# Completing Markets in a One-Good, Pure Exchange Economy

# Without State-Contingent Securities

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### ABSTRACT

**Pareto-efficient consumption in a pure-exchange, one good economy varies over states of nature with respect to only two factors: real aggregate supply and individual utility shocks. One's optimal contract receipts vary with respect to only these two factors and the ratio of one's endowment to real aggregate supply. How one's Pareto-efficient consumption varies with real aggregate supply depends solely on how one's relative risk aversion compares to the average. Complete markets can be approximately achieved by four contracts dealing with these factors. This has implications concerning central banking, efficient insurance contract design, and a possible new financial innovation.**

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Under fairly general assumptions, an Arrow-Debreu economy with state-contingent securities results in a Pareto-efficient consumption allocation. However, for a variety of reasons, state-contingent securities are impractical in the real world. Many financial economists make the presumption that insurance contracts and the financial innovations of options, futures, swaps, and other derivatives are helping to complete markets even though these contracts are not state-contingent securities. This paper reports on a theoretical investigation into whether contracts or securities other than non-state-contingent securities can complete the markets in a pure-exchange economy without storage. We find that markets in such an economy can be approximately completed with four different types of contracts. These contracts are (i) normal contracts, (ii) endowment-sharing contracts, (iii) spending-sharing contracts, and (iv) Real-Aggregate-Supply-Risk-Transfer (RASRT) contracts.

When Arrow (1953) and Debreu (1959) first discussed economies with state-contingent securities, they noted that complete markets would require $n \cdot c \cdot T$ state-contingent securities where n is the number of states, $c$ is the number of commodities that exist, and $T$ is the time horizon of the economy. Arrow (1953) did surmise that, if the state-contingent securities were stated in terms of a numeraire, complete markets could be achieved with $n \cdot T$ such securities. Later, Radner (1972) showed Arrow's surmise to be correct when markets were open in sequential economies. Even so, for large $T$ and large n, $n \cdot T$ securities will be impractically large. In particular if we just considered the characteristic of temperature in 50,000 locations throughout the world and we considered only 10 temperature ranges for each location, the number of temperature-location states would be $10^{50,000}$. If we would consider just three other characteristics of each location with each characteristic having 10 possible ranges, the number of possible states would grow to $10^{200,000}$. Yet, we have only begun to enumerate all the possible states of nature. Other characteristics would include diseases, terrorist acts, volcanic activity, earthquakes, war, and much more. Enumerating all states of nature is clearly impractical. A further problem is that economic agents may not be able to conceive of all possible states of nature. Even if they could conceive of all possible states of nature, the cost of writing the legal documents to clearly define when a state is considered to have occurred very well may be prohibitively expensive and otherwise impractical.

Financial economists presume that insurance, futures, options, swaps, and other derivatives are moving our economies towards having complete markets. However, this is a presumption, not something that has been logically proven theoretically. Answering this question for a completely general economy is likely to be a very complex undertaking. Instead, this paper focuses on pure-exchange economies without storage. In such an economy, Eagle and Domian (2003 and 2004) show that quasi-real bonds by themselves complete markets as long as individuals have the same relative risk aversion, no utility shocks occur, and individuals have ratios of endowment to real aggregate supply that do not vary across states of nature. This paper goes beyond the work of Eagle and Domian to discuss pure-exchange economies without storage in general. In particular the conclusions of this paper do apply to situations where individuals differ in their relative risk aversion, where their ratios of endowment to real aggregate supply are stochastic, and where individual utility shocks occur.

We define **optimal contract receipts** to be the real (inflation-adjusted) amounts for each state of nature that an individual would receive (or pay) if an economy with complete state-contingent security markets existed. We find that the combination of four types of non-state-contingent securities can approximate these optimal contract receipts. Important to this finding are the following realizations reported in this paper:

1. Pareto-efficient consumption only varies with respect to two factors: (i) real aggregate supply and (ii) individual utility shocks.

2. Optimal contract receipts vary with respect to only three factors: (i) real aggregate supply, (ii) individual utility shocks, and (ii) changes in the ratio of endowment to real aggregate supply.

3. When there are no individual utility shocks, the proportionality of an individual's Pareto-efficient consumption to real aggregate supply depends only on how the individual's relative risk aversion compares to average relative risk aversion.

Also, this paper derives the precise relationship between relative risk aversion and how an individual's Pareto-efficient consumption changes when real aggregate supply changes. All of these results are important to help us understand why the four types of contracts discussed in this paper will approximately lead to complete markets.

Section II below reviews the standard Arrow-Debreu pure exchange economy and presents and proves the consumption-aggregate-supply-invariance property, which is the property upon which much of this paper's results rest. This section also proves and derives other important results including the very important relationship between relative risk aversion and how individuals' Pareto-efficient consumption varies with real aggregate supply. Section III then discusses how normal contracts, endowment-sharing contracts, and spending-sharing contracts complete the markets when all consumers have the same relative risk aversion. Section IV discusses how RASRT contracts could be designed to help consumers transfer real-aggregate-supply risk among each other if their relative risk aversions differ. Section V discusses the pricing of RASRT contracts. Section VI then presents an example of how consumers could use all four contracts simultaneously to approximately replicate their optimal contract receipts.

Section VII summarizes this papers results and discusses how these results have implications to real world economies.

## II. Arrow-Debreu Pure Exchange Economy with State-Contingent Securities

This section reviews a standard Arrow-Debreu pure exchange economy without storage consisting of one nonstorable consumption good. Assume each consumer j's time-separable utility function is:

$$U_{j0}(c_{j0}) + \sum_{t=1}^{T} \beta^t \sum_{s=1}^{n_t} \pi_{st} U_{jst}(c_{jst}) \tag{1}$$

where $c_{j0}$ is j's consumption at time 0, $c_{jst}$ is j's consumption in state s at time t, β is the time discount factor, and $\pi_{st}$ is the probability of state s occurring at time t. The functions $U_{j0}(c_{j0})$ and $U_{jst}(c_{jst})$ are continuous, twice differentiable, strictly concave, and strictly increasing. To rule out corner solutions, assume $\lim_{c \to 0} U'_{j0}(|c|) = \lim_{c \to 0} U'_{jst}(|c|) = +\infty$. The time frame for the s subscript is determined by the t subscript next to the s subscript. For example, the s in $c_{js1}$ refers to one of the possible states that can occur at time 1.

At time 0, consumers can buy or sell state-contingent securities. These state-contingent securities are prepaid securities where the buyer pays the seller the price of the security at time 0. Let $x_{jst}$ represent individual j's demand at time 0 for the state-contingent security that delivers one consumption good at time t iff state s occurs at time t. Define $\Omega_{st}$ so that the price of this security equals $P_0 \pi_{st} \Omega_{st}$. With it so defined, $\Omega_{st}$ represents the real pricing kernel.

Each consumer j chooses $x_{jst}$ for all s and t to maximize (1) subject to:

$$P_0 c_{j0} + P_0 \sum_{t=1}^{T} \sum_{s=1}^{S_t} \pi_{st} \Omega_{st} x_{jst} = P_0 y_{j0} \tag{2}$$

$$c_{jst} = y_{jst} + x_{jst} \tag{3}$$

where (3) applies for all s=1,2,…,$S_t$ for all t=1,2,…,T where $S_t$ is the finite number of states of nature at time t.

The market clearing conditions are that $\sum_{j=1}^{n} c_{j0} = Y_0$ , $\sum_{j=1}^{n} c_{jst} = Y_{st}$ , and $\sum_{j=1}^{n} x_{jst} = 0$ for all states s at time t and for t=1,2,…T, where the aggregate supply of the consumption good is represented by $Y_0$ at time 0 and $Y_{st}$ in state s at time t respectively. Consumer j's optimization problem is satisfied when $\frac{U'_{j0}(c_{j0})}{P_0} = \frac{\beta^t \pi_{st} U'_{jst}(c_{jst})}{P_0 \pi_{st} \Omega_{st}}$ for all s=1,2,…,$S_t$ and for all t=1,2,…,T, which implies that

$$\Omega_{st} = \frac{\beta^t U'_{jst}(c_{jst})}{U'_{j0}(c_{j0})} \tag{4}$$

The left side of (4) is the real pricing kernel and the right side is the intertemporal marginal rate of substitution. Some literature mistakenly defines the pricing kernel as the intertemporal marginal rate of substitution (See, for example, Campbell, Lo, and MacKinlay, 1997, p. 294). The equality between the real pricing kernel and the intertemporal marginal rate of substitution shown in (4) is an equilibrium condition not a definition.

Since this is a standard one-good Arrow-Debreu pure-exchange economy with well behaved utility functions, a unique competitive equilibrium exists and that competitive equilibrium is Pareto efficient. Also, the following property holds:

**Consumption-Aggregate-Supply Invariance Property**: Let 1 and 2 represent any two different states of nature. If real aggregate supply and each consumer's utility function $U_{jst}(.)$ is the same in both states of nature (i.e., there are no utility shocks), then every individual's consumption will be the same in both states of nature.

Proof by contradiction. Assume there is some consumption allocation in a competitive equilibrium where for some states 1 and 2, each consumer's utility function is the same for both states 1 and 2, $Y_{1t}=Y_{2t,}$ and there are two individuals j and k such that $c_{j1t} < c_{j2t}$ and $c_{k1t} > c_{k2t}$. Since this is an Arrow-Debreu competitive equilibrium, the consumption allocation must be Pareto efficient. Define $\tilde{c}_{j1t} \equiv \frac{1}{2}\left(c_{j1t} + c_{j2t}\right)$ and $\tilde{c}_{k1t} \equiv \frac{1}{2}\left(c_{k1t} + c_{k2t}\right)$. Define a new consumption allocation where for all consumers, for all states of nature, and for all time periods, the new consumption equals the old consumption except that j's consumption in states 1 and 2 are both $\tilde{c}_{j1t}$ and k's consumption in states and 2 are both $\tilde{c}_{k1t}$. The new consumption allocation is obviously feasible since the original allocation was feasible. Because both j and k are strictly risk averse, they are both better off with this new consumption allocation. However, that contradicts the statement that the original consumption allocation is Pareto efficient. We, therefore, conclude that the consumption allocation must be the same as long as neither aggregate output nor the form of the utility functions changes. Q.E.D.

The consumption-aggregate-supply-invariance property is the foundation for this paper.

Before we discuss implications of this very important property, let us first distinguish between aggregate utility shocks and individual utility shocks. Define **aggregate utility shocks** to be shocks to everyone's utility so that individuals' Pareto-efficient consumption do not change as a result. Individual utility shocks, on the other hand, do affect not only the individual's Pareto-efficient consumption but also other individuals' Pareto-efficient consumption. With this distinction made, we are now ready to discuss two very important corollaries of the consumption-aggregate-supply-invariance property:

**<u>Corollary 1</u>**: Let j represent any particular individual in a pure-exchange economy without storage. Individual j's Pareto-efficient consumption at time t varies across states of nature at time t only if changes occur in one of two and only two factors: These factors are (i) real aggregate supply at time t, and (ii) individual utility shocks at time t, either to individual j or someone else.

The consumption-aggregate-supply-invariance property assumes no utility shocks and states that as long as real aggregate supply remains the same, an individual's Pareto-efficient consumption will remain the same. Therefore, it immediately follows that Pareto-efficient consumption only varies with changes in real aggregate supply or utility shocks. However, by definition, aggregate utility shocks do not affect Pareto-efficient consumption. Hence, Pareto-efficient consumption varies only with changes in real aggregate supply and individual utility shocks.

