) ranging from A- to CCC-. Overpricing (y-axis) denotes the relative pricing error (market price / model price -1).

Panel A



Panel B



result can be explained theoretically because, for deep in-the-money convertibles, the probability of conversion is very high, the time value of the conversion option becomes very small, and the convertible presents less pricing challenges. The large error dispersion for at-the-money convertibles is likely to reflect the difficulties in pricing the option part of a convertible bond, the value of which is particularly large for at-the-money-bonds. For deep out-of-the-money convertibles, the likelihood of exercising the conversion option is near to zero and so is the value of the conversion option. Pricing a deep out-of-the-money convertible is very similar to pricing its straight bond equivalent. We attribute the large error dispersion of out-of-the-money convertibles to difficulties in determining the appropriate credit spread.

Panel B of Figure 5 and Table 7 show the mean relative price deviation and its dispersion for different rating categories. Our sample consists of rating categories ranging from A- to CCC-, but the large majority of data points falls into the range from BBB to CCC+. For both investment-grade bonds and non-investment grade bonds, the two classes with the largest number of observations, BBB and B, have relatively small average pricing errors of 0.42% and -0.32%, respectively. Only CCC+ bonds have a substantially higher average error (9.9%), which is very likely attributable to the necessary approximations in credit-risk measurement. With respect to the error dispersion, it is surprising that there is no clear relationship between the standard deviation of pricing errors and the rating quality. The rather high dispersion for bonds rated CCC and CCC- should be interpreted with caution given the limited number of data points in these rating classes.

To sum up, credit spread accounts for a portion of the observed price dispersions, in particular for out-of-the money convertibles. However, this error dispersion is not larger for bonds with a low rating than it is for investment grade issues.

In Table 8, the relative mispricings are presented for the individual issues in the sample. Out of the 32 issues in the sample, 21 present higher average market prices than model prices. While for sixteen of them, the mispricing is statistically significant at the one percent level, for the other five issues mispricing is not different from zero at the ten percent level. For two out of the eleven issues with on average lower market prices than model prices, the deviation is not statistically significant at the ten percent level.

To test whether the results are biased by certain input parameters or incorrect model specifications, we regress the relative pricing deviation generated by the model on a catalog of potential error sources.

Table 7

Pricing deviation by moneyness and rating

Data points indicates the number of days for which model prices are computed. *Mean pricing deviation* states the extent to which market prices are, on average, above model prices for a given moneyness or rating class. ***, **, and * denote significance levels of 1%, 5%, and 10%, respectively, for the rejection of the null hypothesis that model and market prices are equal in the mean. *Deviation std.* is the standard deviation of observations in the respective class. *Prob.* is the probability that refers to a two-sided test for the null hypothesis that model prices and observed prices are equal in the mean. *RMSE* is the root mean squared error, i.e. the non-central standard deviation of the relative deviation of model prices from market prices.

	Data points	Mean pricing dev.	Dev. std.	Prob. values	RMSE
		Panel A: Classes by Moneyness			
Moneyness					
< 0.50	1242	-0.0156***	0.0822	0.0000	0.0836
$0.50 < 0.80$	1454	0.0008	0.0588	0.6144	0.0588
$0.80 < 0.95$	866	0.0225***	0.0564	0.0000	0.0607
$0.95 < 1.05$	516	0.0229***	0.0442	0.0000	0.0497
$1.05 < 1.20$	447	0.0103***	0.0338	0.0000	0.0353
$1.20 < 2.00$	429	0.0012	0.0292	0.4025	0.0292
> 2.00	59	-0.0032	0.0258	0.3440	0.0258
		Panel B: Classes by Rating			
Rating					
A-	11	0.0183**	0.0244	0.0129	0.0296
BBB+	18	0.0009	0.0742	0.9594	0.0721
BBB	617	0.0042**	0.0525	0.0466	0.0526
BBB-	881	0.0219***	0.0438	0.0000	0.0490
BB+	427	0.0089**	0.0774	0.0177	0.0778
BB	21	0.0228***	0.0256	0.0000	0.0339
BB-	563	-0.0670***	0.0456	0.0000	0.0810
B+	216	0.0353***	0.0715	0.0000	0.0796
B	1751	-0.0032***	0.0432	0.0022	0.0434
B-	255	0.0155***	0.0591	0.0000	0.0610
CCC+	227	0.0980***	0.0584	0.0000	0.1140
CCC	4	0.0084	0.1394	0.9039	0.1210
CCC-	22	0.0440*	0.1112	0.0636	0.1172
Total sample	5013	0.0036***	0.0617	0.0000	0.0618

