

Fractional calculus and continuous-time finance II: the waiting-time distribution

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Abstract

We complement the theory of tick-by-tick dynamics of financial markets based on a continuous-time random walk (CTRW) model recently proposed by Scalas et al [4], and we point out its consistency with the behaviour observed in the waiting-time distribution for BUND future prices traded at LIFFE, London.

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1 Introduction

In financial markets, not only prices can be modelled as random variables, but also waiting times between two consecutive transactions vary in a stochastic

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fashion. This fact is well known in financial research. In his 1973 paper [1], Peter Clark wrote: “Instead of indexing [time] by the integers $0,1,2,\dots$, the [price] process could be indexed by a set of numbers t_1, t_2, t_3, \dots , where these numbers are themselves a realization of a stochastic process (with positive increments, so that $t_1 < t_2 < t_3 < \dots$).”

Till today, there have been various studies on the nature of the stochastic process generating the sequence of the t_j . In Clark’s approach, the variable t is not a physical time but an economic variable, the trading volume, an observable whose increments represent the market intensity.

Lefol and Mercier have written a review [2] on the works inspired by Clark’s seminal paper. In a review by Cont [3], readers can find pointers to the relevant literature and a description of the main research trends in this field.

In a recent paper, Scalas et al. [4] have argued that the continuous time random walk (CTRW) model, formerly introduced in Statistical Mechanics by Montroll and Weiss [5] (on which the reader can find further information in Refs. [6–12]), can provide a phenomenological description of tick-by-tick dynamics in financial markets. Here, we give further theoretical arguments and test the theoretical predictions on the waiting-time distribution against empirical data.

The paper is divided as follows. Section 2 is devoted to the discussion of a new form for the general master equation in the case of non-local and non-Markovian processes. In Section 3, the conditions for the derivation of the time-fractional master equation are given. The Mittag-Leffler function plays a central role in this respect. In Section 4, the theoretical predictions on the waiting-time distribution are compared to market data: high-frequency BUND future prices traded at LIFFE² in 1997. Finally, the main conclusions are drawn in Section 5.

2 The general master equation and the ”memory function”

Throughout this paper the variable x represents the log-price. In other words, if S is the price of an asset, $x = \log S$. The reason for this choice is explained by Scalas et al. [4]; it is essentially due to the fact that, rather than prices, returns are the relevant variable in finance. The physicist will recognize in x the position of a random walker jumping in one dimension. In the following, we shall often use the random walk language.

Let us consider the time series $\{x(t_i)\}$, $i = 1, 2, \dots$, which is characterised

² LIFFE stands for London International Financial Futures (and Options) Exchange. For further information, see <http://www.liffe.com>.

by $\varphi(\xi, \tau)$, the *joint probability density* of jumps $\xi_i = x(t_i) - x(t_{i-1})$ and of waiting times $\tau_i = t_i - t_{i-1}$. The joint density satisfies the normalization condition $\int_0^\infty \left[\int_{-\infty}^{+\infty} \varphi(\xi, \tau) d\xi \right] d\tau = 1$. Relevant quantities are the two probability density functions (*pdf*'s) defined as $\lambda(\xi) := \int_0^\infty \varphi(\xi, \tau) d\tau$, $\psi(\tau) := \int_{-\infty}^{+\infty} \varphi(\xi, \tau) d\xi$, and called *jump pdf* and *waiting-time pdf*, respectively.

The *CTRW* is generally defined through the requirement that the τ_i are identically distributed independent (i.i.d.) random variables. Furthermore, in the following we shall assume that the *jump pdf* $\lambda(\xi)$ is independent of the *waiting-time pdf*, $\psi(\tau)$, so that the jumps ξ_i (at instants t_i , $i = 1, 2, 3, \dots$) are i.i.d. random variables, all having the same probability density $\lambda(\xi)$. Then, we have the factorization $\varphi(\xi, \tau) = \lambda(\xi) \psi(\tau)$. For convenience we set $t_0 = 0$.

The *jump pdf* $\lambda(\xi)$ represents the *pdf* for transition of the walker from a point x to a point $x + \xi$, so it is also called the *transition pdf*. The *waiting-time pdf* represents the *pdf* that a step is taken at the instant $t_{i-1} + \tau$ after the previous one that happened at the instant t_{i-1} , so it is also called the *pausing-time pdf*. Therefore, the probability that $\tau \leq t_i - t_{i-1} < \tau + d\tau$ is equal to $\psi(\tau) d\tau$.

