SEMI-EXPLICIT DELTA AND GAMMA FOR EUROPEAN SWAPTIONS IN HULL-WHITE ONE FACTOR MODEL.

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Abstract. In the framework of the Hull-White model we present a semi-explicit approach to compute the delta and the gamma. The method is faster and more accurate than classical approaches, specially when compared to the Hull-White tree implementation.

1. Introduction

In this note we review the computation of delta and gamma in the Hull-white or extended Vasicek model. Based on the analysis developed in the paper on explicit computation for European swaptions [2], we propose a semi-explicit computation.

Here the type of delta and gamma we analyse is the out-of-the-model one computed by moving each market rate used to construct the yield curve separately.

The traditional way to compute these numbers is to move each grid point up and then down and for each movement to recompute the value of the instrument ($P_\pm$). Then the delta is computed through the symmetrical difference ($(P_+ - P_-)/2$) and the gamma through $P_+ + P_- - 2P_0$. We will denote by $\Delta P/\Delta r_j$ ($\Delta^2 P/\Delta^2 r_j$ for the gamma) that difference when the j-th rate of the yield curve is moved by one basis point. This is done for each market rate than can affect the price (O/N, T/N, 1m, 2m, 3m, 6m, 1y, 2y, 3y, 4y, 5y, 6y,...) .

Using some technique suggested in [2] we propose a semi-explicit method. The method is explicit in the sense that no option price has to be computed and numerically compare. We only work on discount factors and exact coefficients. The *semi* comes from the fact that the sensitivity of the discount factor has to be computed numerically. So we reduce the problem of the sensitivity of an option to the sensitivity of linear cash-flows and a hedging ratio (Delta in the theoretical sense).

We also present a section with numerical results about the computations. We compare its precision and speed with other methods. We show that the method is faster than the usual numerical symmetric difference, even if the explicit formula is used. Compare to the more traditional Hull-White tree implementation, the method is unrivalled both in term of speed and convergence, in particular for the gamma that is almost impossible to compute with the tree approach.

To fix the notations we describe the model to which this note apply. The model is a deterministic volatility Heath-Jarrow-Merton model

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

where the volatility function satisfy the separability condition $\sigma(t,u) = g(t)h(u)$ for some positive functions g and h. The more precise model we have in mind for applications is the Hull and White volatility model [3] where $\sigma(s,t) = \sigma\exp(-a(t-s))$ and a and σ are constant. The results are also valid for the time dependent version of the model where σ depend on t. This model can also be written as a short rate mean reverting model

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t.$$

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2. Delta and gamma formulas

The exact description of the model we use and the notation are the same as in the paper on the pricing formula [2].

The starting point of the analysis is the swaption (receiver) pricing formula given in Theorem 6.1 of [2]

$$S = \sum_{i=0}^{n} c_i P(0, t_i) N(\kappa + \alpha_i) = F(P).$$

We first want to find the derivative (in the mathematical sense) of this price with respect to the (market) rates that compose the yield curve. The market rate do not appear directly in the formula but through the P's. We write everything with the rates as parameters but there is nothing specific about them. They could be any parameter that describes the curve. In particular future prices if the curve is constructed out of futures, or even just one parameter for the parallel *shift* of the curve and one for its *twist*. In line with the idea that the parameters are rates, we call them $r_j (1 \leq j \leq n)$.

Computing the derivative of P with respect to those market rate is a complex task to perform explicitly. The rates have different conventions and there is some interpolation involved. We leave this task to the computer (numerical approximation) and we concentrate on the second part of the chain rule: the derivative of the price with respect to the discount factor. In the formula

$$\frac{\partial S}{\partial r_j} = \sum_{i=0}^{n} D_i F(P) \frac{\partial P_i}{\partial r_j} \tag{1}$$

we are analyse only the $D_i F$ derivative.

That part is very easy at this stage as it was solved already in the paper where the explicit formula was presented. Hidden in the proof of Theorem 5.1 we can see that $D_i F(P) = c_i N(\kappa + \alpha_i)$ (where κ depend it-self of P).

We have now enough information to write our delta theorem.

Theorem 1. *In the HJM one factor model, the delta of a European receiver swaption can be computed by*

$$\frac{\Delta S}{\Delta r_j} = \sum_{i=0}^{n} c_i N(\kappa + \alpha_i) \frac{\Delta P(0, t_i)}{\Delta r_j}.$$

and the delta of a payer swaption by

$$-\frac{\Delta S}{\Delta r_j} = \sum_{i=0}^{n} c_i N(-\kappa - \alpha_i) \frac{\Delta P(0, t_i)}{\Delta r_j}.$$

Remark: With that formula, the computation of the κ has to be done only once for all the grid point. If the delta was numerically computed for each rate movement, we would have to be compute it for each movement up and down.

