

Long Memory Options: LM Evidence and Simulations

Abstract

This paper demonstrates the impact of the observed financial market persistence or long term memory on European option valuation by simple simulation. Many empirical researchers have observed the non-Fickian degrees of persistence or long memory in the financial markets different from the Fickian neutral independence (i.i.d.) of the returns innovations assumption of Black-Scholes' geometric Brownian motion assumption. Moreover, Elliott and van der Hoek (2003) provide a theoretical framework for incorporating these findings into the Black-Scholes risk-neutral valuation framework. This paper provides the first graphical demonstration why and how such long term memory phenomena change European option values and provides thereby a basis for informed long term memory arbitrage. By using a simple mono-fractal Fractional Brownian motion, it is easy to incorporate the various degrees of persistence into the Black-Scholes pricing formula. Long memory options are of considerable importance in corporate remuneration packages, since stock options are written on a company's own shares for long expiration periods. It makes a significant difference in the valuation when an option is "blue" or when it is "red." For a proper valuation of such stock options, the degrees of persistence of the companies' share markets must be precisely measured and properly incorporated in the warrant valuation, otherwise substantial pricing errors may result.

Keywords: Options, Long Memory, Persistence, Hurst Exponent, Identification, Simulation, Executive Remuneration

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1 Introduction

In the past decade, interest in long-term dependence, or "Long Memory" (LM), in the financial markets has increased, mostly due to the availability of more precise empirical measurements (Taqqu, 1986; Robinson, 1994; and Baillie, 1996). These studies of Fractional Brownian Motion (FBM), which models the scaling LM property, experienced an apparent setback when Rogers (1997) proved that the monofractal FBM allowed for "predictability" and thus for arbitrage opportunities, although he also showed that this problem could be corrected by a small alteration in the FBM. Are such arbitrage opportunities real, implying that current market pricing processes are not martingales, or should the FBM be adjusted to being multifractal, by incorporating deforming trading time, to make it a martingale, and to reconcile it with Fama's (1970) martingale theory.

This remains an open research question. Martingales presuppose that new future events in the financial markets fit the historically accumulated probability distribution of pricing events. However, new information may fall outside the accumulated distribution. An extreme event may change the shape of that distribution. An unfair real world may not play according to the accumulated rules of Fama's fair game market model!

Mandelbrot, Calvet and Fisher (1997), Calvet and Fisher (2002), and the popularization by Mandelbrot and Hudson (2004), show that a simple compounding of trading time $\theta(t)$ - which is a fractal accumulation of conventional time t , and the Geometric Brownian Motion (GBM) - results in a Multifractal Model of Asset Returns (MMAR) process, which is a martingale. They also show that such MMAR processes are theoretically superior to the ARMA, ARFIMA, GARCH and FIGARCH and Geometric Brownian Motion (GBM) processes, in the sense that they are consistent with both the empirically observed non-Fickian scaling of pricing processes and with martingale pricing.

Since the multifractal FBM of the MMAR has a Hurst exponent (= modal Lipschitz- α) that

is the same as that of an empirically measured monofractal FBM, Roger's (1997) critique does not negate the importance of the empirically observed LM scaling of financial asset prices. The only difference between the MMAR, or multifractal FBM, and the monofractal FBM is that the value of the empirically observed Hurst exponent H_o is less than that of the Hurst exponent of the trading time H_θ . Mandelbrot's trading time $\theta(t)$ is a fractal function that "warps" the conventional time t , because of the observed "clustering" of news events. Both bad and good news arrives in the financial markets in "clusters," and their effect "lingers on" in the markets, since market participants retain a differentiated memory of these events, resulting in the observed LM effect in financial market pricing

For simplicity, this paper restrict itself to the effects of LM time and frequency scaling on option pricing, when the information process is modeled as a monofractal FBM. It demonstrates that the empirically observed LM effect can be easily incorporated into the Black-Scholes (B-S) options pricing formula (Black and Scholes, 1973). This improved option pricing formula can theoretically be based on a risk-neutral (martingale) valuation, using the multifractal FBM (= MMAR), as Elliott and Van der Hoek (2003) prove.

Black and Scholes (1973) were aware of the problem of the degree of market persistence, but had no proper test for it and therefore maintained their assumption of independent information innovations and thus of a Fickian Hurst exponent $H = 0.5$. They were also aware of the fact that their second important maintained assumption of constant instantaneous return volatility was empirically invalid, but maintained it to keep their option valuation approach simple.

One of the major implications of this paper concerns the valuation of corporative remuneration packages. These packages contain often options written on the companies own stock, These options are long term, *i.e.*, have long expiration times and their values are impacted by the degree of LM or persistence inherent in the market of the underlying company's shares or - in case such shares are not traded - of the market of a comparable company of which the shares are traded. The proper valuation of these options requires the measurement of the degree of persistence of the

underlying asset price forming financial markets.

Options can be priced by using (a) real world probabilities or (b) risk-neutral probabilities. Both methods are completely equivalent, as long as the *shape* of the real world and the risk-neutral probability distributions remains the same. Thus the location and size differences between the real world and the risk-neutral distributions are immaterial for such valuation. Under LM the risk-neutral option valuation method can be used, since the only shape variable is the Hurst time-scaling exponent. The non-Fickian exponent can be easily incorporated in the risk-neutral probabilities. The new scaling binomial distribution with a non-Fickian monofractal Hurst exponent may no longer converge to a Gaussian scaling distribution, but converges to a non-Gaussian scaling, stable distribution.

The following two sections set the stage for such LM option valuation. Section 2 of this paper surveys measurement of LM exponents. In particular, it reviews fractional log-price diffusion and non-Fickian volatility scaling, the corresponding statistical measurement and testing theory, as well as the most salient empirical values in stock, bond and foreign exchange markets for the LM monofractal Hurst exponent as published in the best finance journals. Section 3 on LM option pricing, incorporates the LM assumption in the standard B-S pricing model and compares it with the, now classical, no-memory B-S pricing model.

Section 4 graphically shows the impact of various degrees of LM or persistence on out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) options, both of the call and put variety. Most corporate stock options are initially priced as long term ATM options. In Section 5 we summarize the various related discussions, formulate some conclusions and recommendations for empirical option valuation, and point out some pitfalls of the observed existence of LM for such valuation.

2 Empirical Measurement of LM

In the past two decades the econometric time series literature has demonstrated considerable interest in LM by its focus on unit root time series. Unit root models presume a known degree of memory in the integer (unit) order of differencing, which reduces a series to short memory stationarity and invertibility. But, more recently, empirical financial modelers show a greater acceptance of the flexible, fractional differencing models, which arose from considerations of self-similarity over time and frequencies, when the degree of memory is not *a priori* known or assumed and of non-integer lag length (For extensive surveys, *cf.* Los, 2003, and Robinson, 2003).