The probability that a given interstep interval is greater or equal to τ will be denoted by $\Psi(\tau)$, which is defined in terms of $\psi(\tau)$ by

$$\Psi(\tau) = \int_\tau^\infty \psi(t') dt' = 1 - \int_0^\tau \psi(t') dt', \quad \psi(\tau) = -\frac{d}{d\tau} \Psi(\tau). \quad (2.1)$$

We note that $\int_0^\tau \psi(t') dt'$ represents the probability that at least one step is taken at some instant in the interval $[0, \tau)$, hence $\Psi(\tau)$ is the probability that the diffusing quantity x does not change value during the time interval of duration τ after a jump. We also note, recalling that $t_0 = 0$, that $\Psi(t)$ is the *survival probability* until time instant t at the initial position $x_0 = 0$.

Let us now denote by $p(x, t)$ the *pdf* of finding the random walker at the position x at time instant t . As usual we assume the initial condition $p(x, 0) = \delta(x)$, meaning that the walker is initially at the origin $x = 0$. We look for the evolution equation for $p(x, t)$, that we shall call the *master equation* of the *CTRW*. Montroll and Weiss [5] have shown that the Fourier-Laplace transform of $p(x, t)$ satisfies a characteristic equation, now called the *Montroll-Weiss equation*. However, only based upon the previous probabilistic arguments and without detour onto the Fourier-Laplace domain, we can write, directly in the space-time domain, the required *master equation*, which reads

$$p(x, t) = \delta(x) \Psi(t) + \int_0^t \psi(t-t') \left[\int_{-\infty}^{+\infty} \lambda(x-x') p(x', t') dx' \right] dt'. \quad (2.2)$$

The spatially discrete analogue of this purely integral form of the *master*

equation is quoted in Klafter et al. [13] (see also Ref. [14]). We recognize from Eq. (2.2) the role of the *survival probability* $\Psi(t)$ and of the *pdf's* $\psi(t)$, $\lambda(x)$. The first term in the RHS of (2.2) expresses the persistence (whose strength decreases with increasing time) of the initial position $x = 0$. The second term (a spatio-temporal convolution) gives the contribution to $p(x, t)$ from the walker sitting in point $x' \in \mathbf{R}$ at instant $t' < t$ jumping to point x just at instant t , after stopping (or waiting) time $t - t'$.

Now, passing to the Fourier-Laplace domain, we can promptly derive the celebrated *Montroll-Weiss equation* [5]. In fact, by adopting the following standard notation for the generic Fourier and Laplace transforms:

$$\mathbb{F} \{f(x); \kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{+\infty} e^{i\kappa x} f(x) dx, \quad \mathbb{L} \{f(t); s\} = \widetilde{f}(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

we get from (2.2) the *Montroll-Weiss equation*:

$$\widehat{p}(\kappa, s) = \widetilde{\Psi}(s) \frac{1}{1 - \widehat{\lambda}(\kappa) \widetilde{\psi}(s)} = \frac{1 - \widetilde{\psi}(s)}{s} \frac{1}{1 - \widehat{\lambda}(\kappa) \widetilde{\psi}(s)}. \quad (2.3)$$

Hereafter we present an alternative form to Eq. (2.2) which involves the first time derivative of $p(x, t)$ (along with an additional auxiliary function) so that the resulting equation can be interpreted as an *evolution* equation of *Fokker-Planck-Kolmogorov* type.

For our purposes we re-write Eq. (2.3) as

$$\widetilde{\Phi}(s) [s \widehat{p}(\kappa, s) - 1] = [\widehat{\lambda}(\kappa) - 1] \widehat{p}(\kappa, s), \quad (2.4)$$

where

$$\widetilde{\Phi}(s) = \frac{1 - \widetilde{\psi}(s)}{s \widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{\widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{1 - s \widetilde{\Psi}(s)}. \quad (2.5)$$

Then our master equation reads

$$\int_0^t \Phi(t - t') \frac{\partial}{\partial t'} p(x, t') dt' = -p(x, t) + \int_{-\infty}^{+\infty} \lambda(x - x') p(x', t) dx', \quad (2.6)$$

where the "auxiliary" function $\Phi(t)$, being defined through its Laplace transform in Eq. (2.5), is such that $\Psi(t) = \int_0^t \Phi(t - t') \psi(t') dt'$. We remind the reader that Eq. (2.6), combined with the initial condition $p(x, 0) = \delta(x)$, is equivalent to Eq. (2.4), and then its solution represents the Green function or the fundamental solution of the Cauchy problem.

From Eq. (2.6) we recognize the role of "memory function" for $\Phi(t)$. As a consequence, the CTRW turns out to be in general a non-Markovian process.

