EURODOLLAR FUTURES AND OPTIONS: CONVEXITY ADJUSTMENT IN HJM ONE-FACTOR MODEL

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Abstract. In this note we give pricing formulas for different instruments linked to rate futures (euro-dollar futures). We provide the future price including the *convexity adjustment* and the exact dates. Based on that result we price options on futures, including the mid-curve options.

1. Introduction

This note is dedicated to euro-futures in the HJM framework with deterministic volatility. The futures we intend to price are the Libor-futures as traded on CME for USD and on LIFFE for EUR. The dates related to those futures are based on the third Wednesday of the month[1], which is the *start date* of the Libor rate underlying the future. We denote that rate by L. This rate is fixed at a *spot lag* prior to that date. In EUR and USD this lag is two business days and the fixing take place on the Monday. The *maturity date* of the Libor rate is three month[2] after the *start date*.

On the fixing date at the moment of the publication of the Libor rates the future price is $1 - L$. Before that moment, the price evolves with demand and offer. Every day the *closing price* is used for margining. The margining process consists in receiving the difference in price between the closing price of the day and the closing price of the previous day (or the transaction price on the trade date). The price are multiply by the nominal and divided by four. The one fourth represent the three month period as part of the year, as the Libor rates are quoted on annual basis.

The first task of this note is to compute the *fair* futures price (in the arbitrage free sense) from this description. This question is certainly not new and answers are abundant also. We refer to [4, Section 12.4] and [6, Section 11.5] for the theoretical framework, [5] for a formula in the extended Vasicek model and [7] for a more sophisticated approach including the volatility smile.

The specificity of this note is to provide explicit formula for the general HJM framework with deterministic volatility (than can be found in [7] and [6] under the name of Gaussian HJM) and to include the exact futures dates. All the formulas in the mentioned references do not differentiate between the fixing date and the start date.

The second task will be to price the options on futures as traded on CME. Those options are *not subject to margining* them-self. The premium is paid up-front. If the option is exercised, one enter into a future trade at a price equal to the strike (and no cash is exchanged). The value resulting from the exercise comes through the margining of the futures.

The exercise date is set *before or on* the fixing date of the future. Only the standard quarterly futures (March, June, September and December) are used as underlying of the options. The most popular options, called *quarterly options*, have their exercise date set on the fixing date. Another type of options, called *serial options*, have their expiry one or two months before the fixing date. The last type of options, called *mid-curve options*, have their expiry one or two year before the fixing date.

Those characteristics create a technically quite complex product. Even if we don't take into account the underlying future convexity adjustment, there is a (second) convexity adjustment. the

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[1] All dates mentioned in this note are adjusted in some way in the case they fall on a non-good business day.

[2] Futures exists also on the one month Libor, but the most popular are on three month.

option is on a *forward rate* and not on a deposit. The pay-off is not paid at the deposit maturity, not even at its start or fixing but at the option expiry that can be several years before.

For those options, we also provide explicit formulas in the HJM framework. We don't know any reference with similar formulas. All the exact dates of the options are also used in this case (expiry date, fixing date, start date and end date). In [6] a general theoretical formula is provided when the option exercise date is equal to the future fixing date and is equal to the start of the underlying deposit. We specialize all the results for the extended Vasicek or Hull-White model.

2. Model and hypothesis

We model interest rate products. The base assets are $P(t,u)$, the price in t of the zero-coupon bond paying 1 in u. We describe them for all $0 \leq t, u \leq T$, where T is some fixed constant. We work in a Heath-Jarrow-Morton [2] one factor model framework (see for example the chapter *Dynamical term structure model* in [4]). By this we mean we have a model with the following properties. The function P is positive and regular enough so that it can be written as

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right).$$

Let $A = \{(s,u) \in \mathbb{R}^2 : u \in [0,T] \text{ and } s \in [0,u]\}$. We work in a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}^{\mathsf{real}})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\mathsf{real}})_{0\leq t\leq T}$.

H: There exists $\sigma : [0,T]^2 \to \mathbb{R}^+$ measurable and bounded[3] with $\sigma = 0$ on $[0,T]^2 \setminus A$ such that for some process $(r_s)_{0\leq t\leq T}$, $N_t = \exp(\int_0^t r_s ds)$ forms with some measure $\mathbb{N}$ a numeraire pair[4] (with Brownian motion W_t),

$$\begin{aligned} df(t,u) &= \sigma(t,u)\int_t^u \sigma(t,s)ds\, dt - \sigma(t,u)dW_t \\ dP^N(t,u) &= P^N(t,u)\int_t^u \sigma(t,s)ds\, dW_t \end{aligned}$$

and $r_t = f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Note that ν is increasing in u, measurable and bounded. Moreover for $t > u$, $\nu(t,u) = 0$.

