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Measuring the Degree of Efficiency of Financial Market Efficiency: An Essay

Abstract

This essay discusses first two competing hypotheses of market efficiency: the classical Efficient Market Hypothesis (EMH) of Samuelson and Fama, and the Fractal Market Hypothesis (FMH) of Mandelbrot and Peters and their weaknesses. The EMH depends on the empirically uncorroborated i.i.d. (= independence & stationarity) assumption of market innovations. The time - invariant FMH risk depends on the lengths of time horizons, as measured by the Hurst exponent. By way of empirical examples in the cash, bond and options an futures markets, it is demonstrated that scientifically a much broader concept of financial market risk is needed. This new risk concept should allow for the measurement of the degree of market efficiency, which is time and horizon dependent. The proposed definition of financial market risk is a time - frequency distribution function P, where the shape of the function is determined not only by the second-order moments $\sigma(\omega)$, differentiated by the investment asset return categorizations ω, but also of the length investment horizons, or maturities of the investment securities τ, and of the time period t. In other words, the new concept of financial risk $P(\omega, \tau, t)$ should be able to account for both LT and ST nonlinear time dependence and for strict non-stationarity to be empirically compatible and thus scientifically acceptable. Such a time - frequency distribution $P(\omega, \tau, t)$ can be measured and identified by modern forms of time - frequency signal processing analysis, like windowed Fourier and wavelet multiresolution analysis.

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1 INTRODUCTION

Since Samuelson's (1965) and Fama's (1970) martingale formulation of the Efficient Market Hypothesis (EMH), most textbooks in finance have blindly adopted this theoretical idealization of the financial markets. But there are at least two problems with the EMH: (1) it is not demonstrated that empirical financial pricing processes are random, and (2) even if we would assume that such processes are random, then the problem which theoretical random process accurately and consistently describes the observed rates of return in a financial market in a time-invariant way is still not conclusively resolved. Assuming that financial market prices follow a Geometric Brownian Motion (GBM) is not the same as scientifically establishing they do. A number of competing models had been proposed to explain the following stylized facts of observation:

(i) there is empirical evidence that the tails of measured distributions are fatter than expected for the prevailing model of the GBM, and their modes much higher, i.e., that the kurtosis of the distribution of the rate of return errors is abnormally high, suggesting that a stable, non-Gaussian error distribution is more appropriate ;

(ii) there is empirical evidence that the first and second moments of the relative price changes vary over time, *i.e.*, that there is wide - sense non - stationarity.[1]

Fama (1965) was well aware of competing financial market hypotheses, since in the 1960s he had reviewed and criticized the work on fractal market pricing theory, summarized in the Fractal Market Hypothesis (FMH) by Mandelbrot (1966, 1971, 1982), which has recently been resurrected (scaling laws, stable distributions, Fractional Brownian Motion) in various forms by Müller and Dacorogna, c.s. (1990, 1995), Lo (1991), Peters (1994), Diebold, Hickman, Inoue and Schuermann (1998) and Batten, Ellis and Mellor (1999). The FMH emphasizes the empirically observable and well corroborated self - similarity or power laws of financial pricing, in particular in stock markets.

[1] In Los (2003) I've collected a broader and more extensive literature review.

Such time - dependent self - similarity of empirical financial pricing laws contradicts the EMH based on martingale theory.

While theoreticians have favored Fama's martingale based EMH, and Fama (1991) still supports the martingale based EMH, more than twenty years of empirical research results compiled by Peters (1994) tend to support the FMH. In Los (2003) I've presented new methods of empirical analysis to further corroborate and refine the FMH results. Mantegna (1977) and Mantegna and Stanley (2000) also provide ample additional evidence from the perspective of physics that the EMH is now definitely empirically rejected.

As recently observed, the non-stationarity of the financial rate of return distributions has also serious consequences for investment performance analysis, asset allocation and portfolio monitoring (Klemkosky and Bharati, 1995; Mechan, Yoo and Fong, 1998). But already more than twenty years ago, many analysts observed such non-stationarity within the context of the Capital Asset Pricing Model (CAPM) and asked if it would lead to the "death of beta." Then they desperately tried to explain away the empirically observed non-stationarity, without much success (Roenfeldt, Griepentrof and Pfaum, 1978; Frabozzi and Francis, 1978; Chen, 1981; Theobald, 1981; Chan and Lakonishok, 1993).[2] It is now clear that the endemic non-stationarity is a characteristic of the financial markets that cannot be explained away. We have to explain why it occurs (Los, 1999).

This essay summarizes the current scientific discussion about the conflict between the theoretical EMH, the empirical facts and the newly proposed FMH in very concise but clear terms, with reference only to the most insightful published examples. Several financial experts have already commented on the observed inefficiencies of particular financial markets, since it has consequences for how we should invest for the best results and how we should analyze our investment performance (Haugen, 1999a and b).

[2] As is now clear with 20/20 hindsight, by such and similar articles I was erroneously inspired to write a Ph.D. thesis in theoretical econometrics to explain how one can capture the systematic part of the nonstationarity of the betas (Los, 1984).

The next section discusses the EMH in more detail. The third section discusses the FMH and we propose a few models that better explain the observed empirical facts. On the basis of our comparison of the EMH and FMH, we conclude this essay by proposing that dynamic financial market efficiency is not an absolute concept, but should be measured according to a scale from no efficiency, via moderate efficiency to perfect efficiency. In other words, the various financial markets can be categorized according to their degree of efficiency. Such an efficiency categorization is likely to be very useful for global investors.

2 EFFICIENT MARKET HYPOTHESIS

2.1 Martingales and Fair Games

Before we interpret the efficiency of financial markets in terms of martingales like Fama (1970, 1991), it may be wise to obtain first an intuitive understanding of what martingale processes are. Martingales are very useful concepts for when time series observations are mutually dependent (they usually are!), like:

(1) when the observations result from feedback processes;

(2) when the dynamic process to be identified is nonlinear; and

(3) when the information sets are increasing, because of accumulation of observations.

Intuitively, a discrete martingale can be understood as a sequence of values of conditional (pricing) events, which is as likely to go up as to go down, at each of a series of consecutive time instants.

Definition 1 *For the sequence* $\{X(t) : t = 1, 2, ...\}$*, if*

$$E\{X(t+1) \mid X(1), X(2), ..., X(t)\} = X(t) \tag{1}$$

then the sequence $\{X(t)\}$ *is called a martingale.*

The classical example of a martingale was formulated in game-theoretic terms by a gambler. If $X(t)$ represents the stake in a game at time t held by one of two gamblers, the game between the two gamblers is fair if and only if the martingale property holds. Martingales are one of the simplest kinds of random processes for which a number of convergence results and central limit theorems (CLTs) are available (Chow and Teicher, 1978; Los, 1982). For process identification theory, however, the martingale - difference property is more fundamental, since this property lies somewhere in between the properties of nonlinear independence and uncorrelatedness (= linear independence), respectively.