A continuous time stochastic return process $\{x(\tau); -\infty < \tau < \infty\}$ is self-similar with a Hurst exponent $H \in (0, 1)$ (Hurst, 1951), if for any $a > 0$, the process $\{x(a\tau); -\infty < \tau < \infty\}$ has the same time and frequency distributions as the process $\{a^H x(\tau); -\infty < \tau < \infty\}$. If the first differences of such processes $\Delta x(\tau) = x(\tau) - x(\tau - 1) = \varepsilon(\tau)$, for integer time τ , are covariance stationary, their autocorrelation function (ACF) is

$$\gamma(\tau) = \frac{\gamma(0)}{2} [|\tau + 1|^{2H} - 2|\tau|^{2H} + |\tau - 1|^{2H}] \quad (1)$$

It is easy to show that as the time horizon $\tau \rightarrow \infty$, this ACF decays over time like

$$\gamma(\tau) \sim \sigma_\varepsilon^2 \tau^{2H-2} \quad (2)$$

The formula for the corresponding spectral density can be found in Sinai (1976) and Los (2003) and satisfies a similar power law, with radian frequency $\omega = \frac{2\pi}{\tau}$:

$$P(\omega) = \sigma_\varepsilon^2 \omega^{-(2H-1)} \quad (3)$$

An example of such price diffusion is the monofractal FBM for financial market prices $S(\tau)$. Such "fractional" log price diffusion has been extensively studied by Mandelbrot and Van Ness (1968), Granger and Joyeux (1980), Hosking (1981) and Sowell (1990).

Definition 2 (*Monofractal Fractional Brownian Motion (FBM)*) is defined by the frac-

tionally differenced time series

$$(1 - L)^d x(\tau) = \varepsilon(\tau), \quad d \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \text{with } \varepsilon(\tau) \sim i.i.d.(0, \sigma_\varepsilon^2) \quad (4)$$

where $x(\tau) = \Delta \ln S(\tau) = (1 - L) \ln S(\tau)$, so that $x(\tau)$ is the rates of return, L is the lag operator, and $d = H - 0.5$ is the fractional differencing exponent.

A completely equivalent definition of the monofractal FBM is that $x(\tau)$ is fractionally integrated white noise, since, by inversion

$$x(\tau) = (1 - L)^{-d} \varepsilon(\tau), \quad d \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \text{with } \varepsilon(\tau) \sim i.i.d.(0, \sigma_\varepsilon^2) \quad (5)$$

Alternative forms of the FBM can be found in Marinucci and Robinson (1999). The corresponding spectral density is obtained by the Fourier Transform of this integration FBM (Adenstedt, 1974):

$$F_{FBM}(\omega) = (1 - e^{-j\omega})^{-d} \mathcal{F}[\varepsilon(\tau)] \quad (6)$$

By applying the two exponential series expansions for $e^{j\omega}$ and $e^{-j\omega}$, we obtain the aforementioned power law for the spectral density, from which the differencing exponent d can be identified by measuring the slope of $\log P(\omega)$ versus the frequency ω .

This self-similar generalization of the geometric Brownian Motion (GBM) has also been called an α_Z -stable (Pareto-) Lévy Motion, and has been the subject of three monographs by Samorodnitsky and Taqqu (1994), Janicki and Weron (1994) and Rachev and Mitnik (2000). Such an FBM is "almost certain" (a.c.) dense with discontinuities ("singularities"), in contrast to the GBM, which is a.c. continuous everywhere.

The discontinuities in an α_Z -stable Lévy motion imply that there may be occasionally a drawdown in the stock market with $0.5 < H < 1$ faster than trades can be executed, as occurred, for example in October 1929, and, more recently, in October 1987 in the US, (McCulloch, 1996, p. 397), so that dynamic hedging programs break down. The collapse of Long Term Capital Management in 1998, in response to the sudden and unexpected complete default of Russian debt

payments, comes also to mind. Such possible sudden breakdowns of financial market insurance should be of considerable concern for investment banks and insurance companies, which have usually large percentages of their assets and liabilities exposed to financial market risk (Mandelbrot and Hudson, 2004).

These fractionally differencing models allow for the Hurst exponent, $H = d + 0.5$, to be identifiable from the noisy financial data. It has been argued that statistical inferences and financial modeling based on an incorrect order of differencing are liable to be invalid and may lead to misleading pricing conclusions. (Robinson, 2003, pp. 4 - 25). However, the differencing rule is simple: differentiate by integer numbers, until the residual series has a differencing exponent $d \in (-\frac{1}{2}, \frac{1}{2})$. The current approach is to compute the H exponent from the negative slope coefficient $(2H - 1)$ of the log periodogram $\log P(\omega)$, *i.e.*, the data set based spectral density. The crucial test for financial analysts is the test of the hypothesis of neutral short memory ($H_0 : H = 0.5$, or $d = 0$) against the two alternative hypotheses of LM, persistence ($H_1 : 0.5 < H < 1.0$, or $0 < d < 0.5$) and anti-persistence ($H_2 : 0 < H < 0.5$, or $-0.5 < d < 0$).

The connection of this fractional time series literature with the stable frequency distribution literature is direct (McCulloch, 1996): for $0 < H < 1$, the characteristic or shape (Zolotarev) exponent of a stable distribution $\alpha_Z = \frac{1}{\alpha_L} = \frac{1}{H}$, where α_L is the Lipschitz- α_L . Thus for $H = 0.5$, $\alpha_Z = 2$ and the stable distribution is Gaussian. For $0.5 < H < 1$, we have $2 > \alpha_Z > 1$, and the variance of the distribution is non-existing or "infinite," due to a long upper "Paretian" tail. For example, this is the case for the empirical S&P500 Index, which has a measured $H = 2/3$, so that its stable shape exponent $\alpha_Z = 3/2$. This strongly suggests that the S&P500 Index is statistically unsuitable as an "underlying" index for option valuation, despite the fact that it is traded as such by the Chicago Board of Trade (CBOT).

For the theoretical GBM of Black and Scholes (1973), the one-time integrated log prices $\ln S(\tau)$ themselves have a memory exponent of $1 + H = 1.5$, so that their stability coefficient $\alpha_Z = 2/3$, and, indeed, their mean does not exist, *i.e.*, it is divergent. One differentiation of the GBM

provides the log errors ε , which have a memory exponent of $H = 0.5$, or $\alpha_Z = 2$, so that their mean exists, $E\{\varepsilon(\tau)\} = 0$, as well as their variance σ_ε^2 , and their distribution is equivalent to a Gaussian distribution $N(0, \sigma_\varepsilon^2)$.

