Risk Managing Bermudan Swaptions in the Libor BGM Model[1]

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Abstract. This article presents a novel approach for calculating swap vega per bucket in the Libor BGM model. We show that for some forms of the volatility an approach based on re-calibration may lead to a large uncertainty in estimated swap vega, as the instantaneous volatility structure may be distorted by re-calibration. This does not happen in the case of constant swap rate volatility. We then derive an alternative approach, not based on re-calibration, by comparison with the swap market model. The strength of the method is that it accurately estimates vegas for any volatility function and at a low number of simulation paths. The key to the method is that the perturbation in the Libor volatility is distributed in a clear, stable and well understood fashion, whereas in the re-calibration method the change in volatility is hidden and potentially unstable.

Key words: central interest rate model, Libor BGM model, swaption vega, risk management, swap market model, Bermudan swaption

JEL Classification: G13

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1 Introduction

The Libor BGM interest rate model introduced by Brace, Gątarek & Musiela (1997), Jamshidian (1997) and Miltersen, Sandmann & Sondermann (1997) is presently most popular amongst both academics and practitioners alike. A reason is that the Libor BGM model has the potential for risk managing interest rate derivatives that depend on both the cap and swaption markets, which would render BGM as a *central interest rate model*. It features log-normal Libor rates and almost log-normal swap rates and consequently also the market standard Black formula for caps and swaptions. Approximate swaption volatility formulas exist in the literature (e.g. Hull & White (2000)) and have been shown to be of high quality (e.g. Brace, Dunn & Barton (1998)).

There are however still a number of issues that need to be resolved to achieve the goal of using BGM as a central interest rate model. One of these issues is the calculation of swap vega. A common and usually very successful method for calculating a Greek in a model equipped with a calibration algorithm is to perturb market input, re-calibrate and then re-value the option. The difference in value divided by the perturbation size is then an estimate for the Greek. If however this technique is applied to the calculation of swap vega in the Libor BGM model, then it may (depending on the volatility function) yield estimates with large uncertainty. In other words, the standard error of the vega is relatively high. The uncertainty disappears of course by increasing the number of simulation paths, but the number required for clarity can by far supersede 10,000, which is probably the maximum in a practical environment. This large uncertainty in vega has been illustrated in Section 2, most notably in Figure 2.

For a constant volatility calibration however the vega is estimated with low uncertainty. The number of simulation paths needed for clarity of vega thus depends on the chosen calibration. The cause is that for certain calibrations, under a perturbation, the additional volatility is distributed non-evenly and, may one even say, unstably over time. For the constant volatility calibration of course this additional volatility is naturally distributed evenly over time. It follows that the correlation between the discounted payoff along the original and perturbed volatility is larger. As the vega is the expectation of the difference between these payoffs (divided by the perturbation size) consequently the standard error will be lower.

A method is developed, not based on re-calibration, for computing swap vega per bucket in the Libor BGM model. It may be used to calculate swap vega in the presence of any volatility function, with clarity already for 10,000 simulation paths or less. The strength of the method is that it accurately estimates swap vegas for any volatility function and at a low number of simulation paths. The key to the method is that the perturbation in the Libor volatility is distributed in a clear, stable and well understood fashion, whereas in the re-calibration method the change in volatility is hidden and potentially unstable. The method is based on keeping swap rate correlation fixed while increasing the instantaneous volatility of a single swap rate evenly over time, while all other swap rate volatilities remain unaltered.

It is important to verify that a calculation method reproduces the correct numbers in a situation where the answer is known. We choose to benchmark our swap vega calculation method using Bermudan swaptions. There are two reasons for this: First, a Bermudan swaption is a complicated enough (swap-based) product (in a Libor-based model) that depends non-trivially on the swap rate volatility dynamics; for example, its value depends also on swap rate correlation. Second, a Bermudan swaption is not as complicated as certain other more exotic interest rate derivatives and some intuition exists about its vega behaviour. We show for Bermudan swaptions that our method yields almost identical swap vega as found in a swap market model.

Finally, we mention the paper of Glasserman & Zhao (1999), who provide efficient algorithms for calculating risk sensitivities given a perturbation of Libor volatility. Our problem differs from theirs in that we derive a method to calculate the perturbation of Libor volatility to obtain the correct swap rate volatility perturbation for swaption vega. The Glasserman and Zhao approach may then be applied to efficiently compute the swaption vega given the Libor volatility perturbation found by our method.

The remainder of this paper is organized as follows. First, we present the approach to calculating swap vega per bucket based on re-calibration with a specific volatility function. We show that resulting swap vega may be poorly estimated at a low number of simulation paths. Second, an explanation of this phenomenon is given. Third, the natural definition of swap vega in the canonical swap market model (SMM) is studied. Fourth, the SMM-definition of swap vega is extended to Libor BGM. Fifth, correct numerical swap vega results for a 30 year deal are presented. Sixth, we show that almost identical swap vega are obtained in a swap market model. The article ends with conclusions.

2 Re-calibration Approach

In this section we consider examples of the re-calibration approach of computing swap vega. Three calibration methods are considered. It is shown that, for two of the three methods, resulting vega is hard to estimate, i.e., a large number of simulation paths is needed for clarity. To facilitate our discussion, first some notation is introduced.

Consider a BGM model. Such a model features a tenor structure $0 < T_1 < \cdots < T_{N+1}$ and N forward rates L_i accruing from T_i to T_{i+1}, $i = 1, \ldots, N$. Each forward rate is modelled as a geometric Brownian motion under its forward measure,

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \bar{\sigma}_i(t) \cdot \mathrm{d}\bar{W}^{i+1}(t), \quad 0 \le t \le T_i.$$

Here $\bar{W}^{i+1}$ is a d-dimensional Brownian motion under the forward measure $\mathbb{Q}_{i+1}$. The positive integer d is referred to as the *number of factors* of the model. The function $\bar{\sigma}_i : [0, T_i] \to \mathbb{R}^d$ is the volatility vector function of the i^{th} forward rate. The k^{th} component of this vector corresponds to the k^{th} Wiener factor of the Brownian motion.

A discount bond pays one unit of currency at maturity. The time-t price of a discount bond with maturity T_i is denoted by $B_i(t)$. The forward rates are related to discount bond prices as follows

$$L_i(t) = \frac{1}{\delta_i}\left\{ \frac{B_i(t)}{B_{i+1}(t)} - 1 \right\}.$$

Here δ_i is the accrual factor for the time span $[T_i, T_{i+1}]$.