However, the process is "memoryless", namely "Markovian" if (and only if) the above memory function degenerates into a delta function (multiplied by a certain positive constant) so that $\Psi(t)$ and $\psi(t)$ may differ only by a multiplying positive constant. By appropriate choice of the unit of time we assume $\tilde{\Phi}(s) = 1$, so $\Phi(t) = \delta(t)$, $t \geq 0$. In this case we derive

$$\tilde{\psi}(s) = \tilde{\Psi}(s) = \frac{1}{1+s}, \quad \text{so} \quad \psi(t) = \Psi(t) = e^{-t}, \quad t \geq 0. \quad (2.7)$$

Then Eq. (2.6) reduces to

$$\frac{\partial}{\partial t} p(x, t) = -p(x, t) + \int_{-\infty}^{+\infty} \lambda(x-x') p(x', t) dx', \quad p(x, 0) = \delta(x). \quad (2.8)$$

This is, up to a change of the unit of time (which means multiplication of the R.H.S by a positive constant), the most general *master equation* for a *Markovian CTRW*; it is called the *Kolmogorov-Feller equation* in Ref. [15].

We note that the form (2.6), by exhibiting a weighted first-time derivative, is original as far as we know; it allows us to characterize in a natural way a peculiar class of non-Markovian processes, as shown in the next Section.

3 The time-fractional master equation for "long-memory" processes

Let us now consider "long-memory" processes, namely non-Markovian processes characterized by a memory function $\Phi(t)$ exhibiting a power-law time decay. To this purpose a natural choice is

$$\Phi(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t \geq 0, \quad 0 < \beta < 1. \quad (3.1)$$

Thus, $\Phi(t)$ is a weakly singular function that, in the limiting case $\beta = 1$, reduces to $\Phi(t) = \delta(t)$, according to the formal representation of the Dirac generalized function, $\delta(t) = t^{-1}/\Gamma(0)$, $t \geq 0$ (see e.g. REf. [16]).

As a consequence of the choice Eq. (3.1), we recognize that (in this peculiar non-Markovian situation) our *master equation* (2.6) contains a time fractional derivative. In fact, by inserting into Eq. (2.4) the Laplace transform of $\Phi(t)$, $\tilde{\Phi}(s) = 1/s^{1-\beta}$, we get

$$s^\beta \hat{p}(\kappa, s) - s^{\beta-1} = [\hat{\lambda}(\kappa) - 1] \hat{p}(\kappa, s), \quad 0 < \beta < 1, \quad (3.2)$$

so that the resulting Eq. (2.6) can be written as

$$\frac{\partial^\beta}{\partial t^\beta} p(x, t) = -p(x, t) + \int_{-\infty}^{+\infty} \lambda(x - x') p(x', t) dx', \quad p(x, 0) = \delta(x), \quad (3.3)$$

where $\partial^\beta/\partial t^\beta$ is the pseudo-differential operator explicitly defined in the Appendix, that we call the *Caputo* fractional derivative of order β . Thus Eq. (3.3) can be considered as the time-fractional generalization of Eq. (2.8) and consequently can be called the *time-fractional Kolmogorov-Feller equation*. We note that this derivation differs from the one presented in Ref. [4] and references therein, in that here we have pointed out the role of the long-memory processes rather than that of scaling behaviour in the hydrodynamic limit. Furthermore here the *Caputo* fractional derivative appears in a natural way without use of the *Riemann-Liouville* fractional derivative.

Our choice for $\Phi(t)$ implies peculiar forms for the functions $\Psi(t)$ and $\psi(t)$ that generalize the exponential behaviour (2.7) of the Markovian case. In fact, working in the Laplace domain we get from (2.5) and (3.1)

$$\tilde{\Psi}(s) = \frac{s^{\beta-1}}{1 + s^\beta}, \quad \tilde{\psi}(s) = \frac{1}{1 + s^\beta}, \quad 0 < \beta < 1, \quad (3.4)$$

from which by inversion we obtain for $t \geq 0$

$$\Psi(t) = E_\beta(-t^\beta), \quad \psi(t) = -\frac{d}{dt} E_\beta(-t^\beta), \quad 0 < \beta < 1, \quad (3.5)$$

where E_β denotes an entire transcendental function, known as the Mittag-Leffler function of order β , defined in the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, \quad z \in \mathbf{C}. \quad (3.6)$$

For detailed information on the Mittag-Leffler-type functions and their Laplace transforms the reader may consult e.g. [17–20]. We note that for $0 < \beta < 1$ and $1 < \beta < 2$ the function $\Psi(t)$ appears in certain relaxation and oscillation processes, then called *fractional relaxation* and *fractional oscillation* processes, respectively (see e.g. Refs. [18,19,21,22] and references therein).