Martingales are the partial sums of martingale differences, which we will define shortly. To do so, we need first the definitions of the semi-martinagales, i.e., the sub - and super - martingales, as follows.

Definition 2 *A random process* $\{X(t), G_t : t \in T\}$ *is called a* ***submartingale*** *if*

$$E\{|X(t)|\} \leq \infty \tag{2}$$

and

$$E\{X(t) \mid G_s\} \geq X_s \text{ a.c., } s < t; s, t \in T \tag{3}$$

and a ***supermartingale*** *if, instead,*

$$E\{X(t) \mid G_s\} \leq X_s \text{ a.c., } s < t; s, t \in T \tag{4}$$

and it is thus a ***martingale****, if the process is both a submartingale and a supermartingale.*

Here G_t is the historical information set on which the process is conditioned. We can now easily define the more relevant concept of a martingale - difference.

Definition 3 *A random process* $\{X(t), G_t : t \in T\}$ *is called a* ***martingale - difference*** *(MD) if*

$$X(t) = Y(t) - Y(t-1), \; X(1) = Y(1) \tag{5}$$

where the random process $\{Y(t), G_t : t \in T\}$ *is a martingale.*

Since for a martingale - difference

$$\begin{aligned} E\{X(t) \mid G_{t-1}\} &= E\{Y(t) - Y(t-1) \mid G_{t-1}\} \\ &= E\{Y(t) \mid G_{t-1}\} - E\{Y(t-1) \mid G_{t-1}\} \\ &= Y(t-1) - Y(t-1) = 0 \end{aligned} \tag{6}$$

the martingale - difference process is often defined by the expression

$$E\{X(t) \mid G_{t-1}\} = 0 \tag{7}$$

2.2 Independence and Uncorrelatedness

Compare now this definition of the MD - property with the following classical definitions of independent and uncorrelated random variables (r.v.), to see why the martingale property is different from both and more general.

Definition 4 *Let $\{X(t) : t \in T\}$ be a sequence of r.v.'s on a given probability space (Ω, G, P) with*

$$E\{X(t)\} = 0 \tag{8}$$

*and $\{G_t : t \in T\}$ a current of $\sigma-$algebras on the measurable space (Ω, G), where Ω is the complete universe of all possible events. Then $\{X(t)\}$ is a sequence of **independent r.v.** with respect to $\{G_t\}$ if $X(t)$ is measurable with respect to G_t and is independent of G_{t-1} for all $t \in T$.*

Thus independent r.v.'s have no history! They are not conditioned on historical information. They are immeasurable using any past historical information. Otherwise stated, they are measurable only with respect to current information.

Example 5 *The outcome of a throw of the ideal fair die is an independent r.v..*

However, empirical data are virtually never independent, except when we strictly and ideally "randomize" an experiment, which is almost certainly impossible to achieve. Only the physical process of atomic decay leads to pure random "bits. " (*Cf.* Kalman 1994, 1995, 1996; Pincus and Singer, 1996; Pincus and Kalman, 1997). In empirical reality there is always some form of historical conditionality, some form of dependence on the past, even with a so-called "fair " die. The fundamental question becomes thus: what kind of dependence is that? Is that dependence serial ("short term") or global ("long term")?

Definition 6 *Let $\{X(t) : t \in T\}$ be a sequence of r.v.'s on a given probability space (Ω, G, P) with $E\{X(t)\} = 0$ and $\{H_t : t \in T\}$ a current of **linear** spaces $H_t \supset H_{t-1}$ on the measurable space (Ω, H). Then $X(t)$ is a sequence of **uncorrelated r.v.'s** with respect to $\{H_t\}$ if $X(t) \in H_t$ and is uncorrelated. with all elements of H_{t-1} for all $t \in T$.*

The crucial part of this definition of uncorrelatedness is the *linearity* of the independent information sets. In contrast, martingales allow for both general nonlinear dependence and for correlatedness. Depending on whether the current of data spaces on which they are defined is nonlinear or linear, martingale differences are independent or even uncorrelated, respectively.

2.3 Random Walk Model and GBM: i.i.d. Assumption

We'll need some additional definitions to expand the arsenal of our analytic tools, to sharpen our analytic concepts and make them more specific and detailed, since there is quite some sloppiness in this particular field of research, which has relied on too simple correlation (= linear short term dependence) analysis to try to corroborate the EMH (*cf.* Osborne, 1959, 1962).

Definition 7 *A* ***Random Walk*** *is a particular wide sense Markov process with independent innovations*

$$X(t) - X(t-1) = \varepsilon(t), \text{where } \varepsilon(t) \sim i.i.d. \tag{9}$$

Definition 8 *A* ***Geometric Brownian Motion*** *(GBM) is a random walk of the natural logarithm of the original process $X(t)$. Thus, first we define the rate of return*

$$\begin{aligned} \ln \frac{X(t)}{X(t-1)} &= \ln X(t) - \ln X(t-1) \\ &= \Delta \ln X(t) \\ &= x(t) \end{aligned} \tag{10}$$

so that we have the ratio

$$\frac{X(t)}{X(t-1)} = e^{x(t)} = e^{x(t-1)\varepsilon(t)} \tag{11}$$

or, equivalently, after taking natural logarithms,

$$x(t) = x(t-1) + \varepsilon(t), \text{ where } \varepsilon(t) \sim i.i.d. \tag{12}$$

Consequently, we can also state that the GBM is defined by

$$\begin{aligned} \Delta x(t) &= x(t) - x(t-1) \\ &= \varepsilon(t), \text{where } \varepsilon(t) \sim i.i.d. \end{aligned} \tag{13}$$

The *innovations* $\varepsilon(t)$ of such a logarithmic random walk are identically distributed in the wide sense = wide sense stationary (= with constant mean and variance), and they are mutually independent.

Quite a few of the classical probability theory and estimation results are derived from the Central Limit Theorem of Lindeberg - Lévy, which formulates the convergence in distribution to a standard normal distribution of normalized and centered sums of independently identically distributed (i.i.d.) random variables (Dhrymes, 1974; Chow and Teicher, 1978). The usual approach to prove these asymptotic CLT - based results is to apply an ergodic theorem, which assumes that the ensemble - and time - averages are identical. This allows then the researcher to use time-averages to study frequency behavior. But, as we now understand, this ergodic approach requires such considerable (Platonic) idealization of the empirical situation, that it is unrealizable. We can measure the finite time averages, but we cannot know the complete ensemble set. In fact, modern signal processing engineers, who use windowed Fourier or wavelet multiresolution analysis, have done away with the ergodic theorem an place their identification searches squarely in finite time - frequency spaces.

In finance, one usually assumes a sequence of i.i.d. observations on the innovations of the financial rate of return time series.. In reality, this assumption of independence and wide-sense stationarity is violated because of financial feedback and adaptation processes: financial investors and traders learn from their valuation and trading mistakes. Further, an indispensable condition for the application of an ergodic theorem is that the probability measure must converge almost certainly (a.c.). To achieve convergence a.c., strict stationarity of the observations is assumed. But this assumption can never be checked completely against the data. Thus, we've come now full circle.