In the case of the extended Vasicek model, the volatility function is given by $\nu(s,t) = \frac{\sigma}{a}(1-\exp(-a(t-s)))$. We will analyse this model for σ constant and time-dependent. In the time-dependent version, the volatility parameter is piecewise constant with $\sigma(s) = \sigma_i$ for $s \in [s_{i-1}, s_i)$ where $0 = s_0 < s_1 < \cdots < s_i < \cdots < T$.

3. Preliminary results

We now state two technical lemmas, the proof of which can be found in [3]. Similar formulas can be found in [1, (3.3),(3.4)] in the framework of coherent interest-rate models.

Lemma 1. *Let $0 \leq t \leq u \leq v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)}\exp\left(\int_t^u (\nu(s,v)-\nu(s,u))\, dW_s - \frac{1}{2}\int_t^u \left(\nu^2(s,v)-\nu^2(s,u)\right) ds\right).$$

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[3] Bounded is too strong for the proof we use, some L^1 and L^2 conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

[4] See [4] for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair $(N, \mathbb{N})$.

Lemma 2. *Let $0 \leq u \leq v$. In the HJM one factor model, we have*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds\right) = P(u,v)\exp\left(\int_u^v \nu(s,v)dW_s - \frac{1}{2}\int_u^v \nu^2(s,v)ds\right).$$

4. Futures

The future fixing date is denoted t_0. The fixing is on the rate between t_1 and t_2. The fixing rate is denoted L_{t_0}. If the accrual factor for the period t_1–t_2 is δ, the fixing is linked to the yield curve by

$$1 + \delta L_t = \frac{P(t,t_1)}{P(t,t_2)}.$$

The futures price is Φ_t. On the fixing date, the relation between the price and the rate is

$$\Phi_{t_0} = 1 - L_{t_0}.$$

The futures margining is done on the futures price (multiply by the notional and divided by 4).

Theorem 1. *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the HJM one-factor model, the price of the futures fixing on t_0 for the period t_1–t_2 with accrual factor δ is given by*

$$\Phi_t = 1 - \frac{1}{\delta}\left(\frac{P(t,t_1)}{P(t,t_2)}\gamma(t) - 1\right)$$

where

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2) - \nu(s,t_1))ds\right).$$

Proof. Using the generic pricing future price process theorem [4, Theorem 12.6],

$$\Phi_t = \mathrm{E}_{\mathbb{N}}\left[1 - L_{t_0} \middle| \mathcal{F}_t\right].$$

In L_{t_0}, the only non constant part is the ratio of discount factors. Using Lemma 1 twice, we obtain

$$\frac{P(t_0,t_1)}{P(t_0,t_2)} = \frac{P(t,t_1)}{P(t,t_2)}\exp\left(-\frac{1}{2}\int_t^{t_0} \nu^2(s,t_1) - \nu^2(s,t_2)ds + \int_t^{t_0} \nu(s,t_1) - \nu(s,t_2)dW_s\right).$$

Only the second integral contains a stochastic part. This integral is normally distributed of variance $\int_t^{t_0}(\nu(s,t_1) - \nu(s,t_2))^2 ds$. So the expected value of the ratio of discount factors is reduced to

$$\frac{P(t,t_1)}{P(t,t_2)}\exp\left(-\frac{1}{2}\int_t^{t_0} \nu^2(s,t_1) - \nu^2(s,t_2)ds + \int_t^{t_0}(\nu(s,t_1) - \nu(s,t_2))^2 ds\right)$$

and we have the announced result. □

In the case of the extended Vasicek model, the adjustment factor can be written explicitly

$$\ln\gamma = \frac{\sigma^2}{2a^3}(\exp(-at_1) - \exp(-at_2))(\exp(at_0) - \exp(at))(2 - \exp(-a(t_2 - t_0)) - \exp(-a(t_2 - t))).$$

When the start date of the underlying rate is equal to the fixing date ($t_0 = t_1$) like in the Sterling market, this last formula is equivalent to the one of Kirikos and Novak [5].

In the time-dependent case, if we take $t = 0$ and $s_n = t_0$, we obtain

$$\ln\gamma = \frac{1}{2a^3}(\exp(-at_1) - \exp(-at_2))\sum_{i=1}^n \sigma_i^2(\exp(as_i) - \exp(as_{i-1}))(2 - \exp(-a(t_2 - s_i)) - \exp(-a(t_2 - s_{i-1}))).$$

The impact on differentiating between the fixing date and the start date is the integral in the gamma between t_0 and t_1. For normal volatility level this is below 0.01 basis points. The date impact is obviously larger for the options treated in the next section.

5. Options on futures

We use the notation X for the strike rate, $K = 1 - X$ for the strike price.

