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Why VAR Fails:

Long Memory and Extreme Events in Financial Markets

Abstract

The Value-at-Risk (VAR) measure is based on only the second moment of a rates of return distribution. It is an insufficient risk performance measure, since it ignores both the higher moments of the pricing distributions, like skewness and kurtosis, and all the fractional moments resulting from the long - term dependencies (long memory) of dynamic market pricing. Not coincidentally, the VaR methodology also devotes insufficient attention to the truly extreme financial events, i.e., those events that are catastrophic and that are clustering because of this long memory. Since the usual stationarity and i.i.d. assumptions of classical asset returns theory are not satisfied in reality, more attention should be paid to the measurement of the degree of dependence to determine the true risks to which any investment portfolio is exposed: the return distributions are time-varying and skewness and kurtosis occur and change over time. Conventional mean-variance diversification does not apply when the tails of the return distributions ate too fat, i.e., when many more than normal extreme events occur. Regrettably, also, Extreme Value Theory is empirically not valid, because it is based on the uncorroborated i.i.d. assumption.

Acknowledgement 1 *The author recently published a provocative Letter-to-the-Editor regarding the counter-intuitive effects of long memory on portfolio diversification in* FINANCIAL ENGINEERING NEWS*, January/February 2005. This article explains why long memory is such a crucial phenomenon and why portfolio managers ignore it at their peril*

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1 Introduction

This paper summarizes some of the consequences of the empirical measurement results of non - Gaussianity, irregularity, and non-stationarity of rates of return on cash investment, both for investment in individual assets and for investment in portfolios of assets. In particular, we will focus on the measurement and management of the Value - at -Risk (VaR) of an investment. The VaR measure summarizes only the potential money-loss exposure of an investment to market risk measured by the variance or standard deviation of rates of return. This makes it a popular tool for conveying the magnitude of the market risks of portfolios to senior fund management, directors, sponsors, shareholders and regulators (Hopper, 1996; Hua and Wilmott, 1997; Duffie and Pan, 1997; Jorion, 1997; Dowd, 1998).

However, such a simple potential money loss measure based on only the second moment of a rates of return distribution is insufficient, since it ignores both the higher moments of the pricing distributions, like skewness and kurtosis, and all fractional moments resulting from the long - term dependencies or long memory of dynamic market pricing. The VaR methodology also devotes insufficient attention to the truly extreme financial events, *i.e.,* those events that are *catastrophic* (Embrechts, *et al.*, 1997; Bassi, Embrechts and Kafetzaki, 1998) and that are *clustering*. There exists considerable anecdotal literature on such catastrophic financial events (Kindleberger, 1996), but relatively little rigorous measurement, analysis or theory.

This paper is organized as follows: Section 2 discusses the phenomenon of long memory and the measurement of the degree of long term dependency by the monofractal Hurst exponent. Sections 3 presents the VaR theory for stable distributions based on Zolotarev's parametrization. Section 4 summarizes the parametric VaR and the problem of its implementation due the scarcity of extreme values. Extreme value theory is concisely presented in Section 5, which also discusses the phenomenon of catastrophic collusions of exceedences. Section 6 presents the Fama-Samuleson Theorem and its surprising and counterintuitive results for portfolio diversification, which urge a very accurate measurement of the degree of long term dependence in the financial markets by portfolio managers.

2 Global Dependence of Financial Returns

The pricing processes of financial markets show global dependencies, *i.e.,* they are long - term memory processes, with slowly declining autocovariance functions and with scaling spectra. The reason for this phenomenon is the aggregation in the markets of investment flows of different time horizons and degrees of cash illiquidity. The pricing processes of stocks, bonds and currencies are nonlinear dynamic processes, which show short and long term aperiodic cyclicities, and intermittence, *i.e.,* periods of laminar flows interspersed with periods of turbulent cash flows. Financial turbulence is characterized by successive velocity fluctuations and successive periods of condensation and rarefaction in the frequency of trading transactions. However, there are major differences among these various financial pricing processes:

(1) Foreign exchange (FX) is traded, but FX does not consist of securities. The FX appreciation rates are usually antipersistent with Hurst exponents of the order $0.2 < H < 0.5$. The cash flows in the FX markets are potentially turbulent and may show vortices, in particular when $0.33 < H < 0.41$, as was the case with the D - Mark/US$, now replaced by the Euro/US$, and as is the case with the Yen/US$, when these cash flows are adjacent to much less liquid cash flows in, for example, Asian FX markets.

Long term investment in FX rates is dangerous, since the volatility (= standard deviation) of their appreciation rates does not scale according to the square root of the investment time horizon, $\tau^{0.5}$, as for a Geometric Brownian Motion (GBM). In the short - term FX rates are about as volatile as stock prices. But Fig. 1 shows that when the investment horizon τ increases, the volatility of FX rates tends to increase *slower* than that of a GBM. In popular opinion, in the long term FX markets are considered more risky than stock markets, while the opposite is true.

It depends on the investment horizon. In fact, in the long term FX rates are less risky than stock prices.