Table 8

Pricing deviation by issue

Data points indicates the number of days for which model prices are computed. *Mean pricing deviation* states the extent to which market prices are, on average, above model prices for a given issue. ***, **, and * denote significance levels of 1%, 5%, and 10%, respectively, for the rejection of the null hypothesis that model and market prices are equal in the mean. *Deviation std.* is the standard deviation of observations for each issue. *Probability value* is the probability that refers to a two-sided test for the null hypothesis that model prices and observed prices are equal in the mean. *RMSE* is the root mean squared error, i.e. the non-central standard deviation of the relative deviation of model prices from market prices.

Convertible bond	Data points	Mean pricing dev.	Dev. std.	Prob. values	RMSE
Adaptec	545	-0.0616***	0.0528	0.0000	0.0811
Alpharma	296	0.0045***	0.0249	0.0020	0.0253
Analog Dev.	39	-0.0139***	0.0150	0.0000	0.0203
Charming S.	83	0.0182***	0.0405	0.0000	0.0442
CKE Rest.	248	0.0002	0.0288	0.9304	0.0288
Clear C. C. I	240	0.0215***	0.0233	0.0000	0.0316
Clear C. C. II	144	0.0009	0.0363	0.7619	0.0362
Corning/Oak	22	0.0336***	0.0424	0.0002	0.0534
Cypress S.	124	0.0788***	0.0340	0.0000	0.0857
Genesco	46	0.0397***	0.0301	0.0000	0.0496
Healthsouth	83	-0.0477***	0.0292	0.0000	0.0558
Hexcel	32	0.0225***	0.0426	0.0029	0.0476
Hilton H.	616	0.0260***	0.0229	0.0000	0.0346
Interpubl. G.	46	-0.0543***	0.0152	0.0000	0.0563
Kerr McGee	227	0.0722***	0.0243	0.0000	0.0761
Kulicke & S.	71	-0.0511***	0.0216	0.0000	0.0554
LAM R.	657	-0.0075***	0.0349	0.0000	0.0357
LSI Logic	169	0.0230***	0.0281	0.0000	0.0362
NABI	18	0.0442	0.1293	0.1470	0.1332
Offshore L.	79	-0.0393***	0.0472	0.0000	0.0612
Omnicare	111	0.0401***	0.0431	0.0000	0.0587
Parker Drill.	66	0.0612***	0.0347	0.0000	0.0702
Penn T. A.	65	-0.1280***	0.0746	0.0000	0.1479
Photronics	257	-0.0334***	0.0605	0.0000	0.0690
Pogo Prod.	43	0.0222***	0.0393	0.0002	0.0448
Providian F.	91	-0.0161	0.1171	0.1907	0.1175
Rite Aid	266	-0.0057	0.0789	0.2379	0.0789
Safeguard S.	2	0.1136	0.0982	0.1019	0.1332
Semtech	187	0.1011***	0.0521	0.0000	0.1136
Service C.	9	0.0173	0.0440	0.2391	0.0449
Silicon G.	122	0.0166***	0.0257	0.0000	0.0306
St. Motor Pr.	9	0.1019***	0.0445	0.0000	0.1102
Total sample	5013	0.0036***	0.0617	0.0000	0.0618

We perform the regressions separately for each factor as well as jointly in a multi-factor model, as the correlation coefficients between the regressors are low.

Table 9 shows the results of the cross-sectional regressions. We observe that all coefficients are significant indicating that each of them can explain a portion of the pricing error. The dividend yield has a positive impact on the pricing error. For an increase in the dividend yield of 100 basis points, the pricing error increases on average by 124 basis points in the single-factor regression. The positive impact of the dividend yield is perhaps caused by mean-reverting expectations for dividend yields, which is not taken into account by our model. We assume constant dividend yields. Therefore, if dividend yields are mean-reverting, we overestimate future dividend yields if dividends are high and underestimate future dividends if dividends are low.