Theorem 2. *Let* $0 \leq \theta \leq t_0 \leq t_2 \leq t_3$*,* X*,*

$$\sigma_1^2 = \int_0^\theta (\nu(s,t_2) - \nu(s,t_1))^2 \, ds, \quad \sigma_{12} = \int_0^\theta \nu(s,\theta)\,(\nu(s,t_2) - \nu(s,t_1))\, ds, \quad \sigma_2^2 = \int_0^\theta \nu^2(s,\theta) ds$$

and Σ *be the matrix defined by*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

In the a HJM one-factor model with deterministic volatility, if the matrix Σ *is invertible, the price of the call option with expiry* t_0 *and strike rate* X *on the future with fixing and payment date* t_0 *on the rate between* t_2 *and* t_3 *is given in* 0 *by*

$$C_0 = P(0,\theta)\left((X+1/\delta)N\left(-\kappa + \frac{\sigma_{12}}{\sigma_1}\right) - \frac{\gamma(\theta)}{\delta}\frac{P(0,t_1)}{\delta P(0,t_2)}\exp(\alpha - \sigma_{12})N\left(-\kappa - \sigma_1 - \frac{\sigma_{12}}{\sigma_1}\right)\right)$$

where κ *is defined by*

$$X + 1/\delta = \frac{P(0,t_1)}{\delta P(0,t_2)}\exp(-\sigma_1\kappa - \frac{1}{2}\sigma_1^2 + \alpha),$$

and

$$\alpha = \int_0^\theta \nu(s,t_2)(\nu(s,t_2) - \nu(s,t_1))ds$$

The price of the put is given by

$$P_0 = P(0,\theta)\left(\frac{\gamma(\theta)}{\delta}\frac{P(0,t_1)}{P(0,t_2)}\exp(\alpha - \sigma_{12})N\left(\kappa + \sigma_1 - \frac{\sigma_{12}}{\sigma_1}\right) - (X+1/\delta)N\left(\kappa - \frac{\sigma_{12}}{\sigma_1}\right)\right).$$

Proof. Using the generic pricing theorem [4, Theorem 7.33-7.34] we have

$$\begin{aligned} C_0 &= N_0 \, \mathrm{E}_{\mathbb{N}}\left((\Phi_\theta - K)^+ N_\theta^{-1}\right) \\ &= \mathrm{E}_{\mathbb{N}}\left(\left(X - \frac{\gamma(\theta)}{\delta}\frac{P(t_0,t_1)}{P(t_0,t_2)} + \frac{1}{\delta}\right)^+ N_\theta^{-1}\right). \end{aligned}$$

This expected value can be computed explicitly using standard decomposition and computation of normal distribution. Here those computation are a little bit more involved and required some extra notations. Let

$$X_1 = \int_0^\theta \nu(s,t_2) - \nu(s,t_1) dW_s, \quad X_2 = \int_0^\theta \nu(s,\theta) dW_s.$$

The random variables X_1 and X_2 are jointly normally distributed ([8, Theorem 3.1, p. 60]) with covariance matrix Σ. Using Lemma 1 and Lemma 2, we have

$$\frac{P(\theta,t_1)}{P(\theta,t_2)} = \frac{P(0,t_1)}{P(0,t_2)}\exp(-X_1 - \frac{1}{2}\sigma_1^2 + \alpha)$$

and

$$N_\theta^{-1} = P(0,\theta)\exp(X_2 - \frac{1}{2}\sigma_2^2).$$

In the expectation, the parenthesis is positive when

$$X + 1/\delta > \frac{P(0,t_2)}{\delta P(0,t_3)}\exp(-X_1 - \frac{1}{2}\sigma_1^2 + \alpha)$$

or when $X_1 > \sigma_1\kappa$. With these results, the expected value becomes

(1)

$$A\int_{x_1 > \sigma_1\kappa} P(0,\theta)\left(X + 1/\delta - \frac{\gamma}{\delta}\frac{P(0,t_1)}{P(0,t_2)}\exp(-x_1 - \frac{1}{2}\sigma_1^2 + \alpha)\right)\exp(x_2 - \frac{1}{2}\sigma_2^2)\exp(-\frac{1}{2}x^T\Sigma^{-1}x)dx$$

where $A = \frac{1}{2\pi\sqrt{|\Sigma|}}$. Like in the proof of [3, Theorem 8], we have

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(x_2 - \frac{1}{2}x^T\Sigma^{-1}x\right) dx_2 = \frac{\sqrt{|\Sigma|}}{\sigma_1}\exp\left(-\frac{1}{2\sigma_1^2}\left(x_1^2 - 2x_1\sigma_{12} - |\Sigma|\right)\right).$$