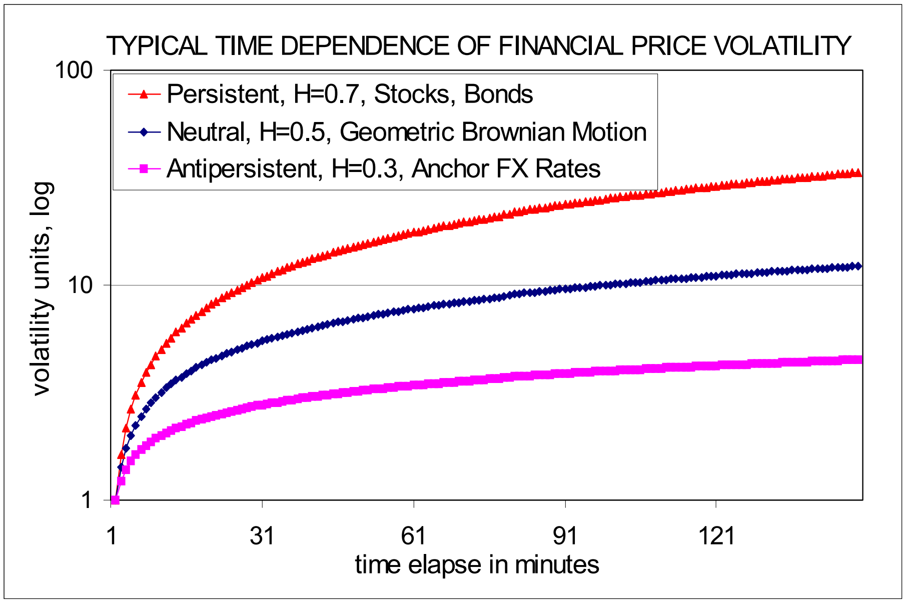

Figure 1: Typical time dependence of financial price volatility, $\log\sigma^2$. The volatility or second moment risk of the persistent stock prices increases faster with the time horizon τ than the volatility of the conventional Geometric Brownian Motion, while the volatility of FX rates increases slower

(2) Stock and bonds are traded securities. Their rates of return are persistent, with Hurst exponents $0.5 < H < 0.8$, *e.g.,* the both the S&P500 stock and Dow Jones Industrials Index Indices exhibit a Hurst exponent value of $H \approx 0.67$. Their rates of return behave closer to that of black noise, with occasional, and essentially unpredictable sharp discontinuities up or down, called *financial catastrophes*. These discontinuous catastrophes cause the frequency distributions of black noise processes to have fat tails. It occurs because the originators of such events are amplified in the financial system. Stock and bond market prices are close to brown noise, and their first differences are persistent pink noise. Thus, their rates of return (= *velocity* of the stock and bond prices) which are first differences normalized on the lagged price, are also persistent pink

noise. The rates of return differences (= the *acceleration* of stock and bond prices) have slightly less power than white noise and are antipersistent, or light blue noise. Since financial volatility or risk has been measured as the standard deviation of the rates of return, it is also antipersistent, or light blue noise.

Long term investments in stocks and bonds are potentially very risky. When the investment horizon τ increases, bond and stock return volatility tends to increase faster than that of a GBM. The *level* of volatility of stock prices is much higher than that of bond prices. Therefore, investments in bonds are less risky than investments in stocks.

3. Real estate investments have equity and bond characteristics, but their liquidity is usually much lower than that of stocks or bonds. Consequently, their prices are extremely persistent and their rates of return truly black noise.[1] Financial catastrophes are a regular occurrence in such consistently persistent and illiquid markets, as the real estate events of the past two decades testify.

Therefore, our advice regarding speculation in the FX, stock, bond and real estate markets runs counter that of both professional and popular investment advice. It is based on the now well-established non-GBM relationship between risk and investment horizon. One cannot judge the riskiness of an investment on the basis of a one - time picture of volatility, but needs to take account of dynamic long - term dependence.

International cash investments of investors with different time horizons simultaneously flow in and out all these international financial markets. Our understanding of these cash flow and pricing processes is still very limited, but it is improving, now that more high frequency data are accumulated and more research efforts are becoming directed towards the measurement and analysis of the various dependence phenomena of financial market risks.

[1] At this moment there are not yet Hurst or Lipschitz exponent measurements of real estate investment returns available in the financial literature, but I expect that they will soon be published.

3 Value - at - Risk for Stable Distributions

3.1 Subjectivity of Value - at - Risk

Informally, the Value - at - Risk or VaR measure summarizes the expected maximum loss (or worst loss) over a limited investment horizon, within a given confidence interval (Wilson, 1998). Thus, measuring VaR involves the choice of two quantitative inputs: the length of the investment horizon τ, and the confidence level. Both are arbitrary, subjective choices. Therefore, by definition, VaR is not an objective, or scientific measure of the exposure to market risk, but a subjective, game type measure, according to some recent theoreticians (Shafer and Vovk, 2001).[2]

Of course, portfolio investors can determine the length of their own investment horizon τ. Commercial banks in the USA currently report their trading VaR over a daily horizon or a horizon of ten days (= two working weeks of five days each), because of the rapid turnover in their portfolios, in agreement with the amended Basle Accord of 1996. In contrast, pension funds tend to report their risk over one - month or one - quarter investment horizons. As Jorion (1997, p. 86) correctly states:

> ”As the holding period should correspond to the longest period needed for an orderly portfolio liquidation, the horizon should be related to the liquidity of the securities, defined in terms of the length of time needed for normal transaction volumes.”

There is much less consensus about the subjective choice of the confidence level. There is a trade - off between the requirements set by the regulators to ensure a safe and sound financial system, and the adverse effects of the requirement for a minimum level of (expensive) capital on bank returns and thus on bank share prices. For example, Bankers Trust sets a 99% confidence level, Chemical and Chase use a 97.5% level, Citibank uses a 95.4% level, while BankAmerica and

[2] I agree with their conclusion that probability theory relates only to (Las Vegas type roulette, card, one-armed-bandit) game situations. But I disagree with their assumption that probability theory has anything to do with the empirical world. Nobody has ever proved that *probability* is an empirically observable real world phenomenon and if and why it exists in the real world, in contrast to *randomness* or *uncertainty*. Probability **theory** does not explain any empirical phenomenon in the real world and is therefore not a *scientific* theory, but only a *philosophical* theory.

J. P. Morgan use a 95% confidence level. These differences are allowed under the current Basle Accord guidelines, since the major, well-capitalized commercial banks are allowed to construct their own internal financial risk management models.