Hereafter, we find it convenient to summarize the features of the functions $\Psi(t)$ and $\psi(t)$ most relevant for our purposes. We begin to quote their series expansions and asymptotic representations:

$$\Psi(t) \begin{cases} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta n + 1)}, & t \geq 0 \\ \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, & t \rightarrow \infty, \end{cases} \quad (3.7)$$

and

$$\psi(t) \begin{cases} = \frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta n + \beta)}, & t \geq 0 \\ \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta + 1)}{t^{\beta+1}}, & t \rightarrow \infty. \end{cases} \quad (3.8)$$

The expression for $\psi(t)$ can be shown to be equivalent to that one obtained in Ref. [14] in terms of the generalized Mittag-Leffler function in two parameters.

In the limit for $\beta \rightarrow 1$ we recover the exponential functions of the Markovian case. We note that for $0 < \beta < 1$ both functions $\psi(t)$, $\Psi(t)$, even if losing their exponential decay by exhibiting power-law tails for large times, keep the "completely monotonic" character. Complete monotonicity of the functions $\psi(t)$, $\Psi(t)$, $t > 0$, means:

$$(-1)^n \frac{d^n}{dt^n} \Psi(t) \geq 0, \quad (-1)^n \frac{d^n}{dt^n} \psi(t) \geq 0, \quad n = 0, 1, 2, \dots \quad (3.9)$$

or equivalently, their representability as (real) Laplace transforms of non-negative functions. In fact it can be shown for $0 < \beta < 1$:

$$\Psi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^{\infty} \frac{r^{\beta-1} e^{-rt}}{r^{2\beta} + 2r^{\beta} \cos(\beta\pi) + 1} dr, \quad t \geq 0, \quad (3.10)$$

and

$$\psi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^{\infty} \frac{r^{\beta} e^{-rt}}{r^{2\beta} + 2r^{\beta} \cos(\beta\pi) + 1} dr, \quad t \geq 0. \quad (3.11)$$

A special case is $\beta = \frac{1}{2}$ for which it is known that

$$E_{1/2}(-\sqrt{t}) = e^t \operatorname{erfc}(\sqrt{t}) = e^t \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-u^2} du, \quad t \geq 0, \quad (3.12)$$

where erfc denotes the *complementary error function*.

It may be instructive to note that for sufficiently small times $\Psi(t)$ exhibits a behaviour similar to that of a stretched exponential; in fact we have

$$E_{\beta}(-t^{\beta}) \simeq 1 - \frac{t^{\beta}}{\Gamma(\beta + 1)} \simeq \exp\{-t^{\beta}/\Gamma(1 + \beta)\}, \quad 0 \leq t \ll 1. \quad (3.13)$$

Hereafter, we consider two relevant forms for the survival probability $\Psi(t)$ (that we shall denote by $f_{\beta}(t)$ and $g_{\beta}(t)$ to distinguish them from $e_{\beta}(t) := E_{\beta}(-t^{\beta})$) which, in exhibiting a decreasing behaviour with a power law decay for large times, represent alternative candidates for long-memory processes.

The simplest function which meets these requirements is expected to be:

$$f_\beta(t) := \frac{1}{1 + \Gamma(1 - \beta)t^\beta}, \quad t \geq 0, \quad (3.14)$$

so that

$$f_\beta(t) \sim \begin{cases} 1 - \frac{\pi\beta}{\sin(\pi\beta)} \frac{t^\beta}{\Gamma(1 + \beta)}, & t \rightarrow 0, \\ \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, & t \rightarrow \infty. \end{cases} \quad (3.15)$$

One can infer from the Eqs (3.7) and (3.15) that $e_\beta(t)$ and $f_\beta(t)$ practically coincide for all $t \geq 0$ if β is sufficiently small, say for $0 < \beta < 0.25$. For greater values of β , that are relevant in our subsequent empirical analysis, their difference is expected to be appreciable in a wide range of time intervals.