**<u>Corollary 2</u>**: Individual j's optimal contract receipts at time t vary across states of nature at time t only if changes occur in one of three and only three factors: (i) real aggregate supply at time t, (ii) individual utility shocks at time t, either to individual j or someone else, and (iii) the ratio of j's endowment to real aggregate supply at time t.

Remember how j's consumption relates to j's endowment and j's holding of the relevant state-contingent security. Equation (3) states that $c_{jst} = y_{jst} + x_{jst}$. Where individual j chooses $c_{jst}$ and $x_{jst}$ optimally, $x_{jst}$ will represent j's optimal contract receipts in state s at time t. Therefore, the optimal contract receipts equal the difference between j's Pareto-efficient consumption and j's endowment. By corollary 1 above, if real aggregate supply does not change and no individual utility shocks occur, then the Pareto-efficient consumption does not change. Therefore, the only other reason by which the optimal contract receipts will change will be if j's endowment changes. Clearly, if real aggregate supply does not change, then the ratio of j's endowment to real aggregate supply will change iff j's endowment changes. Therefore, j's optimal contract receipts can only vary if real aggregate supply changes, individual utility shocks occur, or j's ratio of endowment to real aggregate supply changes. (We state corollary 2 in terms of j's ratio of endowment to real aggregate supply rather than just j's endowment because it fits better with subsequent results concerning the relationship between relative risk aversion and the proportionality of Pareto-efficient consumption to real aggregate supply.)

Another important property of the pure exchange economy without storage is the relationship between relative-risk aversion, Pareto-efficient consumption, and real aggregate supply when there are no individual utility shocks. When no individual utility shocks occur, individual j's Pareto-efficient consumption is a function solely of real aggregate supply. Define the implicit function $\tilde{c}_{jt}(Y_t)$ to be how the Pareto-Efficient consumption by individual j at time t depends on aggregate supply.[1] Note that $\tilde{c}_{jt}(Y_t)$ is a reduced form; it is <u>not</u> the structural consumption function. To help us avoid this confusion, we refer to $Y_t$ as real aggregate supply at time t, not income.

Define $\tilde{a}_{jt}(Y_t) \equiv -\dfrac{U''_{jt}\left(\tilde{c}_{jt}(Y_t)\right)}{U'_{jt}\left(\tilde{c}_{jt}(Y_t)\right)}$, which is the function of how the coefficient of absolute risk aversion varies with real aggregate supply. Define $\tilde{\rho}_{jt}(Y_t) \equiv \tilde{c}_{jt}(Y_t) \cdot \tilde{a}_{jt}(Y_t)$, which is the function of how the relative risk coefficient varies with real aggregate supply. Also, define $\overline{\rho}_t(Y_t) \equiv \sum_{j=1}^{m} \left( \tilde{\rho}_{jt}(Y_t) \cdot \dfrac{d\tilde{c}_{jt}}{dY_t} \right)$, which is the weighted average of the relative risk coefficients using the derivatives of $\tilde{c}_{jt}(Y_t)$ as the weights. Finally, define $\tilde{\alpha}_j(Y_t) \equiv \dfrac{\tilde{\rho}_{jt}(Y_t)}{\overline{\rho}_t(Y_t)}$, which is how j's relative risk coefficient compares to the average relative risk coefficient.

Since equation (4) is true for all j,

$$\frac{U'_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} = \frac{U'_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} \tag{5}$$

[1] There is not just one Pareto-efficient consumption allocation, but rather a continuum of such allocations, each corresponding to a particular allocation of endowments across states. We can think about this Pareto-efficient consumption allocation as the one that corresponds to the existing allocation of endowments.

for j=2..m. Totally differentiating (5) with respect to $Y_t$ gives $\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})}\frac{d\tilde{c}_{jt}}{dY_t}=\frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})}\frac{d\tilde{c}_{1t}}{dY_t}$. Dividing the left and right sides of this by the left and rights sides of (5) respectively gives:

$$\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})}\frac{d\tilde{c}_{jt}}{dY_t}=\frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})}\frac{d\tilde{c}_{1t}}{dY_t} \tag{6}$$

Substituting $\tilde{a}_{jt}(Y_t)\equiv-\frac{U''_{jt}(\tilde{c}_{jt}(Y_t))}{U'_{jt}(\tilde{c}_{jt}(Y_t))}$ into (6) and multiplying both sides by a minus sign and rearranging slightly gives:

$$\frac{\frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}}=\frac{\tilde{a}_{1t}}{\tilde{a}_{jt}} \tag{7}$$

By summing both sides of (7) over all consumers, we get:

$$\frac{\sum_{j=1}^{m}\frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}}=\tilde{a}_{1t}\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}} \tag{8}$$

By equilibrium in the market for the consumption good at time t, $\sum_{j=1}^{m}c_{jt}=Y_t$, which also implies that $\sum_{j=1}^{m}\frac{dc_{jt}}{dY_t}=1$. Therefore, solving (8) for $\frac{d\tilde{c}_{1t}}{dY_t}$ gives $\frac{d\tilde{c}_{1t}}{dY_t}=\frac{1/\tilde{a}_{1t}}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}}$. This and (7) imply that the following is true for all j.

$$\frac{d\tilde{c}_{jt}}{dY_t}=\frac{1/\tilde{a}_{jt}}{\sum_{s=1}^{m}\frac{1}{\tilde{a}_{st}}} \tag{9}$$

This result was first derived by Wilson (1968, see his theorem 5).

Next, we need to determine the value of $\bar{\rho}_t(Y_t)$. The following starts out with the definition of $\bar{\rho}_t(Y_t)$, then substitutes in the definition of $\tilde{\rho}_{jt}(Y_t)$ and the result in (9):

$$\bar{\rho}_t(Y_t) \equiv \sum_{j=1}^{m}\left(\tilde{\rho}_{jt}(Y_t)\cdot\frac{dc_{jt}}{dY_t}\right) = \sum_{j=1}^{m}\left(\tilde{c}_{jt}\tilde{a}_{jt}\cdot\frac{\frac{1}{\tilde{a}_{jt}}}{\sum_{k=1}^{m}\frac{1}{\tilde{a}_{kt}}}\right) = \frac{\sum_{j=1}^{m}\tilde{c}_{jt}}{\sum_{k=1}^{m}\frac{1}{\tilde{a}_{kt}}}$$

However, the sum of consumption across all consumers in this pure exchange economy equals aggregate supply for that period. Therefore,

$$\bar{\rho}_t(Y_t) = \frac{Y_t}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}} \tag{10}$$

From the definition of $\tilde{\alpha}_j \equiv \frac{\tilde{\rho}_{jt}(Y_t)}{\bar{\rho}_t(Y_t)}$, we can write $\tilde{\rho}_{jt} = \tilde{\alpha}_j\bar{\rho}_t$ and then replace $\tilde{\rho}_{jt}$ with $\tilde{c}_{jt}\tilde{a}_{jt}$ and $\bar{\rho}_t$ with (10) to get $\tilde{c}_{jt}\tilde{a}_{jt} = \tilde{\alpha}_{jt}\frac{Y_t}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}}$. Dividing both sides by $\tilde{a}_{jt}Y_t$ gives $\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_{jt}\frac{\frac{1}{\tilde{a}_{jt}}}{\sum_{j=1}^{m}\frac{1}{\tilde{a}_{jt}}}$. Using (9), we can rewrite this as: $\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_{jt}\frac{d\tilde{c}_{jt}}{dY_t}$. Dividing both sides by $\tilde{\alpha}_{jt}$ gives us:

$$\tilde{c}'_{jt}(Y_{it}) = \frac{1}{\tilde{\alpha}_{jt}(Y_{it})}\frac{\tilde{c}_{jt}(Y_{it})}{Y_{it}} \tag{11}$$

Equation (11) is the very important relationship between how consumption changes as real aggregate supply changes and how j's relative risk aversion compares to average relative risk aversion. The derivative of j's Pareto-efficient consumption with respect to real aggregate supply equals a multiplier times the proportion of one's consumption to real aggregate supply. The multiplier is inversely related to how one's relative risk aversion compares to the average relative risk aversion. For example, if aggregate supply decreases by 1%, then (11) says that the Pareto-efficient consumption will decrease by half a percent for someone who has twice the average relative risk aversion, whereas it will decrease by 2% for someone having half the average relative risk aversion. By decreasing their consumption more than proportionately, the lower (relatively) risk-averse consumers are enabling the higher risk-averse consumers to reduce their consumption less than proportionately. In essence, the lower risk-averse consumers are agreeing to absorb more of the risk concerning real aggregate supply so that higher risk-averse consumers can absorb less risk.[2] The RASRT contracts discussed later in this paper are designed to try to meet the need to transfer the risk related to possible changes in real aggregate supply.

## III. Completing Markets When All Have The Same Relative Risk Aversion

In the previous section, we showed that in a pure exchange economy without storage, a consumer j's optimal contract receipts vary across states of nature only when (i) real aggregate supply changes, (2) individual j's ratio of endowment to real aggregate supply changes, and (3) individual utility shocks occur, either to individual j or someone else. We also derived equation (11) which states the precise relationship of how relative risk aversion solely determines how Pareto-efficient consumption and hence the optimal contract receipts vary with real aggregate supply. These results from the previous section are important because if a certain set of contracts were to be able to replicate the optimal contract receipts, the real payments on these contracts would need to respond to individual utility shocks, stochastic changes in endowment ratios, and stochastic changes in real aggregate supply. Also, these contracts would need to enable consumers to transfer the real-aggregate-supply risk from the more relatively-risk-averse consumers to the less relatively-risk-averse consumers.

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[2] The relationship in (11) is related to one derived by Viard (1993), although he assumed that all income was derived from past investments in risky assets whereas we assume all income comes from endowments.

In this section, we show how consumers can use (i) normal contracts, (ii) endowment-sharing contracts, and (iii) spending-sharing contracts to replicate their optimal contract receipts when everyone has the same relative risk aversion. The section following this one then discusses the RASRT contract which would be needed when consumers have the same relative risk aversion.

We define **normal contracts** to be contracts whose real payments are proportional to real aggregate supply. When the central bank of an economy keeps nominal aggregate demand from varying across states of nature, nominal contracts behave as normal contracts. To see this, think about the equation of exchange as $MV=N=PY$ where $M$ is the money supply, $V$ is the "income" velocity[3] of money, $N$ is nominal aggregate demand, $P$ is the price level, and $Y$ is real aggregate supply. If $X_t$ is a cash flow at some future time t, then the real value of this cash flow will be $X_t / P_t$. By the equation of exchange, $P_t = N_t / Y_t$. Therefore, the real value of $X_t$ will be $X_t Y_t / N_t$, which shows that as long as $N_t$ does not vary across states of nature, the real value of $X_t$ will be proportional to real aggregate supply. A central bank that is trying to keep nominal aggregate demand from varying across states of nature is following what is currently called "nominal-income targeting."[4]

If a central bank does not target nominal income or nominal aggregate demand or if the central bank is unable to make nominal aggregate demand invariant to changes in states of nature, nominal contracts will not behave as normal contracts. Quasi-real indexing as proposed by Eagle and Domian (1995) is a way to make contracts behave as normal contracts even when

[3] The term "income velocity of money" is unfortunately the standard terminology here. The term "income" usually refers to aggregate supply not aggregate demand. However, the money supply times velocity equals nominal aggregate demand, which equals nominal aggregate supply only in equilibrium.