Higher confidence levels imply higher VaR figures, which in turn imply higher minimally required equity capital cushion for risk insurance. But higher confidence levels imply also longer testing periods. For example, suppose our investment horizon is 1 day and we accept a confidence level of 95%, we would expect a loss worse than the VaR in 1 day out of 20 days. If we choose a 99% confidence level, we would have to wait on average 100 days, or more than 3 months, to confirm that our risk model conforms to reality! When our investment horizon is 1 month, then a 99% confidence level would force us to observe on average 100 months, or about 8 years of data, before we can confirm our financial risk model. And, when our investment horizon is one year, then a 99% confidence level would force us to observe on average 100 years, or a century, before we can confirm our financial risk model. These confidence levels also assume that the observations in each of those time periods are mutually independent and stationary. These assumptions are almost certainly empirically not true!

The VaR measure can be derived, under the strong assumptions of independence and stationarity, either from actual empirical distributions or from an abstract formal distribution, like the Gaussian distribution, in which case it is based on its second moment only. The Basle Committee, which recommended VaR measures in 1988 and again in 1992 to summarize overall risk exposure, also recommended "back - testing" and "stress - testing," as means to verify the accuracy of VaR figures, as did the landmark $G - 30$ study (*cf.* Chapter 2 of Jorion, 1997, pp. 23 - 39; *cf.* also Hanley, 1998 and Grau, 1999).

3.2 Value - at - Risk as a Quantile Risk Measure

We will now first provide a formal definition of Value - at - Risk (VaR), within the context of our cash flow model of investments of Chapter 10.

Definition 2 *For $X(t - \tau)$ as the initial investment and $x_\tau(t)$ its rate of return over investment* *horizon τ, the investment at the end of the investment horizon is*

$$X(t) = [1 + x_\tau(t)]\, X(t-\tau) \tag{1}$$

Assume that $x_\tau(t)$ is from a stable distribution, i.e., a distribution which maintains its shape, although, perhaps, not its size. The lowest expected portfolio level at the end of the investment horizon τ at a given confidence level c is

$$X^*(t) = [1 + x^*_\tau(t)]\, X(t-\tau) \tag{2}$$

*Then the **VaR relative to the mean** at time t for investment horizon τ is*

$$\begin{aligned} VaR_{mean}(t,\tau) &= E_\tau\{X(t)\} - X^*(t) \\ &= E_\tau\{[1 + x_\tau(t)]\, X(t-\tau)\} - [1 + x^*_\tau(t)])X(t-\tau) \\ &= [\mu_\tau - x^*_\tau(t)]\, X(t-\tau) \end{aligned} \tag{3}$$

*where $\mu_\tau = E_\tau\{x(t)\}$, and the **absolute VaR**, or **VaR relative to zero** at time t for investment horizon τ is*

$$\begin{aligned} VaR_{zero}(t,\tau) &= 0 - X^*(t) \\ &= -x^*_\tau(t)X(t-\tau) \end{aligned} \tag{4}$$

In both cases, finding the VaR is equivalent to determining the quantile cut - off rate of return $x^*_\tau(t)$ from its available empirical distribution of $x_\tau(t)$ and the confidence level $c_\tau(t)$ for time horizon τ, such that

$$c_\tau(t) = P\left[x_\tau(t) > x^*_\tau(t), t\right] = \int_{x^*_\tau(t)}^{+\infty} f\left[x_\tau(t)\right] dx \tag{5}$$

where $f\left[x_\tau(t)\right]$ is the empirical probability density function (pdf). This assumes that the pdf $f\left[x_\tau(t)\right]$ is continuously integrable, which may not be the case with (empirical) fractal distributions. Notice that, in general, the confidence level $c_\tau(t)$ can be time - varying, because the probability distribution $P[., t]$ is time-dependent.

This definition of VaR allows for some kind of non-stationarity and long - term time dependence, *i.e.,* the kind of non-stationarity associated with stable scaling distributions, like Pareto-Lévy distributions. But in the financial literature this distribution is usually assumed to be stationary in the strict sense, implying that the confidence level c is also assumed to be constant (= independent of time t):

$$c_\tau = P[x_\tau(t) > x^*_\tau(t)] = \int_{x^*_\tau(t)}^{+\infty} f[x_\tau(t)]dx \tag{6}$$

For example, $c_\tau = 95\%$ for all t. Or, equivalently, we can express everything in terms of a constant *significance level*:

$$1 - c_\tau = P\left[x_\tau(t) \leq x^*_\tau(t)\right] = \int_{-\infty}^{x^*} f\left[x_\tau(t)\right] dx \tag{7}$$

For example, the significance level $1 - c_\tau = 5\%$ time horizon τ and for all t.

It's important to emphasize that the quantile determination of the VaR is also *valid for any nonstationary stable distribution* of the rates of return on assets, discrete or continuous, skewed or symmetric, leptokurtic or platykurtic, as long as we know how the distribution scales over time! It only has to maintain its shape. See, for example, the investigations by Hull and White (1998a and b) into the impact of non - Gaussian distributions on the VaR, or the VaR bounds for portfolios with assets with non - normal returns (Luciano and Marena, 2001).

Example 3 *The annual report of* 1994 *of J. P. Morgan provides an empirical example in the form of a histogram of its daily revenues* $X(t)$ *(Fig. 2). From the graph, the average revenue*