Another possible choice of major statistical relevance is based on the assumption that the *waiting-time pdf* $\psi(t)$ may be an extremal, unilateral, stable distribution with index β . In this case the Laplace transforms of $\psi(t)$ and $\Psi(t)$ read

$$\tilde{\psi}(s) = \exp(-s^\beta), \quad \tilde{\Psi}(s) = \frac{1 - \exp(-s^\beta)}{s}, \quad 0 < \beta < 1. \quad (3.16)$$

By inversion we obtain for $t \geq 0$

$$\psi(t) = \frac{1}{t} \phi_{-\beta,0} \left(-\frac{1}{t^\beta} \right), \quad \Psi(t) = 1 - \phi_{-\beta,1} \left(-\frac{1}{t^\beta} \right), \quad 0 < \beta < 1, \quad (3.17)$$

where $\phi_{-\beta,0}$, $\phi_{-\beta,1}$ denote entire transcendental functions (depending on two indices), known as the Wright functions, defined in the complex plane by the power series

$$\phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbf{C}, \quad z \in \mathbf{C}. \quad (3.18)$$

For detailed information on the Wright type functions and their Laplace transforms the reader may consult e.g. Refs. [17,22,23]. We note that for $0 < \beta < 1$ and $\mu = 0$ or $\mu = 1 - \beta$ the corresponding Wright functions appear in the fundamental solutions of *time-fractional diffusion* equations (see e.g. Refs. [21,22,24] and references therein).

Hereafter, like for the Mittag-Leffler-type functions (3.5), we quote for the Wright-type functions (3.17) their series expansions and asymptotic representations. For the *waiting-time pdf* we have (see e.g. Ref. [25]),

$$\psi(t) = \frac{1}{\pi t} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(\beta n + 1)}{n!} \frac{\sin(\pi\beta n)}{t^{\beta n}}, \quad t > 0, \quad (3.19)$$

and

$$\psi(t) \sim A t^{-a} \exp(-B t^{-b}), \quad t \rightarrow 0, \quad (3.20)$$

where

$$A = \left[\frac{\beta^{1/(1-\beta)}}{2\pi(1-\beta)} \right]^{1/2}, \quad a = \frac{2-\beta}{2(1-\beta)}, \quad B = (1-\beta)\beta^b, \quad b = \frac{\beta}{1-\beta}. \quad (3.21)$$

For the *survival probability* we obtain

$$g_\beta(t) = \Psi(t) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(\beta n)}{n!} \frac{\sin(\pi\beta n)}{t^{\beta n}}, \quad t > 0, \quad (3.22)$$

and

$$g_\beta(t) = \Psi(t) \sim 1 - C t^c \exp(-B t^{-b}), \quad t \rightarrow 0, \quad (3.23)$$

where

$$C = \left[\frac{1}{2\pi(1-\beta)\beta^{1/(1-\beta)}} \right]^{1/2}, \quad c = \frac{\beta}{2(1-\beta)} = \frac{b}{2}. \quad (3.24)$$

Like for the Mittag-Leffler function, a special case is again $\beta = \frac{1}{2}$ for which we obtain the analytical expressions

$$\psi(t) = \frac{1}{\sqrt{\pi}} t^{-3/2} \exp\left(-\frac{1}{4t}\right), \quad \Psi(t) = \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right), \quad t \geq 0. \quad (3.25)$$

We note that in this particular case the asymptotic representation (3.20) and (3.21) provides the sum of the series (3.19) and henceforth the exact expression for $\psi(t)$ in Eq. (3.25), the so-called *Lévy-Smirnov pdf* (see e.g. Ref. [26]).

Hereafter we would like to point out the major differences between the Mittag-Leffler-type function $e_\beta(t) := E_\beta(-t^\beta)$ and the Wright type function $g_\beta(t) := 1 - \phi_{-\beta,1}(-t^{-\beta})$, that can be inferred by analytical arguments. The first difference concerns the decreasing behaviour before the onset of the common power law regime: whereas $e_\beta(t)$ starts at $t = 0$ vertically (the derivative is $-\infty$) and is completely monotone, $g_\beta(t)$ starts at $t = 0$ horizontally (the derivative is 0) and then exhibits a change in the concavity from downwards to upwards. A second difference concerns the limit for $\beta \rightarrow 1$; whereas $e_\beta(t)$ tends to the exponential $\exp(-t)$ (no memory), $g_\beta(t)$ tends to the box function $H(t) - H(t-1)$ (as directly obtained from the Laplace inversion of Eq. (3.16) for $\beta = 1$). As a consequence, the corresponding *waiting-time pdf* tends to the Dirac delta function $\delta(t-1)$, a peculiar case considered by Weiss [9] in his book in p. 47 as an example of a non-Markovian process.

4 Empirical analysis

In order to corroborate the theory presented above, we have analyzed the waiting-time distribution of BUND futures traded at LIFFE in 1997. BUND is the German word for bond. Futures are derivative contracts in which a party agrees to sell and the other party to buy a fixed amount of an underlying asset at a given price and at a future delivery date. In this case the underlying asset is a German Government bond.