When $0 < H < 0.5$ and $\infty > \alpha_Z > 2$, the resulting "distribution" is not a conventional uni-modal probability distribution, since one or both tails will then lie outside $[0, 1]$. This is the case, for example, in some foreign exchange markets, where it is found that $H = 0.25$. Again, options written in these anti-persistent markets do not make much logical pricing sense, when the standard GBM based option B-S valuation is applied, even though such options are currently traded on the Philadelphia Stock Exchange!

2.1 Published LM Measurements

Harvey and Whaley (1991, 1992) and Dumas, Fleming and Whaley (1998) look for parametric functions in the implied volatility of the S&P500 Index returns. But their approach is spanning the cart before the horse, since their implied volatilities are derived from the B-S option valuation formula based on the now falsified i.i.d. assumption. A similar mistake was made by Xu and Taylor (1994) in the foreign exchange markets, although their analysis confirmed that both the implied *and* historical return volatility are not constant.

Peters (1994) and Cizeau *et al.* (1997) correctly observe and measure that there exists non-Fickian volatility scaling in the S&P500 stock index returns with a persistent $H = 0.67$, which should be incorporated in the B-S formula.¹ Cont, Potters, and Bouchaud (1997), Gopikrishnan *et al.* (1998) and Lo and MacKinlay (1999) also observe non-Fickian volatility scaling in stock markets returns. Batten, Ellis and Mellor (1999), Batten and Ellis (1999) and, Batten, Ellis, and Hogan (1999) observed non-Fickian volatility scaling in Australian Dollar Eurobond and in foreign exchange market returns using more traditional scaled variance measurements.

¹ Interestingly, the Dow Jones Industrial Average index is exceptionally market neutral and have a Fickian Hurst exponent of $H = 0.5$ (Li, 1991). This is like the neutral memory of the River Rhine in Europe, which represents the similar exception to the rule that long rivers are persistent (Mandelbrot and Wallis, 1969).

A seminal comprehensive study of volatility scaling in several financial markets using high frequency data is Müller *et al.* (1995). Müller *et al.* (1990) were early observers of non-Fickian persistent volatility scaling in the smaller pre-Euro foreign exchange markets. But Karupiah and Los (2005) found non-Fickian anti-persistent volatility scaling of $H \approx \frac{1}{3}$ in the ultra-liquid anchor currencies of the Japanese Yen/US dollar and (formerly) German Deutschemark/US dollar foreign exchange markets before and after the Asian Financial Crisis in 1997 by using wavelet multiresolution analysis (MRA). This surprising anti-persistence value suggests that turbulence is, indeed, possible in foreign exchange markets, as already suggested by Ghasgaie *et al.* (1996) and Mantegna and Stanley (1996). This finding by Karupiah and Los in 1998 inspired the theoretical study by Elliott and Van der Hoek (2003; originally presented in 2000). Similar volatility scaling behavior in foreign exchange and stock markets has also been found by Gençay, Selçuk and Whitcher (2001), Kyaw, Los, and Zong (2003), and Lipka and Los (2003) using wavelet MRA. In other words, replication research by complete MRA has now corroborated those initial statistical findings.

More LM financial market results can be found in the compilations of articles by Kondor and Kertesz (1999) and Robinson (2003), and in the monographs by Peters (1994), Mantegna and Stanley (2000) and Los (2003). The existence of these empirical results now warrants a serious addition to the financial literature on option valuation.

In the fast growing interdisciplinary literature on LM time series various notational systems are used, sometimes with the same symbols meaning different concepts, and the opportunity for confusion is substantial. The rational approach to such confusion to adopt one notational system and to translate all other equivalent concepts into it, as is done in Table 4.3 in Los (2003, p. 124). This table shows the equivalence of various critical (ir-)regularity exponents, such as the dependence, difference, spectral, Hurst H , Zolotarev α_Z -stability and Lipschitz- α_L exponents. In this paper we use the monofractal Hurst exponent H , since that has become an accepted LM parameter in the finance and economics literature, even though the Lipschitz- α_L is the most

universally accepted exponent in the mathematics and physics literatures.

These critical exponents are to be identified or computed from the available "noisy" data in the financial markets to determine if these markets are anti-persistent, neutral or persistent. This was pursued, for example, by Beran (1989, 1992, 1994) by conventional covariance stationary time series analysis with constant integer correlations (= linear dependencies) and frequencies and by Flandrin (1989, 1992) and Kaplan and Kuo (1993) by the complete, and therefore superior, wavelet MRA, which can be applied to non-stationary time series with time-varying fractional nonlinear dependencies and frequencies.² Ramsey, Usikov and Zaslavsky (1995) were the first to perform the analysis of stock market volatility using wavelets MRA.

We have collected in Table 1 a summary of the empirical measurements of the Hurst exponent for the various financial markets, as published in first-class finance journals in the last decade: in the *International Review of Financial Analysis*, the *Journal of Banking and Finance*, *Journal of Empirical Finance*, *Journal of Financial Research*, *Journal of Futures Markets*, *Journal of International Money and Finance*, *Multinational Finance Journal*, *Pacific-Basin Finance Journal*, *Review of Quantitative Finance and Accounting*, *Quarterly Review of Economics & Finance*. In addition, there have been a few empirical measurements reported by financial economists in first-class non-finance journals, which traditionally have been interested in long memory: *Chaos, Solitons & Fractals*, *Fractals*, *Physica A*, and *Studies in Nonlinear Dynamics & Econometrics*.

This overwhelming LM evidence refutes the memory-neutral GBM in favor of the more LM FBM. Although most stock markets show persistence with $0.5 < H < 1$, there are quite a few financial markets (mostly liquid foreign currency markets), which are anti-persistent with $0 < H < 0.5$. Table 1 also reveals gaping holes in the recent empirical research efforts, since there are virtually no LM measurements of the cash markets or the derivatives (futures, options, swaps) markets published (yet).³

² Cf. Los (2003) Chapters 7 and 8 for a complete and detailed explanation.