The fundamental empirical problem is that the idealizing assumption of ergodicity, which is based on the unrealistic assumption of an idealized infinite complete ensemble set, is uncheckable on the basis of a single historical realization of the random process, *i.e.*, on a single historical time series. We should check the complete ensemble, but we can't. We assume it exists, while we can't check that it empirically exists. Such an uncheckable assumption doesn't belong in science.

2.4 Dependence - Allowing Efficiency

Samuelson (1965) and, in particular, Fama (1970) used the martingale (difference) property to define *efficient market pricing* so that market efficiency no longer depended on the conventional

$$\boxed{\textbf{i.i.d.} = \text{independence} + \text{stationarity}} \qquad (14)$$

assumption of the innovation series that drives the speculative pricing process as modeled by, say, the original random walk of Bachelier (1900; Cootner, 1964), or the GBM used in Black-Scholes price diffusion equations.[3] Thus, the modern compact definition of financial market efficiency, which allows for other forms of market efficiency than the one defined by pure i.i.d. innovations is as follows:

Definition 9 *A market is* ***Samuelson - Fama - efficient****, when the random market pricing process is a martingale.*

It's important to realize that this kind of market pricing efficiency allows for a particular kind of dependence and that other kinds of dependence are possible. In fact, according to Mandelbrot (1966), martingales and martingale differences are too restrictive to describe efficient empirical speculative markets, since they don't allow for *singularities* in the empirical time series, *i.e.*, the discontinuities and sharp breaks which are observed in empirically efficient speculative markets. Recall that martingales require that the expectation of the absolute values is finite:

$$E\{|X(t)|\} \leq \infty \tag{15}$$

Mandelbrot demonstrated that there are random speculative pricing processes, for example sequences of price singularities, like the incremental changes in FX quotations, which have infinite variances, so that

$$E\{|X(t)|\} = \infty \tag{16}$$

and, therefore, they contradict the assumption of ergodicity on which also the empirical application of martingale theory hinges.

Fortunately, we can still empirically measure the risk in those singular financial markets, but not by using conditional expectations and martingales.[4] Consequently, the efficient convergence

[3] It's historically interesting that Bachelier's mathematical formulation of a random walk - Brownian motion preceded Einstein's (1905) by five years.

[4] The abrupt default of the Russian souvereign bonds and the consequent collapse of Long Term Capital Management (LTCM) demonstrates the occurrence of such discontinuities in the pricing processes, which surprised and confounded the Nobel-price winning partners of LTCM.

to a limiting, dynamic (time-varying) equilibrium distribution is now viewed to be dependent not on the classical i.i.d. random variable assumption, but on a quite different assumption of a particular class of dependent random variables. Such efficient convergence towards a dynamic equilibrium is now viewed to be dependent on either stable distributions or on even more transient phenomena, like the spectrum of singularities of the speculative price series, as in the case of empirical FX quotations.

3 FRACTAL MARKET HYPOTHESIS

3.1 Importance of Investment Horizons

Why do we need a different concept of time - dependence than the serial time - dependence favored by classical time series analysis? Because of the simultaneous existence of a variety of investment horizons in the financial markets (Bierman, 1997). To demonstrate the importance of the various investment horizons in market pricing processes, Holton (1992) contrasts the immateriality of these horizons in the idealized random walk process, with the measured materiality of such horizons for speculative pricing series. As we'll see, Holton's simple graphs clearly demonstrate the incorrectness of the random walk model, even as an approximation, for such empirical pricing processes, since the random walk assumes stationarity of the volatility of the innovations. Before we discuss the details of this new view, we introduce the formal definition of a total rate of return on an investment, as used in this book.

Definition 10 *The* ***total rate of return*** $x(t)$ ***on an investment*** X *made at time* $t-1$ *is represented in various forms, as*

$$x(t) \approx \Delta \ln X(t) \text{ for relatively small numbers}$$

Definition 11 *In finance, the* ***volatility*** *of* $x(t)$ *observed at appropriate moments in time (e.g., in minutes, hours, days, weeks, months, years, etc.) is measured over horizon* τ_i *by the standard deviation, measured by using time - averages:*

$$\begin{aligned} \sigma_{\tau_i} &= \left[E_{\tau_i} \left\{ [x(t) - E_{\tau_i} \{x(t)\}]^2 \right\} \right]^{0.5} \\ &= \left[\frac{1}{\tau_i} \sum_{t=1}^{t=\tau_i} \left[x(t) - \frac{1}{\tau_i} \sum_{t=1}^{t=\tau_i} x(t) \right]^2 \right]^{0.5} \end{aligned} \tag{17}$$

What are the specific consequences of the assumptions of stationarity and independence (*i.e.*, the conventional combined i.i.d. assumption)? Let's spell them out, so that we can find the discrepancies between these theoretical assumptions and empirically observed reality. Under the assumptions of wide sense stationarity, the means are constant:

$$\begin{aligned} E_{\tau_i}\{x(t)\} &= \frac{1}{\tau_i}\sum_{t=1}^{t=\tau_i} x(t) \\ &= \frac{1}{\tau_j}\sum_{t=1}^{t=\tau_j} x(t) \\ &= E_{\tau_j}\{x(t)\} \\ &= \text{ constant, where } \tau_i, \tau_j \in T \end{aligned} \tag{18}$$

and volatility is also constant:

$$\sigma_{\tau_i} = \sigma_{\tau_j} = \sigma_{\tau}, \text{ where } \tau_i, \tau_j \in T \tag{19}$$

In addition, it is assumed that there is complete independence between periods, implying that there is also uncorrelatedness = linear independence between periods. Thus, all cross-covariances between periods are implicitly assumed to be zero and are, consequently, ignored

$$\sigma_{\tau_i \tau_j} = \left[\frac{1}{\tau_i + \tau_j}\sum_{t=1}^{t=\tau_i+\tau_j} \left[x(t) - E_{\tau_i}\{x(t)\}\right]\left[x(t-\tau_i-\tau_j) - E_{\tau_j}\{x(t-\tau_i-\tau_j)\}\right]\right]^{0.5} = 0$$

$$\text{where } \tau_i, \tau_j \in T \tag{20}$$

The result is that the squared volatility (= variance) of the whole observation period, which is the sum of the independent horizons, $T = \tau_1 + \tau_2 + ... + \tau_n$, is measured by the sum of the constant squared volatilities in each of the sub-periods:

$$(\sigma_T)^2 = \sum_{t=1}^{T} (\sigma_\tau)^2 = T(\sigma_\tau)^2 \tag{21}$$

Thus, one reaches the conclusion that the volatility of the whole observation period is an exponential function of time, with an exponent equal to 0.5:

