Fractional calculus and continuous-time finance

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Abstract

In this paper we present a rather general phenomenological theory of tick-by-tick dynamics in financial markets. Many well-known aspects, such as the Lévy scaling form, follow as particular cases of the theory. The theory fully takes into account the non-Markovian and non-local character of financial time series. Predictions on the long-time behaviour of the waiting-time probability density are presented. Finally, a general scaling form is given, based on the solution of the fractional diffusion equation.

Key words: Stochastic processes; random walk; statistical finance; econophysics
PACS: 02.50.Ey, 02.50.Wp, 89.90.+n
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1 Introduction

The importance of random walks in finance has been known since the seminal work of Bachelier [1] which was completed at the end of the XIXth century, nearly a hundred years ago. The ideas of Bachelier were further carried out by Mandelbrot [2], who introduced the concept of Lévy flights and stable distributions [3] in finance, and by the MIT school of Samuelson [4].

Although it was well-known that the distribution of returns or of logarithmic returns approximately followed a stable law, there was a barrier to the appli-

Preprint submitted to Elsevier Preprint *6 December 1999*

cation of these concepts in the financial practice. Indeed, stable distributions have non-finite variance, and this leads to many mathematical difficulties (for a discussion on this point the reader is referred to chapter 3 of Merton's book [4]). Therefore, in mainstream finance, both theoreticians and practitioners prefer to use the more tractable continuous Wiener process instead of discontinuous Lévy flights. A way of overcoming these difficulties has been provided by empirical studies suggesting the use of *truncated* Lévy flights, characterized by probability density distributions with finite moments [5–8].

In financial markets, not only prices and returns can be considered as random variables, but also the waiting time between two transactions varies randomly. So far, a large part of the financial practice is based on daily price changes. However, a company specialized in intra–day transactions and high–frequency data analysis, Olsen & Associates, has published various working papers related to the time behaviour of tick-by-tick data (see, for instance, on fractional time ref. [9] and on mean first passage time ref. [10]).

The purpose of this paper is to present a rather general phenomenological theory of tick-by-tick dynamics in financial markets. Many well-known aspects, such as the Lévy scaling form of ref. [6], follow as particular cases of the theory. The theory fully takes into account the non-Markovian and non-local character of financial time series. Predictions on the long-time behaviour of the waiting-time probability density are presented. Finally, a more general scaling form is given, based on the solution of the fractional diffusion equation.

The paper is divided as follows. In Sec. 2, we discuss the relevance of continuous-time random walks in finance by explicitly performing a mapping from financial data to random walks. In Sec. 3, we present the master equation and we show that it reduces to the fractional diffusion equation in the hydrodynamic limit (corresponding to a long jump-observation scale and long observation times) if some simple scaling assumptions on the jump and waiting-time probability densities hold true. Sec. 4 is devoted to the solutions of the fractional diffusion equation and their natural scaling properties. Finally, in Sec. 5, we point out the main conclusions and outline the direction for future work.

As a final remark, let us stress that the theory of continuous-time random walks is well developed [11,12], and its relation to the fractional diffusion equation and fractional calculus [13] has been recently discussed by various authors [14–18]. However, as far as we know, these concepts have not yet been applied to finance in the form we present here.

2 Continuous-time random walk in finance

The price dynamics in financial markets can be mapped onto a random walk whose properties are studied in continuous, rather than discrete, time [4]. Here, we shall perform this mapping, pioneered by Bachelier [1] and fully exploited by Samuelson and his school [4], in a rather general way.

As a matter of fact, there are various ways in which to embed a random walk in continuous time. Here, we shall base our approach on the so-called continuous-time random walk (henceforth abbreviated as *CTRW*) in which time intervals between successive steps are random variables, as discussed by Montroll and Weiss [11]

Let $S(t)$ denote the price of an asset or the value of an index at time t. In a real market, prices are fixed when demand and offer meet and a transaction occurs. In this case, we say that a trade takes place. In finance, returns rather than prices are considered. For this reason, in the following we shall take into account the variable $x(t) = \log S(t)$, that is the logarithm of the price. Indeed, for a small price variation $\Delta S = S(t_{i+1}) - S(t_i)$, the return $r = \Delta S/S(t_i)$ and the logarithmic return $r_{log} = log[S(t_{i+1})/S(t_i)]$ virtually coincide.