[4] A more effective strategy to get nominal aggregate demand to be unaffected by different states of nature would be to following "nominal aggregate demand targeting" instead of "nominal income targeting". Once again, nominal income is associated with nominal aggregate supply not nominal aggregate demand. Therefore, if disequilibrium does occur, nominal aggregate demand targeting could be more effective than nominal income targeting.

the central bank does not pursue nominal income targeting or nominal aggregate demand targeting. A quasi-real-indexed contract would take a payment $X_t$ at time t and multiply it by $N_t / N_0$. Therefore, the nominal payment would be $X_t N_t / N_0$ and the real payment would be $\frac{X_t N_t}{P_t N_0}$. Since $Y_t = N_t / P_t$ by the equation of exchange, the real payment would be $\frac{X_t Y_t}{N_0}$, which is proportional to real aggregate supply.

Eagle and Domian (2003 and 2004) assume a pure exchange economy without storage but with consumers having the same relative risk aversion and the ratios of endowment to real aggregate supply that do not vary across states of nature. They show that quasi-real bonds by themselves complete the markets under these assumptions. In other words, under these assumptions, normal contracts by themselves enable consumers to replicate their optimal contract receipts.

Eagle and Domian (2003 and 2004) also assume no utility shocks. When individual utility shocks do occur or when consumers' ratios of endowment to real aggregate supply are stochastic, or when consumers have differing relative risk aversion, normal contracts will no longer enable consumers to replicate their optimal contract receipts. This section extends Eagle's and Domian's analysis to show that endowment-sharing contracts can handle stochastic ratios of consumers' endowment to real aggregate supply, and spending-sharing contracts can handle individual spending shocks. The section following this one then shows that RASRT contracts can approximately handle the transfer of real-aggregate-supply risk need when consumers have different relative risk aversion.

To exemplify how normal contracts can replicate the optimal contract receipts when all consumers have the same relative risk aversion and ratios of endowment to real aggregate supply that do not vary across states of nature and no individual utility shocks occur, assume a two-period, pure exchange economy without storage. Assume two individuals exist, named A and B, where each consumer j maximizes his/her logarithmic utility function

$\varepsilon_0 \ln(c_{j0}) + \beta \sum_{s=1}^{n_t} \pi_{s1} \varepsilon_{s1} \ln(c_{js1})$ subject to (2) and (3) where $\varepsilon_0$ and $\varepsilon_{s1}$ represent aggregate utility shocks at time 0 and in state s at time 1 respectively. Assume only two consumers exist, labeled consumer A and consumer B. At time 0 assume real aggregate supply is 100, with consumer 1 being endowed with 20 consumption units and the rest going to consumer 2. At time 1 assume there are five states of nature where real aggregate supply equals 30, 60, 90, 120, and 150 with each state equally likely. Assume the ratio of consumer 1's endowment to real aggregate supply is 40% in at time 1, with the rest of real aggregate supply going to consumer 2.

Table 1 shows equations for the consumption demands and the real pricing kernels as well as the values for the relevant variables, including the demands for the state-contingent securities

**Equations for Logarithmic Example:**

$$c_{j0} = \frac{y_{j0} + E[\Omega_{s1} y_{js1}]}{1 + \beta \frac{E[\xi_{st}]}{\xi_0}}, \quad c_{jst} = \frac{\beta^t \xi_{st}}{\Omega_{st} \xi_0} \left( \frac{y_{j0} + E[\Omega_{s1} y_{js1}]}{1 + \beta \frac{E[\xi_{st}]}{\xi_0}} \right), \quad \Omega_{st} = \frac{\beta^t \xi_{st}}{\xi_0} \frac{Y_0}{Y_{st}}$$

**Example Value Results:**

| | | | consumer A | | | consumer B | | | real agg. supply |
|---|---|---|---|---|---|---|---|---|---|
| | | | endowmt | consumption | $x_{1s1}$ | endowmt | consumption | $x_{2s1}$ | |
| | real | time 0 | 20 | 29.74358974 | --------- | 80 | 70.2564103 | -------- | 100 |
| time 1: state | pricing kernel | prob. | | | | | | | |
| 1 | 3.1667 | 0.2 | 12 | 8.923076923 | -3.08 | 18 | 21.0769231 | 3.08 | 30 |
| 2 | 1.5833 | 0.2 | 24 | 17.84615385 | -6.15 | 36 | 42.1538462 | 6.15 | 60 |
| 3 | 1.0556 | 0.2 | 36 | 26.76923077 | -9.23 | 54 | 63.2307692 | 9.23 | 90 |
| 4 | 0.7917 | 0.2 | 48 | 35.69230769 | -12.3 | 72 | 84.3076923 | 12.3 | 120 |
| 5 | 0.6333 | 0.2 | 60 | 44.61538462 | -15.4 | 90 | 105.384615 | 15.4 | 150 |

Table 1: Equations and Values for Logarithmic Example

($x_{js1}$). Remember that the optimal contract receipts equal the demands for the state-contingent securities.

Figure 1 shows the Pareto-efficient consumption and endowment curves for both A and B as well as the optimal contract receipts for curve for B and a 45-degree line, which represents the magnitude of real aggregate supply must be split between A and B. The Pareto-efficient result is that B exchanges some of his endowment in period 0 for some of A's endowment in period 1. As can be seen the optimal contract receipts are proportional to real aggregate supply. This results because the consumers' ratio of endowment to real aggregate supply are constant and because the logarithmic utility function causes both consumers to have a coefficient of relative risk aversion of one (See equation (11)).

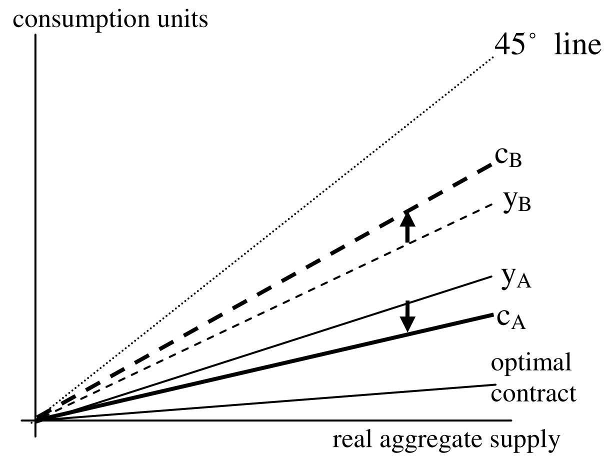

Figure 1: Complete Picture of Optimal Contract in Logarithmic Example

If the ratios of each consumer's endowment to real aggregate supply are stochastic, then endowment-sharing contracts as well as normal contracts would be needed to complete markets. The purpose of endowment-sharing contracts is to handle stochastic variations in the ratio of one's endowment to real aggregate supply. Let $\overline{R}_{jt}$ be the insurance company's contractual endowment ratio for individual j at time t. The insurance company should set $\overline{R}_{jt}$ to be individual j's implicit average endowment ratio at time t, where we define the **implicit average endowment ratio** to be the constant endowment ratio that results in the same expected present value of j's endowment at time t as j's actual endowments. Using the real pricing kernel as the real stochastic discount factor, the expected present value of individual j's endowments at time t is $E[\Omega_{*_t} y_{j*_t}]$, which equals $\sum_{s=1}^{S_1} \pi_{st} \Omega_{st} y_{jst}$. Individual j's implicit average endowment ratio at time t therefore equals:[5]

$$\overline{R}_{jt} = \frac{E[\Omega_{*_t} y_{j*_t}]}{E[\Omega_{*_t} Y_{*_t}]} \tag{12}$$

By entering into an endowment sharing contract, the insurance company would make a payment to individual j when $\frac{y_{jt}}{Y_t} < \overline{R}_{jt}$, and individual j would pay the insurance company when $\frac{y_{jt}}{Y_t} > \overline{R}_{jt}$. The size of the payment would be

$$e_{jt} \equiv \overline{R}_{jt} Y_t - y_{jt} \tag{13}$$

This payment is in real terms. When $e_{jt}$ is positive then the insurance company pays that to individual j. A negative value means individual j pays the insurance company.

Note that if we sum (13) over all individuals, we get zero:

$$\sum_{j=1}^{m} e_{jt} = \sum_{j=1}^{m} \overline{R}_{jt} Y_t - \sum_{j=1}^{m} y_{jt} = Y_t - Y_t = 0$$

(Note that the $\overline{R}_{jt}$ values must add up to one.) This means that if all individuals entered into these endowment-sharing contracts, the insurance company would face no aggregate risk. All risk-averse consumers would choose to enter these endowment-sharing contracts. If more than one insurance company exists, then the insurance companies could reinsure themselves in order

[5] If $\overline{R}_{jt}$ what the ratio for states s then $y_{jst} = \overline{R}_{jt} Y_{st}$, and $E[\Omega_{*_t} y_{j*_t}] = E[\Omega_{*_t} \overline{R}_{jt} Y_{*_t}] = \overline{R}_{jt} E[\Omega_{*_t} Y_{*_t}]$. Solving for $\overline{R}_{jt}$. If this proportionality existed, then $y_{jst} = \overline{R}_{jt} Y_{st}$, and $E[\Omega_{*_t} y_{j*_t}] = E[\Omega_{*_t} \overline{R}_{jt} Y_{*_t}] = \overline{R}_{jt} E[\Omega_{*_t} Y_{*_t}]$. In reality, this proportionality may not exist. Nevertheless, we could determine the value of $\overline{R}_{jt}$ that would result in the same expected present value as the actual endowments. To determine this "equivalent" $\overline{R}_{jt}$, we need to solve the equation $E[\Omega_{*_t} y_{j*_t}] = \overline{R}_{jt} E[\Omega_{*_t} Y_{*_t}]$ for $\overline{R}_{jt}$. This gives the formula for this equivalent $\overline{R}_{jt}$ gives (12).

than none of them face any risk. This absence of risk is consistent with zero economic profit to the insurance company.

Theoretically speaking all risk-averse individuals will choose to enter into an endowment-sharing contract. However, if some individuals choose not to enter into such contracts, the insurance company could structure its payments in the following manner so it still faces no risk:

$$e_{jst} = \overline{R}_{jt} Y_{st} \frac{\sum_{k \in Z_t} \frac{y_{kst}}{Y_{st}}}{\sum_{k \in Z_t} \overline{R}_{kt}} - y_{jst} \tag{14}$$

where $Z_t$ is the set of all insured investors. Summing (14) over all individuals in $Z_t$ gives

$\sum_{j \in Z_t} e_{jst} = \left( \sum_{k \in Z_t} \overline{R}_{jt} \right) Y_{st} \frac{\sum_{k \in Z_t} \frac{y_{kst}}{Y_{st}}}{\sum_{k \in Z_t} \overline{R}_{kt}} - \sum_{k \in Z_t} y_{jst} = Y_{st} \sum_{k \in Z_t} \frac{y_{kst}}{Y_{st}} - \sum_{k \in Z_t} y_{jst} = 0$, which shows that the insurance company will face no risk with the adjusted payments.

