LIQUIDITY RISK ESTIMATION USING FUZZY MEASURE THEORY

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1. Introduction.

One of the most relevant issues in the risk analysis of the financial institutions´ investments is to determine the capital allocation in order to maintain its solvency and liquidity in adverse situations. The portfolio risk analysis is necessary for assuring the right selection of that capital to be allocated.

Each portfolio has a market risk. This risk is directly related to the losses that can be caused by adverse fluctuations of the portfolio asset prices. In this sense, it is necessary to construct a measure able to quantify the potential losses associated with that exposure.

The classical Value-at-Risk[1] measures the pure market risk; therefore, it does not bear some considerations. If a financial institution uses this classical framework to determine the quantity of capital to allocate in order to face its obligations with a certain level of confidence, then the institution does not take into account the partial or total portfolio liquidation consequences at the claim moment. To take into account these consequences is crucial because the number of assets to be sold in the market has an important influence in the price at which the transaction will be made. This influence is determined by the market liquidity at that moment. When these problems take place the financial institution could have liquidity problems to cancel its obligations.

This paper develops and applies a Value-at-Risk model regarding prices fluctuations and potential market liquidity problems. Due to uncertainty of market liquidity in the future, the model includes Fuzzy Measure Theory[2].

The first section of the paper presents some fundamental concepts of Fuzzy Measure Theory and Extreme Value Theory[3]. The second section presents a "fuzzified" risk valuation model under the classical assumption of normal distribution for the investment returns; and, taking into consideration the Argentinean financial

1 See Jorion (2001).
2 See Wang and Klir (1992).
3 See Embrechts, Kluppelberg, Micosch (1997).

crisis, also presents the model under an Extreme Value Theory distribution. Both alternatives are applied to a portfolio of *Repsol-YPF* stocks so as to estimate the risk assumed by the holder.

1.2. "Fuzzy" Measure Theory.

The Fuzzy Measure Theory is a generalization of the classical Measure Theory. The first one changes the additivity assumption and replaces it by two weaker ones: monotonicity and semicontinuity.

When the probabilities distribution function is known, risk situations take place. Otherwise, the risk becomes into uncertainty. When uncertainty arises a fuzzy measures can be constructed:

$$g_{\lambda}(v) = \frac{1 - F_V(v)}{1 + \lambda . F_V(v)} ,$$

where $\lambda \in [0, \infty)$ is an indicator of uncertainty. Given λ , there exists a one to one relationship between a measure and a probability distribution function. Associated to each value of λ , there exists a dual value $\lambda^* = -1/(1+\lambda)$ (with $\lambda^* \in [-1, 0)$) such that it defines a set of probability distribution functions:

$$\Gamma(F, \lambda) = \{P : g_{\lambda}(v) \leq F_V(v) \leq g_{\lambda^*}(v)\}$$

This set limits the values of the possible distribution functions in uncertainty cases and propose probability intervals.

In the pure risk situations, the known probability distribution function is the only element of the set mentioned below. In this way, it can be seen that the classical Measure Theory is a particular case of this theory.

1.3. Extreme Value Theory

The Extreme Value Theory studies and models a random variable distribution tail. This theory determines an excess probability distribution beyond a threshold. This distribution can be expressed as:

$$F_u(y) = P(X - u \leq y / X > u) \quad ,$$

where u is the threshold.

A theorem developed by Extreme Value Theory says that the probability distribution function of the values beyond the threshold is well approximated by a Pareto´s function, as the threshold grows to infinity. The Pareto´s function is defined as:

$$G_{\xi,\sigma}(x) = 1 - \left(1 + \frac{\xi \,.(\mathrm{x} - \mathrm{u})}{\sigma}\right)^{-1/\xi} \quad .$$

The selection of u is an important issue. There exist a trade-off between the selection of a high value and a low value. If it is too high, then the sample that will be used to estimate the parameters is too small. Otherwise, if is too low, then the approximation of the distribution is poor. It is difficult to find the equilibrium and there is not an objective way to find that value. The selection of the threshold value is an Extreme Value Theory field.

2. Liquidity risk estimation. Application to a Repsol-YPF´s shares portfolio.

In this section the methodology applied will be described, starting from a classical analysis and introducing, gradually, some characteristics of the financial markets reality. For practical proposes of the description, every step is presented with an application of an Argentinean financial market case. For this, a portfolio compounded by a financial asset is analyzed. This portfolio, during a period of time, is exposure to a price fluctuation risk. Moreover, there exists a risk related to the investment asset liquidation. This latter risk can affect the institution´ s liquidity to carry out its obligations.