$$\sigma_T = \sigma_\tau T^{0.5} \tag{22}$$

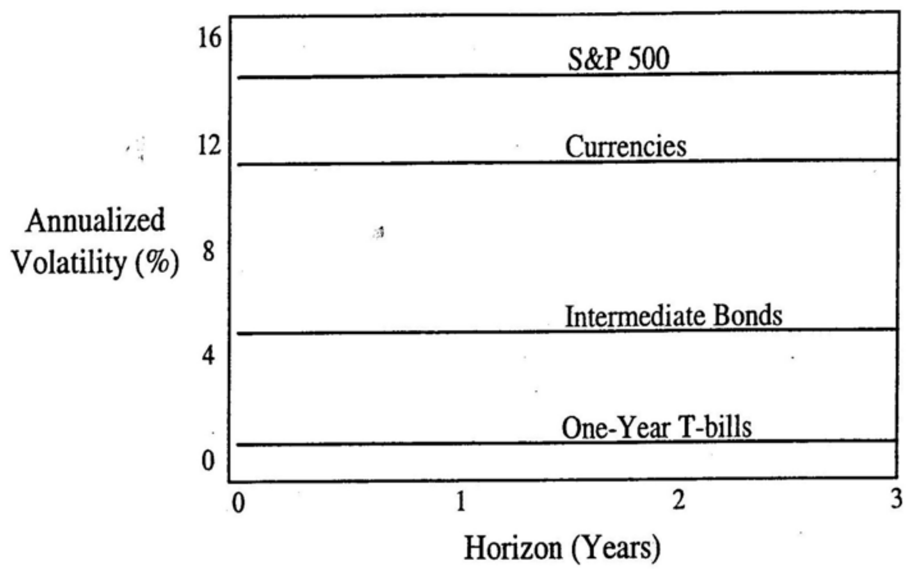
Therefore, the GBM assumption, which relies on the combination of the two assumptions of stationarity and independence, states that the volatility of the rates of return behaves as a square root function of time. Otherwise formulated, when we normalize these volatilities by the square root of their time horizon, all normalized volatilities equal the same constant volatility.

When the time periods of observation or horizons are equal to each other and $\tau_i = \tau$ for all i, the observation period is proportional $T = n\tau$. For given T = length of the complete set of observations, when the time periods τ of observation become smaller, *i.e.*, when the scale of time observation becomes smaller and is reduced to $\tau \rightarrow 1$, the fundamental time unit of observation increases, $n \rightarrow T$, and we observe relative smaller risks.

Definition 12 ***Normalized random walk volatility*** *is*

$$\sigma_\tau = \sigma_T (n\tau)^{-0.5} = \sigma_T T^{-0.5} \text{ } for \text{ } \tau = 1 \tag{23}$$

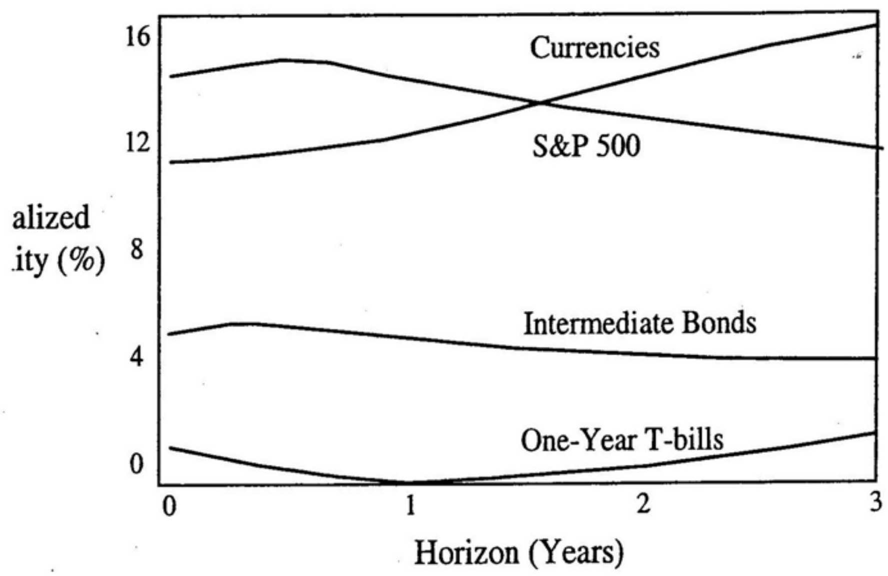
Thus, under the simplifying random walk assumption, we only have to compute the volatility of an asset class once over the complete time period T, and then we can extrapolate all other time horizon volatilities for these asset classes by appropriate square root scaling adjustments (Greer, 1997). On a normalized basis, still each asset class has its own constant volatility, as in Fig. 1. This is what we meant by the financial market risk being dependent only on the asset class categorization. Under the random walk model, the temporal dimension of the risk of an asset is irrelevant.



The random walk model of constant, normalized, asset return volatilities

Remark 13 *This i.i.d. assumption implies* ***self-similarity of the risk****, since the risk σ_T over the horizon T, is similar to the risk σ_τ of the base observation period τ, except for the so-called* ***Fickian scaling factor*** *$T^{0.5}$.*

But is this random walk property empirically true? Fig. 2 shows the empirically observed reality of normalized volatility, *i.e.*, of the empirically measured volatility for each maturity τ is normalized by dividing it by $\tau^{0.5}$. Fig. 2 shows that empirical financial market risk is not only dependent on market risk, but also on the investment horizon τ. The normalized volatility of currencies continuously increases when the investment horizon is extended. The volatility of the rates of return of the S&P500 index increases within the horizon of one year, but then declines the longer the investment horizon. Similar behavior can be observed for the intermediate bonds, except that the initial horizon is about a quarter of a year. The volatility of the rates of return of one - year T - bills declines within a year, is, of course, zero at the horizon of one year (and is thus used as Tobin's risk-free cash factor), and thereafter it increases again.



Risk is a function of horizon: time dependent, standardized, asset return volatilities

Thus when we vary the investment horizons and compute for each horizon separately the corresponding empirical volatility normalized by the Fickian scaling factor, it appears that these volatilities are not constant, but time - dependent. This is a clear refutation of the i.i.d. assumptions of the GBM model of financial risk. This time - dependence is systematic and not stochastic, as has been suggested by i.a. Hull and White (1987, 1988), but their stochastic volatility also rejects the wide - sense stationarity part of the i.i.d. assumption. We have to conclude that time - dependence plays empirically a much more important role in the financial markets, than financial analysts have accounted for. The term *horizon analysis* is currently used in the financial markets for the analysis of such time dependencies of financial risk.

3.2 Fractal Market Hypothesis

We can now also provide a simple definition of the Fractal Market Hypothesis (FMH). This definition will provide us with an analytic framework for precise financial market measurements.