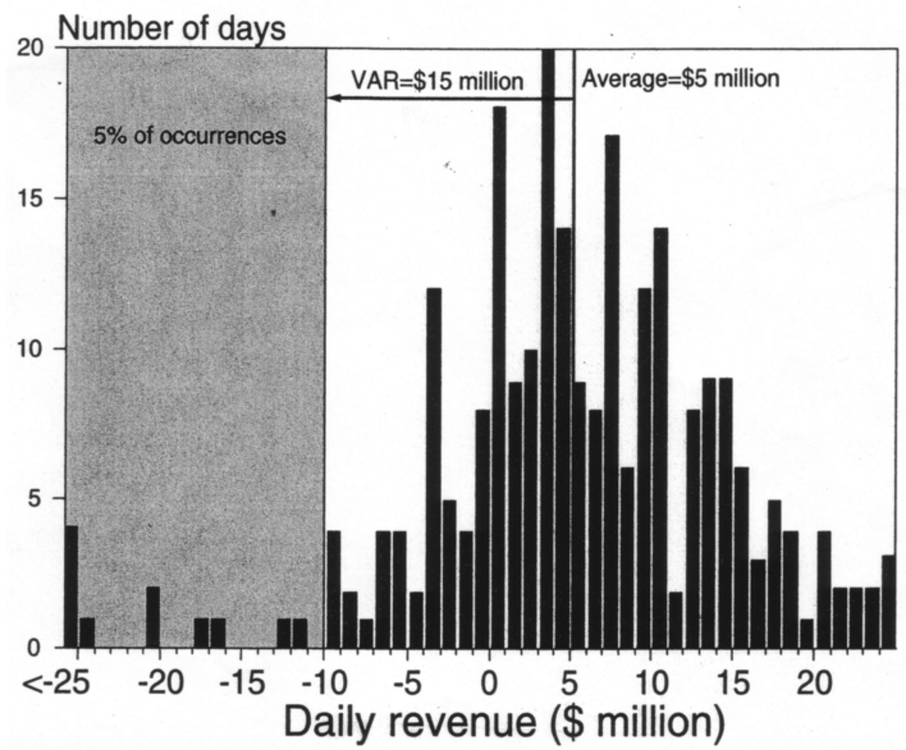

Figure 2: Empirical distribution of daily revenues of JP Morgan in 1994. Notice some extreme daily losses in the left tail, as indicated by the grey area, left of the $10 million daily loss.

$\mu = US\$5.1million = US\$5.1m$. There are $T = 254$ daily observations. We try to find $X(t)$, such that the number of observations to its left is $T \times (1-c) = 254 \times 5\% = 12.7$ days. Because of the coarseness of the histogram we need to interpolate. There are 11 daily observations to the left of $-US\$10m$, and 15 observations to the left of $-US\$9m$. Interpolation gives

$$X^*(t) = US\$ \left[-9 - \frac{12.7 - 15}{11 - 15} \right] m = -US\$9.575m \tag{8}$$

Thus, the VaR of daily revenues measured relative to the mean is

$$\begin{aligned} VaR_{mean}(t) &= E\{X(t)\} - X^*(t) \\ &= US\$5.1m - (-US\$9.575m) \\ &= US\$14.57m \end{aligned} \tag{9}$$

and the VaR of daily revenues in absolute dollar loss is

$$\begin{aligned} VaR_{zero}(t) &= 0 - X^*(t) \\ &= US\$9.575m \end{aligned} \tag{10}$$

4 Value - at - Risk for Parametric Distributions

4.1 Gaussian Value - at - Risk

The VaR computation is simplified considerably when the returns are assumed to adhere to a constant parametric Gaussian distribution is assumed: $x(t) \sim N(\mu, \sigma^2)$. Then the VaR can be derived directly from the portfolio volatility σ, using a multiplicative factor depending on the confidence level.

First, we transform the general stationary density function $f[x(t)]$ into a standardized normal distribution $g[z(t)]$, which has mean zero and a unitary standard deviation: $z(t) \sim N(0,1)$, so that the standardized variable is defined by

$$z(t) = \frac{x(t) - \mu}{\sigma} \tag{11}$$

and its cut - off rate by

$$z^* = \frac{x^* - \mu}{\sigma} \tag{12}$$

The VaR can be expressed in standard fashion by

$$\begin{aligned} 1 - c = P(x(t) \le x^*) &= \int_{-\infty}^{x^*} f[x(t)]\, dx(t) \\ &= \int_{-\infty}^{z^*} g[z(t)]\, dz(t) \end{aligned} \tag{13}$$

Now we only have to examine the tables of the *cumulative standard normal distribution*

$$F_{z^*}\left[z(t)\right] = \int_{-\infty}^{z^*} g\left[z(t)\right] dz(t) \tag{14}$$

which provide the integrated area left of the value $z(t) = z^*$. We can always return to the original parametrized Gaussian distribution, since $x(t) = z(t)\sigma + \mu$ and $x^* = z^*\sigma + \mu$.

4.2 Statistical Problem: Scarcity of Extreme Values

The main problem facing the statistical VaR practitioners is that the VaR is an extreme quantile of a rate of return distribution (Bassi, Embrechts and Kafetzaki, 1998). Therefore, we have relatively few historical observations with which to estimate it (Hendricks, 1996). VaR estimates are usually imprecise, and become even more so, the more we move further out onto the tail of the distribution. Practitioners have responded by relying on assumptions to make up for the lack of data. The common, but decidedly unrealistic, assumption is that the empirical rates of return are from a constant parametric Gaussian distribution.

However, financial returns are usually fat - tailed and assuming Gaussianness can lead to serious under - estimates of VaR (Hull and White, 1998a; Ju and Pearson, 1999). This has lead to the suggestion of adaptive updating of a time - varying volatility (Hull and White, 1998b). A more satisfactory assumption is that the returns follow a heavy - tailed stable distribution (Rachev and Mittnik, 2000). Even then we still face the problem that most observations are central ones. The estimated distribution fits the central observations best, and therefore remains ill - suited to the extreme observations with which financial risk analysts are mainly concerned. The estimation of low frequency events with finite data remains highly problematic and that is the true reason why *stress testing* is often recommended (Grau, 1999).

Market makers are very much interested in large moves in the prices of stocks, bonds or other traded assets, since the largest price moves cause market makers to lose money. Only for small price moves, market makers make money on commission trading. The loss from large price moves result from the option's *gamma*. Market makers are *delta - hedgers*. If prices move sufficiently, their delta - hedged positions become unhedged. For example, when a market - maker delta - hedges a stock position, he is short a call option and a large move generates a loss. As the stock price rises, the delta of the call option increases and it loses money faster than the stock makes money. Vice versa, as the stock price fall, the delta of the call option decreases and it makes money more slowly than the fixed stock position loses money. In effect, the market maker becomes unhedged net long as the stock price falls and unhedged net short as the stock price rises.