As we mentioned before, in financial markets, not only prices can be modelled as random variables, but also waiting times between two consecutive transactions vary in a stochastic fashion. Therefore, the time series $\{x(t_i)\}$ is characterised by $\varphi(\xi, \tau)$, the *joint probability density* of jumps $\xi_i = x(t_{i+1}) - x(t_i)$ and of waiting times $\tau_i = t_{i+1} - t_i$. The joint density satisfies the normalization condition $\int \int d\xi d\tau \varphi(\xi, \tau) = 1$.

Montroll and Weiss [11] have shown that the Fourier-Laplace transform of $p(x, t)$, the probability density function, *pdf*, of finding the value x of the price logarithm (which is the diffusing quantity in our case) at time t, is given by:

$$\widetilde{\widehat{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} \frac{1}{1 - \widetilde{\widehat{\varphi}}(\kappa, s)} , \tag{1}$$

where

$$\widetilde{\widehat{p}}(\kappa, s) = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} dx \, \mathrm{e}^{-st + i\kappa x} \, p(x, t) \, , \tag{2}$$

and $\psi(\tau) = \int d\xi \, \varphi(\xi, \tau)$ is the waiting time pdf.

Let us now consider the situation in which the waiting time and the size of the step are *independent*. In this case the joint density function, φ, can be factorized, namely written as the product of a "spatial" part and a temporal part: $\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau)$. Here $\lambda(\xi)$ is the probability for a displacement ξ

in each single step (transition probability density). Now, the normalization condition for the transition pdf: $\int d\xi \lambda(\xi) = 1$ must be added to that for the probability density of the waiting time $\int d\tau \psi(\tau) = 1$.

As a consequence we get:

$$\widetilde{\widehat{p}}(\kappa, s) = \frac{1 - \widetilde{\psi}(s)}{s} \, \frac{1}{1 - \widehat{\lambda}(\kappa)\, \widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{1 - \widehat{\lambda}(\kappa)\, \widetilde{\psi}(s)} \,, \tag{3}$$

where $\widehat{\lambda}(\kappa)$, the Fourier transform of the transition probability density, is usually called the *structure function* of the random walk and $\tilde{\Psi}(s) = (1 - \tilde{\psi}(s))/s$ is the Laplace transform of

$$\Psi(t) = \int_t^{\infty} \psi(t')\, dt' = 1 - \int_0^t \psi(t')\, dt' \,. \tag{4}$$

$\Psi(t)$ is the *survival probability* at the initial point position ($t_0 = 0$) [15]. $\int_0^t \psi(t')\, dt'$ represents the probability that at least one step is taken in the interval $(0, t)$, hence $\Psi(t)$ is the probability that the diffusing quantity does not change during the time interval of duration t after a jump [18].

According to Weiss [19], $\Psi(t)$ can be viewed as the probability that the duration of a given interval between successive steps is strictly greater than t and is the peculiar function needed to specify the probability of the displacement at time $t^* + t$ in a CTRW, where t^* is the instant of the last jump. The waiting-time pdf is related to $\Psi(t)$ by the formula: $\psi(t) = -d\Psi(t)/dt$.