In this way, it is analyzed a portfolio compounded by a "Repsol YPF" stock, this asset is trading in the Argentinean financial market. To the application purposes, the portfolio exposure period is one day. This means that once that length of time is achieved, the institution plans to cancel part of its obligations.

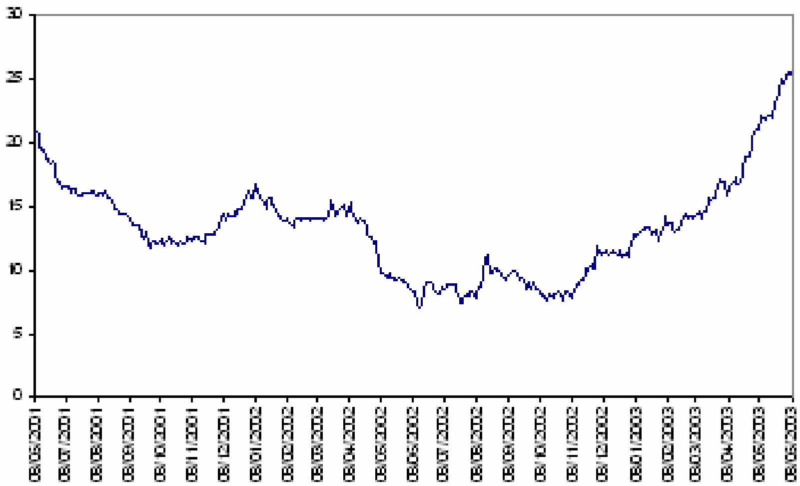
At the beginning, it is considered a portfolio with only one unit of the stock. Later, the consequences of the increment of the units of the stock are taking into consideration. This is not trivial because that increment affects, depending on the circumstances, to the quantity of assets to be liquidated, and this latter has an important influence in the price at which the transaction will be made, and in the market ability to absorb those assets.

2.1. Description and analysis of the sample.

The initial point of the estimation is connected with the sample to be used. This sample is the prices asset series, $\{S_t\}$ $(t=1,2,...,n)$, compounded by n elements. In the example, the "Repsol YPF" sample goes from 8 June of 2001 to 9 June of 2003. In this way, the sample has 497 elements, those can be seen in Graphic 2.1.

Because the aim is to quantify a potential loss, the sample of prices has to be transformed into a sample of relative losses (or negative returns) in each exposure period, $\{L_t\}$.

It is important to make two comments about some characteristics of the new generated sample. First, it can be observed that some elements are negative values. Those values must be interpreted as earnings (positive returns). Second, each element of the new sample is representative of partial relative losses. The fact that they are not considered as total relative losses is because they are calculated with the market prices of the original sample and does not consider the real price at which the liquidation would be made in the case of the portfolio liquidation. This is very important and gives information about the kind of errors that can be made if it is considered no more than the asset prices fluctuations cased only by market risk.



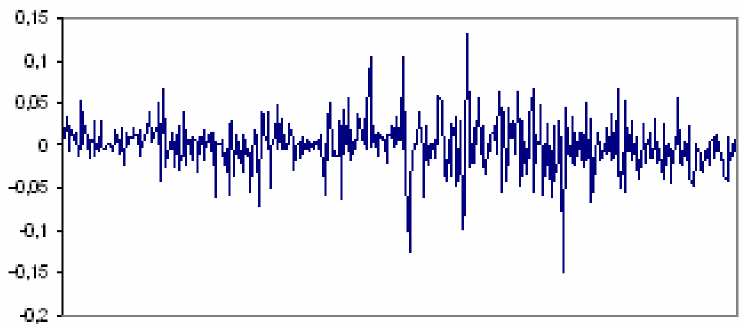
Graphic 2.1. Sample of “Repsol-YPF” prices,
from 06/08/2001 to 06/09/2003.

The relative loss (or the negative return) variable is defined as:

$$L_t = \lim_{\Delta t \to 0} -\frac{S_{t+\Delta t} - S_t}{S_t} = -\lim_{\Delta t \to 0} \ln\left(\frac{S_{t+\Delta t}}{S_t}\right),$$

where Δt is the proportion that represents the exposure period length over a year. Using the limits properties in the relative loss variable, it is assuming that Δt tends to zero. In the “Repsol YPF” case $\Delta t = \frac{1}{250} = 0{,}004$.