We have considered two different delivery dates: June 1997 and September 1997. Usually, for a future with a certain maturity, transactions begin some months before the delivery date. At the beginning, there are few trades a day, but closer to the delivery there may be more than 1 000 transactions a day. For each maturity, the total number of transaction is greater than 160 000.

In Figs. 1 and 2 we plot $\Psi(\tau)$ for the June and September delivery dates, respectively. The circles refer to market data and represent the probability of a waiting time greater than the abscissa τ . We have determined about 600 values of $\Psi(\tau)$ for τ in the interval between 1 and 50 000s, neglecting the intervals of market closure. The solid line is a two-parameter fit obtained by using the Mittag-Leffler-type function

$$\Psi(\tau) = e_{\beta}(\gamma\tau) = E_{\beta} \left[-(\gamma\tau)^{\beta} \right], \quad (4.1)$$

where β is the index of the Mittag-Leffler function and γ is a time-scale factor, depending on the time unit. For the June delivery date we get an index $\beta = 0.96$ and a scale factor $\gamma = \frac{1}{12}$, whereas, for the September delivery date, we have $\beta = 0.95$ and $\gamma = \frac{1}{12}$. The fit in Fig. 1 has a reduced chi square $\tilde{\chi}^2 \simeq 0.26$, whereas the reduced chi square of the fit in Fig. 2 is $\tilde{\chi}^2 \simeq 0.25$. The chi-square values have been computed considering all the values of Ψ .

In Figs. 1 and 2, the dash-dotted line is the stretched exponential function $\exp\{-(\gamma\tau)^{\beta}/\Gamma(1+\beta)\}$ (see Eq. (3.13)), whereas the dashed line is the power-law function $(\gamma\tau)^{-\beta}/\Gamma(1-\beta)$ (see the second equation in Eq. (3.7)). The Mittag-Leffler function interpolates between these two limiting behaviours: the stretched exponential for small time intervals, and the power-law for large ones.

Even if the two fits seem to work well, some words of caution are necessary. The Mittag-Leffler-type function $e_{\beta}(\gamma\tau)$ naturally derives from our assumption on the "memory function" in the CTRW model; however, as previously observed, it is not the unique possibility compatible with a long-memory process with a power-law decay. As a consequence, hereafter, we shall also consider for $\Psi(\tau)$ the two alternative functions discussed in Section 3, namely the rational function $f_{\beta}(\gamma\tau)$ (see Eqs. (3.14) and (3.15)), and the Wright function $g_{\beta}(\gamma\tau)$

(see Eqs. (3.22)-(3.24)).

In Figs. 3 and 4, by taking the same data as in Figs. 1 and 2 respectively, we compare the functions $e_\beta(\gamma\tau)$ (solid line), $f_\beta(\gamma\tau)$ (dash-dotted line) and $g_\beta(\gamma\tau)$ (dashed line). Whereas in the previous figures we have adopted a log-log scale to point out the power-law decay by a straight-line, now we find it convenient to use a linear scale for the ordinates to point out the behaviour of the functions for small values of τ . From these figures we can infer that the Mittag-Leffler function fits the data of the empirical analysis much better than the other two chosen functions, thus corroborating our approach to CTRW based on the *fractional-time derivative*. However, the Mittag-Leffler fit significantly differs from the empirical data for small values of τ .

5 Conclusions

The CTRW is a good phenomenological description of the tick-by-tick dynamics in a financial market. Indeed, the CTRW can naturally take into account the pathological time-evolution of financial markets, which is non-Markovian and/or non-local. From this point of view, by a proper choice of a (perhaps non-stationary) joint *pdf* $\varphi(\xi, \tau)$, one could accurately reproduce the statistical properties of market evolution. In this respect, the model can be useful for applications where Monte-Carlo simulations of market settings are needed.

With additional assumptions, the CTRW hypothesis can be tested against empirical data, thus providing useful information on the restrictions of the premises. In this paper, we have assumed a particular form for the time-evolution kernel, leading to a time-fractional Kolmogorov-Feller equation. In its turn, this implies that $\Psi(\tau)$, the probability of finding a waiting-time interval greater than τ , is a Mittag-Leffler function. There is a satisfactory agreement between this prediction and the empirical distributions analyzed in Figs. 1-4, but not for small time intervals.