The coefficient of the credit spread is 0.50 and highly significant. Moreover, the R-squared of the credit spread, at a value of 0.140, explains substantially more of the error variance than any of the other variables. As the results in Table 7 suggest, the distortional impact of credit spread in our sample is mainly concentrated on CCC+ bonds. Nevertheless, a certain bias due to credit-risk measurement is not surprising as issue-specific credit spreads are inferred from industry-average credit spreads of the corresponding rating category. Apparently, this approximation introduces a slight pricing bias, especially for CCC+ rated bonds. A potential improvement of the pricing precision might be achieved by extracting credit information from market prices of bonds of the same issuer and similar characteristics (seniority, maturity, coupon, etc.). Such data requirements, however, are difficult to satisfy because most firms do not have publicly traded straight debt issues outstanding.

The coefficient of maturity is 0.35. Discounting bonds with long maturities has a stronger effect on the price of the bond, and therefore, discounting errors have a stronger impact on the pricing errors. This is consistent with the positive coefficient for the credit spread, as estimation biases from the credit spreads is amplified by longer maturities. Additionally, we also observe a negative coefficient for the coupon, although it is not significant at the one percent level. The coupon reduces the duration of the bond and therefore again the impact of discounting on the price. Finally, the coefficient for moneyness is positive but small, indicating that moneyness has only a limited systematic effect on pricing errors. This confirms the findings in Table 7 but stands in contrast to the results in Ammann et al. (2003) and Carayannopoulos and Kalimipalli (2003). Surprisingly, these authors report that in their samples,

Table 9
Cross-sectional analysis

In this table, the percentage pricing deviations (market price / model price -1) are regressed against some input variables affecting the value of the convertible bond: the *dividend yield*, the *coupon*, the *risk-free interest rate*, the *credit spread*, the *maturity* in years, and the *moneyness* as the ratio of the conversion value and the investment value. t-values from testing the coefficients for difference from zero are given in parentheses.

Constant Term	Dividend Yield (%)	Coupon (%)	Credit Spread (%)	Maturity (years)	Moneyness (%)	Adjusted R-squared
0.00	1.25	-	-	-	-	0.027
(0.03)	(11.86)					
1.47	-	-0.19	-	-	-	0.001
(3.76)		(-2.42)				
-2.09	-	-	0.50	-	-	0.140
(-17.13)			(28.56)			
-1.10	-	-	-	0.35	-	0.017
(-5.61)				(9.34)		
-0.30	-	-	-	-	1.06	0.007
(-1.79)					(5.89)	
-2.22	1.92	-1.16	0.73	0.57	0.01	0.286
(-6.22)	(17.78)	(-15.60)	(41.42)	(14.00)	(8.31)	

observed market prices of in-the-money (out-of-the-money) convertible bonds tend to be higher (lower) than prices generated by their theoretical model.

With the exception of the credit spread, the explanatory power as measured by R-squared is small. In the multi-factor regression, while the magnitude of the coefficients varies, their signs are unchanged compared to the single-factor regression. [7] The adjusted R-squared is 0.286, indicating that some systematic errors exist, perhaps caused by estimation error or approximations such as the extraction of the credit spread from ratings. However, the mean pricing accuracy achieved in this study is higher than in previous studies, as discussed in the beginning of this section.

[7] We also estimated the model using orthogonalized regressors. With the exception of the coefficient for the coupon, which is lower, the coefficients are of similar magnitude and are therefore not reported

6. Conclusion

We propose a simulation-based pricing method for convertible bonds. Extending existing approaches, the method is able to account for complex real-world convertible-bond characteristics such as embedded call features with various path-dependent trigger conditions. The method uses parametric representations of the early exercise decisions and consists of two stages aimed at reducing the Monte Carlo pricing bias. Pricing convertible bonds with Monte Carlo Simulation is more flexible than previous lattice-based methods because it permits to implement more accurate dynamics for the stock price and to capture the contractual specifications of actually traded convertible bonds.

We implement the model and undertake the so far most extensive empirical pricing study for the US convertible bond market, covering daily prices for an entire period of 69 months. We find that theoretical values for the analyzed convertible bonds are on average 0.36% lower than observed market prices, with a RMSE of 6.8%. A partition of the sample according to the moneyness indicates that pricing accuracy, measured by the standard deviation of the pricing error or RMSE, is rather high for in-the-money convertibles while it is lower for at- and out-of-the-money bonds. Whereas we still observe some systematic pricing biases, mostly caused by the credit-spread estimation, the average pricing errors obtained with the proposed simulation-based approach are smaller than those reported in previous studies. In particular the average overvaluation (model prices higher than market prices) and the positive relationship between overvaluation and moneyness found in previous articles are not confirmed in this study.