Let us finally remark that, in general, the CTRW is a non-Markovian model [19], as at any time one has to know the value of the diffusing quantity as well as the time at which the last step took place in order to predict the further course of the walk. The non-Markovian property arises because the time of the previous step does vary and could be even $t = 0$, so that the complete history of the process must be taken into account at all times. The only Markovian version of the CTRW is the one in which the waiting time pdf, $\psi(\tau)$, is a negative exponential:

$$\psi(\tau) = \frac{1}{T} \exp(-\tau/T),$$

where T is the average time between successive steps. Only for this form of the density, the probability that a step of the random walk will take place in $(t\,,\, t + dt)$ is dt/T, as $dt \to 0$, independent of the time at which the immediately preceding step occurred. This is not true of any other form of $\psi(\tau)$.

3 Master equations and fractional diffusion

The *master equation* governing the probability density profile in a CTRW can be derived by inverting the Fourier-Laplace transform in eq. (3). Rewriting (3) as

$$\widehat{\widetilde{p}}(\kappa, s) = \widetilde{\Psi}(s) + \widetilde{\psi}(s)\widehat{\lambda}(\kappa)\widehat{\widetilde{p}}(\kappa, s)$$

we obtain

$$p(x,t) = \delta_{x\,0}\Psi(t) + \int_0^t dt'\, \psi(t-t') \int_{-\infty}^{+\infty} dx'\, \lambda(x-x')\, p(x',t')\,. \qquad (5)$$

This form of the master equation is quoted, e.g., in Klafter et al. [20] and Hilfer and Anton [15]. However, equivalent forms can be found in the literature. The following form shows the non-local and non-Markovian character of the CTRW [12,18]:

$$\frac{\partial}{\partial t} p(x,t) = \int_0^t dt'\, \phi(t-t') \left[-p(x,t') + \int_{-\infty}^{+\infty} dx'\, \lambda(x-x')\, p(x',t') \right] ; \qquad (6)$$

here, the kernel $\phi(t)$ is defined through its Laplace transform

$$\widetilde{\phi}(s) = \frac{s\, \widetilde{\psi}(s)}{1 - \widetilde{\psi}(s)}\,.$$

The above equations allow to compute $p(x,t)$ from the knowledge of the jump pdf $\lambda(\xi)$ and of the waiting-time pdf $\psi(\tau)$. In principle, both these quantities are empirically accessible from high-frequency market data, even if, recently, within the physics community, emphasis has been given to the jump pdf [6].

The time-evolution equation for $p(x,t)$ has a remarkable limit, if some scaling conditions on the structure function and on the waiting time pdf are verified.

Let us assume the following scaling behaviour in the hydrodynamic limit (long-jump scale and long observation times):

$$\widehat{\lambda}(\kappa) \sim 1 - |\kappa|^{\alpha}\,, \quad \kappa \to 0\,, \quad 0 < \alpha \leq 2\,, \qquad (7)$$

and

$$\widetilde{\psi}(s) \sim 1 - s^{\beta}\,, \quad s \to 0\,, \quad 0 < \beta \leq 1\,. \qquad (8)$$

The above approximations are consistent with the following explicit expressions for the Fourier and Laplace transforms:

$$\widehat{\lambda}(\kappa) = \exp(-|\kappa|^{\alpha})\,, \quad 0 < \alpha \leq 2\,, \qquad (9)$$

and

$$\widetilde{\psi}(s) = \frac{1}{1+s^{\beta}}\,, \quad 0 < \beta \le 1\,. \tag{10}$$

We note that eq. (9) represents the characteristic function for the symmetric Lévy stable pdf of index α; for $0 < \alpha < 2$ the pdf decays like $|x|^{-(\alpha+1)}$ as $|x| \to \infty$, for $\alpha = 2$ the Gaussian pdf is recovered.