The swap rate corresponding to a swap starting at T_i and ending at T_{j+1} is denoted by $S_{i:j}$. The swap rate is related to discount bond prices as follows

$$S_{i:j}(t) = \frac{B_i(t) - B_{j+1}(t)}{\mathrm{PVBP}_{i:j}(t)}.$$

Here PVBP denotes the *present value of a basis point*,

$$\mathrm{PVBP}_{i:j}(t) = \sum_{k=i}^{j} \delta_k B_{k+1}(t).$$

It is understood that $\mathrm{PVBP}_{i:j} \equiv 0$ whenever $j < i$. We will consider the swap rates $S_{1:N}, \ldots, S_{N:N}$ corresponding to the swaps underlying a co-terminal Bermudan swaption[4]. Swap rate $S_{i:N}$ is a martingale under its forward swap measure $\mathbb{Q}_{i:N}$. We may thus implicitly define its volatility vector $\bar{\sigma}_{i:N}$ by

$$\frac{\mathrm{d}S_{i:N}(t)}{S_{i:N}(t)} = \bar{\sigma}_{i:N}(t) \cdot \mathrm{d}\bar{W}^{i:N}(t), \quad 0 \le t \le T_i. \tag{1}$$

[4] A co-terminal Bermudan swaption is an option to enter into an underlying swap at several exercise opportunities. In other words the holder of a Bermudan swaption has the right at each exercise opportunity to either enter into a swap or hold on to the option; all underlying swaps that may possibly be entered into share the same end date.

In general $\bar{\sigma}_{i:N}$ will be stochastic because swap rates are not log-normally distributed in the BGM model. These are however distributed very close to log-normal as shown for example by Brace et al. (1998). Because of near log-normality the Black formula approximately holds for European swaptions. Closed-form formulas exist in the literature for the swaption Black implied volatility, see for example Hull & White (2000). The swap rate volatility formula will be treated in more detail in Section 5.

We choose to model the Libor instantaneous volatility as constant in between tenor dates (piece-wise constant). A volatility structure $\{\bar{\sigma}_i(\cdot)\}_{i=1}^N$ is *piece-wise constant* if

$$\bar{\sigma}_i(t) = (\text{const}), \quad t \in [T_{i-1}, T_i).$$

The volatility will sometimes be modelled as time homogeneous. To define this, first define a fixing to be one of the time points $T_1, \ldots, T_N$. Define $\iota : [0, T] \to \{1, \ldots, N\}$,

$$\iota(t) = \#\big\{ \text{ fixings in } [0, t) \big\}.$$

A volatility structure is said to be *time homogeneous* if it depends only on the index to maturity $i - \iota(t)$.

Three volatility calibration methods are considered:

1 (THFRV) *Time homogeneous forward rate volatility.* This approach is based on ideas of Rebonato (2001). Because of the time homogeneity restriction, there remain as many parameters as market swaption volatilities. A Newton Rhapson type solver may be used to find the exact calibration solution (if such exists).

2 (THSRV) *Time homogeneous swap rate volatility.* The algorithm for calibrating with such volatility function is a two stage bootstrap. The first and second stage are described for example in equation (6.20) and Section 7.4, respectively, of Brigo & Mercurio (2001).

3 (CONST) *Constant forward rate volatility.* Note that constant forward rate volatility implies constant swap rate volatility. The corresponding calibration algorithm is similar to the second stage of the two stage bootstrap.

All calibration methods have in common that the forward rate correlation structure is calibrated to a historic correlation matrix via principal components analysis (PCA), see Hull & White (2000). Correlation is assumed to evolve time homogeneously over time.

Table 1: Market European swaption volatilities.

Expiry (Y)	1	2	3	...	28	29	30
Tenor (Y)	30	29	28	...	3	2	1
Swaption Volatility	15.0%	15.2%	15.4%	...	20.4%	20.6%	20.8%

We considered a 31NC1 co-terminal Bermudan payer's swaption deal struck at 5% with annual compounding. The notation xNCy denotes an 'x non-call y' Bermudan option, which is exercisable into a swap with a maturity of x years from today but is callable only after y years. The option is callable annually. The BGM tenor structure is $0 < 1 < 2 < \cdots < 31$. All forward rates are taken to equal 5%. The time-zero forward rate instantaneous correlation is assumed given by the form of Rebonato (1998) p. 63,

$$\rho_{ij}(0) = e^{-\beta|T_i - T_j|}.$$

Here β is chosen to equal 5%. The market European swaption volatilities were taken as displayed in Table 1.

To determine the exercise boundary the Longstaff & Schwartz (2001) least squares Monte Carlo method was used. Only a single explanatory variable was considered, namely the swap net present value (NPV). Two regression functions were employed, a constant and linear term.

For each bucket a perturbation $\Delta\sigma$ ($\approx 10^{-8}$) was applied to the swaption volatility in the calibration input data[5]. The model was re-calibrated and it was checked that the calibration error for all swaption volatilities was a factor 10^6 smaller than the volatility perturbation. The Bermudan swaption was re-priced through Monte Carlo simulation using the exact same random numbers. Denote the original price by V and the perturbed price by $V_{i:N}$. Then the re-calibration method of estimating swap vega $\mathcal{V}_{i:N}$ for bucket i is given by

$$\mathcal{V}_{i:N} = \frac{V_{i:N} - V}{\Delta\sigma}. \tag{2}$$

[5]It was verified that the resulting vega is stable for a whole wide range of volatility perturbation sizes. For very extreme perturbation sizes the vega however deviates: If the volatility perturbation size is chosen too large, then vega-gamma terms affect the vega. If the volatility perturbation size is chosen too small, then floating number round-off errors affect the vega.