Appendix A. Numerical Implementation

This appendix addresses specific issues related to the numerical implementation of the proposed pricing model. We implement all the optimization-based pricing routines in C and use, as source for normally distributed random numbers, the *Box-Muller* method. Correlated random numbers are obtained by Cholesky decomposition. Equally distributed random deviates are generated by the linear congruential generator proposed by Park and Miller (1988) as described in Press et al. (1992). For the purpose of comparison, the random number generator of L'Ecuyer (1988) was implemented as well, but no effect on the results could be noticed. Each pricing point within one model run is computed with a different starting point of the random number sequence (seed). In order to compare the results of different pricing runs with different model specifications, the seed attributed to one pricing point (one convertible bond at one specific date) is held constant across these pricing runs.

For the optimization task needed in the first stage of the simulation method, i.e. maximizing or minimizing the value of the convertible bond given a simulation set for the state variables, we employ a variant of a minimization method originally proposed by Nelder and Mead (1965) and described in Press et al. (1992). This method is based on a simplex, which is a geometric figure consisting of $N+1$ vertices (with all interconnected segments) in an N-dimensional space. This minimization technique is particularly convenient because it is a self-contained method that requires only function evaluations but no derivatives. Once $N+1$ initial points are defined, the function to be minimized is evaluated at each vertex of the simplex and subsequently transformed following several standard geometric iterations. The point with the highest functional value may be reflected through the opposite face of the simplex, or may be reflected and projected farther. Alternatively, the simplex can be contracted on one or more of its vertices. If none of the transformations results in a decrease of the convertible-bond value larger than a predefined tolerance, the procedure is terminated. Thus, the simplex is iterated until any additional change of the conversion (call) boundary cannot increase (decrease) the value of the convertible bond by an amount larger than a tolerance of 0.1. To check the validity of the minimization, the simplex procedure is restarted with one point corresponding to the previously found minimum and representing an N-dimensional vector $\mathbf{Z}_0$. The other N initial vertices are calculated by adding a fixed value α in each dimension of the space to $\mathbf{Z}_0$:

$$\mathbf{Z}_i = \mathbf{Z}_0 + \alpha \mathbf{e}_i,$$

where $\mathbf{e}_i$'s are N orthogonal unit vectors.

As mentioned in Section 2.2, the exercise rule for any of the options embedded in the convertible bonds is numerically modeled in form of a parametric function $G(t;\theta)$ that defines the exercise boundary and delimits the exercise region. The function $G(t;\theta)$ is defined through a tuple of threshold points $(\theta_0, \theta_1, ..., \theta_K)$ in such a way that each θ_k refers to the critical stock price for the option-exercise decision at a different date. θ_0 refers to the first possible exercise date and θ_K refers to the last possible exercise date ($\mathcal{T}$). Since the most important variations in the shape of the exercise boundaries occur closest to maturity, we choose to concentrate the majority of threshold points in this region. More specifically, each intermediate θ_k ($k = 1, ..., K-1$) refers to date $t = \mathcal{T} \times (2 \times (2^k - 1))/(2 \times 2^k)$. Usually, $\mathcal{T}$ is equal to the maturity of the instrument, T. However, in certain cases, it is possible to rule out early exercise after a given date. This is for instance the case when the put price is less than the principal. Since at maturity the investor will get at least the principal, no exercise will happen as long as the discounted principal is higher than the put price. For the empirical analysis K is chosen to be equal to ten. The threshold applied to each exercise date between two threshold points is determined by cubic Hermite interpolation. This approach has the advantage of allowing the American-style conversion option to be applied to every time step, which in our setting is one day. Consequently, even a limited number of parameters for representing the exercise strategies still allows for early exercise at every time step. Although the choice of the parametric representation of the exercise boundary might appear somewhat arbitrary, the numerical results are found to be surprisingly robust to changes in the parametric form of the chosen function.

References

[1] Ait-Sahalia, Y. (1996): ”Nonparametric pricing of interest rate derivative securities”, *Econometrica*, 64 (3), 527-560.

[2] Ammann, M., Kind, A., and Wilde, C. (2003): ”Are convertible bonds underpriced? An analysis of the French market”, *Journal of Banking and Finance*, 27 (4), 87-105.

[3] Andersen, L. (2000): ”A simple approach to the pricing of Bermudan swaptions in the multifactor LIBOR market model”, *Journal of Computational Finance*, 3 (2), 5-32.

[4] Andersen, L., Buffum, D. (2004): ”Calibration and implementation of convertible bond models”, *Journal of Computational Finance*, 7 (2), 1-34.