A third type of contract to help individuals replicate their optimal contract receipts are spending-sharing contracts. These contracts are needed to handle individual utility shocks that cause either increases or decreases in an individual's spending needs. From the previous section, we learned that in a pure-exchange economy without storage, one's Pareto-efficient consumption varies with respect to only two factors: real aggregate supply and individual utility shocks. Normal contracts in conjunction with RASRT contracts handle variations in real aggregate supply as we will discuss later. Spending-sharing contracts will enable consumers to handle individual utility shocks.

Since individual j's Pareto-efficient consumption at time t varies across states of nature solely with real aggregate supply and individual utility shocks, we can write j's Pareto-efficient consumption as the following function of real aggregate supply and individual utility shocks: $\tilde{c}_{jt}(Y_{st}, \xi_{1st}, \xi_{2st}, ..., \xi_{mst})$, where $\xi_{jst} \in (0, \infty)$ is the utility shock to individual j's utility constructed so that $\xi_{jst} = 1$ represents no utility shock.[6]

Note that j's Pareto-efficient consumption is not only affected by his/her individual utility shocks but by other individual utility shocks as well. For example if individual 1's experienced a positive utility shock at time t causing individual 1 to need to spend more that period; the Pareto-efficient solution is for other individuals such as individual j (assuming $j \neq 1$) to spend less so that individual 1 can spend more. A real life example of such a utility shock would be if individual 1 had a health-care emergency causing her spending to sharply increase. The Pareto-efficient solution is for individuals who did not have health-care issues to help individual 1. To do so, other individuals will need to spend less so that individual 1 can spend more on the health-care emergency.

Theoretically, an individual j's spending-sharing contract receipts will equal:

$$\omega_{jst} = \tilde{c}_{jt}(Y_{st}, \xi_{1st}, \xi_{2st}, ..., \xi_{mst}) - \tilde{c}_{jt}(Y_{st}, 1, 1, ..., 1) \tag{15}$$

where $\omega_{jst}$ is j's spending-sharing contract receipts in state s at time t, and $\tilde{c}_{jt}(Y_{st}, 1, 1, ..., 1)$ is j's Pareto-efficient consumption with no utility shocks. (This assumes that the utility shocks are defined so that a utility shock of one means no utility shocks.)

Figures 2 and 3 help show some differences between endowment-sharing contracts and spending-sharing contracts. Figure 2 shows the effect of the endowment-sharing contract receipts or payments which move the individual from the kinked curve with the endowment shocks to the straight line without the endowment shocks. The endowment-sharing contract

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[6] In order for these utility shocks to be well defined, some condition needs to exist. I surmise that this condition would be something like that $E[\Omega_{*_t} \tilde{c}_{j*_t}]$ must be the same whether with or without utility shocks.

receipts or payments result with the resources of the individual being moved to where those resources are proportional to real aggregate supply. The reason for this is that unexpected increases in individual endowments is not a justification for an individual to consume more or less in Pareto-efficient sense. If markets are complete, individuals will agree in advance to make some payments when their endowments in some states are greater than expected so that they will receive payments when their endowments in other states are less than expected. The effect of the endowment-sharing contract receipts or payments is to make the sum of one's endowment with these receipts or payments to be proportional to real aggregate supply.

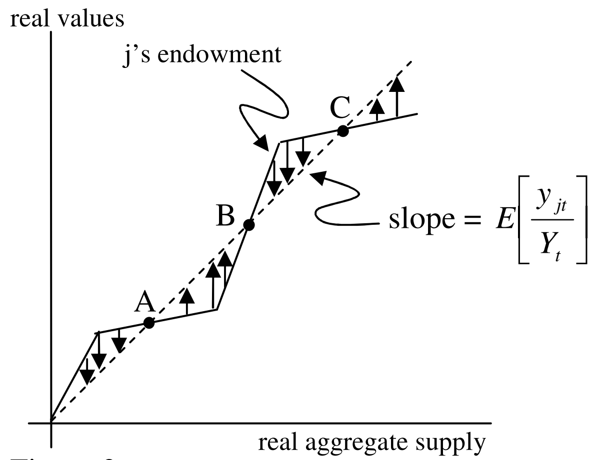

Figure 2: Endowment-Sharing Contract Receipts (Payments)

On the other hand, spending-sharing contract receipts move one from smooth consumption to kinked consumption as shown in Figure 3. Sometimes an individual's Pareto-efficient consumption will be greater than other times because of utility shocks. If markets are complete, an individual would be willing to make some payments when his/her Pareto-efficient consumption is less than normal so that he/she can receive payments when his/her Pareto-efficient consumption is greater than normal.

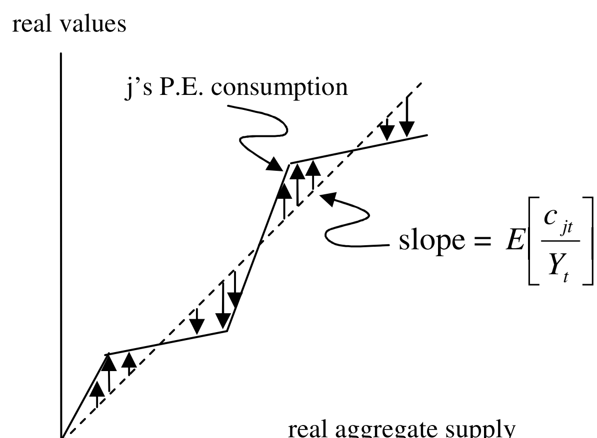

Figure 3: Spending-Sharing Contract Receipts (Payments)

With true spending-sharing contracts, adverse selection is not theoretically an issue. Because a spending-sharing contract is based on one's Pareto-efficient consumption with and without utility shocks, if those Pareto-efficient consumption levels are accurately assessed, all risk averse individuals will choose to participate in the insurance. Such universal spending-sharing contracts would pose no risk to the insurance company. To see this, note:

$$\sum_{j=1}^{m} \omega_{jst} = \sum_{j=1}^{m} \tilde{c}_{jt}(Y_{st}, \xi_{1st}, \xi_{2st}, \ldots, \xi_{mst}) - \sum_{j=1}^{m} \tilde{c}_{jt}(Y_{st}, 1, 1, \ldots, 1) = Y_{st} - Y_{st} = 0 .$$

Thus, if one insurance company provided everyone with these spending-sharing contracts, the aggregate spending-sharing-contract receipts would equal 0. This result is based on the equilibrium condition that the sum of Pareto-efficient consumption over all consumers equals real aggregate supply regardless whether that Pareto-efficient consumption is with or without utility shocks.

## IV. RASRT contracts

When consumers have different relative risk aversion, no longer will their Pareto-efficient consumption be proportional to real aggregate supply. Figure 5 shows how the Pareto-efficient consumption will vary for individual A and B when A has greater than average relative risk aversion and B has less than average relative risk aversion. (More precisely, this example assumes A and B have coefficients of relative risk aversion of 2 and ½ respectively.) As equation (11) predicts, A's

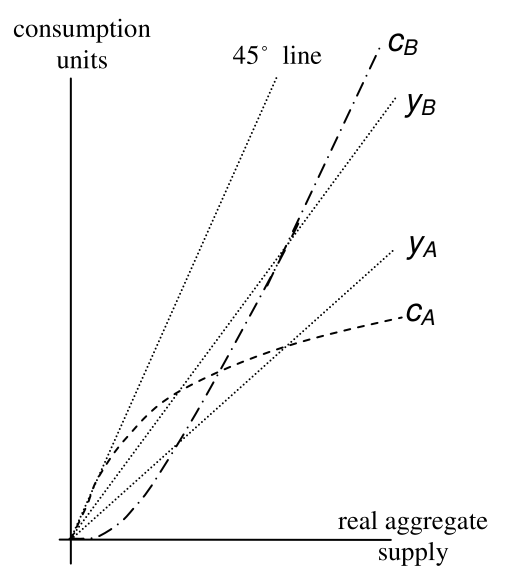


Figure 5: A's and B's consumption when A has greater than average relative risk aversion and B has less than average relative risk aversion

Pareto-efficient consumption will change less than proportionally to changes in real aggregate supply whereas B's Pareto-efficient consumption will change more than proportionally to changes in real aggregate supply. This is clearest at the intersection of A's and B's consumption curves. If real aggregate supply decreases, B decreases his consumption more than proportionately so that A can decrease her consumption less than proportionately. Because of her high relative risk aversion, individual A appreciates that B is willing to do this. In return for having to decrease his consumption more than proportionately when real aggregate supply decreases, B will increase his consumption more than proportionately when real aggregate supply increases. Basically, A has agreed through complete markets to benefit less than proportionately when real aggregate supply increases in return for her being able to decrease her consumption less than proportionately when real aggregate supply increases. Since A has higher relative risk aversion than does B, both A and B are better off with this arrangement than if they proportionately shared in changes in real aggregate supply.

To try to enable consumers to transfer real-aggregate-supply risk among themselves, this paper invents RASRT contracts. (RASRT stands for Real-Aggregate-Supply-Risk-Transfer.) We worked with two different types of RASRT contracts, straight and curved. Because these RASRT contracts do not perfectly meet each individual's needs, we will discuss two methods consumers could use to approximate their risk-transfer needs. If an individual uses the **tangency method**, they will enter into RASRT contracts so their resulting consumption is tangent to their Pareto-efficient consumption. A second method an individual could use is the **minimum-variance** method, which minimizes the expected squared deviations of their resulting consumption from their Pareto-efficient consumption.

The simplest RASRT contract is a straight RASRT contract. The seller of the straight RASRT contract agrees to pay the buyer an amount equal to $b(F_t - Y_t)$ where b is some positive constant specified in the contract, $F_t$ is the price of the RASRT contract, and $Y_t$ is real aggregate supply. If $b(F_t - Y_t)$<0, then the buyer will pay the seller the amount. The amount, $b(F_t - Y_t)$, is in real terms since optimal contract receipts are in real terms. The straight RASRT contract has similarities with many futures contracts currently in use, except that no futures contracts currently deal with real aggregate supply.

There are some problems with straight RASRT contracts. The first problem is bankruptcy. In Figure 5, we would plotted the Pareto-efficient consumption as a function of real aggregate supply for individuals A and B who had coefficients of relative risk aversion of two and ½ respectively. Figures 6, 7, and 8 continue with this example, but incorporate different RASRT contracts. Figure 6 shows A's consumption for both the tangency method and the minimum-variance method of determining the number of RASRT contracts. Regardless whether she uses the tangency or minimum-variance methods, individual A will consume a positive amount when real aggregate supply is zero if she relies on straight RASRT contracts. However, that cannot be possible in an economy without storage. When we look at Figure 7 we see that consumer B's use of a straight RASRT contract will cause him to consume a negative amount when real aggregate supply equals zero also regardless if he uses the tangency method or the minimum-variance method. Since it is impossible to consume a negative amount, this means that B will be unable to fulfill his straight RASRT contractual obligations when real aggregate supply is quite small. In other words, straight RASRT contracts will lead to B becoming bankrupt at low levels of real aggregate supply, which will then impact A as well.

Another problem with straight RASRT contracts is that straight RASRT contracts usually result with an individual's consumption differing significantly from one's Pareto-efficient consumption. Figures 6 and 7 shows that that the consumption resulting from the straight RASRT contracts for A and B depart significantly from their Pareto-efficient consumption for both methods..