It can be shown that the market maker's profit depends on the squared change in the stock price, *i.e.,* on the magnitude and not the direction of the stock price move. It can also be shown that a market maker breaks even for a one standard deviation move in the stock price, $\epsilon_\tau = \sigma \tau^H X$, assuming that the maket prices $X(t)$ follow Fractional Brownian Motions (FBM, with $0 < H < 1$. This is equivalent to the GBM when $H = 0.5$). The market maker makes money within that price range and loses money outside that price range. The market-maker's regular profit and occasional large loss can thus be explained by the preponderance of small price movements in the financial market and the occasional extreme moves.

Differently stated, the leptokurtosis of the distribution of the financial market price increments explains the leptokurtosis of the distribution of the market-makers' profits. When the distributions are close to normal, a market maker expects to make small profits about two - thirds of the time, and large losses about one - third of the time and on average to break even (McDonald, 2002, pp. 13 - 8 and 13 - 16).[3]

Extreme value problems are not unique to financial risk management, but occur also in other scientific and engineering disciplines. They are particularly prominent in hydrology, where statisticians and engineers, like "Father - of - the - Nile" hydrologist Hurst in the 1950s, have long struggled with the problem of how high dams should be to contain flood probabilities within reasonable limits (Mandelbrot and Wallis, 1969). These hydrologists have usually even less data than

[3] Even very recently, McDonald of Northwestern University still assumes in his book (McDonald, 2002) that the distribution is normal, so that the Hurst exponent is $H = 0.5$, despite all the empirical evidence to the contrary. Different financial markets exhibit different degrees of persistence, as I brought to his attention, when I officially reviewed his book a few months before its publication..

financial risk managers and often have to estimate quantiles well out the range of their historical data (Whitcher *et al.*, 2002). Because of the scarcity of observations on extreme market prices, the approach has been to develop extreme value theory based on a few minimal assumptions.

5 Extreme Value Theory

Thus, almost two decades ago, a small group of theorists developed an extreme value (EV) theorem based on the (strong) assumption of i.i.d. returns, which tells us that the limiting distribution of extreme returns has always the same form, whatever the unknown i.i.d. distribution from which the data are drawn.[4]

Theorem 4 *(**Extreme Value**) Subject to the i.i.d. condition, the density of extreme returns converges asymptotically to*

$$H(x;\mu,\sigma,\xi) = \left\{ \begin{array}{c} e^{\left(-\left[1+\xi\frac{(x-\mu)}{\sigma}\right]^{-\frac{1}{\xi}}\right)} \ \ if\ \xi \neq 0 \\ e^{\left(-e^{-\frac{(x-\mu)}{\sigma}}\right)} \ \ if\ \xi = 0 \end{array} \right\} \tag{15}$$

The parameters μ and σ correspond to the mean and standard deviation, respectively, and the third parameter, the *tail index* ξ, indicates the heaviness of the tails. The bigger ξ, the heavier the tail (Longin, 1996; McNeil, 1996, 1998; Lauridsen, 2000). For some applications, *cf.* Koedijk, Schafgans and deVries, 1990, and, most recently, Blum and Dacorogna, 2002). For an expert's critique of the use of the tail index measure of the fatness of the tails of distributions to identify the stability exponent α_Z, *cf.* McCulloch (1997).

Remark 5 *The Extreme Value Theorem is related to the classical Central Limit Theorem (CLT), but applies to the extremes of observations rather than their means (= the concentrations). It allows to estimate the asymptotic distribution of extreme values, without making assumptions other than the i.i.d. assumption, about the unknown empirical distribution. The first step is to estimate these parameters and there is a choice between semi - parametric methods, like the Hill estimator, which focus on the estimation of the tail index ξ, and parametric methods, like the ML method, which estimate all three parameters μ, σ and ξ simultaneously. However, the estimation is complicated by non - linearities and the statistical properties of these parametric estimators are still not well understood.*

[4] EV theory, which was discovered by Stephan Resnick (1987), seems to be first applied to VaR by François Longin in 1996, followed by Jon Danielson, Casper de Vries and their collaborators at the Tinbergen Institute in The Netherlands and at the London School of Economics (LSE) in London, by Paul Embrechts and Alexander McNeil at the ETH Zentrum in Zürich, and by Francis Diebold and his associates at the Wharton School in Pennsylvania, USA

EV theory faces, at least, two complications. First, the choice of the tail size of the distribution of our rate of return observations affects the VaR estimates through the effect on the estimate of the tail index ξ. Second, the EV theorem assumes that the rates of return are i.i.d.. But we know from the empirical research reported in the financial literature that empirical financial rates of return show forms of clustering, with periods of alternating high and low volatility, due to global dependencies. These empirically observed global dependencies violate this key assumption of EV theory!

5.1 Increased Inter - Correlation of Financial Exceedences

A paper by Dacorogna *et al.* (2001), presented at University of Konstanz provides vivid evidence that the extreme values or so-called *exceedences* (*i.e.,* the values exceeding certain confidence boundaries, like 95% of the distribution) of international rate of return distributions tend to cluster and to highly positively correlate at times of financial distress (*cf.* also Blum and Dacorogna, 2002)! Thus, in times of distress, portfolio diversification tends to be defeated by increased positive inter - correlations between the extreme rates of return of the various portfolio investments. This severely diminishes the value of the VaR approach to financial risk management, since it appears that portfolios behave very differently in times of distress compared with times of normality. The i.i.d. (= identically distributed, independently distributed) assumption does not hold in empirical reality, in particular not when there is financial market stress. In other words, portfolio variances and covariances are time - varying and they are varying in such a way that they defeat conventional risk diversification rules.