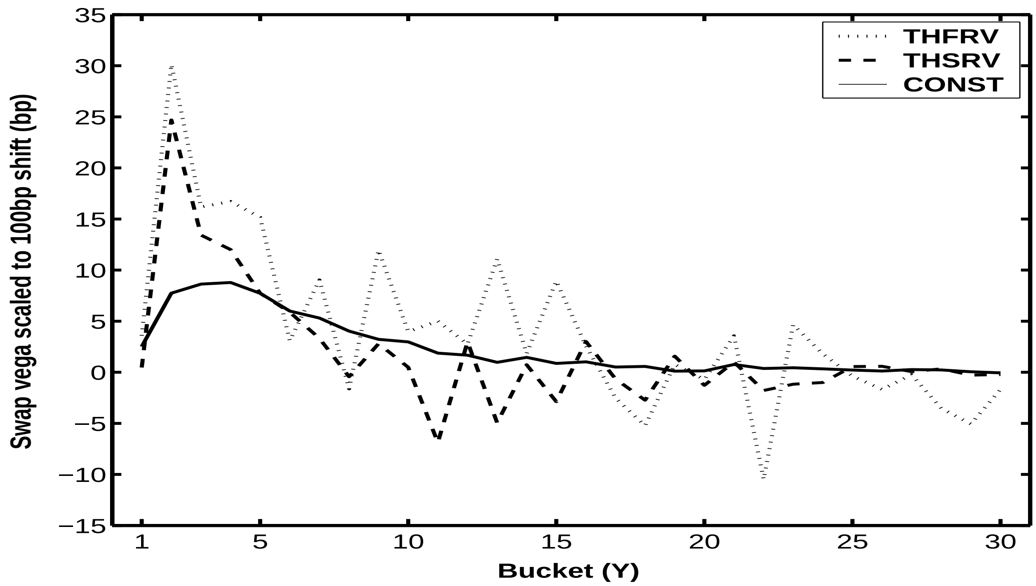

Figure 1: Re-calibration swap vega results for 10,000 simulation paths. The vega is a scaled numerical derivative and we verified that it is insensitive to the actual size of the small volatility perturbation used.

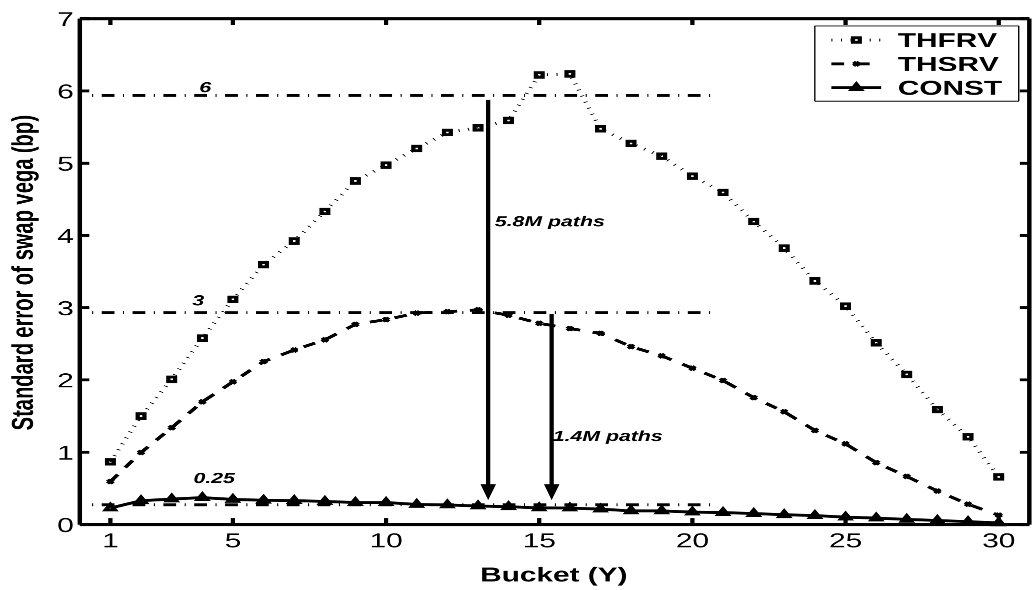

Figure 2: Empirical standard errors of the vega for 10,000 simulation paths.

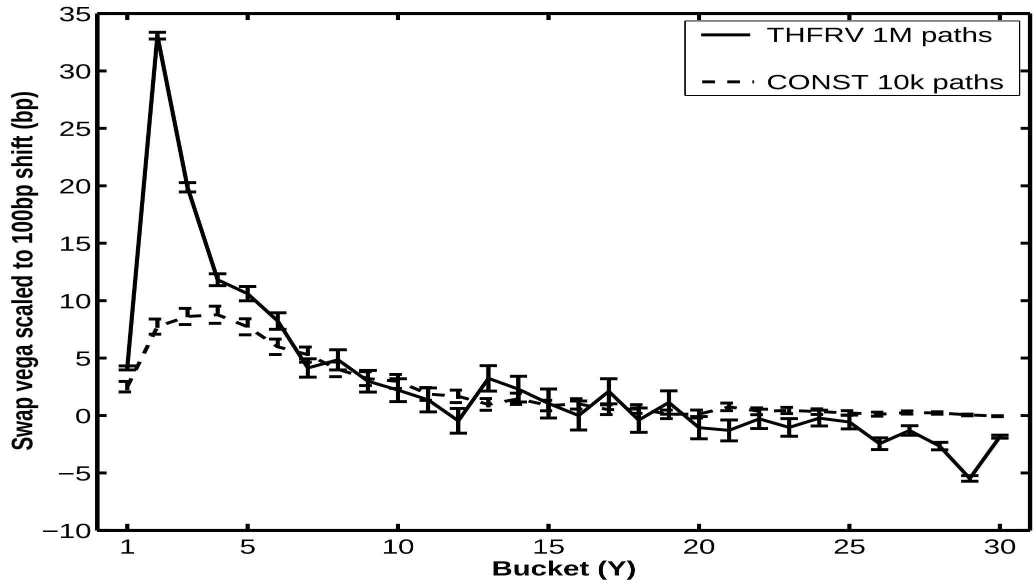

Figure 3: Re-calibration THFRV vega results for 1,000,000 simulation paths. Error bars denote a 95% confidence bound based on twice the standard error. The vega is a scaled numerical derivative and we verified that it is insensitive to the actual size of the small volatility perturbation used.

Usually the swap vega is denoted in terms of a shift in the swaption volatility. For example, consider a 100 basis point (bp) shift in the swaption volatility. The swap vega scaled to a 100 bp shift $\mathcal{V}_{i:N}^{100\text{bp}}$ is then defined by

$$\mathcal{V}_{i:N}^{100\text{bp}} = (0.01) \cdot \mathcal{V}_{i:N}.$$

Swap vega results for a Monte Carlo simulation of 10,000 scenarios have been displayed in Figure 1. The standard errors (SEs) have been displayed separately in Figure 2. The level of SE for THFRV and CONST are 6 and 0.25, respectively. The number of paths needed for THFRV to obtain the same SE as CONST is thus $(6/0.25)^2 \times 10{,}000$=5.8M. Similarly for THSRV we find that 1.4M paths are needed. Figure 3 displays the THFRV vega for 1M simulation paths.