³ The details of Table 1 are collected in a detailed survey table in the corresponding working paper, downloadable from the following web site: http://papers.ssrn.com/sol3/cf_dev/AbsByAuth.cfm?per_id=41027

[PLACE TABLE 1 ABOUT HERE]

2.2 Precision of LM Exponent Identification

How precise is the current identification of the degree of market persistence? Statisticians have tried to find an LM test statistic with a limiting probability distribution which can be easily computed and that has good power for hypothesis testing. Based on some admittedly “slightly defective heuristics” (Robinson, 2003, p.14), Geweke and Porter-Hudak (1983) argued that, asymptotically, the distribution of the log-periodogram regression estimate of d satisfies

$$\tau^{0.5}(\widehat{d} - d) \xrightarrow{d} N\left(0, \frac{\pi^2}{24}\right) = N(0, 0.41123) \quad (7)$$

giving rise to apparently simple inferential procedures, which are prejudiced towards rejecting the LM hypothesis. Subsequently, Robinson (1995a, b) rigorously established a more precise result, based on a slightly different differencing estimator of the whole range $-0.5 < d < 0.5$:

$$\tau^{0.5}(\widehat{d} - d) \xrightarrow{d} N\left(0, \frac{1}{4}\right) = N(0, 0.25) \quad (8)$$

This result provides also simple asymptotic interval estimates, as well as a test of neutrality, $d = 0$, *i.e.*, $H = 0.5$. But both these limiting statistical results are scientifically flawed, since they are based on the uncorroborated presumption of independence, on inapplicable infinity (= limiting) arguments applied to the finite support for d or H , and on prejudicial unidirectional projections.

A more appropriate statistical theory for the H exponent based on wavelet multiresolution analysis (MRA), which takes account of the fact the Hurst exponent must have an identifiable a lower limit, $0 < H$, is now available. Flandrin (1992) and Flandrin and Gonçalvès (1996) prove that the detailed wavelet resonance coefficients of a scalogram, which correlate wavelets with particular segments of the time series $x(t)$,

$$W(\tau, a) = \int_{-\infty}^{+\infty} x(t)\psi_{\tau,a}^*(t)dt \quad (9)$$

- where $\psi_{\tau,a}^*(t)$ is a particularly localized wavelet - are Gaussian with mean zero and a variance that is their own value squared:⁴

$$W(\tau, a) \sim N(0, |W(\tau, a)|^2) \quad (10)$$

The wavelet scalogram consist, by definition, of these coefficients of determination. We know then that each coefficient of the finitely tiled scalogram $P_W(\tau, a)$ is chi-squared distributed with one degree of freedom and non-centrality parameter $P_W(a)$:

$$P_W(\tau, a) \sim \chi^2(1, P_W(a)) \quad (11)$$

where

$$P_W(a) = \int_{-\infty}^{+\infty} |W(\tau, a)|^2 d\tau \quad (12)$$

The properly sized chi-squared distributions for each of the scalogram coefficients $P_W(\tau, a)$ can be obtained by bootstrapping (*Cf.* Los, 2003, p. 252, Remark 338). The parameter $P_W(a)$ is by itself chi-squared distributed with $\frac{T}{a}$ degrees of freedom and has itself as non-centrality parameter:

$$P_W(a) \sim \chi^2\left(\frac{T}{a}, P_W(a)\right) \quad (13)$$

The mean/dispersion ratio of this chi-squared distribution is then given by:

$$\frac{\mu\{P_W(a)\}}{\sigma\{P_W(a)\}} = \frac{[\frac{T}{a} + P_W(a)]}{\sqrt{2[\frac{T}{a} + 2P_W(a)]}} \approx \sqrt{\frac{T}{2a}} \quad (14)$$

which is larger when the number of observations T increases, $T \uparrow \infty$ or the scale a vanishes, $a \downarrow 0$.

Thus, the scalegram $P_W(a)$ is better identified when there are more observations at finer scales of data resolution. For example, when $T = 10,000$ observations, for scale $a = 2$, we have $\frac{\mu}{\sigma} \approx 50$ or a 2% variation in the scalegram $P_W(a)$, but for scale $a = 8$, we have $\frac{\mu}{\sigma} \approx 25$ or 4% variation. For one month of minute-by-minute observations $T = 40,320$, the variation is 1% and 2% respectively. As we saw earlier, the variation of the scalegram resides primarily in the identification uncertainty of its identifiable exponent $(2H + 1)$.

⁴ Despite the ∞ signs in the integral, the empirical support for these wavelet coefficients is empirically finite, as it is for similar (windowed) Fourier coefficients, since the wavelet support is empirically finite: $1 < \tau < T$ and $1 < a < T$.

Moreover, the skewness of the chi-squared distribution of the scalegram is very low:

$$skewness\{P_W(a) \approx 2^{3/2} \left(\frac{a}{T}\right)^{1/2} \quad (15)$$

and vanishes, when the number of observations increases, $T \uparrow \infty$, or the scale becomes vanishingly small, $a \downarrow 0$. The kurtosis of this chi-squared distribution of the scalegram is:

$$kurtosis\{P_W(a) \approx 3 + 12 \left(\frac{a}{T}\right) \quad (16)$$

which approaches that of normality ($= 3$), when $T \uparrow \infty$, or $a \downarrow 0$.

Therefore, together with all engineers and scientists, who have confidently used wavelet MRA since Mallat (1989a, b and c) provided its theory, we are no longer concerned about a possible lack of "statistical insignificance" of the slope $(2H + 1)$ and thus of the Hurst exponent H . In all empirical research on persistence in the financial markets reported using scalegrams we have observed that the actual statistical variation of slope $(2H + 1)$ is extremely small, since most MRA research uses large numbers of high frequency financial observations.

For example, when Karuppiah and Los (2005) measured the Hurst exponent for eight Southeast Asian currency markets and checked how much it actually varied from month to month over a four month period, with ca. 40,000 minute-by-minute observations/month, its variability was in the order of less than 8% either way over the first eight scales: $H = 0.25 + / - 0.02$. This established anti-persistence in the FX markets beyond reasonable doubt. This new wavelet MRA methodology is in agreement with the intuition that the smallest scale levels (= highest data resolution) should correspond with the highest degree of accuracy of exponent identification, while the largest scale levels (= lowest data resolution) should correspond with the lowest degree of accuracy. In other words, the degree of measurement accuracy should correspond with the scale of the data resolution.

3 LM Option Pricing

Let's now put the measurement of the degree of LM in the context of option pricing. Suppose there is a non-dividend paying stock S_0 with an expected rate of return $\mu > r$, the risk-free rate.