Definition 14 ***Fractal Market Hypothesis (FMH)***: *The magnitude of a risky (= random)* *market asset pricing process is both frequency (asset class) dependent and time - dependent and shows global dependencies via its fractality, i.e., via its self - similarities in both the frequency and time domains. Such a process can be homogeneous = mono-fractal (= exhibiting one form of self - similarity), or non - homogeneous = multi - fractal (= exhibiting many co - existing self - similarities).*

We have not yet provided a precise definition of the term fractal in the FMH. Here it is:

Definition 15 *A geometric object is* ***fractal*** *when it has a fractional (= non - integer) dimension D.*

Contrast this definition with that of the well - known Euclidean objects, with which we've become so familiar. because of the classical mathematics education of our childhood.

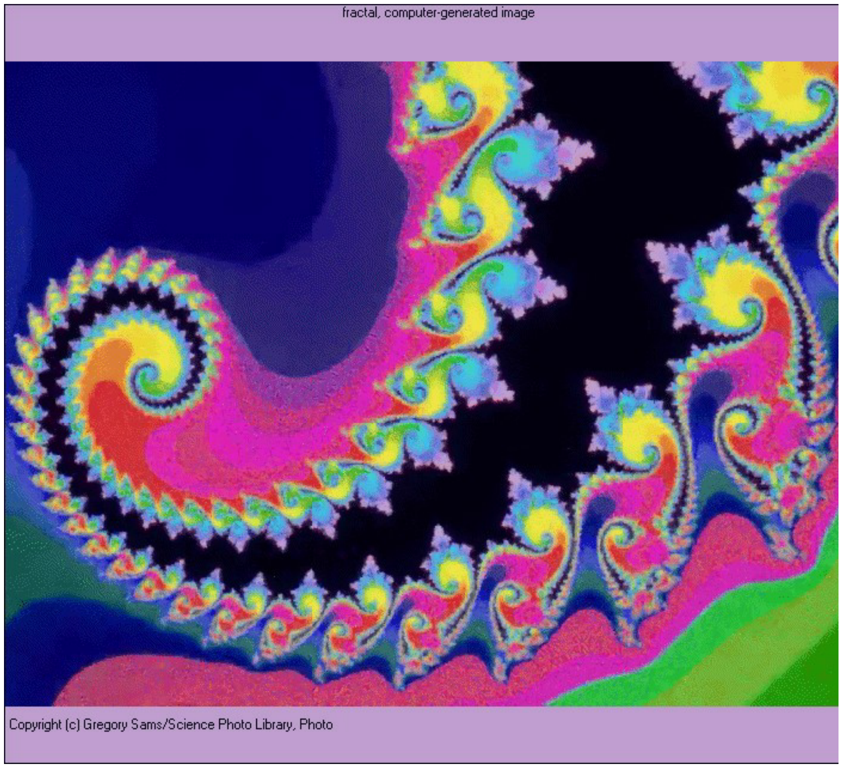
Definition 16 ***Euclidean*** *geometric objects have discrete integer dimensions, such as* $D(point) = 0$, $D(straight\ line) = 1$, $D(plane) = 2$, $D(cube) = 3$.

Euclidean geometry is the study of points, lines, planes, and other geometric figures, using a modified version of the assumptions of Euclid. In contrast, a *fractal* geometric object has a fractal, non - integer dimension. Such a fractal dimension indicates the extent to which the fractal object fills the Euclidean dimension in which it is embedded.

For example, the random process $x(t)$ of rates of return process on the S&P500 stock price index has the measured fractional dimension $D = 1.4$ within the $2-$dimensional (x, t) data plane. Such fractional dimensions can be measured by various methods, such as Range - Scale (R/S) analysis, Roughness - Length (RL) analysis, Variograms, Spectrograms (based on Fourier Transforms) and Scalograms (based on Wavelet Transforms).

Fractal geometry describes objects that are *self - similar* (Mandelbrot, 1982). This means that when such objects are magnified, their parts are seen to bear an exact resemblance to the whole, the likeness continuing with the parts of the parts and so on to infinity, as in the following Fig. 3 of Mandelbrot's famous Julia set.[5]

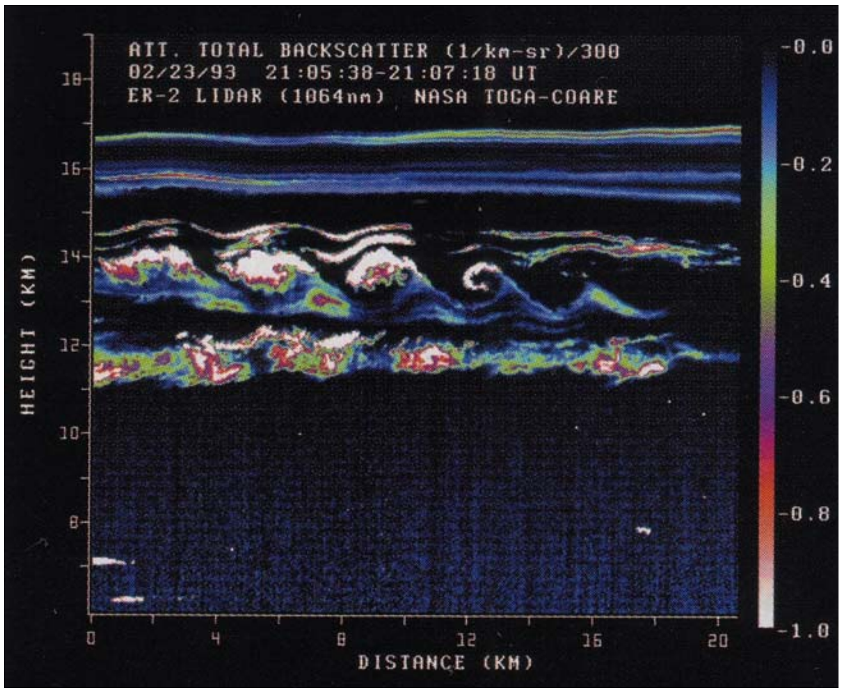
[5] Fractional dimensions were not discussed until 1919, when the German mathematician Felix Hausdorff launched the idea in connection with the small-scale structure of mathematical shapes. Other mathematicans of the time considered such shapes as "pathologies" that had no empirical significance. This attitude persisted until the mid - 20th century and the work of Polish - born French mathematician Benoit Mandelbrot (1924 - present), who developed fractal geometry. Mandelbrot's 1961 study of similarities in large - and small - scale fluctuations of the stock market was followed by work on phenomena involving nonstandard scaling, including the turbulent motion of fluids and the distribution of galaxies in the universe. By 1975, Mandelbrot had developed a complete theory of fractals, and publications by him and others made fractal geometry accessible to a wider audience and the subject began to gain importance in the sciences.



Mandelbrot's Julia set. This set's major element of stability is a budlike shape.

Mandelbrot's Julia set resembles the empirical breaking - wave patterns in clear - air turbulence, due to changes in the wind within and around the jet stream, in Fig. 4 (Clear - air turbulence, *IEEE Spectrum*, September 2000, p. 40). Such empirical turbulence measurements have been made possible by a burst of recent research in technologies for real - time detection of the time-frequency phenomenon of turbulence in all its varieties, like the on-board Doppler laser radar, or lidar, used to produce the multi - colored graph of Fig. 4, predicting various persistence phenomena.