3 Explanation

The vega results of the re-calibration approach of Section 2 will be explained. The key to this explanation is the change in swap rate instantaneous variance after re-calibration. We observed this change and noted that for the THFRV

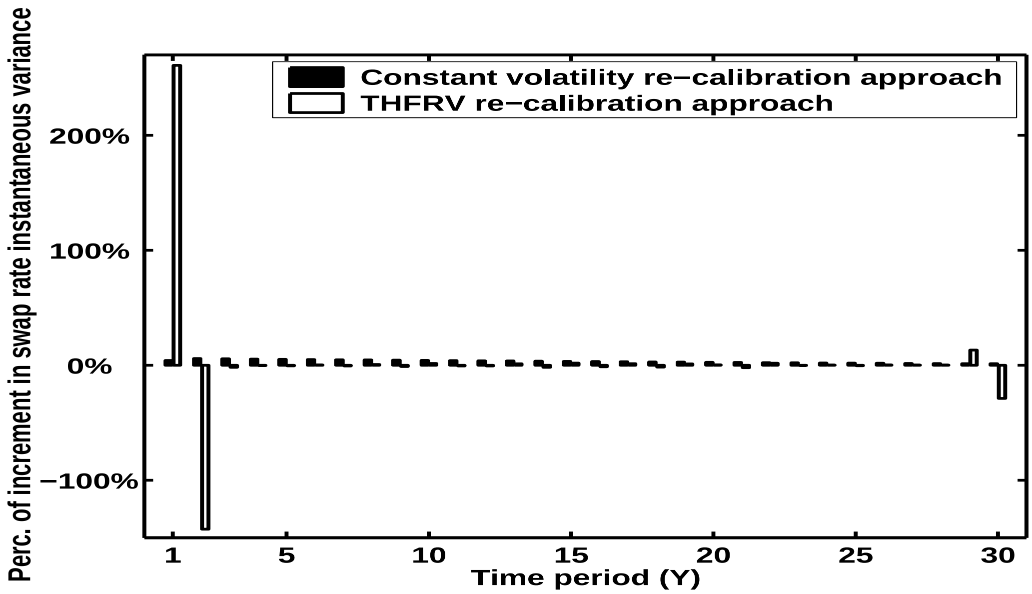

Figure 4: The observed change in swap rate instantaneous variance for the THFRV and CONST re-calibration approach for the deal setup of Section 2. Concern here is the calculation of swap vega corresponding to bucket 30. To accomplish this, the price differential has to be computed in the limit of the 30×1 swaption implied volatility perturbation $\Delta\sigma$ tending to zero. This implies a swap rate instantaneous variance increment of $30\Delta\sigma^2$. The total variance increment has to be distributed over all time periods. Note that for both data sets the sum of the variance increments equals 100%.

and THSRV re-calibration approaches the instantaneous variance increment (in the limit) is completely different from a constant volatility increment. This holds for all buckets. For illustration we restrict to the exhibit in Figure 4. As can be seen from the Figure, for THFRV the distribution of the variance increment is concentrated on the begin and end time periods and is even negative for the second time period. This deviates away from the natural and intuitive evenly distribution as dictated by the CONST re-calibration.

An explanation of the results found in Section 2 may now be given. From equation (2) it follows that the simulation variance of the vega is given by

$$\begin{aligned} \mathrm{Var}\big[\mathcal{V}_{i:N}^{100\mathrm{bp}}\big] &= c^2 \mathrm{Var}\big[P_{i:N} - P\big] \\ &= c^2 \Big\{ \mathrm{Var}\big[P_{i:N}\big] - 2\mathrm{Cov}\big[P_{i:N}, P\big] + \mathrm{Var}\big[P\big] \Big\}, \end{aligned} \tag{3}$$

with P and $P_{i:N}$ the payoff along the path of the original and perturbed model, respectively. Here $c := 0.01/\Delta\sigma_{i:N}$. The vega standard error is thus minimized if the covariance between the discounted payoff in either the original and the perturbed model is largest. This does *not* occur for a perturbation such as dictated by THFRV since the stochasticity in the simulation is basically moved around to other time periods (in our particular case from period 2 to 1). Because the rate increments over different time periods are *independent* this leads to a decrease in the covariance, in turn leading to a larger standard error of the vega.

The covariance between the payoffs is higher under the perturbations of variance implied by the CONST calibration, because then each independent time period maintains approximately the same level of variance; no stochasticity is moved to other random sources. From equation (3) it then follows that the standard error is lower.

An alternative method for calculating swap vega is developed below. The advantage of the method is that the estimates of vega have a low standard error for any volatility function. The first step is to study the definition of swap vega in the swap market model.

4 Swap Vega and the Swap Market Model

The definition of swap vega in the swap market model will be extended to Libor BGM, which will provide us with an alternative method of calculating swap vega per bucket.

How much European swaptions our dynamically managed hedging portfolio should hold is essentially determined by the swap vega per bucket. The latter is the derivative of the exotic price with respect to the Black implied swaption volatility. Consider a swap market model $\mathcal{S}$. In the latter, swap rates are log-normally distributed under their forward swap measure. This means that all swap rate volatility functions $\bar{\sigma}_{i:N}(\cdot)$ of equation (1) are deterministic. The Black implied swaption volatility $\sigma_{k:N}$ is given by

$$\sigma_{k:N} = \sqrt{\frac{1}{T_k}\int_0^{T_k} |\bar{\sigma}_{k:N}(s)|^2 \mathrm{d}s}.$$

As may be seen from the above equation, there are an uncountable number of perturbations of the swap rate instantaneous volatility to obtain the very same perturbation of the Black implied swaption volatility. There is however a natural 1-dimensional parameterized perturbation of the swap rate instantaneous volatility, namely a simple proportional increment. This has been illustrated in Figure 5.

The definition of swap vega in the swap market model is as follows. Denote the price of an interest rate derivative in a swap market model $\mathcal{S}$ by V.

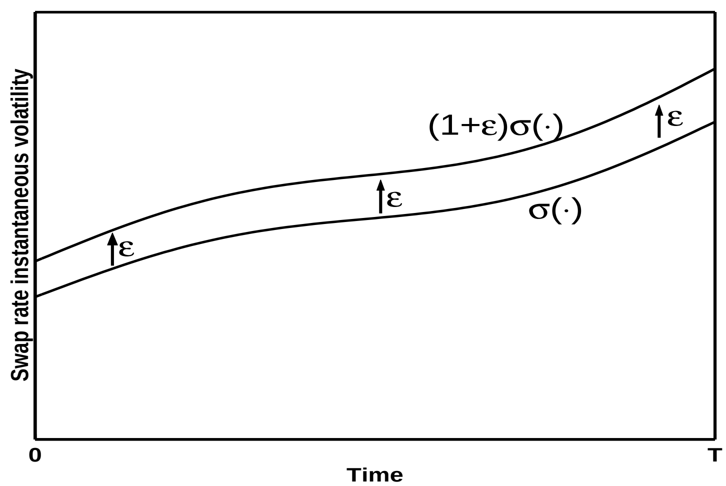

Figure 5: Natural increment of Black implied swaption volatility through proportional increment of swap rate instantaneous volatility: $\sigma(\cdot)$ becomes $(1+\varepsilon)\sigma(\cdot)$.