6 Fractional Brownian Motion and Value - at - Risk

The VaR can be put in a dynamic context, with nonstationary distributions, as long as the risk is measurable by the second moment only. Thus, for the FBM (Elliott and van der Hoek, 2000), which has i.i.d. *increments* and a second-order risk measure depending on the investment horizon

τ, since:

$$\sigma_\tau = \sigma_\varepsilon \tau^H \tag{16}$$

the VaR relative to the mean for investment horizon τ is

$$\begin{aligned} VaR_{mean}(t,\tau) &= [\mu - x^*(t)]\, X(t-\tau) \\ &= -z^* \sigma_\varepsilon \tau^H X(t-\tau) \end{aligned} \tag{17}$$

since

$$[\mu - x^*(t)] = -z^* \sigma_\tau \tag{18}$$

and the corresponding absolute loss VaR is thus

$$\begin{aligned} VaR_{zero}(t,\tau) &= -x^*(t) X(t-\tau) \\ &= -(\mu\tau + z^* \sigma_\varepsilon \tau^H) X(t-\tau) \end{aligned} \tag{19}$$

For example. for the GBM with $H = 0.5$, the VaR relative to the mean for investment horizon τ is easily to:

$$VaR_{mean}(t,\tau) = -z^* \sigma_\varepsilon \tau^{0.5} X(t-\tau) \tag{20}$$

and the corresponding absolute loss VaR to:

$$VaR_{zero}(t,\tau) = -(\mu\tau + z^* \sigma_\varepsilon \tau^{0.5}) X(t-\tau) \tag{21}$$

But the remaining fundamental question is: can we really measure financial risk by only the second moment of a distribution, in particular in a dynamic portfolio situation with global time dependence? The next section will provide some tentative, and, perhaps, discouraging, answers.

6.1 Value - at - Risk and Fractal Pricing Processes

6.1.1 Concerns and Doubts About Value - at - Risk

In this section we will formulate why we have deep concerns and doubts about the use of the VaR as an overall measure of the exposure to market risk, when VaR is based on assumed simple parametric distributions, like the Gaussian, and why we insist on measuring the stability of the empirical distributions of the rates of return $x_\tau(t)$, in addition to measuring their various forms of long - term time dependence in the various financial markets. We have observed, and reasoned throughout this book, that the Gaussian distribution is inadequate to describe financial market returns, since empirical financial market returns show skewed, leptokurtic, non - normal distributions and, most importantly, non-stationarity in the strict sense.[5] For example, from Chapter 3 we already know that some nonlinear market pricing systems may produce nonstationary distributions without a definable ("existing") mean or variance!

The classical Modern Portfolio Theory (MPT) of Markowitz and Sharpe is the basis for the VaR theory. It presupposes stationary (Gaussian) rates of return distributions and that will be the starting point for the following discussion for didactic purposes. Gauss showed about two hundred years ago that the limiting distribution of a set of independent, identically distributed (i.i.d.) random variables is the normal distribution. This is the classical Central Limit Theorem. But we now know that there are instances where amplification occurs at extreme values and that may lead to heavy, long - tailed distributions, such as the Pareto income distribution. These long - tailed distributions led Lévy to formulate a generalized stable density function, of which the normal as well as the Gauchy distributions are special cases.

However, there exists a Generalized Central Limit Theorem (GCLT) for stable distributions, properly parametrized by Zolotarev (Rachev and Mittnik, 2000). In this parametrization, the *stability exponent* α_Z determines the kurtosis of the distribution, *i.e.,* the peakedness at its central location δ and the fatness of the tails. When $\alpha_Z = 2$, the distribution is normal with variance $\sigma^2 = 2\gamma^2$, where γ is the new statistical measure of dispersion in this parametrization. However, when $\alpha_Z < 2$, the second moment, or (population) variance, becomes infinite or undefined. When $1 < \alpha_Z < 2$, the first moment exists in the sense that there is second moment convergence, but

[5] Because often it is implicitly assumed that the distributions are Gaussian, "stationarity" is often taken to mean "stationarity in the wide sense."

when $0 < \alpha_Z \leq 1$, the theoretical (population) average μ becomes infinite or undefined too. For example, the Cauchy distribution has infinite, undefined mean and variance. This means that the Cauchy distribution has no limiting mean or variance and cannot be used for the usual VaR!

Remark 6 *Of course, we can always compute the (sample) average over time or the variance over time of a finite data set. Undefined theoretical (population) averages and variances only mean that there is no convergence to fixed finite moment values, when we enlarge the data set. The sequential mean and variance of that data set, which calculate the mean, respectively the variance, as observations are added to the data set one at a time, will then never converge to a specific mean and variance, but will continue to "wander."*

Thus, if the distribution of rates of return is not Gaussian and $\alpha_Z < 2$, the variance of the finite data set can say nothing about the theoretical (population) variance, because it does not even exist in the limit! This is, of course, what is meant by a n*on - ergodic* data set: the ensemble avareages are not the same as the time-averages and the usual time-series analysis, which is based on ergodicity, cannot be properly applied. The variances of our finite financial data sets are potentially unstable and don't tend to any value, even as the data set increases in size. More observations do no longer improve our "statistical parameter estimates," but may actually deteriorate them!

For example, we found that the rate of return series $x(t)$ of the S&P500 stock index shows $1 < \alpha_Z = \frac{1}{H} = \frac{1}{0.6} = 1.67 < 2$. In that case $x(t)$ is fractal and globally dependent, and has infinite memory. It also has a stable mean, like a stable Lévy distribution, but it has an undefined or "infinite" variance.[6] This non - convergence or "wandering path" of the variance of stock and stock index returns has entered the finance literature under the scientific misnomer of "stochastic volatility" (*cf.* Hull and White, 1987, 1988, 1998a and b). But there is no stochasticity involved in indefiniteness! Probability cannot be substituted for ignorance! In such a case it may not be prudent to base a risk measure, such as VaR on the computed standard deviation, since that standard deviation remains undefined over time. The sequential return variance σ_τ^2 will never converge!