From eq. (10), we observe that

$$\widetilde{\Psi}(s) = \frac{1-\widetilde{\psi}(s)}{s} = \frac{s^{\beta-1}}{1+s^{\beta}}\,, \quad 0 < \beta \le 1\,, \tag{11}$$

so that the survival probability turns out to be

$$\Psi(t) = E_{\beta}(-t^{\beta})\,, \quad 0 < \beta \le 1\,, \tag{12}$$

where

$$E_{\beta}(-t^{\beta}) = \sum_{n=0}^{\infty}(-1)^n\,\frac{t^{\beta n}}{\Gamma(\beta n+1)}$$

is the Mittag-Leffler function of order β [21,22]. Thus the pdf for the waiting time is

$$\psi(t) = -\frac{d}{dt}\Psi(t) = -\frac{d}{dt}E_{\beta}(-t^{\beta})\,, \quad 0 < \beta \le 1\,, \tag{13}$$

which is in agreement with the expression obtained in [15] in terms of the generalized Mittag-Leffler function in two parameters.

For $0 < \beta < 1$ the Mittag-Leffler function $E_{\beta}(-t^{\beta})$ is known to be, for $t > 0$, a completely monotonic function of t, decreasing from 1 (at $t = 0$) to 0 like $t^{-\beta}$ as $t \to \infty$ [22]. As a consequence the pdf for the waiting time is strictly positive and monotonically decreasing to zero like $t^{-(\beta+1)}$. For $\beta = 1$ the Mittag-Leffler function reduces to $\exp(-t)$ and we recover from eqs. (11 – 13) the Markovian CTRW.

If we insert eqs. (7) and (8) into eq. (1), we get the limiting relation:

$$s^{\beta}\widehat{\widetilde{p}}(\kappa,s) + |\kappa|^{\alpha}\widehat{\widetilde{p}}(\kappa,s) = s^{\beta-1}\,. \tag{14}$$

Inverting eq. (14), we obtain the time-evolution equation for $p(x,t)$ in the hydrodynamic limit. If $0 < \beta \le 1$ and $0 < \alpha \le 2$, we have, for $x \in R$:

$$\frac{\partial^{\beta}p(x,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha}p(x,t)}{\partial|x|^{\alpha}} + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(x), \quad (t > 0). \tag{15}$$

In eq. (15), we have introduced the fractional derivatives $\partial^\beta/\partial t^\beta$ and $\partial^\alpha/\partial|x|^\alpha$ defined as the inverse Laplace and Fourier transforms of s^β and $-|\kappa|^\alpha$, respectively [13,17]. Fractional derivatives are non-local operators belonging to the larger class of pseudo-differential operators [23,24], which allow power-law effects. In particular, the "time" operator in eq. (15) is the Riemann-Liouville fractional derivative of order β defined as (if $0 < \beta < 1$):

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} \, d\tau \right\} ,$$

whereas the "jump" operator is the Riesz fractional derivative of order α which, if $0 < \alpha < 2$ can be represented as [13]:

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \Gamma(1+\alpha) \frac{\sin(\alpha\pi/2)}{\pi} \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\alpha}} \, d\xi \, .$$

Finally, let us mention that eq. (14) was derived by Weiss [19] and by Afanas'ev and co-workers [25]. Moreover, the above derivation of eq. (15) was implicit in a paper by Fogedby [14] and was explicitly presented by Compte [16] and by Saichev and Zaslavsky [17].

4 Lévy flights and scaling of solutions

We start this section with the analysis of a particular case of eq. (15), the limit $\beta \to 1$, where we have (in the weak sense) [17]:

$$\lim_{\beta \to 1} \frac{t^{-\beta}}{\Gamma(1-\beta)} = \delta(t),$$

and eq. (15) becomes equivalent to the following initial value problem:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial^\alpha p(x,t)}{\partial |x|^\alpha}, \quad p(x,0) = \delta(x). \tag{16}$$

The Cauchy problem (16) can be solved by Fourier-transforming both sides of the equation with respect to x. After integrating and inverse Fourier-transforming, one gets:

$$p(x,t) = \frac{1}{t^{1/\alpha}} L_\alpha \left(\frac{x}{t^{1/\alpha}} \right), \tag{17}$$

where $L_\alpha(u)$ is the Lévy standardized probability density function:

$$L_\alpha(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-iqu - |q|^\alpha} \, dq. \tag{18}$$