Consider a perturbation of the swap rate instantaneous volatility given by

$$\bar{\sigma}^{\varepsilon}_{k:N}(\cdot) = (1+\varepsilon)\,\bar{\sigma}_{k:N}(\cdot), \tag{4}$$

the shift applied only to $k:N$. Denote the corresponding swap market model by $\mathcal{S}_{k:N}(\varepsilon)$. Note that the implied swaption volatility in $\mathcal{S}_{k:N}(\varepsilon)$ is given by $\sigma^{\varepsilon}_{k:N} = (1+\varepsilon)\sigma_{k:N}$. Denote the price of the derivative in $\mathcal{S}_{k:N}(\varepsilon)$ by $V_{k:N}(\varepsilon)$. Then the swap vega per bucket $\mathcal{V}_{k:N}$ is defined as

$$\mathcal{V}_{k:N} = \lim_{\varepsilon\to 0} \frac{V_{k:N}(\varepsilon) - V}{\varepsilon \sigma_{k:N}}. \tag{5}$$

Equation (5) is the derivative of the exotic price with respect to the Black implied swaption volatility; using suggestive notation we may write

$$\mathcal{V}_{k:N} = \frac{\partial V}{\partial \sigma_{k:N}} = \lim_{\Delta\sigma_{k:N}\to 0} \frac{V(\sigma_{k:N} + \Delta\sigma_{k:N}) - V(\sigma_{k:N})}{\Delta\sigma_{k:N}}. \tag{6}$$

In equation (5) $\varepsilon\sigma_{k:N}$ is equal to the swaption volatility perturbation $\Delta\sigma_{k:N}$ and $V_{k:N}(\varepsilon)$ and V denote the prices of the derivative in models in which the k^{th} swaption volatility equals $\sigma_{k:N} + \Delta\sigma_{k:N}$ and $\sigma_{k:N}$, respectively.

The swap rate volatility perturbation of equation (4) defines a relative shift. It is also possible to apply an absolute shift in the form of

$$\bar{\sigma}^{\varepsilon}_{k:N}(\cdot) = \left(1 + \frac{\varepsilon}{|\bar{\sigma}_{k:N}(\cdot)|}\right) \bar{\sigma}_{k:N}(\cdot), \tag{7}$$

the shift applied only to $k:N$. This ensures that the absolute level of the swap rate instantaneous volatility is increased by an amount ε. Note that the relative and absolute perturbation are equivalent when the instantaneous volatility is constant over time. The method for calculating swap vega per bucket is largely the same for both the relative and absolute perturbation. If there are any differences then these will be pointed out in the text. The first difference is in the change in swaption implied volatility $\Delta\sigma_{k:N}$ of equation (6), namely straightforward calculations reveal that the perturbed volatility satisfies

$$\sigma^{\varepsilon}_{k:N} = \sigma_{k:N} + \varepsilon\,\frac{\frac{1}{T_k}\int_0^{T_k} |\bar{\sigma}_{k:N}(s)|\mathrm{d}s}{\sigma_{k:N}} + \mathcal{O}\big(\varepsilon^2\big).$$

5 Alternative Method for Calculating Swap Vega

In this section an alternative method for calculating swap vega in the BGM framework is presented. It may be applied to any volatility function to yield accurate vega at a low number of simulation paths. The method is based on a perturbation in the forward rate volatility to match a constant swap rate volatility increment. The method is briefly hinted at in Section 10.6.3 of Rebonato (2002) in terms of covariance matrices. In comparison, our derivation below is written more insightfully in terms of the volatility vectors.

Swap rates are not log-normally distributed in Libor BGM. This means that swap rate instantaneous volatility is stochastic. The stochasticity is however almost not apparent as shown empirically for example by Brace et al. (1998). In Section 1.5.5 of D'Aspremont (2002) it is shown that the swap rate is uniformly close to a log-normal martingale.

In Hull & White (2000) it is shown that the swap rate volatility vector is a weighted average of forward Libor rate volatility vectors;

$$\bar{\sigma}_{i:N}(t) = \sum_{j=1}^{N} w_j^{i:N}(t)\bar{\sigma}_j(t), \quad w_j^{i:N}(t) = \frac{\delta_j \gamma_j^{i:N}(t) L_j(t)}{1+\delta_j L_j(t)}, \tag{8}$$

$$\gamma_j^{i:N}(t) = \frac{B_i(t)}{B_i(t) - B_{N+1}(t)} - \frac{\mathrm{PVBP}_{i:(j-1)}(t)}{\mathrm{PVBP}_{i:N}(t)},$$

where the weights $w^{i:N}$ are in general state-dependent.

Hull and White derive an approximating formula for European swaption prices based on evaluating the weights in equation (8) at time zero. The quality of this approximation is high in virtue of the near log-normality of swap rates in Libor BGM. We will denote the resulting swap rate instantaneous volatility by $\bar{\sigma}^{\mathrm{HW}}_{i:N}$, thus

$$\bar{\sigma}^{\mathrm{HW}}_{i:N}(t) = \sum_{j=i}^{N} w_j^{i:N}(0)\bar{\sigma}_j(t). \tag{9}$$

Write $w_j^{i:N} := w_j^{i:N}(0)$ and make the convention that

$$\bar{\sigma}_i(t) = \bar{\sigma}_{i:N}(t) = 0 \quad \text{when} \quad t > T_i,$$

then an insightful presentation of equation (9) is:

$$\begin{array}{ccccccc} \bar{\sigma}^{\mathrm{HW}}_{1:N}(t) & = & w_1^{1:N}\bar{\sigma}_1(t) & + & \dots & + & w_N^{1:N}\bar{\sigma}_N(t) \\ \vdots & & & & \ddots & & \vdots \\ \bar{\sigma}^{\mathrm{HW}}_{N:N}(t) & = & & & & & w_N^{N:N}\bar{\sigma}_N(t) \end{array} . \tag{10}$$