Among the various questions for future research on this topic, two are particularly relevant in our opinion. The first one concerns the behaviour of other assets. Prices of liquid stocks should have a completely different time scale. Indeed, for the futures here considered, at the beginning of their lifetime several hours passed between two consecutive trades, a feature which is not likely to be shared by liquid stocks. The second problem concerns the uniqueness of the kernel. The Mittag-Leffler kernel yields the elegant time-fractional Kolmogorov equation, but there might be other possibilities for interpolating between the small waiting-time and the large waiting-time behaviour of $\Psi(\tau)$.

Finally, there is an implication for microscopic market models, a realm where

many physicists have started researching. We believe that a microscopic model should, at least phenomenologically, take into account that agents in the market decide to sell and buy an asset at randomly distributed instants. It would be a success to derive the “right” waiting-time distribution from first principles, whatever these first principles will be.

Appendix. The Caputo fractional derivative

For the sake of convenience of the reader here we present an introduction to the *Caputo* fractional derivative starting from its representation in the Laplace domain and pointing out its difference with respect to the standard *Riemann-Liouville* fractional derivative. So doing, we avoid the subtleties lying in the inversion of fractional integrals. If $f(t)$ is a (sufficiently well-behaved) function with Laplace transform $L\{f(t); s\} = \tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$, we have

$$L\left\{\frac{d^\beta}{dt^\beta}f(t); s\right\} = s^\beta \tilde{f}(s) - s^{\beta-1} f(0^+), \quad 0 < \beta < 1, \quad (A.1)$$

if we define

$$\frac{d^\beta}{dt^\beta}f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\beta}. \quad (A.2)$$

We can also write

$$\frac{d^\beta}{dt^\beta}f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t [f(\tau) - f(0^+)] \frac{d\tau}{(t-\tau)^\beta} \right\}, \quad (A.3)$$

$$\frac{d^\beta}{dt^\beta}f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau \right\} - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0^+). \quad (A.4)$$

A modification of Eqs. (A.1)-(A.4) holds any non integer $\beta > 1$ (see [19]). We refer to the fractional derivative defined by Eq. (A.2) as the *Caputo* fractional derivative, since it was formerly applied by Caputo in the late 1960s for modelling the dissipation effects in *Linear Viscoelasticity* (see e.g. Refs. [27,22]). The reader should observe that this definition differs from the usual one named after Riemann and Liouville, which is given by the first term in the R.H.S. of (A.4) see e.g. [28]. For more details on the *Caputo* fractional derivative we refer to Refs. [19,29,30].

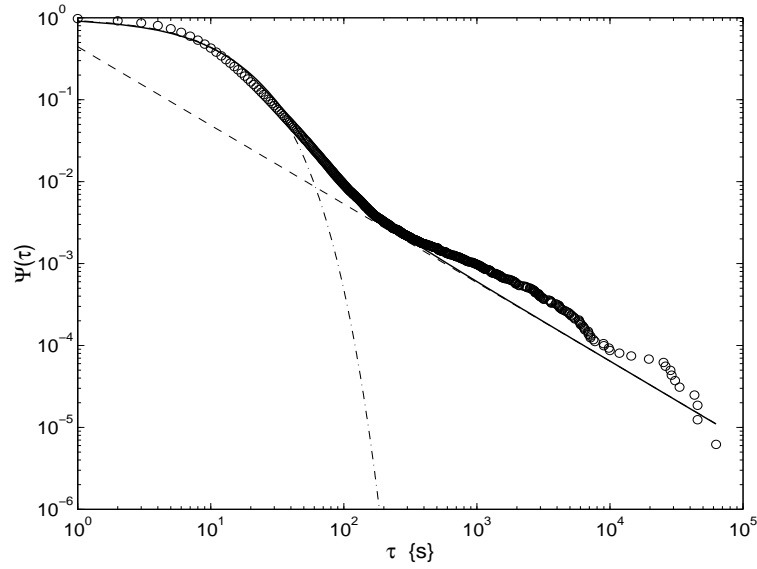


Fig. 1

Survival probability for *BUND* futures with delivery date: June 1997. The Mittag-Leffler function (solid line) is compared with the stretched exponential (dash-dotted line) and the power (dashed line) functions. ($\beta = 0.96$, $\gamma = 1/12$)