Note that $x_1^2 - 2x_1\sigma_{12} - |\Sigma| + \sigma_1^2\sigma_2^2$ can be written as $(x_1 - \sigma_{12})^2$ and $x_1^2 - 2x_1\sigma_{12} - |\Sigma| + 2x_1\sigma_1^2 + \sigma_1^4 + \sigma_1^2\sigma_2^2$ can be written as $(x_1 + \sigma_1^2 - \sigma_{12})^2 + 2\sigma_{12}\sigma_1^2$. So the double integral in (1) is equal to

$$\begin{aligned} P(0,\theta) \qquad & \left((X + 1/\delta)\frac{1}{\sqrt{2\pi}}\int_{x_1>\sigma_1\kappa} \exp(-\frac{1}{2\sigma_1^2}(x_1 - \sigma_{12})^2)\frac{1}{\sigma_1}dx_1\right. \\ & \left. -\frac{\gamma}{\delta}\frac{P(0,t_1)}{P(0,t_1)}\exp(\alpha - \sigma_{12})\frac{1}{\sqrt{2\pi}}\int_{x_1>\sigma_1\kappa} \exp(-\frac{1}{2\sigma_1^2}(x_1 + \sigma_1^2 - \sigma_{12})^2)\frac{1}{\sigma_1}dx_1\right) \end{aligned}$$

Writing the integrals as normal cumulative distributions gives the result. □

For option on futures, it is very important to differentiate between the option expiry date and the fixing of the future, specialy for mid-curve options.

In the extended Vasicek model, the coefficients can be written explicitly. When the volatility is constant the results are

$$\begin{aligned} {\sigma_1}^2 &= \frac{\sigma^2}{2a^3}\left(\exp(-at_1) - \exp(-at_2)\right)^2\left(\exp(2a\theta) - 1\right), \\ {\sigma_2}^2 &= \frac{\sigma^2}{2a^3}\left(2a\theta - 3 + 4\exp(-a\theta) - \exp(-2a\theta)\right), \\ \sigma_{12} &= \frac{\sigma^2}{2a^3}\left(\exp(-at_2) - \exp(-at_1)\right)\left(\exp(-a\theta) + \exp(a\theta) - 2\right) \quad \text{and} \\ \alpha &= \frac{\sigma^2}{2a^3}\left(\exp(-at_1) - \exp(-at_2)\right)\left(2\exp(a\theta) - \exp(-a(t_2 - 2\theta)) - 2 + \exp(-at_2)\right). \end{aligned}$$

When the volatility is piecewise constant the formulas are a little bit more involved and include a sum but are similar. We write the formulas with $0 = s_0$ and $\theta = s_n$.

$$\begin{aligned} {\sigma_1}^2 &= \frac{1}{2a^3}\left(\exp(-at_1) - \exp(-at_2)\right)^2\sum_{i=1}^n \sigma_i^2\left(\exp(2as_i) - \exp(2as_{i-1})\right), \\ {\sigma_2}^2 &= \frac{1}{2a^3}\sum_{i=1}^n \sigma_i^2\Big(2a(s_i - s_{i-1}) - \left(\exp(-a(\theta - s_i)) - \exp(-a(\theta - s_{i-1}))\right) \\ &\qquad \left(4 - \exp(-a(\theta - s_i)) - \exp(-a(\theta - s_{i-1}))\right)\Big), \\ \sigma_{12} &= \frac{1}{2a^3}\left(\exp(-at_1) - \exp(-at_2)\right) \\ &\qquad \sum_{i=1}^n \sigma_i^2(\exp(as_i) - \exp(as_{i-1}))(2 - \exp(-a(\theta - s_i)) - \exp(-a(\theta - 2s_{i-1}))) \quad \text{and} \\ \alpha &= \frac{1}{2a^3}\left(\exp(-at_1) - \exp(-at_2)\right) \\ &\qquad \sum_{i=1}^n \sigma_i^2(\exp(as_i) - \exp(as_{i-1}))(2 - \exp(-a(t_2 - s_i)) - \exp(-a(t_2 - 2s_{i-1}))). \end{aligned}$$

6. Conclusion

We review the pricing of euro-dollar futures in the HJM framework with deterministic volatility. Even if this note is not the first one to deal with this convexity adjustment problem, it improves existing results by taking care of all the relevant dates, including the spot-lag between the fixing and the underlying start date. The formula obtained is specialized for the case of the time-dependent extended Vasicek model with piecewise constant volatility parameter.

The note also provide explicit formulas for exchange traded options on futures. In a similar way all the relevant date are taken into account, including the expiry, fixing and start date. The explicit results are also specialized for the extended Vasicek model. To our knowledge this is the first note that gives and explicit formula for those products. The resulting formula is somehow involved in its formulation. this is due to the margining characteristic of the underlying future that requires an convexity adjustment factor. On top of it, the option is on the rate, not on the deposit which creates a second and different convexity adjustment.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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Contents

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