If W is this upper triangular non-singular weight matrix (with upper triangular inverse W^{-1}) then these volatility vectors can be jointly related through the matrix equation

$$\left[\ \bar{\sigma}_{\cdot:N}\ \right] = W \left[\ \bar{\sigma}_{\cdot}\ \right].$$

The swap rate volatility under relative perturbation (equation (4)) of the k^{th} volatility is

$$\left[\ \bar{\sigma}_{\cdot:N}\ \right] \rightarrow \left[\ \bar{\sigma}_{\cdot:N}\ \right] + \varepsilon \left[\ 0 \quad \dots \quad 0 \quad \bar{\sigma}_{k:N} \quad 0 \quad \dots \quad 0\ \right]^{\top}.$$

Note that the swap rate correlation is left unaltered. The corresponding perturbation in the BGM volatility vectors is given by

$$\left[\ \bar{\sigma}_{\cdot}\ \right] \rightarrow \left[\ \bar{\sigma}_{\cdot}\ \right] + \varepsilon W^{-1} \left[\ 0 \quad \dots \quad 0 \quad \bar{\sigma}_{k:N} \quad 0 \quad \dots \quad 0\ \right]^{\top}. \tag{11}$$

Note that only the volatility vectors $\bar{\sigma}_k(t), \dots, \bar{\sigma}_N(t)$ are affected (due to the upper triangular nature of W^{-1}), which are the vectors that underly $\bar{\sigma}_{k:N}(t)$ in the Hull and White approximation. With the new Libor volatility vectors, prices can be recomputed in the BGM model and the vegas calculated.

6 Numerical Results

In this section the algorithm of Section 5 is applied to the deal setup of Section 2. The results for a simulation with 10,000 scenarios have been displayed in Figure 6. Note that the approach yields slightly negative vegas for buckets 17-30. In the appendix this negativity is shown not to be a spurious result.

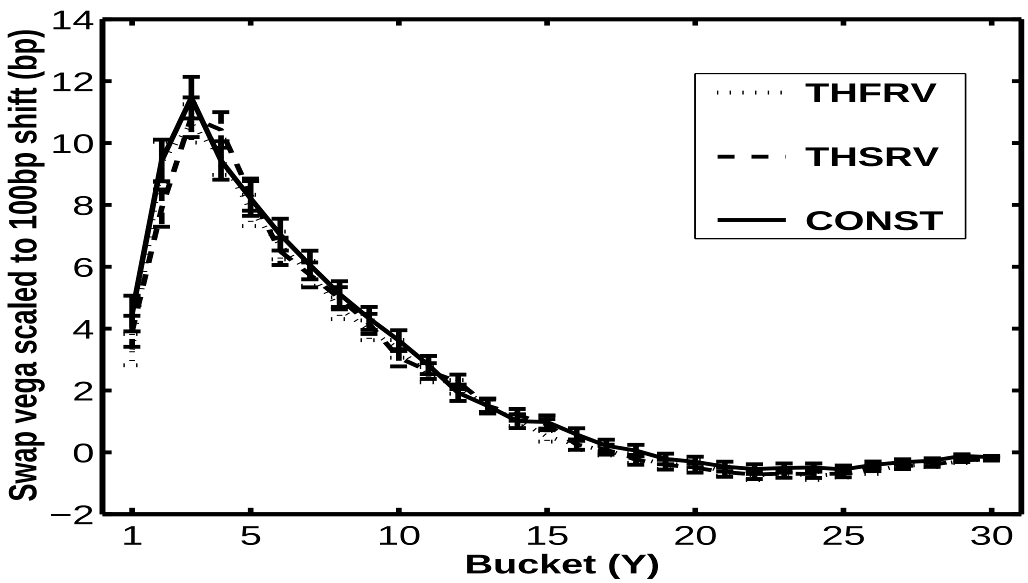

Figure 6: Method of Section 5 swap vega results for 10,000 simulation paths. Error bars denote a 95% confidence bound based on the standard error. The vega is a scaled numerical derivative and we verified that it is insensitive to the actual size of the small volatility perturbation used.

Namely for the analytically tractable setup of a two stock Bermudan option it is shown that negativity of vega occurs with correlation $\approx$1 and with the volatility for short expiries higher than the volatility at longer expiries – this of course is a typical interest rates setting.

The vegas have been displayed for the relative perturbation method. These vegas have been calculated as well using the absolute perturbation method. The differences in the vegas for the two methods are minimal; for any vega with absolute value above 1 bp the difference is less than 4%, and for any vega with absolute value below 1 bp the difference is always less than a third of a basis-point.

7 Comparison with the Swap Market Model

This section reports the results of an empirical comparison with the swap market model, which is the canonical model for computing swap vega per bucket. The key is to compare the Libor BGM model against a swap market model with the very same swap rate quadratic cross-variation structure. This (approximate) equivalence between the two models was established by Joshi

& Theis (2002), equation (3.8).

The test was performed for an 11NC1 pay-fixed Bermudan option on a swap with annual fixed and floating payments. A single-factor Libor BGM model was taken with constant volatility calibrated to the cap volatility curve of 10 October 2001. The zero rates were taken to be flat at 5%. In the Monte Carlo simulation of the SMM we applied the discretization suggested in Lemma 5 of Glasserman & Zhao (2000). Results may be found in Table 2 and have been displayed partially in figures 7 and 8. In this particular case the BGM Libor model reproduces the swap vegas of the swap market model with high accuracy.

8 Conclusions

This article presented a novel approach for calculating swap vega per bucket in the Libor BGM model. We showed that for some forms of the volatility an approach based on re-calibration may lead to a large uncertainty in estimated swap vega, as the instantaneous volatility structure may be distorted by re-calibration. This does not happen in the case of constant swap rate volatility. We then derived an alternative approach, not based on re-calibration, by comparison with the swap market model. The strength of the method is that it accurately estimates swaption vegas for any volatility function and at a low number of simulation paths. The key to the method is that the perturbation in the Libor volatility is distributed in a clear, stable and well understood fashion, whereas in the re-calibration method the change in volatility is hidden and potentially unstable. We also showed for a Bermudan swaption deal that our method yields almost identical swap vega as found in a swap market model.