[5] Avramidis, A., Hyden, P. (1999): ”Efficiency improvements for pricing American options with stochastic mesh”, Proceedings of the winter simulation conference, ACM Press, New York.

[6] Ayache, E., Forsyth, P.A., Vetzal, K.R. (2003): ”Valuation of convertible bonds with credit risk”, *Journal of Derivatives*, 11 (1), 9-29.

[7] Bardhan, I., Bergier, A., Derman, E., Dosemblet, C., Kani, I., Karasinski, P. (1993): ”Valuing convertible bonds as derivatives”, *Goldman Sachs Quantitative Strategies Research Notes*, July.

[8] Barone-Adesi, G., Bermudez, A., Hatgioannides, J. (2003): ”Two-factor convertible bonds valuation using the method of characteristics/finite elements”, *Journal of Economic Dynamics and Control*, 27, 1801-1831.

[9] Barraquand, J., Martineau, D. (1995): ”Numerical valuation of high dimensional multivariate American securities”, *Journal of Financial and Quantitative Analysis*, 30, 383-405.

[10] Bollerslev, T. (1986): ”Generalized Autoregressive Conditional Heteroskedasticity”, *Journal of Econometrics*, 31, 307-327.

[11] Bossaerts, P. (1989): ”Simulation estimators of optimal early exercise”, *Working Paper*, Carnegie Mellon University.

[12] Boyle, P., Broadie, M., Glasserman, P. (1997): ”Monte Carlo methods for security pricing”, *Journal of Economic Dynamics and Control*, 21, 1267-1321.

[13] Brennan, M.J., Schwartz, E.S. (1977): ”Convertible bonds: Valuation and optimal strategies for call and conversion”, *Journal of Finance*, 32 (5), 1699-1715.

[14] Brennan, M.J., Schwartz, E.S. (1980): ”Analyzing convertible bonds”, *Journal of Financial and Quantitative Analysis*, 15 (4), 907-929.

[15] Broadie, M., Cao, M. (2003): ”Improved Monte Carlo methods for tighter price bounds of American options”, *Working Paper*, Columbia University.

[16] Broadie, M., Detemple, J., Ghysels, E., Torres, O. (2000): ”American options with stochastic dividends and volatility: A non-parametric investigation”, *Journal of Econometrics*, 94, 53-92.

[17] Broadie, M., Glasserman, P. (1997a): ”Monte Carlo methods for pricing high-dimensional American options: An overview”, *NetExposure*, December, 15-37, 1997.

[18] Broadie, M., Glasserman, P. (1997b): ”Pricing American-style securities using simulation”, *Journal of Economic Dynamics and Control*, 21, 1323-1352.

[19] Broadie, M., Glasserman, P. (2004): ”A stochastic mesh method for pricing high-dimensional American options”, *Journal of Computational Science*, 7, 35-72.

[20] Broadie, M., Glasserman, P., Jain, G. (1997): ”Enhanced Monte Carlo estimates for American option prices”, *The Journal of Derivatives*, 5, 25-44.

[21] Buchan, M.J. (1997): ”Convertible bond pricing: Theory and evidence”, *Unpublished Dissertation*, Harvard University.

[22] Buchan, M.J. (1998): ”The pricing of convertible bonds with stochastic term structures and corporate default risk”, *Working Paper*, Amos Tuck School of Business, Dartmouth College.

[23] Bühler, W., Koziol C. (2002): ”Valuation of convertible bonds with sequential conversion”, *Schmalenbach Business Review*, 54, 302-334.

[24] Carayannopoulos, P. (1996): ”Valuing convertible bonds under the assumption of stochastic interest rates: An empirical investigation”, *Quarterly Journal of Business and Economics*, 35 (3), 17-31.

[25] Carayannopoulos, P., Kalimipalli, M. (2003): ”Convertible bond prices and inherent biases”, *Journal of Fixed Income*, 13, 64-73.

[27] Carrière, J.F. (1996): ”Valuation of early-exercise price of options using simulations and nonparametric regression”, *Insurance Mathematics and Economics*, 19, 19-30.

[27] Clément, E., Lamberton, D., Protter, P. (2002): ”An analysis of a least squares regression method for American option pricing”, *Finance and Stochastics*, 6, 449-471.

[28] Cox, J.C., Ingersoll, J.E., Ross, S.A. (1980): ”An analysis of variable rate loan contracts”, *Journal of Finance*, 35 (2), 389-403.