An option is a leveraged investment in the stock and is thus riskier than the stock. Consequently, it must be discounted at an expected rate $\gamma > \mu > r$. Indeed, stock, bond and derivatives traders and investors prefer to talk in terms of expected returns and risk premia added to risk-free rates, *i.e.*, in terms of μ and γ , and, thus, in terms of real world stochastic discount factors. However, the equivalence with the theoretical risk-neutral valuation using only the risk-free rate r is an established logical fact and easy to prove. The option pricing formula using real world probabilities (= incorporated into the stochastic discount factors or *pricing kernel*) gives the same option price as the risk-neutral probability valuation (McDonald, 2003, pp. 337 - 338 and 358 - 359).

Since the value of the expected stock return μ does not matter in risk-neutral pricing, *any* consistent pair of μ and γ will give the same option price. The real world task of traders is to find out by tentative trading what a consistent pair of expected return μ and stochastic discount rate γ is. Why then is theoretical risk-neutral pricing valuable? Because setting the expected stock return equal to r , the risk-free rate, results in the *simplest* pricing procedure that does not rely on trial-and-error trading to find out what the real world probabilities are. Thus, even so-called "soft options" on non-traded or non-tradable underlying assets, like on the equity in a family-owned business, can be valued.

This risk-neutral pricing equivalence hinges on the linearity of the asset combinations, in conjunction with and assumed i.i.d. Wiener information processes. For example, Black and Scholes (1973) assume that the instantaneous return innovations followed a simple neutral i.i.d. Wiener process. For the B-S assumption of a GBM, the rate of stock return is, in continuous time:

$$d \ln S_\tau = \mu d\tau + \sigma dz_\tau, \quad dz_\tau \sim i.i.d.(0, \tau) \quad (17)$$

In this model, the Hurst exponent has the Fickian value of $H = 0.5$ so that

$$Var\{d \ln S_\tau\} = \sigma^2 \tau^{2H} = \sigma^2 \tau \quad (18)$$

Thus, the expiration time-adjusted, instantaneous variance of the rate of return is, indeed, assumed

by Black and Scholes (1973) to be a constant:

$$\frac{Var \{d \ln S_\tau\}}{\tau} = \sigma^2 \quad (19)$$

Consequently, under this i.i.d. assumption, the (Cox, Ross and Rubinstein, 1979) binomial uptick $u = e^{\sigma\tau^{0.5}}$ and downtick $d = e^{-\sigma\tau^{0.5}}$ remain constant for a particular expiration time (= horizon) τ , and risk-neutral valuation is, indeed, extremely simple to implement.⁵

We can now generalize these results. For the assumption of a monofractal FBM, the rate of stock return is similarly:

$$d \ln S_\tau = \mu d\tau + \sigma dz_\tau^*, \quad dz_\tau^* \sim i.i.d.(0, \tau^{2H}) \quad (20)$$

but now the Hurst exponent has the non-Fickian value of $0 < H < 1$, $H \neq 0.5$ so that

$$Var \{d \ln S_\tau\} = \sigma^2 \tau^{2H} \neq \sigma^2 \tau \quad (21)$$

With the existence of LM, the expiration time-adjusted variance of the stock rate of return is not constant, except when $H = 0.5$, since

$$\frac{Var \{\ln(S_\tau/S_0)\}}{\tau} = \sigma^2 \tau^{2H-1} \neq \sigma^2 \quad (22)$$

Indeed Holton (1992) called time τ the second dimension of risk, the first being the instantaneous return variance σ^2 .

The B-S European option value based on a mono-FBM is then as follows. The call option value is, with $0 < H < 1$, where g is the dividend yield:

$$C_0 = S_0 e^{-g\tau} SD(d_1) - K e^{-r\tau} SD(d_2) \quad (23)$$

$$d_1 = \frac{\ln(S_0/K) + (r-g)\tau + \frac{1}{2}\sigma^2\tau^{2H}}{\sigma\tau^H} \quad (24)$$

$$d_2 = d_1 - \sigma\tau^H \quad (25)$$

⁵ Notice that the Black-Scholes assumptions do NOT include a specific assumption regarding the shape of the distribution of the stochastic return shocks: i.i.d.= independent, identically distributed (= strict-sense stationarity, although in their methodology wide-sense stationarity suffices). The Gaussianness equivalence stems from the neutral memory assumption of $H = 0.5$, which is implied by their not so innocuous assumption of an i.i.d., "white noise" or "flat spectrum" information process.

The put value follows from the put-call parity. Strictly speaking, the d_1 variable is no longer a standardized Gaussian variable, since it scales in a non-Gaussian way. Therefore the accumulations $SD(d_1)$ and $SD(d_2)$ do no longer represent cumulative Gaussian distributions indicated by $N(\cdot)$, but cumulative (non-Gaussian) stable distributions, indicated by $SD(\cdot)$. With the standard parametrization, they represent cumulative stable (scaling) distributions, *e.g.*, self-similar Lévy, Cauchy, Beta, Gamma, etc. distributions, something that Mandelbrot (1971) had already observed in early computer generation of LM time series by aggregation.

Unfortunately, such a "closed form" representation does not hold true for most stable distributions, although there exist explicit Zolotarev parametrizations for their characteristic functions. This means that most stable distributions can only be numerically integrated by simulation (McCulloch, 1996). Such numerical integration still allows a calculation of numerical probability distribution tables and therefore the use of hand-held option value calculators for option traders.

4 Simulation of LM Option Valuations

But how much does the empirically observed LM affect the option valuation? Let's look at the numerical impact of time decay within the whole range of degrees of persistence of a mono-fractal price diffusion process ($0 < H < 1$) on the values of out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) European call and put options. We model the price diffusion by a simple mono-fractal FBM and use the corresponding memory LM B-S formula to derive some qualitative and quantitative statements regarding their correct LM pricing, relative to their theoretical neutral memory pricing. For the sake of these examples, at all times we assume a constant instantaneous risk-free rate of $r = 0.06$, an instantaneous volatility of $\sigma = 0.30$, and a strike price of $K = \$60$. We also assume a non-dividend-paying stock S (with dividend yield $g = 0$), but that assumption can easily be relaxed. Moreover, the underlying can be an asset, a commodity, a futures (resulting in an LM Black option formula), etc., or a foreign exchange rate.

4.1 LM Call Option Values

The following pictographic presentation begins with an LM out-of-the-money (LM OTM) call option, with the underlying stock price at \$40, while we let the expiration time vary over $\tau = 1, 2, 5, 10$ and 20 years, for various degrees of persistence $0 < H < 1$, as in Fig. 1. All GBM B-S call option values are measured along the vertical line above the neutral memory $H = 0.5$ exponent in the middle of each figure. To the left of the vertical line are the values of the LM options in anti-persistent markets, while the values of the options in persistent markets lie to the right of this line. Each horizontally curved line indicates all such option values for all degrees of persistence for the specifically indicated expiration time τ .