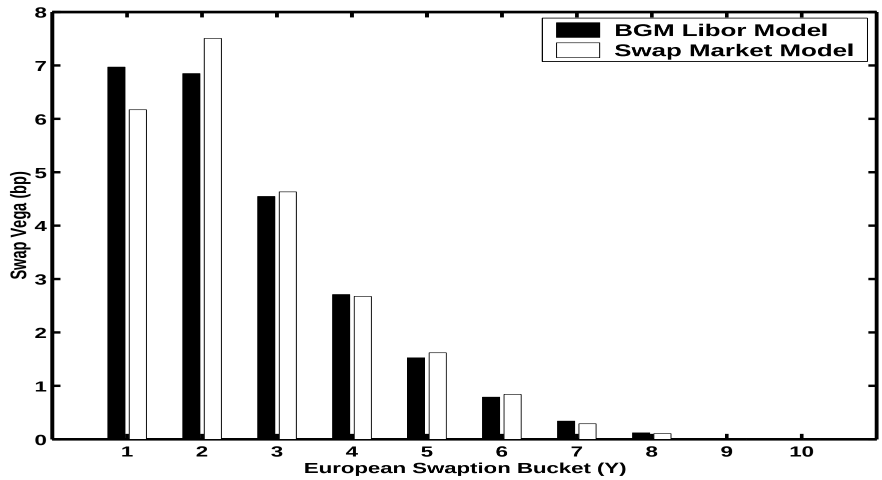

Figure 7: Comparison of LMM and SMM for swap vega per bucket. The fixed rate of the swap is 5%.

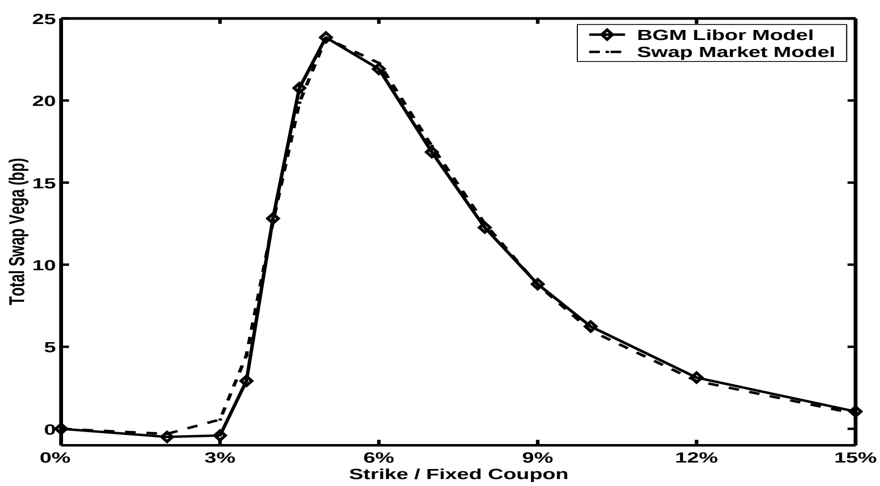

Figure 8: Comparison of LMM and SMM for total swap vega against strike.

Table 2: Swap vega per bucket test results for varying strikes (fixed rate of the swap). Prices and vegas are stated in basis points. The standard error is denoted within parentheses. 10,000 simulation paths.

BGM LIBOR MODEL

Fixed Rate	2%	3%	3.5%	4%	4.5%	5%	6%	7%	8%	9%	10%	12%	15%
Value	2171	1476	1138	829	585	410	210	112	64	36	21	8	2
	(4)	(5)	(5)	(5)	(5)	(4)	(3)	(2)	(2)	(1)	(1)	(1)	(0)
1Y	-2.0	-2.0	2.6	10.9	11.1	7.0	1.2	0.1	0.0	0.0	0.0	0.0	0.0
2Y	1.5	1.6	1.0	2.6	5.7	6.8	4.0	1.0	0.0	0.0	0.0	0.0	0.0
3Y	0.0	0.0	-0.3	0.1	2.5	4.5	4.1	2.1	1.0	0.3	0.0	0.0	0.0
4Y	0.0	0.0	-0.1	-0.1	1.1	2.7	4.4	3.6	2.0	1.1	0.5	0.2	0.1
5Y	0.0	0.0	-0.1	-0.2	0.4	1.5	3.7	3.6	2.7	1.5	1.0	0.3	0.1
6Y	0.0	0.0	-0.1	-0.2	0.1	0.8	2.1	2.5	2.0	1.7	1.2	0.3	0.2
7Y	0.0	0.0	-0.1	-0.2	0.0	0.3	1.3	1.8	1.8	1.6	1.1	0.5	0.0
8Y	0.0	0.0	0.0	-0.1	-0.1	0.1	0.7	1.3	1.5	1.3	1.3	0.9	0.3
9Y	0.0	0.0	0.0	-0.1	-0.1	0.0	0.3	0.7	0.8	0.8	0.8	0.6	0.3
10Y	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.3	0.4	0.4	0.3	0.2
Total Vega	-0.5	-0.4	2.9	12.8	20.8	23.8	21.9	16.9	12.3	8.8	6.2	3.1	1.0

SWAP MARKET MODEL

Fixed Rate	2%	3%	3.5%	4%	4.5%	5%	6%	7%	8%	9%	10%	12%	15%
Value	2172	1480	1146	841	592	411	204	109	61	34	19	7	1
	(6)	(6)	(6)	(5)	(5)	(4)	(4)	(3)	(2)	(1)	(1)	(1)	(0)
1Y	-1.9	-0.7	4.4	11.3	11.5	6.2	0.4	0.0	0.0	0.0	0.0	0.0	0.0
2Y	1.6	1.6	1.1	2.2	5.2	7.5	3.6	0.5	0.0	0.0	0.0	0.0	0.0
3Y	0.0	-0.1	-0.4	0.0	2.0	4.6	4.7	2.2	0.6	0.2	0.0	0.0	0.0
4Y	0.0	-0.1	-0.2	-0.1	0.9	2.7	4.8	3.7	1.7	0.8	0.3	0.1	0.0
5Y	0.0	0.0	-0.2	-0.2	0.4	1.6	3.7	3.0	2.3	1.2	0.5	0.1	0.0
6Y	0.0	0.0	-0.1	-0.2	0.1	0.8	2.6	3.3	3.1	2.3	1.2	0.2	0.0
7Y	0.0	0.0	-0.1	-0.2	-0.1	0.3	1.3	2.0	1.9	1.3	1.4	0.8	0.1
8Y	0.0	0.0	0.0	-0.1	-0.1	0.1	0.8	1.3	1.5	1.5	1.2	0.6	0.2
9Y	0.0	0.0	0.0	-0.1	-0.1	0.0	0.4	0.9	1.0	1.0	0.9	0.7	0.3
10Y	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.4	0.5	0.5	0.4	0.3
Total Vega	-0.3	0.6	4.5	12.6	19.9	23.8	22.3	17.2	12.5	8.8	6.0	2.9	0.9

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Appendix: Negative vega for a two stock Bermudan option

In this appendix a two stock Bermudan option is studied; in particular its vega per bucket is analyzed and it is shown that it is negative for certain situations. The holder of a two stock Bermudan has the right to call the first stock S_1 at strike K_1 at time T_1; if he or she decides to hold on to the option then the right remains to call the second stock S_2 at strike K_2 at time T_2; if this right is not exercised then the option becomes worthless. Here $T_1 < T_2$.