As soon as the sample is transformed into an appropriate one, a relative loss probability distribution function has to be obtained. Regarding the transformation, the new sample is compounded by $n-1$ elements. In the "Repsol YPF" case the sample is compounded by 496 elements. Those relative losses can be observed in the Graphic 2.2.



Graphic 2.2. Sample of the losses of "Repsol-YPF" portfolio.
from 06/08/2001 to 06/09/2003.

As a first approximation, the empirical distribution function of the variable L is constructed. For this, the elements of the sample, $\{L_t\}$, must be ordered in an ascendant way: $L^i > L^j \Leftrightarrow i > j$, with $j,i=1,2,...,n-1$ *and* $j\neq i$.

In these sense, it is defined a function that assign a cumulate probability value to each element of the sample. This function is:

$$F_{n-1}\left(L^i\right) = \frac{i}{n-1} \qquad \left(i=1,\ 2,\dots,n-1 \right).$$

Because the sample space of the variable to be analyzed is wider than the empirical variable sample space, it is assumed that the density between each element of the sample is uniform. Considering this, the empirical distribution can be expressed as:

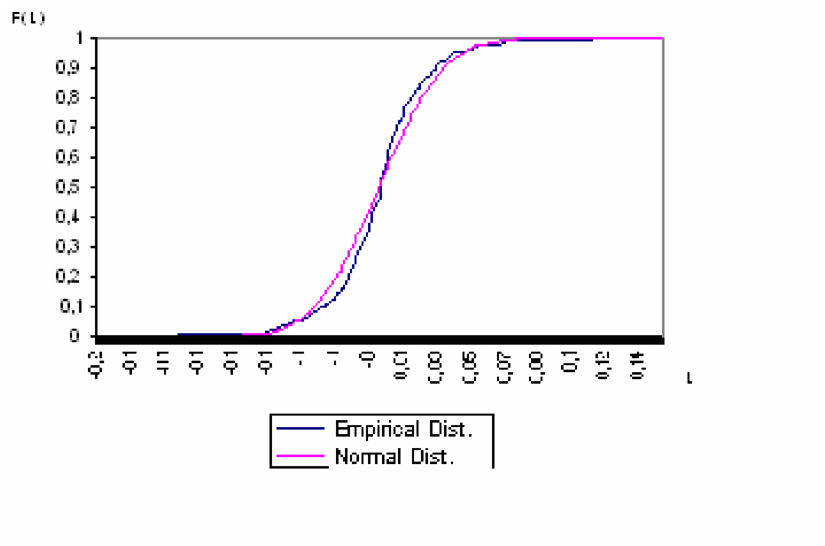
$$F_{n-1}^{*}(L) = \begin{cases} \dfrac{(L - L_{MIN})}{(n-1).(L^{1} - L_{MIN})} & if \quad L_{MIN} \leq L < L^{1} \\[2ex] \dfrac{1}{n-1}.\left[i + \dfrac{(L - L^{i})}{(L^{i+1} - L^{i})} \right] & if \quad L^{i} \leq L < L^{i+1} \quad (with \ i = 1,\ 2, \ldots n-2) \\[2ex] 1 & if \quad L = L^{n-1} \end{cases},$$

where, respectively, L_{MIN} and L^{n-1} are the minimum and the maximum value that the variable can assume.

2.2. Parametrical adjustment using Normal distribution.

In order to get analytical convenience, the next step is to find a parametrical probability distribution function that represents the behavior of the variable L . It is worth to say that representing these real world phenomena using parametrical functions is subject to some errors. In this way, it is crucial to be aware of the right selection of the best parametrical function in order to avoid this kind of errors.

As a classical view, the criterion selection of the distribution function is based on three assumptions about the behavior of the financial markets. The first is that movement prices are stochastically independent in time. The second is that the movement in the prices arises when the information set is modified and there is a constant incorporation of new information in that set. And firth, the process analyzed is stationary.



Graphic 2.3. Normal distribution adjustment

Assuming that the movement prices set up a sum of a huge number of independent and identical distributed random variables (the reaction of the traders due to the incorporation of new information) and using the Central Limit Theorem, the probability distribution of L tends to a Normal distribution defined as:

$$^{Normal}F_L(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2.\pi}.\sigma} . e^{-\frac{1}{2}.\left(\frac{y-\mu}{\sigma}\right)^2} dy \ ,$$

where $-\infty < x < \infty$.