Inspired by this technology and these visualization results, a major research question for me regarding financial market risk is: are such fractal turbulence patterns also observable in financial time - series, such as the rate of return data in various asset markets and in foreign exchange markets? We'll have already collected some evidence there can be, since most currency markets show a degree of anti-persistence that allows for turbulence and we have even already found already some pricing "eddies."



Clear-air turbulence, in which air currents move in strong breaking-wave patterns, shows up starkly in this image observed with a downward-looking Doppler laser radar, or lidar, aboard the NASA ER-2 high-altitude research aircraft.

It is important to emphasize that such fractal processes have two important properties:

(1) Fractals are *scale - symmetric* (= self - similar): the fractal process can be magnified, but keeps the same shape. In financial risk - theoretic terms, a marginal distribution of a fractal speculative pricing process maintains its shape when observations are taken over time horizons of different length.

(2) Fractals are *translational - asymmetric*: the fractal process does not keep the same shape when shifted. Fractals are devoid of translational symmetry, *i.e.*, they don't exhibit the smoothness associated with Euclidean lines, planes, and spheres. Instead, a rough, jagged quality is maintained

at every scale at which an object can be examined (*cf.* Fig. 3). We should keep in mind that there are both upper and lower limits to the size range over which empirical fractal objects are self - similar. Above and below that size range, the shapes are either rough (but not self - similar), or smooth, *i.e.*, conventionally Euclidean.

In financial risk - theoretic terms, a fractal rate of return process is non - periodic, although it does show a form of cyclicity over a particular time range, because of the observed scale - symmetry.

Consider now a finite time series $S(t) = \{x(t1), x(t2), x(t2), ..., x(tT)\}$ where the occurrence of incremental movement is unrestricted with respect to the direction of the movement: it can go up or down and the amplitude of the movements is unrestricted. When a particular trace of this time series is measured with respect to time (along the horizontal time axis), Mandelbrot (1974) shows that the function $S(t)$ is self - affine. Moreover, when the time scale of this function is changed by the ratio $\alpha < 1$, the required change in the amplitude (measured along the vertical axis) of the series is shown to be α^H for such a self - affine function, where the $H-$exponent is the statistic proposed by the hydrologist Hurst.

Definition 17 *The **Hurst exponent** H is defined for a whole set of horizons τ as*

$$0 < H = \lim_{\tau \to \infty} \frac{\ln RS_H(\tau)}{\ln \tau} < 1 \tag{24}$$

*where **Hurst's Range Scale Statistic***

$$RS_H(T) \equiv \frac{1}{c_2^{0.5}} \left[Max_{1 \leq \tau \leq T} \sum_{t=1}^{\tau} [x(t) - m_1] - Min_{1 \leq \tau \leq T} \sum_{t=1}^{\tau} [x(t) - m_1] \right] \geq 0 \tag{25}$$

with empirical mean

$$m_1 = \frac{1}{T} \sum_{t=1}^{T} x(t) \tag{26}$$

and empirical variance

$$c_2 = \frac{1}{T} \sum_{t=1}^{T} [x(t) - m_1]^2 \tag{27}$$

Thus Mandelbrot related the scaling of the amplitude of uncertain time series to the scaling of the time moments at which it is observed. We will now see that the Fractional Brownian Motion time series model is such an affine time series.

3.3 Fractional Brownian Motion

The FBM is, perhaps, the most useful generic research models for a random process currently in existence in the financial markets literature. This random process model encompasses virtually all of the observed empirical phenomena in the time series of financial markets. A recent theoretical paper by Elliott and van den Hoek (2000) discusses the theoretical niceties of the FBM and how it easily extends what we already know about the theoretical GBM into a more realistic direction.

How does the FBM relate to the various investment time horizons in the market? The Fractional Brownian Motion (FBM) is a representative mathematical model for the FMH, given the empirical fact that coexisting financial market participants, operating in the same interlinking financial markets, have different investment horizons. This is more or less acknowledged in the finance literature on the term structure of interest rates, which introduces the theory of bond market segmentation according to maturity "buckets," representing different time horizons, in addition to the theory of rational expectations and liquidity preferences. At the long end of the market one finds more the pension fund and life insurance managers and at the short end retailers and manufacturers looking for easy access to short term cash. But the market segmentation literature suggests that these "maturity buckets" don't commute, while the Fractal Brownian motion model implies they do. Here is how the FBM looks like.

Definition 18 *The **Fractional Brownian Motion (FBM)** is defined by the fractionally differenced time series*

$$(1-L)^d x(t) = \varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \text{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{28}$$

where $x(t) = \ln X(t) - \ln X(t-1) = (1-L)\ln X(t)$. *A completely equivalent representation is that Fractionally Brownian Motion* $x(t)$ *is also fractionally integrated white noise, since*

$$x(t) = (1-L)^{-d}\varepsilon(t),\ d \in (-\frac{1}{2}, \frac{1}{2}),\ \text{with}\ \varepsilon(t) \sim i.i.d.(0, \sigma_\varepsilon^2) \tag{29}$$

Within the range $-0.5 < d < 0.5$, the order of the difference operator d is related to the measurable Hurst exponent H as follows:

$$d = H - 0.5 \tag{30}$$

For serially, or short term dependent time series, such as strong - mixing processes, $H \to 0.5$ when $\tau \to \infty$, but for globally dependent time series $H \to 0.5 + d$. In fact, the fractionally - differenced random processes satisfy the equality $H = 0.5 + d$. Thus one can plot $\ln RS(\tau)$ against $\ln \tau$ to compute H from the slope of the resulting plot. Mandelbrot calls any time series $x(t)$ for which shows the R/S statistic time -scaling, $RS_H(\tau) \propto \tau^H$: "*Hurst noise*."

Remark 19 *Hurst (1951) found that, based on the water-level minima recorded in the period 622 - 1469, the annual water flow of the Nile river in Egypt shows a strong long term persistence with* $H = 0.91$, *that requires unusually high barriers, such as the Aswan High Dam, to contain damage and rein in the floods. For the rivers Saint Lawrence in Canada, Colorado in the USA, and the Loire in France, the persistence is considerably lower with* $0.5 < H < 0.9$. *The river Rhine (at the Swiss - French - German triple point near Basel) is exceptional with a long term exponent of* $H = 0.5$, *indicating that its water flow changes like white noise (Mandelbrot and Wallis, 1969).*

Thus the risk inherent in a market pricing process depends on the different lengths of the investment horizons of the various market participants, who either don't have uniform investment horizons, or have investment horizons than can be easily normalized. In addition, the financial risk depends also on the particular frequency distribution ($\approx$ asset class), as usual.