[PLACE FIG. 1 ABOUT HERE]

There is, of course no exponential effect when $\tau = 1$ year, when the call price is \$0.90, no matter what the degree of persistence of the market of the underlying stock. But for a two-year call option, the difference in value of a call option on a stock trading in an ultra-persistent market ($H \uparrow 0$) and in an ultra-anti-persistent (= ultra-fast mean-reversing) market is already ca. \$3.00.⁶

That extreme LM difference in call value grows to ca. \$17.00 when the expiration time is $\tau = 5$ years and ca. \$26.00 when $\tau = 10$ years.

Such ten-year options are used in corporative remuneration packages. They are written on the shares of the own corporation and called "warrants". The question whether the underlying shares are trading or being valued in an anti-persistent, neutral or persistent market is thus a very relevant question for a manager, who finds a substantial part of his or her remuneration awarded in the form of such stock options. The differences between option values based on the B-S GBM and based on the monofractal FBM are largest in persistent markets.

Since most corporate shares are issued by medium-sized companies, these markets are always smaller, less liquid, and traded by fewer traders. They are thus likely to be much more persistent

⁶ It's interesting to note that the human heart beats in an ultra-anti-persistent fashion, with $H = 0.02$.

than the perfect competition model of stock markets suggests. For example, a small or medium sized stock trading on the NASDAQ is likely to show more persistent pricing than a blue chip technology stock trading on the New York Stock Exchange. Notice that for 20-year options this extreme difference has again been reduced to ca. \$18.00, because of the theta effect.

What happens with the intra-year option values? Is there still a LM effect? The answer is a resounding "yes," as demonstrated by Fig. 2. This Fig. 2 is the enlarged lower part of Fig. 1, for intra-year expiration times $\tau = 1/365$ (= 1 day), $30/365$ (= 1 month), $90/365$ (= three months), and $180/365$ (=six months).

[PLACE FIG. 2 ABOUT HERE]

But now the LM effect works "in reverse." There is virtually no impact on the one-day options, except in the empirically unobserved area of the blue noise, ultra-anti-persistence $H \downarrow 0$. The empirically observed range of the Hurst exponent (thus far) is about $0.2 < H < 0.8$. There is, for example, a \$0.20 90-day call value difference between a $H = 0.2$ anti-persistent (ultra-efficient, since ultra-fast reversing) market and a slightly persistent (= slightly inefficient) market. This grows to a \$0.30 difference for a 180-day call. We found that the LM effect is less pronounced with the ATM options.

Interestingly, we find that the differences in degrees of persistence are almost non-existent in the anti-persistent $0 < H < 0.5$ stock markets for LM ITM calls. In other words, for LM ITM call options the B-S GBM formula will provide the (almost) correct value. However, the different degrees of persistence do matter in the persistent $0.5 < H < 1$ stock markets for expiration times $\tau > 2$ years. For a ten-year LM ITM option at the extreme $H \uparrow 1$ value, the value difference compared with the GBM valuation is about \$8.50.

4.2 LM Put Option Values

Let's now look at the supra-year put options. As Fig. 3 shows, the LM effect is important almost exclusively in the persistent stock markets for the OTM puts and is virtually unimportant in

the anti-persistent markets. This is clearly different from the OTM calls, where the effect was noticeable in both anti-persistent and persistent markets.

[PLACE FIG. 3 ABOUT HERE]

Another interesting phenomenon is the difference between the ten-year and 20-year options in the extreme persistence markets where $H \uparrow 1$. This is caused by the deep discounting of the strike price (equivalent to a zero-coupon bond), which imposes a maximum value constraint on the present put option value P_0 , since when $S_0 = 0$, the present put value is maximally

$$P_0 = Ke^{-r\tau} \tag{26}$$

Now, for $\tau = 10$, $Ke^{-r\tau} = 20 \times e^{-0.06 \times 10} = \10.98 , but for $\tau = 20$, $Ke^{-r\tau} = 20 \times e^{-0.06 \times 20} = \6.03 .

The intra-year effect on puts for the various degrees of persistence applies only to the ultra-anti-persistent markets with $0 < H < 0.3$, and is therefore not relevant for the (thus far observed) empirically observed markets. In fact, put options show an interesting dichotomy: for the OTM put valuation the supra-year effects are very important for the empirically relevant LM range of $0.3 < H < 1$, while the intra-year effects are only relevant for the empirically irrelevant $0 < H < 0.3$ anti-persistence range.

For the LM ATM put options the degrees of persistence do matter over the whole range of $0 < H < 1$, as can be seen in Fig. 4. In other words, while this LM effect is important for the OTM put options in the persistent markets, but not in the anti-persistent markets, anti-persistence begins to matter for put valuation only with the LM ATM put options.

[PLACE FIG. 4 ABOUT HERE]

For an ITM put option four LM effects can be observed in Fig. 5. First, all intra-year put option values lie, of course, above the one year put option values. Second, the supra-year ITM put option values are smaller than the one-year values for the relevant empirical range $0 < H < 0.7$. Thus for this persistence range, the put option becomes more valuable as the time to expiration decreases. Third, the differences between ITM put option values for these empirically relevant

degrees of persistence are striking: in an anti-persistent ($H = 0.3$) market a ten-year ITM put option is worth \$5.00, but in a persistent market the same put option is worth \$17.50, a \$12.50 difference. Fourth, the supra-year ITM put option values are larger than the one-year values in the empirically irrelevant range $0.7 < H < 1$.

[PLACE FIG. 5 ABOUT HERE]

5 Conclusion

This paper demonstrates the impact of observed financial market persistence, *i.e.*, Long Memory (LM) on European option valuation. The degree of market persistence has a significant impact on the LM option values via the time-dependent volatility and thus via the risk-neutral probabilities used in their valuation. Some of these effects we find to be rather counter-intuitive. Therefore, we conclude that option traders should be very much aware of these important LM phenomena and the arbitrage opportunities they entail between persistent and anti-persistent markets.

Proper LM option valuation is also of considerable importance in corporate remuneration packages, since such options are written on a company's own shares with long expiration periods. Therefore, we recommend that, for a proper valuation of such stock options, the degrees of persistence of the companies' share markets are measured and properly incorporated in the stock option valuation.