We can now look at the computation of the gamma, which is a little bit more involved. As mentioned above we have to be careful with the fact that κ depend it-self on P. If we apply the chain rule again to the the equation (1) we have (after a slight reordering of the factors)

$$\frac{\partial^2 R}{\partial r_k \partial r_j} = \sum_{i=0}^{n} c_i N(\kappa + \alpha_i) D^2_{k,j} P_i + \left(\sum_{i=0}^{n} c_i N'(\kappa + \alpha_i) D_j P_i \right) \left(\sum_{l=0}^{n} D_l \kappa D_k P_l \right).$$

Like for the delta we leave all the derivative (first and second order) of the discount factor with respect to the rates to the computer.

All the terms are as explicit as we can hope except N' and $D_l \kappa$. For the first one, we can write it explicitly as

$$\frac{1}{\sqrt{2\pi}} \exp(-(\kappa + \alpha_i)^2).$$

Using the implicit function theorem, we can write the second one as

$$\frac{c_l \exp(-\frac{1}{2}\alpha_l^2 - \alpha_l \kappa)}{\sum_{i=0}^n \alpha_i c_i P(0,t_i) \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa)}$$

The formula may look a little bit ugly but the contains only (relatively) simple operations and improve the computation time.

Theorem 2. *In the HJM one factor model, the gamma of a European receiver swaption can be computed by*

$$\begin{aligned}\frac{\Delta^2 S}{\Delta r_k \Delta r_j} &= \sum_{i=0}^n c_i N(\kappa + \alpha_i) \frac{\Delta^2 P(0,t_i)}{\Delta r_k \Delta r_j} \\ &\quad + \frac{1}{\sqrt{2\pi}} \frac{\exp(-\kappa^2/2)}{\sum_{i=0}^n \alpha_i c_i P(0,t_i) \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa)} \left(\sum_{i=0}^n c_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa) \frac{\Delta P(0,t_i)}{\Delta r_j} \right) \\ &\quad \left(\sum_{i=0}^n c_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa) \frac{\Delta P(0,t_i)}{\Delta r_k} \right)\end{aligned}$$

and the one of a payer swaption by

$$\begin{aligned}\frac{\Delta^2 S}{\Delta r_k \Delta r_j} &= -\sum_{i=0}^n c_i N(-\kappa - \alpha_i) \frac{\Delta^2 P(0,t_i)}{\Delta r_k \Delta r_j} \\ &\quad + \frac{1}{\sqrt{2\pi}} \frac{\exp(-\kappa^2/2)}{\sum_{i=0}^n \alpha_i c_i P(0,t_i) \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa)} \left(\sum_{i=0}^n c_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa) \frac{\Delta P(0,t_i)}{\Delta r_j} \right) \\ &\quad \left(\sum_{i=0}^n c_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa) \frac{\Delta P(0,t_i)}{\Delta r_k} \right)\end{aligned}$$

3. Numerical implementation, speed and convergence

In this section we show that this approach not only speed-up considerably the time to compute the first and second order sensitivities but also improve significantly the precision, particularly of the second order one. For this we study a USD 1y x 5y receiver ATM swaption. The yield curve is the one of 28-Oct-2004. The Vasicek parameters are (arbitrarily) chosen at $a = 0.01$ and $\sigma = 0.01$.

3.1. **Speed...** We first compare the speed of different approaches. We do this for the two possibilities indicated earlier in this note (numerical computation by symmetrical difference and semi-explicit), but also for a third possibility which is a traditional Hull-White trinomial tree approach (as described in Brigo and Mercurio [1] with long term discount factors recovered form the one-step one as described in Hull [4]).

The delta and gamma computed are *single grid point* delta-gamma. This means that each grid point is moved up (PV_u) and down (PV_d). The delta is the symmetrical difference ($PV_d - PV_u$). For the gamma the amount $PV_u + PV_d - PV_0$ is computed. We don't imply that those numbers are the most relevant, we only address the speed and the precision of the computation.

We obtain one time for the semi-explicit computation and one for the numerical one. For the Hull-White tree we compute the time for the symmetrical difference with 5, 10, 20, 50, 100, 200 and 500 steps. The results are graphed in Figure 1.

The numerical approach require 36% more time than the semi-explicit one. For the Hull-White tree as soon as a meaningful number of steps is used, the time of the tree is several order of magnitude larger. For example for 50 steps, the ratio is 5.5, for 100 steps it is close to 25, for 200 it is above 200 and for 500 it is above 4,250!