[6] Similarly, Fama (1965) and Peters (1994, pp. 210 - 212) compute an approximate value of $\alpha_Z = 1.66$ for the Dow Jones Industrials Index. Peters clearly demonstrates the nonconvergence of the volatility of the DJIA.

6.2 Fama - Samuelson MPT Proposition

If two distributions are stable with the same value of α_Z, their sum also is stable with the same stability exponent α_Z. This mathematical result has applications in modern portfolio theory (MPT), which are, or at least should be, rather disturbing for global portfolio managers (Lucas and Klaassen, 1998; Sornette, 1998).

Proposition 7 ***(Fama - Samuelson)*** *If the securities in a portfolio have rates of return $x(t)$ with the same stability exponent α_Z, then the portfolio itself has a rate of return $x(t)$ that is stable, with the same value of α_Z.*

Proof. In Zolotarev's parametrization, we have the logarithm of the characteristic function of the non-standardized stable distribution of the random variable $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 0)$ as

$$\ln\left[E\left\{e^{j\omega x}\right\}\right] \begin{aligned} &= \left(-\gamma^{\alpha}\left|\omega\right|^{\alpha}\left[1 + j\beta\tan\tfrac{\pi\alpha}{2}sign(\omega)(\gamma\left|\omega\right|^{1-\alpha} - 1)\right] + j\delta\omega\right) \text{ if } \alpha \neq 1 \\ &= \left(-\gamma\left|\omega\right|\left[1 + j\beta\tfrac{2}{\pi}sign(\omega)(\ln\left|\omega\right| + \ln\gamma)\right] + j\delta\omega\right) \text{ if } \alpha = 1 \end{aligned} \quad (22)$$

with the four parameters: (1) *stability exponent* $\alpha_Z \in (0, 2]$, (2) *skewness parameter* $\beta \in [-1, 1]$, (3) *scale parameter* $\gamma > 0$, and (4) *location parameter* $\delta \in \mathbb{R}$. For simplicity, we'll discuss the case of symmetric distributions when $\beta = 0$, so that

$$\ln\left[E\left\{e^{jx\omega}\right\}\right] = j\delta\omega \ - \gamma^{\alpha_Z}\left|\omega\right|^{\alpha_Z} \quad (23)$$

For the stable distributions of two rates of return $x_i(t), i = 1, 2$, the distribution of the weighted portfolio sum $x_p(t) = w_1x_1(t) + w_2x_2(t)$, with $w_1 + w_2 = 1$, has the characteristic function:

$$\begin{aligned} \ln E\left\{e^{jx_p\omega}\right\} &= \ln E\left\{e^{j(w_1x_1 + w_2x_2)\omega}\right\} \\ &= \ln\left[E\left\{e^{jw_1x_1\omega}\right\}E\left\{e^{jw_2x_2\omega}\right\}\right], \ \textit{i.e.}, \text{ stable distributions} \\ &= \ln E\left\{e^{jx_1w_1\omega}\right\} + \ln E\left\{e^{jx_2w_2\omega}\right\} \\ &= [j\delta_1w_1\omega \ - \gamma_1^{\alpha_Z}\left|w_1\omega\right|^{\alpha_Z}] + [j\delta_2w_2\omega \ - \gamma_2^{\alpha_Z}\left|w_2\omega\right|^{\alpha_Z}] \\ &= j(w_1\delta_1 + w_2\delta_2)\omega \ - [w_1^{\alpha_Z}\gamma_1^{\alpha_Z} + w_2^{\alpha_Z}\gamma_2^{\alpha_Z}]\left|\omega\right|^{\alpha_Z} \\ &= j\delta_p\omega - \gamma_p^{\alpha_Z}\left|\omega\right|^{\alpha_Z} \end{aligned} \quad (24)$$

so that the location parameter, or mean, of the stable portfolio distribution

$$\delta_p = w_1\delta_1 + w_2\delta_2 \tag{25}$$

and its scale parameter

$$\gamma_p^{\alpha_Z} = w_1^{\alpha_Z}\gamma_1^{\alpha_Z} + w_2^{\alpha_Z}\gamma_2^{\alpha_Z} \tag{26}$$

It is easy to see that this bivariate return result generalizes, so that for stable distributions with the same stability parameter in general, for a portfolio with $i = 1, 2, ..., n$ assets, the portfolio location parameter or mean

$$\delta_p = \sum_{i=1}^{n} w_i\delta_i, \text{ where } \sum_{i=1}^{n} w_i = 1 \tag{27}$$

and the portfolio scale parameter

$$\gamma_p^{\alpha_Z} = \sum_{i=1}^{n} w_i^{\alpha_Z}\gamma_i^{\alpha_Z} \tag{28}$$

or

$$\gamma_p = \left(\sum_{i=1}^{n} w_i^{\alpha_Z}\gamma_i^{\alpha_Z}\right)^{\frac{1}{\alpha_Z}} \tag{29}$$

■

Fama (1965) and Samuelson (1967) used this proposition to adapt the portfolio theory of Markowitz (1952) for infinite or undefined variance distributions of rates of return on investments. It is a peculiar fact of history that this Proposition of Fama and Samuelson has disappeared from the standard textbooks on investments and portfolio analysis and management, although it has considerable empirical value! S&P500 stock index is often used as the market index in the Capital Asset Pricing Model (CAPM). But, as we just saw, the S&P500 stock index has no finite limiting variance, and this fact alone undermines most if not all of the stock and bond pricing results from the CAPM.