As Elliott and van der Hoek (2003) show, this LM financial market analysis can be extended to multifractal Fractional Brownian Motions (FBMs). The solution favored by Mandelbrot, Calvet and Fisher (1997), is the one with a compounding process between the GBM and persistent trading time (= the time warping of various tick-by-tick trading intervals) to produce persistent multifractal pricing processes. However, the monofractal FBM shares the LM scaling phenomenon with the MMAR and is easier to use for a pedagogical explanation of the effect of LM scaling on the option pricing, as we demonstrate in this paper.

Indeed, the current empirical corroboration of the existence of multifractal FBM markets points

to a corroboration of the idea that a market for a particular financial instrument of a particular maturity is actually segmented into buyers and sellers according to the differences in time horizon of the various market participants, as was suggested by Peters (1989 and 1994, p. 272). In other words, what looks like one financial market may actually be a set of sub-markets, differentiated according to different time-horizons, since the trading participants bring memories of different length to the trading floor, based on their own personal experience. For example, some traders still recall both crashes of 1929 and 1987, others recall only the crash of 1987. In the case of the GBM this does not matter, since time is neutral in the GBM and the GBM differences are independent. But time is not neutral in an empirical FBM, as both the accumulated empirical evidence in the finance literature in Section 2, and our exposition of the valuation of LM options, clearly demonstrate.

This also points to an indicator of potential emerging market malfunctioning. The wider the multifractal spectrum, the more varied the investment horizons of the market participants. The narrower the multifractal spectrum, the narrower the spectrum of market participant memories. When the multifractal spectrum narrows too much, more and more market participants tend to have the same investment horizon or investment view and there are less buyers and sellers with different investment perspectives. This may lead to the occasional market breakdown or financial crisis.⁷

Finally, we would like to mention that currently the U.S. Financial Accounting Standards Board (FASB) is grappling with the problem of how to value options, because the B-S formula is not doing the job properly. After hearings on the subject in early 2004, the U.S. FASB is now poised to broaden the list of approved valuation methods beyond the standard B-S formula. The LM option formula would clearly fit in that approved extension of the pricing domain, since it takes account of the LM phenomena observed in the financial markets.

⁷ This idea is explored by Rossitsa Yalamova in her successfully defended PhD thesis "Wavelet MRA of Index Patterns Around Stock Market Shocks," Kent State University, 2003.

6 FIGURES AND TABLE

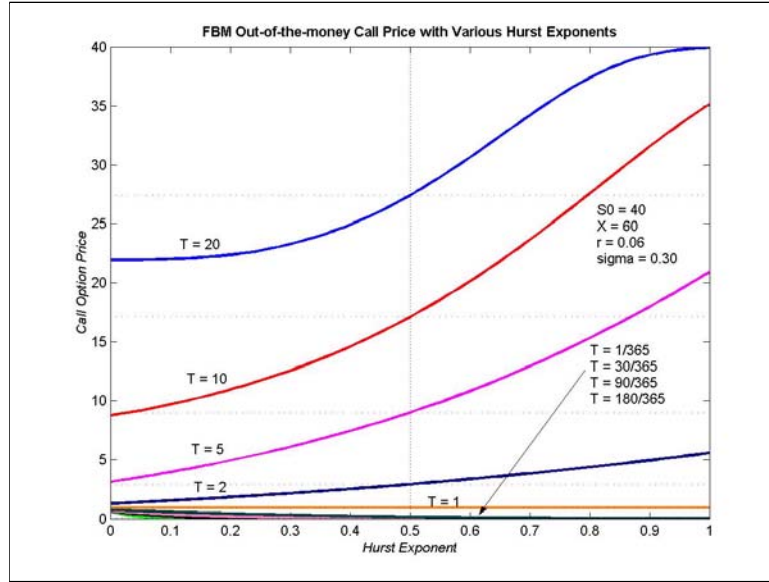


Figure 1: This graph shows various prices of the Fractional Brownian Motion (FBM) out-of-the-money (OTM) call options, with an emphasis on long expiration times. For demonstration purposes, the underlying stock price, $S_0 = \$40$. The strike price, $X = \$60$. The risk-free rate, $r = 6\%/yr$ and the volatility $\sigma = 30\%/yr$. On the vertical axis the various intrinsic call values C_0 are measured, for the various degrees of persistence as measured by the whole range of Hurst exponent on the horizontal axis, $0 < H < 1$. Various option value lines are drawn for various values of the expiration time, τ : for 1 day, 30 days, 90 days, 180 days, 1 year, 2 years, 5 years, 10 years, and 20 years, respectively, as indicated. For the extra-year options, the call option value is higher when the persistence of the underlying asset market is higher.

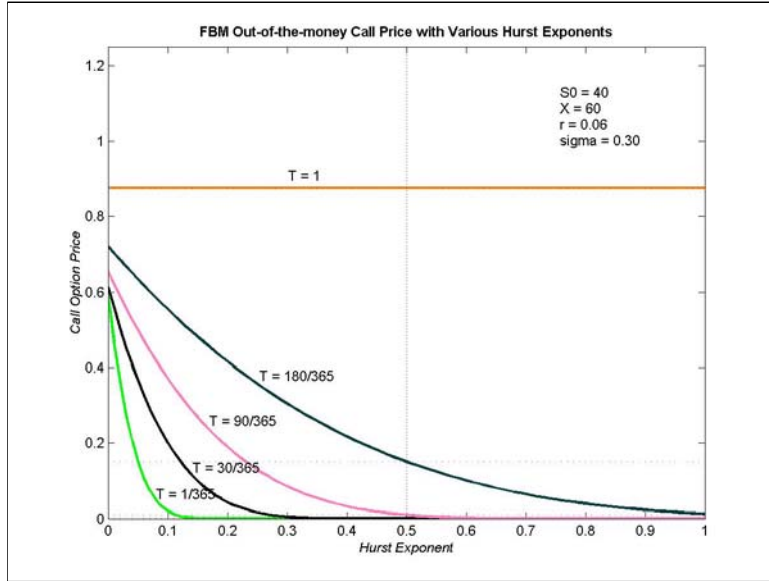


Figure 2: This graph shows various prices of the Fractional Brownian Motion (FBM) out-of-the-money (OTM) call options, with short, intra-year, expiration times, with the same parameters as in Fig. 1. For intra-year options, the call option value is lower when the persistence of the underlying asset value is higher.