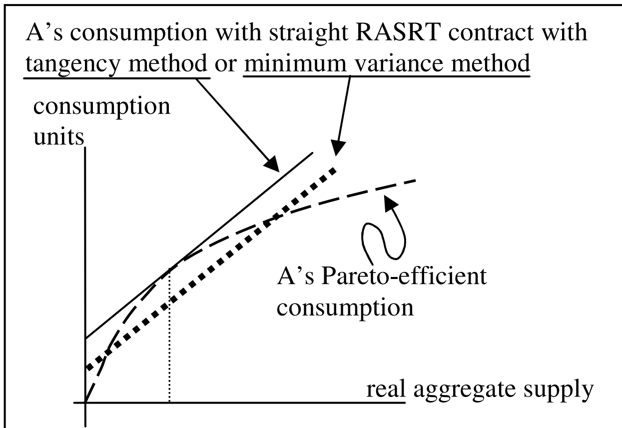


Figure 6: A's consumption with straight RASRT contract compared to Pareto-efficient consumption under tangency method

The curved RASRT contracts deal with both of these problems. In general, a curved RASRT contract takes the form of $b\left(f(F_t) - f(Y_t)\right)$ where b is a positive constant and f is an increasing function. A special case is $f(Y_t) = Y_t$ which is the straight RASRT contract. Two other special cases are $f(Y_t) = \sqrt{Y_t}$ , which we will call the SQRT RASRT contract, and is $f(Y_t) = LN\left(1 + \frac{Y_t}{\overline{Y}_t}\right)$, which we will call the LOG RASRT contract, where $\overline{Y}_t$ is initially set equal to $E[Y_t]$.

Figure 8 shows A's consumption using the SQRT RASRT contract using the tangency method of determining the amount of these RASRT contracts to demand. Also shown is A's consumption using the

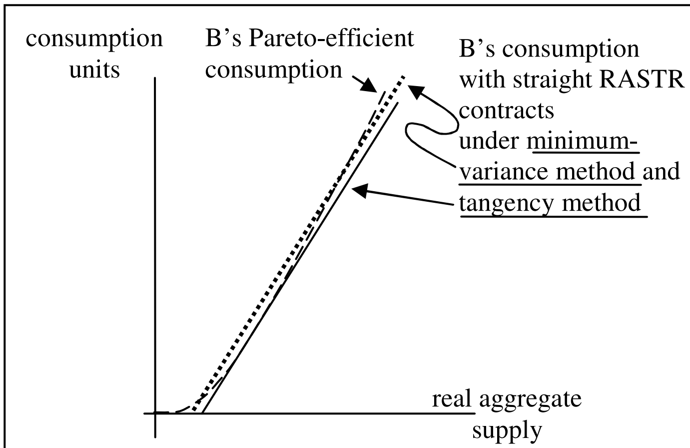


Figure 7: B's consumption with straight RASRT contract compared to Pareto-efficient consumption under tangency method

LOG RASRT contract. While the SQRT RASRT contract still results with A consuming more than Pareto-efficient when real aggregate supply differs from its expected value, the overstatement is substantially reduced. The LOG RASRT contract does even better than the SQRT RASRT contract when real aggregate supply is less than its expected value, but does very poorly when real aggregate supply is greater than expected.

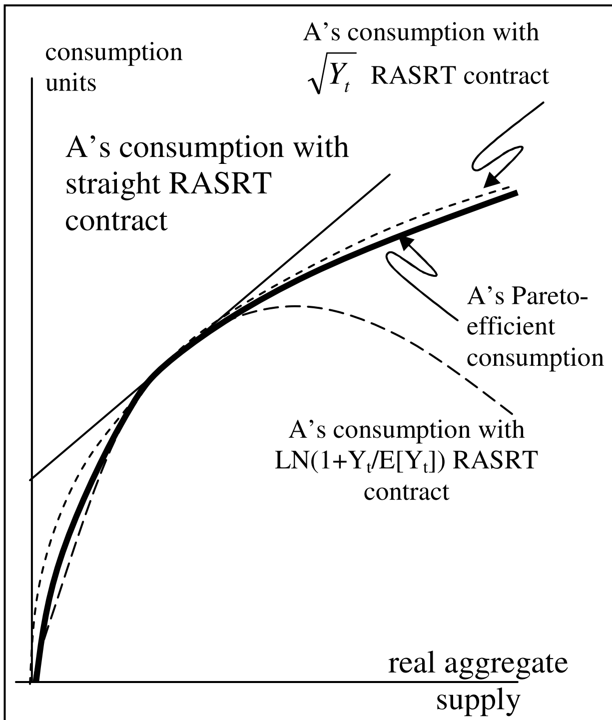

Figure 8: Pareto-efficient consumption compared with consumption using Straight and curved RASRT contracts

To see how different RASRT contracts can perform for a variety of different distributions of endowment and relative-risk aversions, I studied several examples involving 18 individuals each of whom had CRRA utility functions,[7] but with different coefficients of relative risk aversion and with different endowment ratios.[8] I looked at six different scenarios, which are described in Table 2. For all scenarios, the possible values of real aggregate supply were 3, 6, 9, …, 147, 150 each with a 0.02 probability. I used numerical techniques to determine the expected variance of the combination of the RASRT contract receipts (or payments) from the optimal contract receipts (or payments). These results are presented in Table 3.

[7] Some may criticize my use of CRRA utility functions. However, what is important according to equation (11) is how one's relative risk aversion compares to average relative risk aversion. That comparison based on individuals having CRRA utility functions is likely to be similar to what would occur if consumers have other reasonably behaved utility functions.

[8] That I assumed individuals had endowments that were a constant proportional of real aggregate supply was necessary so that individuals not need endowment-sharing contracts to obtain their optimal contract receipts. The issue of endowment-sharing contracts will be discussed in the next chapter.

**Scenario I**

| end. ratio | coefficient of relative risk aversion: 0.6 | 0.8 | 1 | 1 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|
| 0.8 | A | B | C | J | K | L |
| 1 | D | E | F | M | N | O |
| 1.2 | G | H | I | P | Q | R |

**Scenario III**

| coeff. of r.r.a | endowment ratio: 0.6 | 0.8 | 1 | 1 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|
| 0.8 | A | D | G | J | M | P |
| 1 | B | E | H | K | N | Q |
| 1.2 | C | F | I | L | O | R |

**Scenario II**

| end. ratio | coefficient of relative risk aversion: 0.25 | 0.5 | 1 | 1.5 | 2 | 2.5 |
|---|---|---|---|---|---|---|
| 0.8 | A | B | C | J | K | L |
| 1 | D | E | F | M | N | O |
| 1.2 | G | H | I | P | Q | R |

**Scenario IV**

| coeff. of r.r.a | endowment ratio: 0.2 | 0.3 | 0.5 | 1 | 1.5 | 2.5 |
|---|---|---|---|---|---|---|
| 0.8 | A | D | G | J | M | P |
| 1 | B | E | H | K | N | Q |
| 1.2 | C | F | I | L | O | R |

**Scenario VI**

| end. ratio | coefficient of relative risk aversion: 0.1 | 0.7 | 1 | 1.3 | 3 | 5 |
|---|---|---|---|---|---|---|
| 0.6 | A | B | C | J | K | L |
| 1 | D | E | F | M | N | O |
| 1.4 | G | H | I | P | Q | R |

**Scenario V**

| coeff. of r.r.a | endowment ratio: 0.2 | 0.3 | 0.5 | 1 | 1.5 | 2.5 |
|---|---|---|---|---|---|---|
| 0.5 | A | D | G | J | M | P |
| 1 | B | E | H | K | N | Q |
| 2 | C | F | I | L | O | R |

Table 2: Scenario Details

For each RASRT contract, the percentages represent the expected resulting variance as a percentage of the variance that would have resulted if no RASRT contracts existed; only nominal contracts existed. The first percentage without the parentheses is the result when I used the minimum variance method to determine the contracts purchased or sold. The percentage in parentheses is the expected variance when I used the tangency method. For example, if following the minimum variance method and consumers use only straight RASRT contracts, they will be able to reduce the expected variance to about 27% of what it would have been with no RASRT contracts. On the other hand, the individuals following the tangency method using straight RASRT contracts would have only reduced this expected variance to 59% of what that expected variance would have been without any RASRT contracts.

The SQRT RASRT contract was able to do much better with the expected variance falling to 6.84% under the minimum variance method and 13.08% under the tangency method. Better still were the RASRT contracts where $f(Y_t) = LN\left(1+\frac{Y_t}{\bar{Y}_t}\right)$ and $f(Y_t) = Y_t^{0.99}$. For both of these contracts, individuals in Scenario I were able to reduce the expected variance between their consumption with RASRT contracts and their Pareto-efficient consumption to less than 1% the expected variance had no RASRT contracts existed, assuming they followed the minimum variance approach. Even if they followed the tangency approach, they would be able to reduce that expected variance to less than 2.5% of what would have existed without RASRT contracts.

| | Scenario I | Scenario II | Scenario III | Scenario IV | Scenario V |
|---|---|---|---|---|---|
| Straight Futures | 27.01% (59.0%) | 34.76% (83.4%) | 22.95% (49.5%) | 22.95% (49.5%) | 24.34% (53.5%) |
| SQRT(Y) | 6.84% (13.08%) | 12.56% (23.41%) | 4.30% (7.88%) | 4.30% (7.88%) | 5.45% (8.94%) |
| $LN\left(1+\frac{Y_t}{\overline{Y}_t}\right)$ | 0.92% (2.06%) | 2.74% (7.84%) | 1.13% (2.16%) | 1.13% (2.16%) | 2.37% (6.03%) |
| Y^0.99 | 0.98% (2.13%) | 4.35% (8.93%) | 0.14% (0.30%) | 0.14% (0.30%) | 1.36% (2.86%) |

Table 3: Minimum Variance Compared to Variance With Only Normal Contracts (Percentage in Parentheses represents ratio under Tangency Method)

For the other Scenarios, the results were similar, although for Scenario II the reduction of the expected variance was less and for Scenarios III and IV, the reduction was more especially for the RASRT contract where $f(Y_t) = Y_t^{0.99}$.