The estimation of the parameters μ and σ is made using Maximum Likelihood method, taking the value of the parameters that make maximum the function:

$$^{Normal}l(\mu, \sigma^2 \mid L) = -\frac{n-1}{2}.\ln(2.\pi) - \frac{n-1}{2}.\ln(\sigma^2) - \frac{1}{2.\sigma^2}.\sum_{i=1}^{n-1} (L^i - \mu)^2 \ .$$

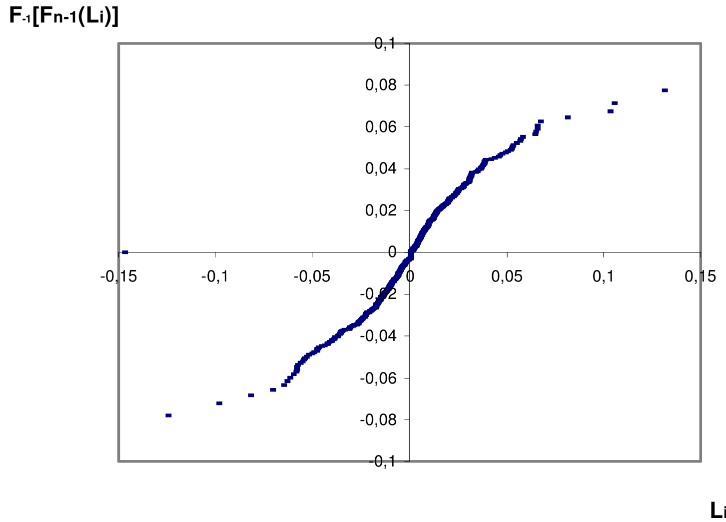
The estimated parameters are observed in Table 2.1 and the adjustment under Normal distribution whit respect to the empirical distribution can be seen in Graphic 2.3.

μ	-0,00041368
σ	0,02704855

Table 2.1. Parameters estimated.

2.3. Contrasting Normal distribution assumption with market reality.

There exists a problem at using the Normal distribution: in the market is observed that the financial relative losses present some characteristics that diverge from the results that are obtained under that hypothesis. The market reality shows that the value of empirical relative losses situated away from the mean occurs with more frequency than the frequency that is assigned by the Normal distribution. This fact is much more likely in the emergent financial markets, in which the extreme risks are larger. This phenomenon can be seen, in the case of "Repsol YPF", in a graphic such as the QQ-Plot (Graphic 2.4) in which the percentiles of a parametrical adjusted (Normal distribution) and the percentiles of an empirical distribution are contrasted. That graphic is constructed by the ordered pares $\left\{ L^{i} ; {}^{Normal}F_{L}^{-1}\left(F_{n-1}\left(L^{i}\right)\right) \right\}$(with $i = 1, 2, \ldots, n-1$). In the case in which an empirical function is well adjusted by a parametrical function, the graphic takes the form of a 45° line. In the graphic, it can be observed the "S" shape. This is a typical characteristic of the financial relative losses. This shows the problems that can be raised if the Normal distribution is used to explain financial facts.



Graphic 2.4. QQ-Plot. Contrast between the percentiles of a Normal distribution adjustment and the percentiles of the empirical distribution.

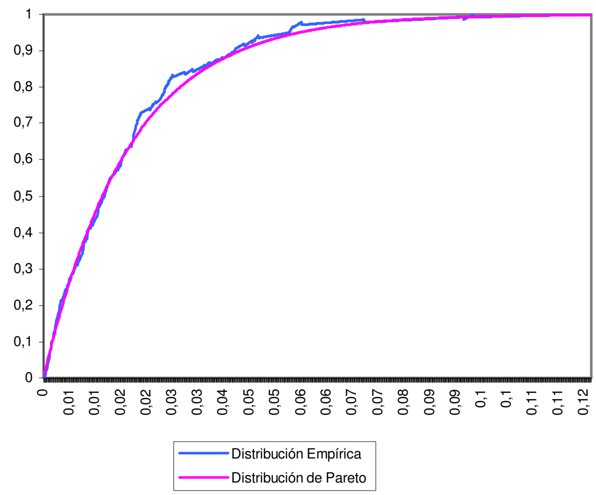
2.3. Parametrical adjustment using Extreme Value Theory.

Due to the imperfections of the Normal distribution adjustment it is necessary to use an alternative method that gives better results, minimizing the errors that can be made under the use of a hypothesis far to the reality. For this reason, the alternative is the use of the Extreme Value Theory.