Using this broader definition of financial market risk, generally expressed in probability distribution terms, we broadly define such "financial market risk" as the time - varying, horizon - dependent *shape* of the frequency distribution of investment returns. This frequency distribution P can be written as a function of the investment asset class categorization ω (for frequency), the particular investment horizon τ, and time period t:

$$\boxed{\text{"financial risk"} = P(\omega, \tau, t)} \tag{31}$$

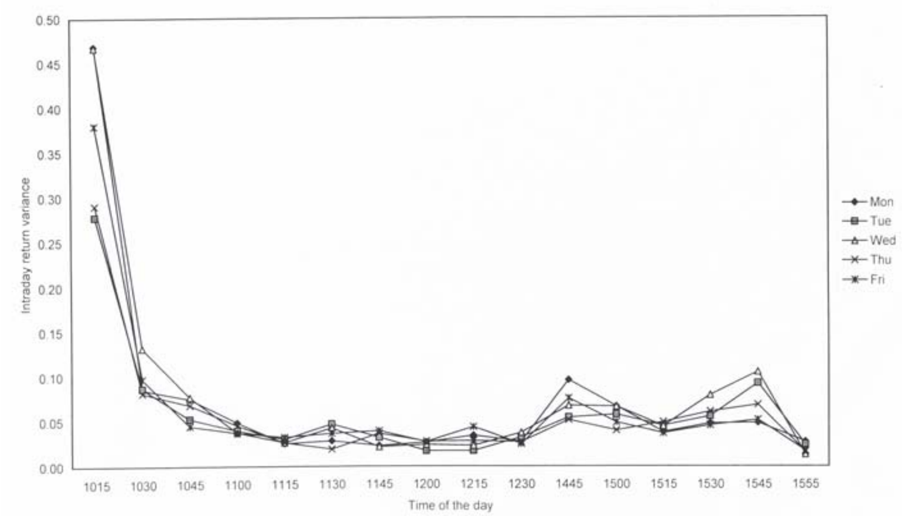
In other words, the most recent measurable concept of financial market risk is not independent and stationary, but it is is time - scaling *and* time - varying.

Example 20 *Sophisticated financial market participants are fully cognizant of the phenomenon that financial risk is frequency -, maturity -, and time - dependent (Harvey and Whaley, 1991; Harvey and Whaley, 1992). For example, real world option market traders use the* ***volatility term structure*** *when pricing options. They recognize that the volatility used to price an at - the - money option depends on the maturity* τ *of the option. They use so - called* ***volatility matrices****, which combine volatility smiles with the volatility term structure to tabulate volatilities appropriate for pricing an option with any strike price and any maturity. Table 1 provides an* *example of such a volatility matrix, which forms a contour plot of the options volatility, and, via the Black - Scholes model, of the options price (Hull, 2001, p. 291).*

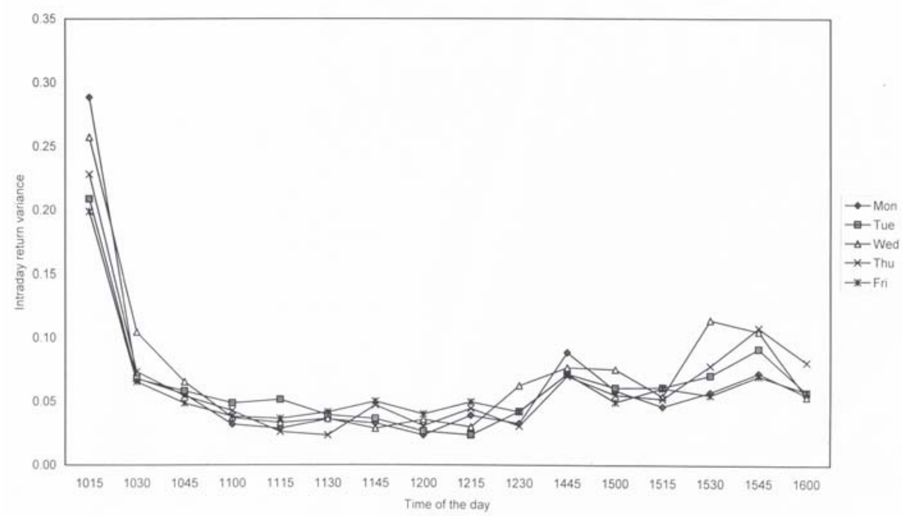
Table 1	***Volatility***	***Matrix***			
		Strike	*Price X*		
Maturity τ	0.90	0.95	1.00	1.05	1.10
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

Volatility tends to be an increasing function of maturity τ*, when, at a particular time* t*, short - dated volatilities are historically low. This is because then there is the expectation that volatilities will increase. Similarly, volatility tends to be a decreasing function of maturity* τ*, when, historically at a particular time* t*, short - dated volatilities are high. Then there is the expectation that volatilities will decrease. Thus the changing term structure of volatilities reflects a nonlinear feedback mechanism in the financial markets based on adjusting expectations.*

Example 21 *Recently more empirical research has been done on the intra-day and intraweek volatility patterns of various stock and futures market indices. Also this research shows the time - dependence of market volatility at least within the time horizon of a day or a week. Usually one tests the null hypothesis of arbitrage equality of the cost - of -carry model between the futures and spot markets,* $H_0 : \sigma_f = \sigma_s$*, which is almost always rejected, against one of two alternative hypotheses based on information asymmetry in the markets: (1) the "wait - to trade" hypothesis,* $H_1 : \sigma_f > \sigma_s$*, or the "noise - traders" hypothesis,* $H_1 : \sigma_f < \sigma_s$*. Using* $15-$ *minute data covering July 1, 1994 to June 28, 1996, Tang and Liu (2001), who provide also a comprehensive literature survey on these issues, find that the first alternative hypothesis is true for all weekdays in Hong Kong: interday data for the Hang Seng Index Futures (HSIF) are more volatile than those of the Hang Seng Index (HSI). However, for the intra-day returns, the HSI is significantly more volatile than the HSIF for the first 15 - 20 minutes after the markets open on all weekdays, except on Mondays. Intra - day and intra - week volatility patterns exist for both markets. Fig. 5 shows the variation of the intra-day variance of the HSI by weekdays and Fig. 6 does the same for the HSIF.*



Intraday return variance of the Hang Seng Index by weekdays based on data for July 1, 1994 to June 28, 1996



Intraday return variance of the Hang Seng Index Futures by weekdays based on data for July 1, 1994 to June 28, 1996

Example 22 *Of course, not all computational results for the valuation of derivatives are sensitive to the time - variation of risk. For example, the* ***put - call parity*** *is true for any set of distributive assumption, since it is based on simple arbitrage. It does not depend on the lognormal assumption underlying the Black - Scholes model (Black and Scholes, 1972). The put - call parity holds true even when the underlying distribution is time - varying, because at a particular moment t the financial risk of a put option, $P(\omega, \tau, t)$, is the same as that of a corresponding call option with the same maturity τ written on the same underlying asset category ω. Most of the research has therefore concentrated on the information content of options to see if that could be used to test if a market was efficient (Black and Scholes, 1972; Galai, 1977; Chiras and Manaster, 1978; Klemkosky and Resnick, 1979).*

Contrast our new time - dependent, horizon - dependent and moment - dependent definition of financial market risk, with the more narrow definition of financial risk, expressed exclusively in terms of an invariant term structure of volatility (= second order moments), as, for example, used in Dumas, Fleming and Whaley (1998):

$$\sigma_\tau = g(\omega, \tau) \tag{32}$$

Batten, Craig and Mellor (1999) provide also an application of this second definition, which allows the measurement of time - invariant volatility scaling, in the financial markets. This second - order definition of financial market risk is dependent on the investment horizon τ, but not on the time period t.