Table 3 shows that when we look at the aggregate of these expected variances, curved RASRT contracts can be very successful at enabling individuals to approximately replicate their optimal contract receipts and therefore their Pareto-efficient consumption. However, it is important to consider how each individual was able to use these RASRT contracts to meet their needs. Table 4 shows how each individual fared with straight RASRT contracts or with the Y^0.99 RASRT contracts compared to no RASRT contracts at all when the individuals used the minimum variance method. The curved RASRT contracts enabled most individuals to reduce their expected variance between their with-RASRT consumption and their Pareto-efficient consumption. One exception is individual B in Scenario II, who was able to use straight RASRT contracts to reduce his expected variance to 22.08% of what it would have been without any

| | Straight RASRT contracts | | | | | Y^0.99 RASRT Contracts | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| ind. | Scenario I | Scenario II | Scenario III | Scenario IV | Scenario V | Scenario I | Scenario II | Scenario III | Scenario IV | Scenario V |
| A | 29.97% | 37.54% | 24.00% | 24.00% | 25.88% | 0.95% | 3.33% | 0.06% | 0.06% | 0.29% |
| B | 13.65% | 22.08% | 44.42% | 44.42% | 57.59% | 2.23% | 39.83% | 6.41% | 6.40% | 14.18% |
| C | 45.23% | 60.04% | 20.72% | 20.72% | 15.96% | 6.81% | 16.60% | 0.04% | 0.04% | 1.25% |
| D | 29.97% | 37.54% | 24.00% | 24.00% | 25.88% | 0.95% | 3.33% | 0.06% | 0.06% | 0.29% |
| E | 13.65% | 22.08% | 44.42% | 44.42% | 57.59% | 2.23% | 39.83% | 6.41% | 6.40% | 14.18% |
| F | 45.23% | 60.04% | 20.72% | 20.72% | 15.96% | 6.81% | 16.60% | 0.04% | 0.04% | 1.25% |
| G | 29.97% | 37.54% | 24.00% | 24.00% | 25.88% | 0.95% | 3.33% | 0.06% | 0.06% | 0.29% |
| H | 13.65% | 22.08% | 44.42% | 44.42% | 57.59% | 2.23% | 39.83% | 6.41% | 6.39% | 14.18% |
| I | 45.23% | 60.04% | 20.72% | 20.72% | 15.96% | 6.81% | 16.60% | 0.04% | 0.04% | 1.25% |
| J | 45.23% | 32.90% | 24.00% | 24.00% | 25.88% | 6.81% | 1.87% | 0.06% | 0.06% | 0.29% |
| K | 24.35% | 21.89% | 44.42% | 44.42% | 57.59% | 0.08% | 0.64% | 6.41% | 6.39% | 14.18% |
| L | 18.29% | 15.78% | 20.72% | 20.72% | 15.96% | 0.30% | 1.53% | 0.04% | 0.04% | 1.25% |
| M | 45.23% | 32.90% | 24.00% | 24.00% | 25.88% | 6.81% | 1.87% | 0.06% | 0.06% | 0.29% |
| N | 24.35% | 21.89% | 44.42% | 44.42% | 57.59% | 0.08% | 0.64% | 6.41% | 6.39% | 14.18% |
| O | 18.29% | 15.78% | 20.72% | 20.72% | 15.96% | 0.30% | 1.53% | 0.04% | 0.04% | 1.25% |
| P | 45.23% | 32.90% | 24.00% | 24.00% | 25.88% | 6.81% | 1.87% | 0.06% | 0.06% | 0.29% |
| Q | 24.35% | 21.89% | 44.42% | 44.42% | 57.59% | 0.08% | 0.64% | 6.41% | 6.39% | 14.18% |
| R | 18.29% | 15.78% | 20.72% | 20.72% | 15.96% | 0.30% | 1.53% | 0.04% | 0.04% | 1.25% |

Table 4: Individual Minimum Variance Results Compared to Normal Contracts Only

RASRT contracts. Individual B did even worse in Scenario II when he used the Y^0.99 RASRT contract as he was only able to reduce this expected variance to 39.83% of the no-RASRT level.

Figure 9 plots several individual's optimal contract receipts including individual B's. The curvature of B's optimal contract receipts changes. For low levels of real aggregate supply, B's optimal contract receipts are convex, but for higher levels they are concave. I traced the reason to how the average relative risk aversion for the economy changed as real aggregate supply changed. Remember that the average relative

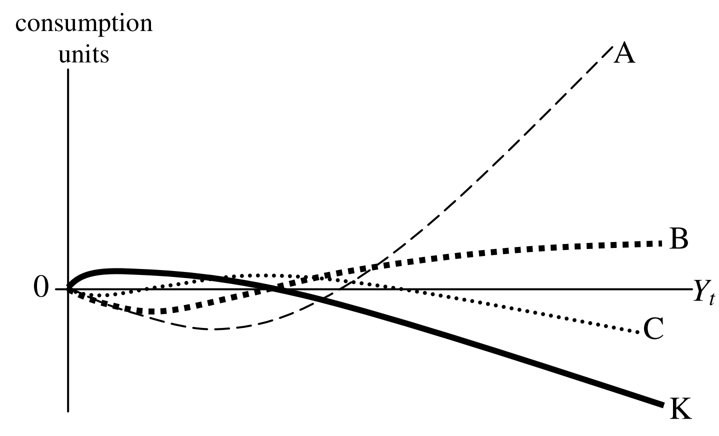


Figure 9: Selected Individual's Optimal Contract Receipts in Scenario II

risk aversion is a weighted average of the individuals' coefficients of relative risk aversion.

My investigation found that the weighted average relative risk aversion fell from 1.88 to 0.43 as real aggregate supply increased from 6 to 147. This resulted in those individuals with coefficients of relative risk aversion of 0.5, 1.0, and 1.5 changing from being below average relative risk aversion to being above average relative risk aversion. This was particularly difficult for individual B as he switched from having his 0.5 coefficient of relative risk aversion initially being well below average to becoming above average where real aggregate supply exceeded 109. This changed his optimal contract receipts from being convex to being concave with respect to real aggregate supply. Since the curved RASRT contracts are either concave or convex depending whether one sells or buys them, the curved RASRT contracts worked worse for individual B than would have straight RASRT contracts. This problem of changing from being below average to above average relative risk aversion was also true for individuals C and J who had coefficients of relative risk aversion

Individual C also experiences the switch but at a lower level of real aggregate supply (40), which means that the range of real aggregate supply for which C's optimal contract receipts are convex is relatively insignificant. This insignificance is even more for individual j whose switch occurs when real aggregate supply is 19.

The issue of switching is less an issue for Scenarios I, III, IV and V. For scenario I, the average relative risk aversion drops from 1.09 to 0.87 resulting with individuals C, F, I, J, M, and P experiencing only modest shifting. For Scenarios III and IV, the relative risk aversion only drops from 1.04 to 0.95, again causing only modest shifting among individuals B, E, H, K, N, and Q. For Scenario V, the relative risk aversion does drop from 1.61 to 0.70, which causes

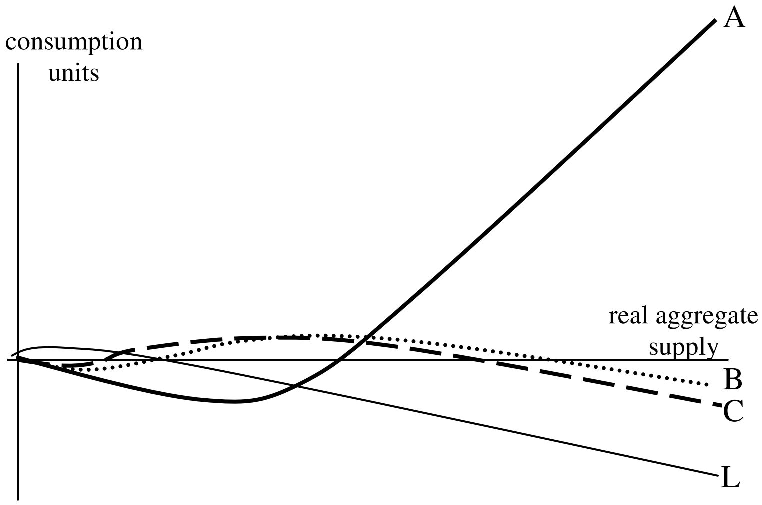


Figure 10: Individual Optimal Contract Receipts in Scenario VI

| individual | Straight RASRT contracts | Y^0.99 RASRT Contracts |
|---|---|---|
| A | 41.15% | 5.47% |
| B | 82.88% | 43.37% |
| C | 59.38% | 16.83% |
| D | 41.15% | 5.47% |
| E | 82.88% | 43.37% |
| F | 59.38% | 16.83% |
| G | 41.15% | 5.47% |
| H | 82.88% | 43.37% |
| I | 59.38% | 16.83% |
| J | 44.84% | 6.90% |
| K | 15.84% | 1.88% |
| L | 7.38% | 6.02% |
| M | 44.84% | 6.90% |
| N | 15.84% | 1.88% |
| O | 7.38% | 6.02% |
| P | 44.84% | 6.90% |
| Q | 15.84% | 1.88% |
| R | 7.38% | 6.02% |
| Overall | 41.95% | 7.25% |

Table 5: Variance between consumption with RASTR contract and Pareto-efficient consumption compared to no RASTR contract under Scenario VI

shifting to affect individuals B, E, H, K, N, and Q causing these individuals to retain 14.18% of the expected variance between RASRT consumption and Pareto-efficient consumption.

To further investigate the performance of curved RASRT contracts, I constructed another scenario, Scenario VI, which is defined in Table 2 at the end of this chapter. Here the range of coefficients of relative risk aversion was even greater than in Scenario II, ranging from 0.1 to 5. Table 5: shows how the expected variance between with-RASRT consumption and Pareto-efficient consumption compare between straight RASRT contracts and the Y^0.99 RASRT contract. In Scenario VI, the curved RASRT contract outperforms the straight RASRT contract for all individuals, although individuals B and C, E and F, and H, and individual I do experience problems of the shifting average relative risk aversion. Figure 10 shows how the optimal contract receipts for B and C change with real aggregate supply.

Even though some individuals switch from being below average to being above average, the aggregate expected variance is still substantially reduced by curved RASRT contracts. In scenario 6, the aggregate expected variance is 7.25% of the level with no RASRT contracts, compared to a ratio of 41.95% with straight RASRT contracts. As seen in Table 3, even in Scenario II, the Y^0.99 curved RASRT contracts were able to reduce the aggregate expected variance to 4.35% compared to 34.76% with straight RASRT contracts. For the other scenarios in Table 3, the Y^0.99 curved RASRT contract was able to reduce the aggregate expected variance to around 1% compared to over 20% with straight RASRT contracts. While curved RASRT contracts do pose problems to individuals who may experience switching between being below and above average relative risk aversion, the individuals who really need the RASRT contracts are either substantially above or substantially below the average and do not experience switching. As seen in Table 3, for individuals not experience this switching, the curved RASRT contracts enable those individuals to reduce their expected variance to a level of less than 2%.

## Pricing of RASRT Contracts

This section discusses the pricing of RASRT contracts in general under the assumption that markets are completed by the RASRT contracts in conjunction with normal, endowment-sharing, and spending-sharing contracts. Under the complete markets assumption, we can use the real pricing kernel to determine the price of the RASRT contracts.

Remember that a general RASRT contract pays the buyer an amount equal to $b\big(f(F_t) - f(Y_t)\big)$ where $F_t$ is the price of the RASRT contract. Assume that $k_{jt} \equiv \frac{y_{jst}}{Y_{st}}$ is the ratio of individual j's endowment to real aggregate supply which does not vary across states of nature.[9] Let $Q_{jt}$ be the amount of the normal contracts that mature at time t that individual j demands at time 0. Without loss of generality, assume the real payment on one of these normal contracts will equal $aY_{st}Q_{jt}$. Then at time 0, j's budget constraint will be:

$$P_0 c_{j0} + V_0 Q_{jt} = P_0 y_{j0} \tag{16}$$

At time 0, individual j chooses between consumption at time 0 and the normal contracts. While individual j may enter into RASRT contracts, he/she does not exchange any money at time 0; hence, the RASRT contracts do not enter into the budget constraint at time 0. Individual j will be to sell normal contracts as well as buy them which then could enable j to consume more than his/her endowment at time 0 if that is what j chooses.