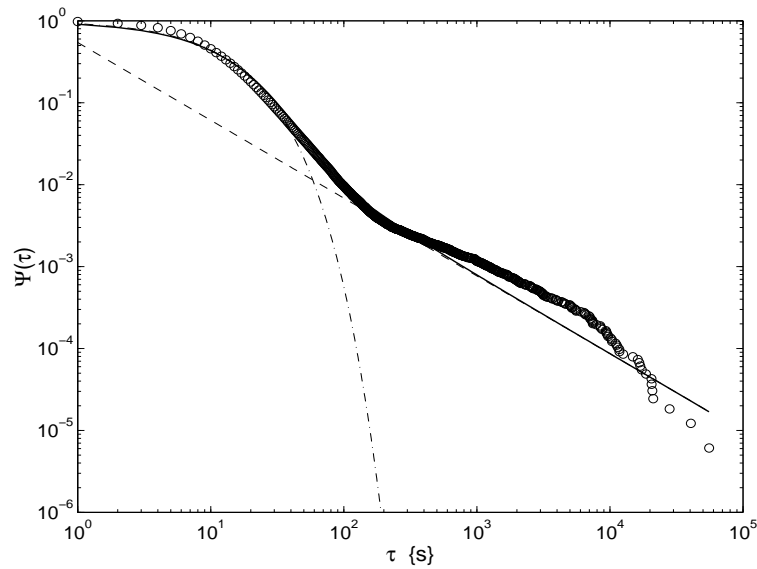


Fig. 2

Survival probability for *BUND* futures with delivery date: September 1997. The Mittag-Leffler function (solid line) is compared with the stretched exponential (dash-dotted line) and the power (dashed line) functions. ($\beta = 0.95$, $\gamma = 1/12$)

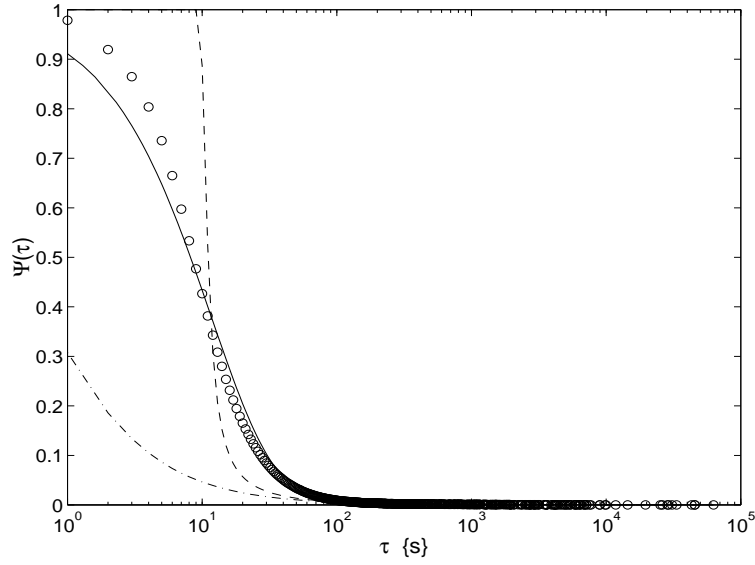


Fig. 3

Survival probability for *BUND* futures with delivery date: June 1997.
 The Mittag-Leffler function (solid line) is compared with the rational
 (dash-dotted line) and the Wright (dashed line) functions.
 $(\beta = 0.96, \gamma = 1/12)$

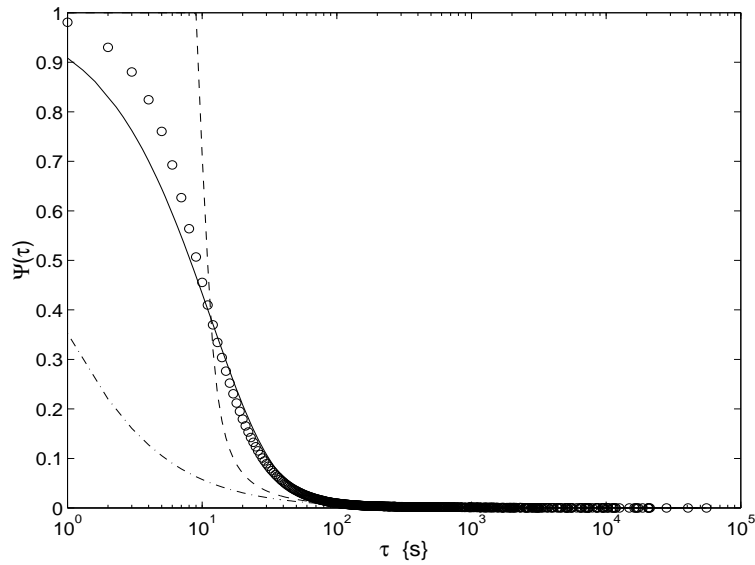


Fig. 4

Survival probability for *BUND* futures with delivery date: September 1997.
 The Mittag-Leffler function (solid line) is compared with the rational
 (dash-dotted line) and the Wright (dashed line) functions.
 $(\beta = 0.95, \gamma = 1/12)$

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