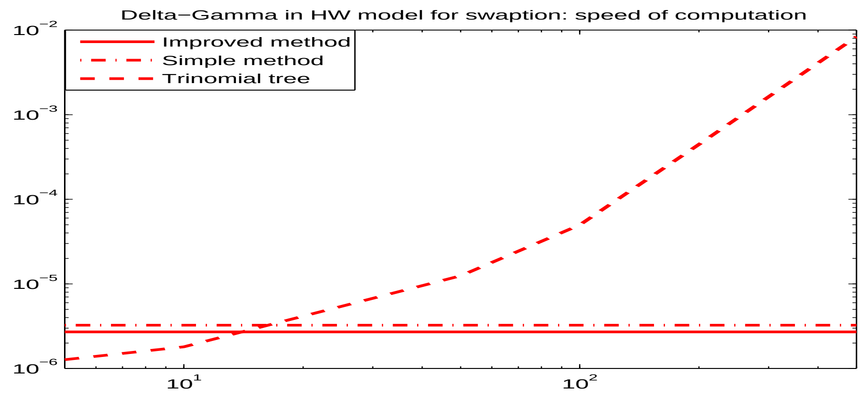

FIGURE 1. Computation time for the semi-explicit method, the numerical method based on explicit formula and Hull-White tree

3.2. **Convergence...** It is known that the tree approach perform very badly for gamma computation. Increasing the number of steps increases the precision of the price. The first order approximation is also improved, but in a irregular way. But unfortunately the second order approximation failed to be even vaguely close to where it should be.

We show some results related to this problem. We compute yield-curve delta and gamma with different methods. By this we mean we compute the sensitivities for a parallel move of the yield curve (one single delta and gamma for each method). The precision number, for a shift of one basis point, are reported in Figures 2 and 3.

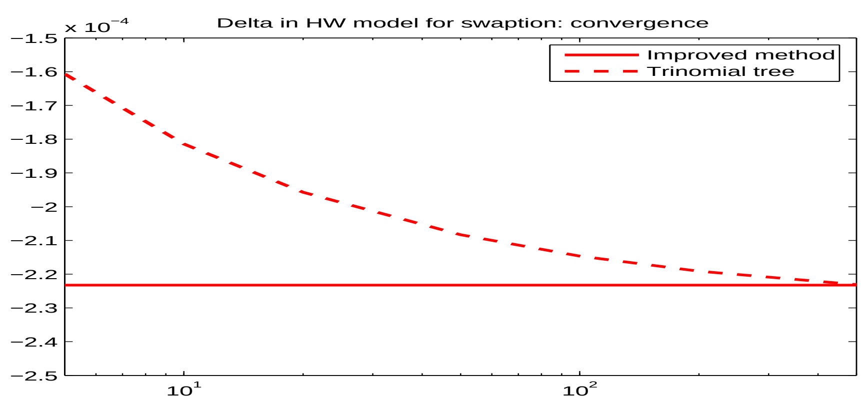

FIGURE 2. Convergence of the delta computation for the tree method.

For delta there is a clear improvement with the number of steps. The difference between the two other methods is insignificant (0.00002 % difference for delta and 0.001% for gamma). It take around 100 steps to have a acceptable delta (4% difference). To have a one percent difference, one has to go above 200 steps.

For the gamma the two methods again provide very close results. The gamma results for the tree is not even vaguely correct. In this case it is almost 0. The increase number of steps is of no help. The fact the computed gamma is very low is somehow standard. But for some particular cases it will be very large. The next paragraphs gives a little bit more insight in this.

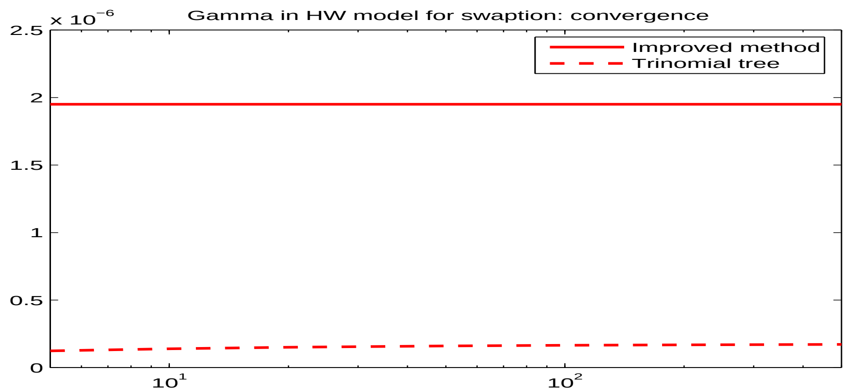

FIGURE 3. Convergence (!) of the gamma computation for the tree method.

Another interesting way to look at the problem is to fix the swaption, fix the number of steps (arbitrarily 100 in our case) and look at the delta and gamma for different levels of rates. Here we move the curve by parallel increment of 1 basis point up to 150 basis points away from the initial curve. The results are given in Figures 4, 5 and 6.