Individual j's budget constraint at time t is:

$$c_{jst} = k_{jt} Y_t + aY_{st} Q_{jt} + b\big(f(F_t) - f(Y_t)\big)x_{jt} \tag{17}$$

At time t, j's consumption will equal his real endowment ($k_{jt}Y_t$) plus his real payments on his normal contracts maturing at time t plus the real payments on the RASRT contracts maturing at time t. The variable $x_{jt}$ represents the amount of the RASRT contracts maturing at time t that j owns (if negative, then j would have sold these RASRT contracts). Once again, individual j would have entered into the RASTR contracts at time 0, but not paid anything for them.

At time 0, individual j will maximize his/her expected utility function,

$$U_{j0}(c_{j0}) + \sum_{t=1}^{T} \beta^t \sum_{s=1}^{S_t} \pi_{st} U_{jt}(c_{jst}) \tag{18}$$

I wrote this utility function to allow the utility function to vary by individual and by time t. However, I am not allowing it to vary by state of nature to preclude individual utility shocks.

[9] If this endowment ratio did vary, then the endowment-sharing contracts discussed in Chapter VI would be needed.

Individual j will maximize (18) subject to (16) and (17) where (17) applies for s=1,2,…,$S_t$ and for t=1,2,…,T.[10] The First Order Necessary Conditions (FONCs) are:

$U_{j0}'(c_{j0}) - P_0\lambda_0 = 0$ which implies that:

$$U_{j0}'(c_{j0}) = P_0\lambda_0 \tag{19}$$

$\beta^t \pi_{st} U'(c_{jst}) - \lambda_{st} = 0$ which implies that:

$$\beta^t \pi_{st} U'(c_{jst}) = \lambda_{st} \tag{20}$$

$$-V_0\lambda_0 + \sum_{t=1}^{S_t} bY_{st}\lambda_{st} = 0 \tag{21}$$

$$\sum_{s=1}^{S_t} a\big(f(F_t) - f(Y_{st})\big)\lambda_{st} = 0 \tag{22}$$

Substituting (19) and (20) into (21) gives $-V_0 \dfrac{U'(c_{j0})}{P_0} + \sum_{t=1}^{S_t} bY_{st}\beta^t \pi_{st} U'(c_{jst}) = 0$, which can be simplified as follows:

$$V_0 \frac{U'(c_{j0})}{P_0} = b\beta^t \sum_{t=1}^{S_t} \pi_{st} Y_{st} U'(c_{jst})$$

$$V_0 \frac{U'(c_{j0})}{P_0} = b\beta^t E[Y_{st} U'(c_{jst})]$$

$$\frac{V_0}{P_0 b} = \frac{\beta^t E[Y_{st} U'(c_{jst})]}{U'(c_{j0})} \tag{23}$$

Substituting (20) into (33) gives $\sum_{s=1}^{S_t} a(F_t - Y_{st})\beta^t \pi_{st} U'(c_{jst}) = 0$, which can be simplified as follows:

$$aF_t\beta^t \sum_{s=1}^{S_t} \pi_{st} U'(c_{jst}) = a\beta^t \sum_{s=1}^{S_t} \pi_{st} U'(c_{jst}) Y_{st}$$

$$f(F_t) = \frac{E\big[U'(c_{jst}) f(Y_{st})\big]}{E\big[U'(c_{jst})\big]}$$

Under the assumption that markets are complete, the real pricing kernel is:

$$\Omega_{st} = \frac{\beta^t U'(c_{jst})}{U'(c_{j0})}$$

[10] Additional assumptions are needed in order that nominal aggregate demand be determined in this model. One approach to do so is to assume Eagle and Domian's (2003 and 2004) temporary money. A second approach is to assume a cash-in-advance constraint that applies before the goods market opens but after consumers learn all the information for that period.

Therefore, $U'\left(c_{jst}\right) = \dfrac{\Omega_{st} U'(c_{j0})}{\beta^t}$. Substituting this into (22) gives

$f(F_t) = \dfrac{E\left[\beta^t \Omega_{st} U'(c_{j0}) f(Y_{st})\right]}{E\left[\beta^t \Omega_{st} U'(c_{j0})\right]}$ or $f(F_t) = \dfrac{E\left[\Omega_{st} f(Y_{st})\right]}{E\left[\Omega_{st}\right]}$. Therefore:

$$F_t = f^{-1}\left(\frac{E\left[\Omega_{st} f(Y_{st})\right]}{E\left[\Omega_{st}\right]}\right) \tag{24}$$

Since the pricing kernel is uniquely determined, then $F_t$ must also be uniquely determined.

## V. Consolidating the Four Types of Contracts

We previously discussed the four different types of contracts in isolation. In this section, we present an example where individuals used all four types of contracts to approximately duplicate their optimal contract receipts. This example involved five individuals with CRRA utility functions, each with a different coefficient of relative risk aversion. They lived three periods and interacted in a closed pure-exchange economy of one good without storage.

Each individual j maximized $\dfrac{c_{j0}^{1-\gamma_j}\xi_{j0}}{1-\gamma_j} + \sum_{t=1}^{T} \beta^t \sum_{s=1}^{S_t} \dfrac{c_{jst}^{1-\gamma_j}\xi_{jst}}{1-\gamma_j}$ subject to (2) and (3) where (3) applies for all states s at time t and for t=1,2,…,T. The $\xi_{j0}$ and $\xi_{jst}$ were utility shocks to j's utility function. Individual j's consumer optimization problem was satisfied when the following conditions held:

$$c_{js1} = \left(\frac{\beta^t}{\Omega_{s1}}\right)^{\frac{1}{\gamma_j}} c_{j0} \tag{25}$$

$$c_{j0} = \frac{y_{j0} + \sum_{t=1}^{T} E[\Omega_{*_t} y_{j*_t}]}{1 + \sum_{t=1}^{T} \beta^{\frac{t}{\gamma_j}} E\left[\Omega_{*_t}^{\,1-\frac{1}{\gamma_j}}\right]} \tag{26}$$

For our example we assumed a pure exchange economy without storage but with three time periods: time 0, time 1, and time 2. The economy consisted of five individuals named A, B, C, D, and E. For time periods 1 and 2, five equally likely states of nature existed. All consumers had a common time discount factor of $\beta = 0.95$ and the real aggregate supply at time 0 was 100. Table 5 presents the individual's coefficients of relative risk aversion (c.r.r.a.) and their time-0 endowments we assumed. For each state of nature possible in time 1 and time 2, Table 6 gives the levels of real aggregate supply, the real pricing kernel,[11] and the individuals' endowments. Table 7 gives the assumed Pareto-efficient consumption with the utility shocks in each state. (Rather than stating the utility shocks explicitly, we just give the Pareto-efficient consumption resulting from the utility shocks.)

| | A | B | C | D | E |
|---|---|---|---|---|---|
| relative risk aversion coeff. | 0.6 | 0.8 | 1 | 1.2 | 1.4 |
| time 0 endowment | 15 | 20 | 10 | 30 | 25 |

Table 5: Assumed Coefficients of relative risk aversion and time-0 endowments

| time 1: prob. | $Y_t$ | real pricing kernel | endowments A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| 0.2 | 80 | 1.182894 | 4.79626 | 13.60489 | 14.40016 | 23.20367 | 23.99501 |
| 0.2 | 90 | 1.052959 | 10.79579 | 11.70550 | 21.60018 | 22.50413 | 23.39439 |
| 0.2 | 100 | 0.950000 | 7.99532 | 19.00611 | 20.00020 | 21.00459 | 31.99377 |
| 0.2 | 110 | 0.866400 | 15.39486 | 16.50673 | 17.60022 | 29.70505 | 30.79314 |
| 0.2 | 120 | 0.797159 | 11.99439 | 13.20734 | 26.40024 | 27.60551 | 40.79252 |

| time 2: prob. | $Y_t$ | real pricing kernel | endowments A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| 0.2 | 80 | 1.123749 | 24.80238 | 6.39962 | 7.19041 | 24.00355 | 17.60404 |
| 0.2 | 90 | 1.000311 | 17.10268 | 8.99957 | 15.28921 | 32.40400 | 16.20454 |
| 0.2 | 100 | 0.902500 | 27.00298 | 15.99952 | 12.98801 | 24.00444 | 20.00505 |
| 0.2 | 110 | 0.823080 | 25.30328 | 4.39947 | 16.48681 | 35.20488 | 28.60555 |
| 0.2 | 120 | 0.757301 | 30.00358 | 14.39942 | 25.18561 | 33.60533 | 16.80606 |

Table 6: Assumed Endowments, Real Aggregate Supply and Resulting Real Pricing Kernel

[11] We determined the real pricing kernel through numerical techniques based on no individual utility shocks occurring.

We assumed the only existing RASRT contract was a Y^0.99 RASRT which paid the buyer of the contract $0.2\left(F_t^{0.99} - Y_t^{0.99}\right)$ at time t.

| time 1: prob. | $Y_t$ | real pricing kernel | Pareto-efficient consumption with utility shocks A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| 0.2 | 80 | 1.182894 | 14.67657 | 13.15474 | 10.38330 | 23.58227 | 18.20312 |
| 0.2 | 90 | 1.052959 | 10.33272 | 16.96374 | 15.26204 | 24.18349 | 23.25803 |
| 0.2 | 100 | 0.950000 | 16.53924 | 11.19805 | 18.92103 | 30.30984 | 23.03185 |
| 0.2 | 110 | 0.866400 | 17.08371 | 17.05274 | 11.96077 | 34.96855 | 28.93424 |
| 0.2 | 120 | 0.797159 | 24.55543 | 16.52393 | 17.78187 | 27.96571 | 33.17306 |

| time 2: prob. | $Y_t$ | real pricing kernel | Pareto-efficient consumption with utility shocks A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| 0.2 | 80 | 1.123749 | 5.87657 | 9.15474 | 11.98330 | 29.18227 | 23.80312 |
| 0.2 | 90 | 1.000311 | 20.23271 | 16.06374 | 7.16204 | 23.28349 | 23.25803 |
| 0.2 | 100 | 0.902500 | 13.53924 | 15.19805 | 21.92103 | 31.30984 | 18.03185 |
| 0.2 | 110 | 0.823080 | 22.58371 | 9.35274 | 13.06077 | 30.56855 | 34.43424 |
| 0.2 | 120 | 0.757301 | 22.15543 | 27.32393 | 21.38187 | 24.36571 | 24.77306 |

Table 7: Assumed Paretio Efficient Consumption With Utility Shocks

Technically speaking, we should rely on consumers maximizing utility to determine the amounts of each contracts each individual would demand. However, I have not worked out the mathematics for this. Instead, I have taken an easier approach, perhaps less rigorous approach, albeit not significantly less rigorous. In a previous section, we discussed two methods to determine the number of nominal and RASRT contracts, the tangency method and the minimum variance method. These methods were not based on individuals' maximizing utility, but rather were attempts to approximate the consumers' optimal contract receipts. If a method results in perfect replication of an individual's optimal contract receipts, then the method should give the same answer as a utility maximizing approach. If a method results in a good approximation of an individual's optimal contract receipts, the presumption I make is that the method is fairly close to the result of a utility-maximizing approach.

For this example, we used a different method for computing the number of normal contracts and RASRT contracts. Neither the tangency method or the minimum-variance method discussed previously insures that consumption in time 0 will be the same as it would be under an

Arrow-Debreu economy. For this chapter's example, we use a method that is similar to the tangency method but that does insure that consumption at time 0 will remain its Pareto-efficient level.