The Bermudan option will be valued in the standard Black-Scholes world. Under the risk-neutral measure the stock prices satisfy the following SDEs

$$\frac{\mathrm{d}S_i}{S_i} = r\mathrm{d}t + \sigma_i \mathrm{d}W_i, \quad i = 1, 2,$$

$$\mathrm{d}W_1 \mathrm{d}W_2 = \rho \mathrm{d}t.$$

Here σ_i is the volatility of the i^{th} stock. W_i, $i = 1, 2$, are Brownian motions under the risk-neutral measure, with correlation ρ. It follows that the time-T_1 stock prices are distributed as follows

$$S_i(T_1) = F(S_i(0), 0; T_1) \exp\left\{ \sigma_i \sqrt{T_1} Z_i - \frac{1}{2}\sigma_i^2 T_1 \right\}, \quad i = 1, 2, \tag{12}$$

where the pair (Z_1, Z_2) is standard bivariate normally distributed with correlation ρ and where

$$F(S, t; T) := S \exp\left\{ r\big(T - t\big) \right\}$$

is the time-t forward price for delivery at time T of a stock with current price S. At time T_1 the holder of the Bermudan will choose whichever of the two following alternatives has a higher value: either calling the first stock or holding onto the option on the second stock; the value of the latter is given by the Black-Scholes formula. Therefore the (cash-settled) payoff $V(S_1(T_1), S_2(T_1), T_1)$ of the Bermudan at time T_1 is given by

$$\max\left\{ \big(S_1(T_1) - K_1 \big)_+, \ \mathrm{BS}_2\big(S_2(T_1), \ T_1 \big) \right\},$$

where BS is the Black-Scholes formula,

$$\begin{aligned}
\mathrm{BS}_i(S, T) &= e^{-r(T_i - T)}\left\{ F(S, T; T_i) N(d_1^{(i)}) - K_i N(d_2^{(i)}) \right\} \\
d_{1,2}^{(i)}(S, T) &= \frac{\ln\big(F(S, T; T_i)/K_i\big) \pm \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}.
\end{aligned}$$

Table 3: Deal setup for examples where a vega per bucket for the two stock Bermudan option is negative.

spot price for stock 1	$S_1(0)$	150
spot price for stock 2	$S_2(0)$	140
strike price for stock 1	K_1	100
strike price for stock 2	K_2	100
exercise time for stock 1	T_1	1Y
exercise time for stock 2	T_2	2Y
volatilities	σ_i	Variable
correlation	ρ	0.9
risk-free rate	r	5%

Here $N(\cdot)$ is the cumulative normal distribution function. The time-zero value $V(S_1, S_2, 0)$ of the Bermudan option may thus be computed by a bivariate normal integration of the discounted version of the above payoff

$$V(S_1, S_2, 0) = e^{-rT_1}\,\mathbb{E}\Big[\,V\big(\,T_1, S_1(T_1), S_2(T_1)\,\big)\,\Big].$$

The vega per bucket $\mathcal{V}_i$ is defined as

$$\mathcal{V}_i := \frac{\partial V(S_1, S_2, 0)}{\partial \sigma_i}, \quad i = 1, 2.$$

The vega may be numerically approximated by finite differences

$$\mathcal{V}_i = \frac{V(S_1, S_2, 0; \sigma_i + \Delta\sigma_i) - V(S_1, S_2, 0; \sigma_i)}{\Delta\sigma_i} + \mathcal{O}(\Delta\sigma_i^2), \quad i = 1, 2,$$

for a small volatility perturbation $\Delta\sigma_i \ll 1$. We claim that the vega per bucket may possibly be negative; this may occur for both the first and the second bucket. To provide examples of vega negativity, the vega per bucket has been computed for the deal setup described in Table 3. Results have been displayed in Table 4. The volatility was perturbed by a small amount. It was verified that the resulting vega was insensitive to either the perturbation size or the density of the 2D integration grid. The results clearly establish instances where a vega per bucket is negative, both for the first and second bucket. To ensure that the negative vega was not due to an implementation error, an alternative valuation of the two stock Bermudan option was developed in a private paper of the authors, available from the authors upon

Table 4: Examples where a vega per bucket for the two stock Bermudan option is negative. The vega has been re-scaled to a 100 bp volatility shift (1%), $\mathcal{V}_i^{100\text{bp}} = (0.01)\mathcal{V}_i$.

	σ_1	σ_2	price	$\mathcal{V}_1^{100\text{bp}}$	$\mathcal{V}_2^{100\text{bp}}$
scenario 1	10%	30%	64.53	-0.45	0.56
scenario 2	30%	10%	65.11	0.56	-0.44

request. The alternative valuation is based on conditioning and involves a one-dimensional numerical integration over the Black formula. Because the alternative valuation method is different in nature, it may be viewed as an independent implementation. Indeed, the alternative independently implemented method yielded the exact same results. Moreover, we provide in the developments below an economic explanation for the occurrence of negative vega.

An economic explanation of the negative vega is given as follows. Note in Table 4 that the negative vegas occur in case of high correlation and for the bucket with lowest volatility. In the case of high correlation and one stock having significantly higher volatility than the other, we contend that the only added value of the additional option on the low volatility stock lies in offering protection against a down move of both stocks (recall that the stocks are highly correlated). Namely, consider the following two scenarios:

- *Up move.* Both stocks move up. Because the high volatility stock moves up much more than the low volatility stock, the high volatility call will be exercised.

- *Down move.* Both stocks move down. Because the high volatility stock moves down much more than the low volatility stock, the high volatility call becomes out of the money and the low volatility call will be exercised.

If now the volatility of the low volatility stock is increased by a small amount, then in the above scenarios the exercise strategy remains unchanged. Also, in case of an up move, the payoff remains unaltered. However, in case of a down move, the low volatility stock (volatility slightly increased) moves down more than in the unperturbed case. Therefore the payoff of the protection call decreases. In total, the value of the Bermudan option thus decreases.