[29] Cox, J.C., Ingersoll, J.E., Ross, S.A. (1985): ”A theory of the term structure of interest rates”, *Econometrica*, 53, 385-407.

[30] Davis, M., Lishka, F. (1999): ”Convertible bonds with market risk and credit risk”, *Working Paper*, Tokyo-Mitsubishi International.

[31] Duan (1995): ”The GARCH option pricing model”, *Mathematical Finance*, 5, 13-32.

[32] Duffie, D., Singleton, K. (1999): ”Modelling Term Structures of Defautable Bonds”, *Review of Financial Studies*, 12, 687-720.

[33] Figlewski, S. (1997): ”Forecasting Volatility”, *Financial Markets, Institutions, and Instruments*, 6 (1), Blackwell Publishers, Boston.

[34] Fu, M.C., Laprise, S.B., Madan, D.B., Su, Y., and Wu, R. (2001): ”Pricing American options: A comparison of Monte Carlo simulation approaches”, *Journal of Computational Finance*, 4, 39-88.

[35] García, D. (2003): ”Convergence and biases of Monte Carlo estimates of American option prices using a parametric exercise rule”, *Journal of Economic Dynamics and Control*, 26, 1855-1879.

[36] Grant, D., Vora, G., Weeks, D. (1997): ”Path-dependent options: Extending the Monte Carlo simulation approach”, *Management Science*, 43, 1589-1602.

[37] Haugh, M., Kogan, L. (2004): ”Pricing American options: A duality approach”, *Operations Research*, 52 (2), 258-271.

[38] Hung, M.-W., Wang, J.-Y. (2002): ”Pricing convertible bonds subject to default risk”, *Journal of Derivatives*, 10 (2), 75-87.

[39] Ingersoll, J.E. (1977a): ”A contingent claim valuation of convertible securities”, *Journal of Financial Economics*, 4, 289-322.

[40] Ingersoll, J.E. (1977b): ”An examination of corporate call policy on convertible securities”, *Journal of Finance*, 32, 463-478.

[41] King, R. (1986): ”Convertible bond valuation: An empirical test”, *Journal of Financial Research*, 9 (1), 53-69.

[42] Lewis, C.M. (1991): ”Convertible debt: Valuation and conversion in complex capital structures”, *Journal of Banking and Finance*, 15, 665-682.

[43] Li, B., Zhang, G. (1996): ”Hurdles removed”, *Working Paper*, Merrill Lynch.

[44] Longstaff, F.A., Schwartz, E.S. (2001): ”Valuing American options by simulation: Simple least-squares approach”, *Review of Financial Studies*, 14, 1, 113-147.

[45] McConnell, J.J., Schwartz, E.S. (1986): ”LYON taming”, *The Journal of Finance*, 41 (3), 561-576.

[46] Milstein, G.N. (1978): ”A method of second-order accuracy integration of stochastic differential equations”, *Theory of Probability and its Applications*, 23, 396-401.

[47] Nelder, J.A., Mead, R. (1965): ”A simplex method for functional minimization”, *Computer Journal*, 7, 308-313.

[48] Press, W., Teukolsky, S., Vetterling, W., and Flannery B. (1992): ”Numerical recipes in C: The art of scientific computing”, *Cambridge University Press*.

[49] Raymar, S., Zwecher, M. (1997): ”Monte Carlo valuation of American call options on the maximum of several stocks”, *Journal of Derivatives*, 5, 7-20.

[50] Rogers, L.C.G. (2002): ”Monte Carlo valuation of American options”, *Mathematical Finance*, 12 (3), 271-289.

[51] Takahashi, A., Kobayashi, T., Nakagawa N. (2001): ”Pricing convertible bonds with default risk”, *Journal of Fixed Income*, 11, 20-29.

[52] Tilley, J.A. (1993): ”Valuing American Options in a path simulation Model”, *Transactions of the Society of Actuaries*, 45, 83-104.

[53] Tsitsiklis, J., Van Roy, B. (1999): ”Optimal stopping of Markov processes: Hilbert space theory, approximation algorithms, and an application to pricing high dimensional financial derivatives”, *IEEE Transactions on Automatic Control*, 44, 1840-1851.

[54] Tsiveriotis, K., Fernandes, C. (1998): ”Valuing convertible bonds with credit risk”, *Journal of Fixed Income*, 8 (3), 95-102.