Remember that state-contingent securities are prepaid securities. One pays for the state-contingent securities at time 0. Of our four types of contracts only normal contracts can be prepaid contracts. RASRT contracts, endowment-sharing contracts, and spending-sharing contracts involve no payments until the time for which the contracts apply. Therefore, in order for the consumption at time 0 with normal, RASRT, endowment-sharing, and spending-sharing contracts to equal the Pareto-efficient consumption at time 0, an individual's holding of prepaid normal contracts entered into at time 0 must equal the prepaid value of all state-contingent securities that would have existed in an Arrow-Debreu economy. This is the approach we took in this example to determine the amount of normal contracts. We then chose the number of RASRT contracts to match the slope of the Pareto-efficient consumption at the expected real aggregate supply of 100.

In order to apply this new method, we needed to be able to determine the price of a prepaid normal contract. Such a prepaid normal contract is a type of discount bond. By definition of a normal contract, the real payments on the normal contract will equal $bY_{st}$ for some constant b. Assume that the constant b is the same for all standard normal contracts. That the real pricing kernel is a real stochastic discount factor implies that the price at time 0 of this standard normal contract is $E[\Omega_{*_t} bY_{*_t}] = bE[\Omega_{*_t} Y_{*_t}]$. For this example, we assumed that b=0.2.

In order that the time-0 consumption be the same as the time-0 Pareto-efficient consumption, the value an individual invests in prepaid normal contracts at time 0 must equal the expected present value of the state-contingent securities that would exist in an Arrow-Debreu economy. In other words, $P_0 \sum_{t=1}^{T} \sum_{s=1}^{S_t} \Omega_{st} \pi_{st} x_{jst} = P_0 \sum_{t=1}^{T} \sum_{s=1}^{S_t} \Omega_{st} \pi_{st} \left( c_{jst} - y_{jst} \right)$. We can actually be more specific than this. First, we can discuss this in real terms rather than nominal terms. Second, the real value each individual invests in each normal contract expiring at time t must equal the real value each individual would have invested in state-contingent securities that would have matured at time t. In other words, the real value individual j invests in normal contracts maturing at time t should equal $\sum_{s=1}^{S_t} \Omega_{st} \pi_{st} \left( c_{jst} - y_{jst} \right) = E\left[ \Omega_{*t} \left( c_{j*t} - y_{j*t} \right) \right]$.

We used this approach to determine the number of prepaid normal contracts each individual bought or sold. To determine the number of RASRT contracts each consumer bought or sold, we set the slope of his/her individual consumption as real aggregate supply changes equal to the slope of his/her Pareto-efficient consumption with respect to real aggregate supply when no individual utility shocks occur. Let $\tilde{c}'_{jt}(Y_{st})$ be the slope of j's Pareto-efficient consumption with no individual utility shocks. Let b be the slope of a standard normal contract, which in this example equals 0.2. Assuming the proportion of endowments to real aggregate supply is always the implicit average endowment ratio $\overline{R}_{jt}$, then the resources available for j's consumption at time t will equal j's endowment plus his/her normal contract receipts plus his/her RASRT contract receipts or:

$$\overline{R}_{jt} Y_{st} + bY_{st} z_{jt} + a_t \left( f(F_t) - f(Y_{st}) \right) \rho_{jt} \tag{27}$$

where f(.) is the RASRT contract function, $z_{jt}$ is j's quantity demanded of time-t normal contracts, and $\rho_{jt}$ is j's quantity demand of time-t RASRT contracts. Taking the derivative of (27) and setting it equal to $\tilde{c}'_{jt}(Y_{st})$ gives $\overline{R}_{jt} + bz_{jt} - af'(Y_{st})\rho_{jt} = \tilde{c}'_{jt}(Y_{st})$. Solving the above for $\rho_{jt}$ gives $\rho_{jt} = \dfrac{\tilde{c}'_{jt}(Y_{st}) - \overline{R}_{jt} - bz_{jt}}{-af'(Y_{st})}$. With the method for determining the number of RASRT contracts that I am currently following, the objective is to get the slope of j's consumption to match j's Pareto-efficient consumption when $Y_{st}$=E[$Y_{*t}$]. Substituting $Y_{st}$=E[$Y_{*t}$] into the above gives the formula we used to determine:

$$\rho_{jt} = \frac{\tilde{c}'_{jt}(Y_{st}) - \overline{R}_{jt} - bz_{jt}}{-af'(E[Y_{*t}])} \tag{28}$$

where the denominator of equation (28) is the slope of the RASRT contract.

To determine the individual demands for the endowment-sharing contracts and the spending-sharing contracts, we used the same approach discussed previous in this paper.

| state 1 at time 1: | A | B | C | D | E |
|---|---|---|---|---|---|
| endowment | 4.796259 | 13.60489 | 14.40016 | 23.20367 | 23.995[illegible] |
| normal-contract receipts | 5.18194 | 0.123425 | -4.07224 | 2.67806 | -3.911 |
| RASRT-contract receipts | -1.85325 | -0.59338 | 0.075021 | 0.977741 | 1.393[illegible] |
| endowment-sharing receipts | 3.203741 | -1.60489 | 1.599839 | -3.20368 | 0.0049[illegible] |
| spending-sharing receipts | 3.2 | 1.6 | -1.6 | 0 | -[illegible] |
| endowments plus contract receipts | 14.52869 | 13.13004 | 10.40278 | 23.6558 | 18.282[illegible] |
| P.E. consumption w/ utility shocks | 14.67657 | 13.15474 | 10.3833 | 23.58227 | 18.203 |
| residual market imperfections | 0.14788 | 0.024698 | -0.01948 | -0.07353 | -0.079[illegible] |
| imperfections as % of P.E. Cons. w/ utility shocks | 1.01% | 0.19% | -0.19% | -0.31% | -0.44 |

Table 8: Contract receipts in state 1 at time 1

Table 8 shows how the receipts of the different contracts led to consumption levels for state 1 at time 1 that closely approximated the Pareto-efficient consumption with utility shocks. Also, shown in Table 8 is the residual market imperfections as a percent of Pareto-efficient consumption. Table 9 shows these imperfection percentages

| state | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 1.01% | 0.19% | -0.19% | -0.31% | -0.44% |
| 2 | -0.71% | -0.07% | 0.07% | 0.15% | 0.16% |
| 3 | -0.86% | -0.19% | 0.10% | 0.23% | 0.32% |
| 4 | -0.42% | -0.07% | 0.08% | 0.10% | 0.13% |
| 5 | 0.51% | 0.10% | -0.10% | -0.22% | -0.19% |

Table 9: Imperfections as % of Pareto-efficient consumption for all states at time 1:

not only for state 1at time 1 but for all states for time 1.  Results for time 2 were similar.

That these contracts cannot perfectly replicate the optimal contract receipts are solely due to the RASRT contracts being unable to perfect handle the needs of differences in relative risk aversion.  The endowment-sharing and spending-sharing contracts in this example perfectly handle the issues created by stochastic endowment ratios and individual utility shocks.  However in the real Alien and Earth worlds, I suspect the theoretical imperfections of RASRT contracts to be of much less magnitude that the practical problems of implementing endowment-sharing and spending-sharing contracts.

## VI. Summary and Conclusion

In a pure-exchange economy without storage Pareto-efficient consumption only varies with respect to two factors: (i) real aggregate supply and (ii) individual utility shocks.  Optimal contract receipts vary with respect to only three factors: (i) real aggregate supply, (ii) individual utility shocks, and (ii) changes in the ratio of endowment to real aggregate supply.  The derivative of an individual j's Pareto-efficient consumption with respect to real aggregate supply equal the proportion of j's consumption to real aggregate supply divided by the ratio of j's relative risk aversion over average relative risk aversion.  Given these results, this paper was able to use (i) normal contracts, (ii) endowment-sharing contracts, (iii) spending-sharing contracts to approximately replicate individuals' optimal contract receipts.  We therefore conclude that these four types of contracts can enable consumers to very closely reach their Pareto-efficient consumption that would be reachable under complete markets.

It is important to note that only the residual imperfections from these four contracts are due solely from the inability of RASRT contracts to perfectly transfer risk among consumers; the endowment-sharing contracts and the spending-sharing contracts perfectly dealt with the issues of stochastic endowment ratios and individual utility shocks. However, in the real world, RASRT contracts would be fairly straight forward to implement as real aggregate supply is relatively objectively measured. However, the *post hoc* determination of the endowment-sharing contracts and spending sharing contracts will face individuals who want to exaggerate their endowment shortfalls and exaggerate their spending needs.

The results of this paper have implications for the real world. First, in order for nominal contracts to behave as normal contracts, the central bank needs to pursue nominal-income targeting or nominal-aggregate-demand targeting. However, the current fad in central banking is inflation targeting, and monetary economists almost universally agree on the primary objective of central banking be price stability, which would be in conflict with nominal-income or nominal-aggregate-demand targeting when real aggregate supply changes.

A second implication of this work has to do with insurance contract design. In this paper the endowment-sharing contracts and spending-sharing contracts pose no aggregate risk to insurance companies. However, current insurance contracts in the real world do expose insurance companies to aggregate risk. An important assumption difference between this paper and reality is that no real capital exists in this paper's model. If real capital exists, then insurance companies could rely on capital reserves as a way to protect the insurance companies for aggregate risk. However, in some extreme situations, these reserves would prove insufficient for insurance companies. In particular, if a flu epidemic killed half the population, who all had life insurance, life insurance companies would become bankrupt. The way endowment-sharing contracts and spending-sharing contracts are designed may prove useful in real-world redesigning of insurance contracts.

In the current real world, RASRT contracts do not exist. Perhaps this is because everyone has the same relative risk aversion. If that were the case, then normal contracts, endowment-sharing contracts, and spending-sharing contracts could complete the markets. However, if individuals do have differing relative risk aversions then RASRT contracts may have a place in the real world.

## References


Arrow, Kenneth J., (1953), "The Role of Securities in the Optimal Allocation of Risk Bearing," *Colloques Intternationaux du Centre National de la Recherche Scientifique*, XL: 41-48; translated in the 1964 *Review of Economic Studies*, 31(#2):91-96.

___________, (1965), *Aspects of the Theory of Risk-Bearing* (Helsinki: Yrjo Jahnssonin).

Debreu, Gerard, (1959), *Theory of Value* (Wiley – New York).

Eagle, D. and D. Domian, (1995), "Quasi-Real Bonds - A Financial Innovation to Eliminate a Danger Inherent in Inflation-Indexed Bonds," *Applied Economic Letters*, December 1995, 487-490.

Eagle, David and Dale Domian, (2003), "Quasi-Real Contracts: The Pareto-Efficient Solution to Inflation Indexing," working paper, http://www.cbpa.ewu.edu/~deagle/banking/journey.doc accessed on December 23, 2003.

Eagle, D. and D. Domian, (2004), "Quasi-Real Bonds – The Pareto-Efficient Solution to Inflation Indexing," Working Paper, http://ideas.repec.org/p/wpa/wuwpma/0312010.html accessed on November 14, 2004.

Viard, Alan D. (1993), "The Welfare Gain from the Introduction of Indexed Bonds," Journal of Money, Credit, and Banking, 25(#3):469-708.

Wilson, Robert (1968), "The Theory of Syndicates," *Econometrica*, 36(#1):119-132.

R. Radner (1972) "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets", *Econometrica*, Vol. 40 (2), p.289-303.

Campbell, Lo, and MacKinlay, (1997), *The Econometrics of Financial Markets* (Princeton University Press – Princeton).