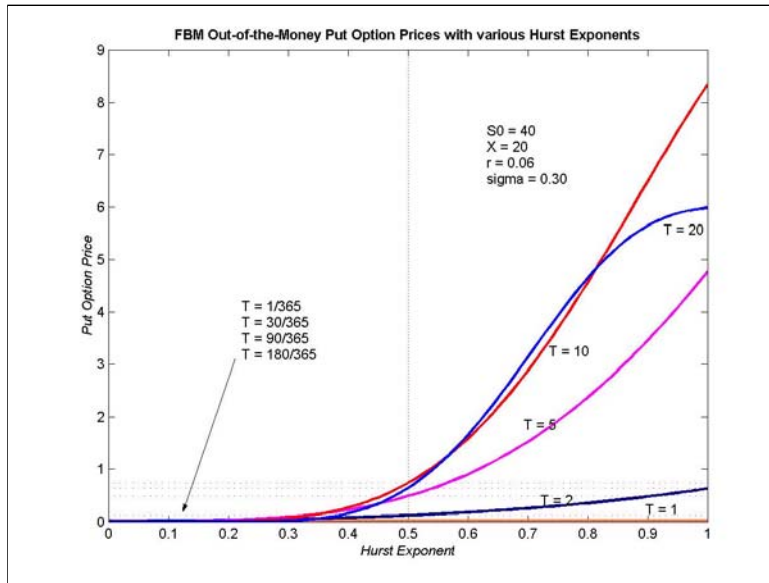


Figure 3: This graph shows various prices of the Fractional Brownian Motion (FBM) out-of-the-money (OTM) put options, with an emphasis on long expiration times, with the same parameters as in Fig. 1. Notice that the effect of persistence, H , when $0.5 < H < 1$, on the put option values is quite dramatic. Notice also that for extremely long memory options, when $\tau = 20$, the effect is diminished for the "red" put options, when $H \uparrow 0$.

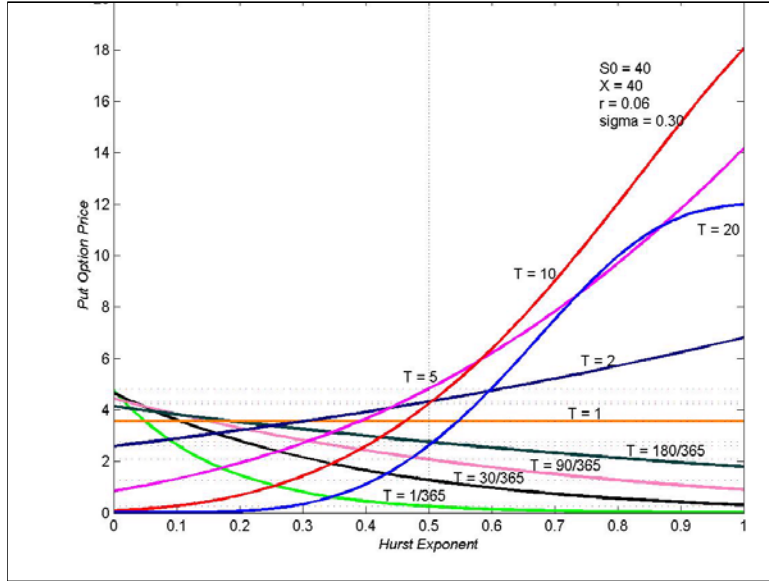


Figure 4: This graph shows various prices of the Fractional Brownian Motion (FBM) at-the-money (ATM) put options, for all expiration times, with the same parameters as in Fig. 1. For the intra-year put options the value effect is lower when the degree of market persistence is higher, while it is higher for the extra-year put options.

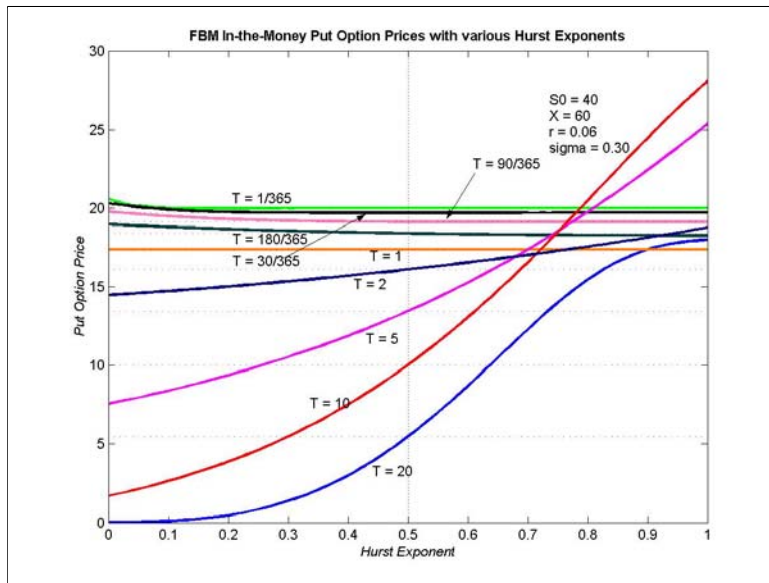


Figure 5: This graph shows various prices of the Fractional Brownian Motion (FBM) in-the-money (ITM) put options, for all expiration times, with the same parameters as in Fig. 1. For the intra-year put options the value effect is about the same for the full range of the Hurst exponent, while it increases for the extra-year put options, when the Hurst exponent moves from the "light blue" ($0 < H < 0.5$) to the "white" ($H = 0.5$) and then into the "pink" ($0.5 < H < 1$) regions.

TABLE 1		HURST EXPONENTS				
AUTHOR	YEAR	STOCK	TERM RATES	FUTURES	CASH	FOREX
Karuppiah and Los	2005					X
Gil-Alana	2004				X	
Lillo and Farmer	2004	X				
Matteo <i>et al.</i>	2004	X	X			X
McCarthy <i>et al.</i>	2004		X			
Morana and Beltratti	2004					X
Mulligan	2004	X				
Mulligan and Lombardo	2004	X				
Corazza and Malliaris	2002			X		X
Henry and Olekalns	2002					X
Cheung and Lai	2001					X
Lee <i>et al.</i>	2001	X				
Crato and Ray	2000			X		
Grau-Carles	2000	X				
Lien and Tse	1999	X		X		
Opong <i>et al.</i>	1999	X				
Hauser	1998				X	
Barkoulas and Baum	1998	X	X			X
Barkoulas et al	1997				X	
Jacobsen	1996	X				
Cheung and Lai	1995	X				
Evertsz	1995a,b	X				X
Evertsz and Berkner	1995	X				

Table 1: A summary of Long memory Literature. This table summarizes the investigation of Long Memory effect in the financial price or return series as they appeared in the high quality finance journals (highly regarded as "A" level journals) and in first-class non-finance journals in the past 10 years. Further details can be found in the Appendix. The author names and years of publications match those in the references. The studies in the table are categorized according to the time series investigated.

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