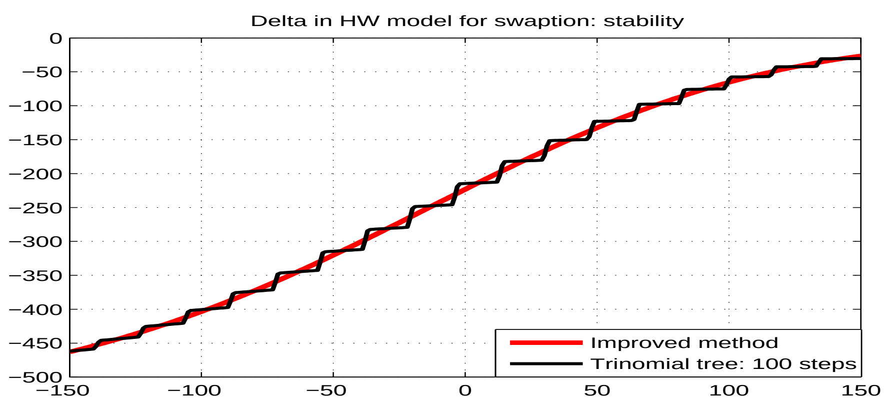

FIGURE 4. Delta for different levels of interest rate for a 100 steps tree and the improved method.

The results are a lot more smooth with the explicit approach. This is certainly a desirable feature in practice as one does not want to hedge a trading book on unstable or noisy numbers. The transactions cost in unnecessary trades would be huge. The gamma picture is quite surprising. The gamma is very low and from time to time bursts, probably when some point in the tree come in or out of the money. For example for 100 steps, the last step has a 17.3 basis points rate discretisation. The burst in the delta appear every 17 or 18 basis points, very much in line. To have this effect disappearing with 10 points covering the one basis point movement, we need a rate discretisation of 0.1 basis point. It would take 3,000,000 steps (and also 9.10^{12} grid computations) to achieve this! Or we could use a 170 basis points rate movement but it would not be a delta for *small* changes anymore.

3.3. **Conclusion.** The delta-gamma computation for the Hull-White model presented here performs better in term of speed than a simple numerical symmetrical difference. This improvement is from the already very good one obtain from the explicit computation approach. If one compare

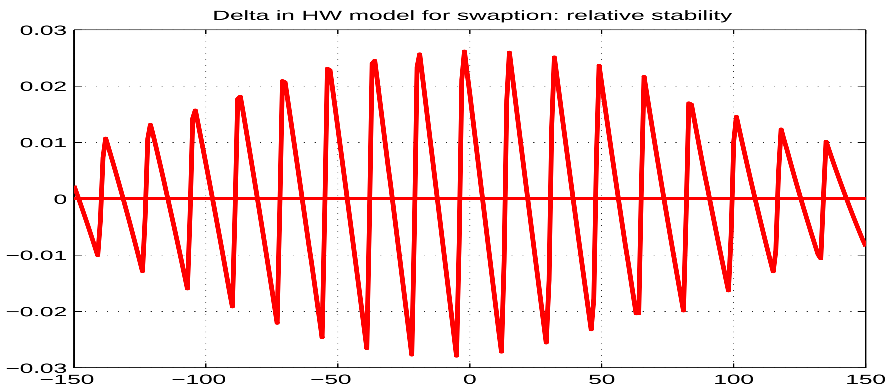

FIGURE 5. Delta difference in percent between a 100 steps tree and the improved method.

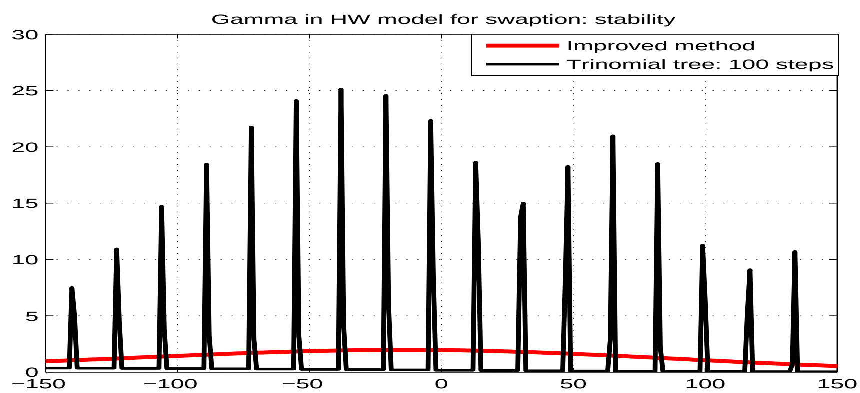

FIGURE 6. Gamma for different levels of interest rate for a 100 steps tree and the improved method.

it with a Hull-White trinomial tree approach, the method is several order of magnitude faster. Moreover the precision is also significantly enhanced, In particular for the gamma the tree figures are meaningless but obtained very fast and accurately in the proposed method.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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