One of the most important results of Extreme Value Theory is an approximation theorem that was enunciated in the first section of this paper. In this way, it is necessary to determine the threshold, u , where the probability distribution of the excess values beyond it, can be can be well approximated by a Pareto´s distribution. That threshold is determined by techniques that belong to this theory. In the "Repsol YPF" case, the threshold is 0,01.

In this sense, the parameters are obtained by the Maximum Likelihood method. The parameters are such that make maximum the function:

$$ {}^{E.V.T.}l\left(\xi,\sigma \mid L\right) = -(n-1).\ln(\xi) + (n-1).\frac{1}{\xi}.\ln\left(\frac{\sigma}{\xi}\right) - \left(\frac{1+\xi}{\xi}\right).\sum_{i=1}^{n-1}\ln\left(\frac{\xi.L^{i}-\sigma}{\xi}\right) $$



Graphic 2.5. Pareto´s distribution adjustment with respect to the empirical distribution.

The estimated parameters are shown in the Table 2.2 and the adjustment can be seen in Graphic 2.5.

σ	0,01880347
ξ	1,1465E-07

Table 2.2. Pareto´s function parameters

2.4. The classical Value-at-Risk calculation.

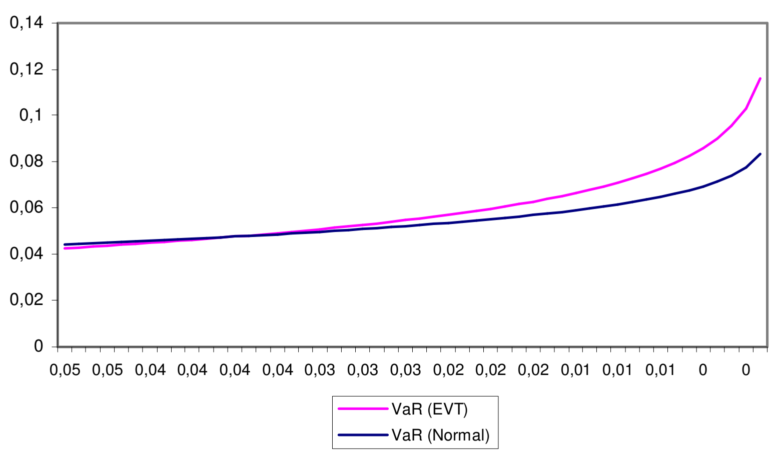
Once the adjustments of the two distributions are made it is necessary to make the Value-at-Risk calculation in order to determine the capital allocation. As it was mentioned, the Value-at-Risk (VaR) measures the maximum probable partial relative loss of the portfolio during an exposure period and under a level of significance:

$$\Pr\left[L \leq VaR\right] = \alpha \ ,$$

where α is the level of significance of the VaR.

α	${}^{EVT}VaR$	${}^{Normal}VaR$
0,05	0,042410233	0,044077218
0,04	0,046606107	0,046939823
0,03	0,052015529	0,05045903
0,02	0,059639682	0,055137208
0,01	0,072673258	0,062510628

Table 2.3. ${}^{Normal}VaR$ and ${}^{EVT}VaR$ for different α .



Graphic 2.6. ${}^{Normal}VaR$ and ${}^{EVT}VaR$ for different α .

The Graphic 2.6 shows the differences between ${}^{Normal}VaR$ (Normal distribution) and ${}^{EVT}VaR$ (Pareto´s distribution). Table 2.3 summarized this result.

In the graphic can be seen that both Value-at-Risk are similar for high values of α . This is explained because the Normal distribution assumption gives a good adjustment for the values of L near to the mean. In contrast, it can be appreciated the divergence of both Value-at-Risk calculated for lower values of α . This fact shows the sub estimation of the portfolio market risk under the normal distribution assumption.

2.5. Value-at-Risk calculation taking into account the liquidity risk.

Until here it has be made an analysis of a portfolio risks without taking into account the consequences of the investment liquidation. As mentioned before, the inclusion of these consequences is extremely important because this risk could make the financial institution to have liquidity problems at the time to carry out its obligations. It is worth to say that until this point it was considered the analysis of an investment compounded by one monetary unit of a stock, for this reason the inclusion of the liquidity risk had not much sense. The question now is: how to improve the Value-at-Risk model in order to include the liquidity risk of the institution when the portfolio number of assets is significant? Due to the uncertainty of the ability of the market to absorb the stocks to be liquidated, to answer the question it has to be considered some techniques that takes into account that uncertainty. For this reason the Fuzzy Measure Theory is used.