Taking the limit $\beta \to 1$ in eq. (15) corresponds to considering independent time increments. Continuous-time random walks whose pdf $p(x,t)$ is given by eq. (17) are called *symmetric Lévy flights* or better *symmetric α-stable Lévy processes* [3,26]. In 1963, analysing the scaling properties of financial time-series, Mandelbrot [2] found that the empirical pdf $p(x,t)$ could be well fitted by the Lévy density function (17) with $\alpha = 1.7$. As we mentioned in the introduction, the main difficulty in dealing with the Lévy distribution is that its moments diverge. For $0 < \alpha < 2$, the only bounded finite moments have index γ satisfying $-1 < \gamma < \alpha$. For this reason, the results of Mandelbrot were well-known but not much used in mainstream quantitative finance [4]. The recent empirical analysis of Mantegna and Stanley [6] suggests that *truncated* Lévy flights should be used instead, as good models for financial price dynamics [5]. Koponen [7] introduced a class of truncated Lévy flights, which was successively generalized by Boyarchenko and Levendorskii [8]. However, all these studies somehow neglected the waiting-time pdf.

In the general case, the Cauchy problem of eq. (15) can be solved by the same technique used above. There is, however, a mathematical subtlety. In order to give a meaning to the Cauchy problem, the Riemann-Liouville operator must be replaced by the Caputo fractional derivative of order β [22,27]:

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, d\tau \right\} - \frac{t^{-\beta}}{\Gamma(1-\beta)}\, f(0)\,.$$

Now, the solution is:

$$p(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \tag{19}$$

and $W_{\alpha,\beta}(u)$ is the following scaling function:

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-iqu} E_\beta(-|q|^\alpha) dq, \tag{20}$$

where E_β the Mittag-Leffler function of order β and argument $z = -|q|^\alpha$

Further empirical studies on high-frequency financial data may reveal the scaling form (20), if the waiting-time pdf satisfies the asymptotics (8).

5 Conclusions and outlook

In this paper, we argued that the continuous-time random walk (CTRW) is a good phenomenological model for high-frequency price dynamics in financial markets, as, in general, this dynamics is non-Markovian and/or non-local.

CTRW naturally leads to the so-called fractional diffusion equation in the hydrodynamic limit if some scaling properties of the waiting time pdf $\psi(\tau)$ and of the jump pdf $\lambda(\xi)$ hold true in that limit. This point needs a further discussion. Indeed, the scaling regime of eqs. (8) and (7) breaks down for very large jumps. For this reason, truncated Lévy flights have been introduced [5–8]. Preliminary investigations [28] on high frequency financial data show that a similar problem is present for the waiting time pdf. Nevertheless, we can view the fractional diffusion equation (15) as a model for approximating the true behaviour of returns in financial markets.

In the region where the dynamics is well approximated by eq. (15), we expect the following scaling for the waiting-time pdf (see the discussion in Sec. 3):

$$\psi(\tau) \sim \tau^{-\mu}, \tag{21}$$

where $\mu = \beta + 1$ varies in the range $1 < \mu < 2$. Consequently, the more complex scaling form (20) should hold true.

Empirical analyses on market high-frequency data will be necessary in order to verify these predictions. In any case, we expect that the concepts of CTRW and of fractional calculus will be of help in practical applications such as option pricing, as they provide an intuitive background for dealing with non-Markovian and non-local random processes.

In this paper, the mathematical apparatus has been kept to a minimum, the interested reader will find full mathematical details in a forthcoming paper [29].

Acknowledgements

This work was partially supported by the Italian INFM and INFN, and by the Research Commission of the Free University in Berlin. R. G. is grateful to the Italian "Istituto Nazionale di Alta Matematica" for supporting his visit in Italy. E. S. wishes to thank CERN for its wonderful library open 24 hours a day: an ideal place to study in a quiet atmosphere after shifts at the colliders. Discussions on high-frequency financial data with Marco Raberto inspired this work.

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