Remark 8 *For Gaussian distributions, when $\alpha_Z = 2$, we have the familiar portfolio variance relationship from classical Markowitz mean - variance analysis, except that Markowitz' important diversifying correlation term is missing:*

$$\gamma_p^2 = w_1^2\gamma_1^2 + w_2^2\gamma_2^2 \tag{30}$$

For Gaussian distributions, the variance $\sigma_i^2 = 2\gamma_i^2$ *(i.e.,* $\gamma_i^2 = \sigma_i^2/2$*), so that for stable distributions with the same* α_Z *we have also:*

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 \tag{31}$$

The Proposition implies that the distribution of the portfolio returns is *self – affine* and scales with stability exponent α_Z as scaling exponent. In other words, the shape of the stable distribution of portfolio returns is the same as that of the underlying asset returns, no matter what the scale of portfolio variance. Only the value of the location parameter changes.

How does the existence of stable non - Gaussian rates of return distributions affect portfolio diversification? For example, when we use uniform weights $w_i = \frac{1}{n}$,

$$\gamma_p^{\alpha_Z} = \left(\frac{1}{n}\right)^{\alpha_Z} \sum_{i=1}^{n} \gamma_i^{\alpha_Z} \tag{32}$$

Samuelson (1967) showed that we can discern three important cases:

(1) When $1 < \alpha_Z \leq 2$, the portfolio risk, as measured by the scaling parameter

$$\gamma_p = \frac{1}{n}\left(\sum_{i=1}^{n} \gamma_i^{\alpha_Z}\right)^{\frac{1}{\alpha_Z}} \tag{33}$$

decreases, as the number of assets in the portfolio, n, increases. In other words, there is a diversification effect: including more assets in the portfolio reduces the portfolio risk, despite the empirically established fact that there exists no finite limiting variance.

Remark 9 *Since most (but not all!) empirical stocks appear to have a stability exponent of* $\alpha_Z \approx 1.67$*, diversification does reduce the non - market risk of an empirical stock investment portfolio, including that of the portfolio underlying the S&P500 Index. But this risk reduction through diversification has nothing to do with correlations, as in Markowitz' (1952) original theory.*

(2) When $\alpha_Z = 1$

$$\gamma_p = \frac{1}{n}\sum_{i=1}^{n} \gamma_i \tag{34}$$

there is no diversification effect: adding more assets to the portfolio does not reduce the portfolio risk.

(3) When $0 < \alpha_Z < 1$, increasing the number of assets in the portfolio may actually increase the portfolio risk.[7] In this case, neither the means nor the variances of the rates of return of the

[7] This range of $\alpha_Z = \frac{1}{\alpha_L}$ cannot be measured by the Hurst exponent H, but can be measured by the Lipschitz α_L.

assets in the portfolio exist. Neither their means nor their variances converge. In other words, when asset return rates behave like black noise, increasing the portfolio size only increases the portfolio risk! This has the counter-intuitive consequence that adding assets to a portfolio adds to its risk.

For example, Kevin Dowd claims that a risk measure must be sub-additive: "If our risk measure is non-sub-additive, there is a danger it might suggest that diversification is a bad thing, and that would imply the laughable conclusion that putting all your eggs in one basket might be good risk management!" (*Financial Engineering News*, November/December 2004, p. 7).

But the sub-additivity of a risk measure doesn't depend on a subjective choice, but on the empirical long memory of the rates of return of the assets. In some cases that does not guarantee sub-additivity and, ridiculous as it may sound to modern portfolio managers, the Fama - Samuelson Proposition demonstrates that sometimes it may actually be good risk management to put all your eggs in one basket! (Los, 2005).

Of course, MPT-diversification to reduce non-market risk is still useful when the asset returns are non - Gaussian, but they have stable distributions with the same stability $1 < \alpha_Z \leq 2$, despite the fact that these stable distributions have undefined variances. However, when $\alpha_Z = 1$, there is no diversification and when $0 < \alpha_Z < 1$, the portfolio risk can actually increase when more assets are included in the portfolio. Thus, it is very important for portfolio managers to compute the homogeneous Zolotarev alpha $\alpha_Z = \frac{1}{\alpha_L}$, to determine the degree of achievable diversification. Also, portfolio risk managers should compute the multifractal spectrum of heterogeneous of stock return stability exponents $\alpha_{Zi} = \frac{1}{\alpha_{Li}}$, which may lie outside the range of the usual measurement of the homogeneous Hurst exponent H.

It is also very important to realize that, since there is no correlation under parametrized stable distributions, Markowitz - type portfolio diversification and optimization, which exploits such correlation among the assets, simply does not work. However, this does not necessarily mean that there does not exist a *Tobin liquidity preference theorem.* As we will see, we can still reduce the risk in a portfolio by including more risk - free cash, even when the distributions are nonstationary but stable. In other words, it is dynamic *liquidity management* that ultimately determines the investment portfolio risk exposure of a fund manager (Bawa, Elton and Gruber, 1979).

6.3 Skewed - Stable Investment Opportunity Sets

The Fama - Samuelson Proposition shows why it is important to determine the stability parameters of the rates of return $x_\tau(t)$ for the assets in a portfolio and to see if they are the same. However, if the stability parameters are different,heterogenous, α_{Zi}, this simple generalization of Markowitz mean - variance analysis, or Modern Portfolio Theory (MPT) and its derivatives, does no longer hold true. Or, as Peters (1994, p. 208) states:

> ".…different stocks can have different Hurst exponents and different values of α_Z. Currently, there is no theory on combining distributions with different alphas. The EMH, assuming normality for all distributions, assumed $\alpha_Z = 2.0$ for all stocks, which we know [now] to be incorrect."

Huston McCulloch of Ohio State University has done some empirical work on what happens when the stability parameters α_{Z_i} for the rates of return of the assets in a portfolio are heterogeneous, *i.e.,* they are different from each other. In particular, he has produced interesting $3-$dimensional visualizations of the resulting Markowitz efficiency frontiers, which are no longer $2-$dimensional (McCulloch, 1986, 1996). In accordance with these findings, McCulloch (1996) also developed an alternative to the Black - Scholes option pricing formula, using stable distributions.

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