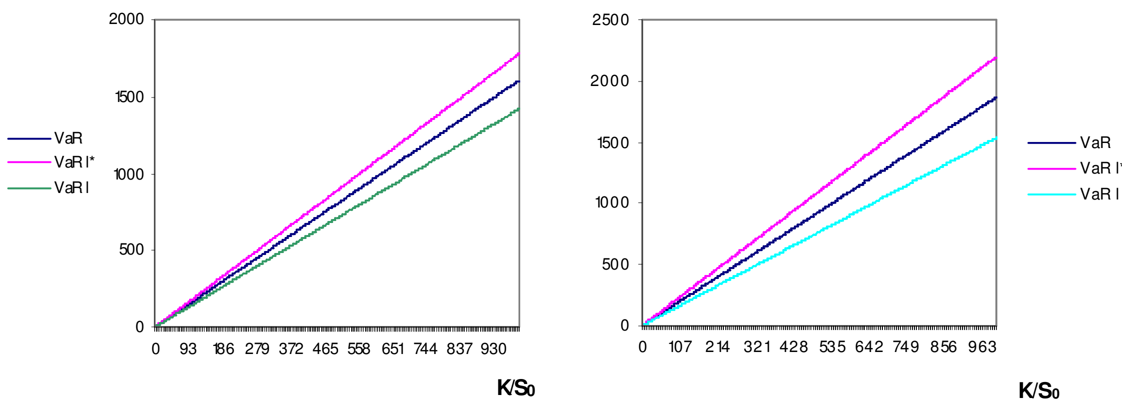
In this way, it is considered that each event is not associated to a unique probability. Each event is associated to a probability interval. So, when the portfolio is liquidated, the realization of each loss, x , is associated with a set of probability defined in the first section.

For all this, given a level of significance, the Value-at-Risk, $VaR_{\frac{K}{S_0}}$, regarding market risk and liquidity risk estimations, for an investment compounded by K monetary units is contained in the set:

$$V(F_L,\lambda)=\left\{VaR^{\lambda}_{\frac{K}{S_0}}=\frac{K}{S_0}.VaR^{\lambda}\le VaR^{\lambda=0}_{\frac{K}{S_0}}=\frac{K}{S_0}.VaR^{\lambda=0}\le VaR^{\lambda*}_{\frac{K}{S_0}}=\frac{K}{S_0}.VaR^{\lambda*}\right\}$$

where $VaR^{\lambda*}$, $VaR^{\lambda=0}$ and VaR^{λ} follow the next equations:

$$\alpha=g_{\lambda}(VaR^{\lambda});\quad \alpha=g_{\lambda*}(VaR^{\lambda*});\quad \alpha=1-F_L(VaR^{\lambda=0})$$



Graphic 2.8. Value-at-Risk intervals.

Normal distribution on the left, and Pareto´s distribution on the right.

These calculations are shown in Graphic 2.8, where it can be seen the Value-at-Risk with respect to the number of assets of the portfolio, assuming both alternatives. For the calculation purposes the parameter λ takes the value of 1. The role of this parameter is crucial in the results of the portfolio Value-at-Risk. In this way, it is necessary to find some method to estimate it.

It is remarkable the increment of the Value-at-Risk amplitude interval as the portfolio number of assets grows. This fact has a reason: the more number of assets, the more uncertainty of the liquidation price at the end of the exposure period.

Once the Value-at-Risk interval is constructed it must be determined which value included in the interval is the Value-at-Risk figure. Due to the uncertainty mentioned before, this figure is the upper limit of the interval. In this way, the capital to allocate will guarantee in an appropriate way the solvency and the liquidity of the financial institution.

3. Conclusions.

During this paper it was developed a methodology that takes into account, in addition of the market risk of an investment, the liquidity risk of a financial institution. These risks can affect the carrying out of a financial institution ´ s obligations. As it was observed, the Extreme Value Theory is more appropriate than the classical (Normal) assumption to represent a portfolio market risk. Regarding the liquidity risk of the institution, it can be said that the Fuzzy Measure Theory represents a way determine the capital to allocate in order to reduce the impact of the consequences associated to this risk.

It is extremely important to say that there is one aspect that represents an issue of futures research. This is connected with the assumption of $\lambda=1$, during the paper. In this way, it is necessary to find some method capable to estimate this parameter, in order to get with the actual level of uncertainty regarding the liquidity risk.

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