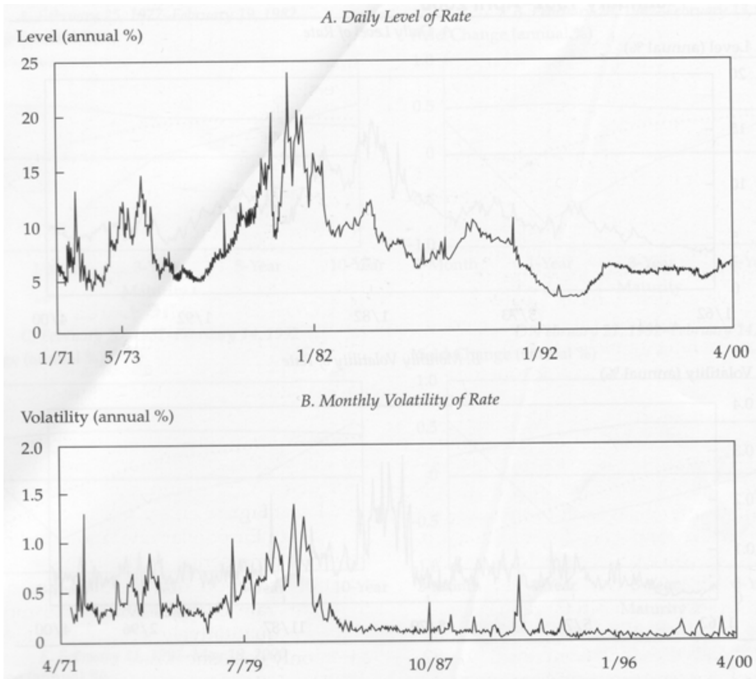
Finally, compare these two definitions with the older and, now considered, very unrealistic classical i.i.d. definition of financial market risk representative of most financial textbooks, as expressed by the simple expression:

$$\sigma = g(\omega) = \text{constant for each asset class } \omega \tag{33}$$

In other words, this older definition assumes strict stationarity and independence of the asset return distributions. Notice that our new definition of financial market risk encompasses both preceding functional definitions and extents the issue from measuring just the second - moment to measuring the shape - determining higher order moments of the financial risk distribution.

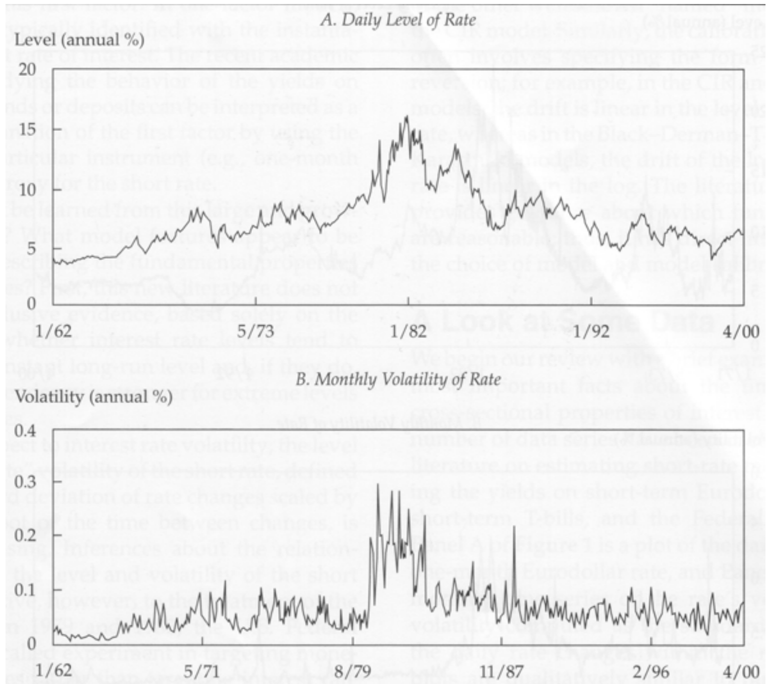
The following two examples shows that the empirically measured financial risk in the term structure of interest rates at various maturities τ is, indeed, time - varying, or dependent on time t.

Example 23 *Fig 7 shows the one - month Eurodollar yield $x_{one-month}(t)$ together with its volatility $\sigma_{one-month}(t)$. The following stylized facts emerge: (1) This short - rate series is a* ***persistent*** *time series. It spends long, consecutive periods above and below the (sample estimate of) the unconditional, or long - run mean. (2) In the 1979 - 82 period, the average level and volatility of the short rate was substantially higher than for other years in the 1971 - 200 period. (3) The volatility of the short - rate level appears to be both time varying and persistent (Chapman and Pearson, 2001, pp. 78 - 79).*



One - Month Eurodollar Yield and Volatility, April 1971 - April 2000

Example 24 *Fig. 8 plots again the level (Panel A) and volatility (Panel B) of the five - year constant - maturity Treasury (CMT) yield from January 1962 through April 2000. This longer maturity series has a lower mean and volatility than the short rate in Fig. 7, the overall movements in levels and volatility are qualitatively similar. In particular, the five - year rate is also a persistent series, and the 1979 - 82 period was characterized by substantially higher yield levels and volatility than any other years in the data set. These observations appear to be also true for CMT yields of maturities from 1 year to 30 years (Chapman and Pearson, 2001, pp. 78, 80).*



Five - Year Constant Maturity Treasury Yield and Volatility, January 1962 - April 2000

Example 25 *Based on the recognition that some volatilities are time - varying and not constant, a recent innovation in the swap markets is the so-called* ***volatility swap*** *(cf. Hull, 2001, p. 407). Just like a* ***vanilla swap*** *exchange floating for fixed payments, a volatility swap exchanges varying volatility or financial risk for fixed volatility or financial risk. Suppose the nominal principal is L. On each payment date, one side in the volatility swap pays* $L\sigma_\tau$*, where* σ_τ *is the historical volatility measured in the usual way, by taking daily observations on the underlaying asset during the immediately preceding accrual period or horizon* τ *and the other side pays* $L\sigma_K$*, where* σ_K *is a pre - specified constant volatility.* ***Variance swaps****,* ***correlation swaps****, and* ***covariance swaps*** *are defined similarly.*

4 DEGREE OF MARKET EFFICIENCY

The identification of the degree of market efficiency is important not only for the accurate identification, measurement, analysis and management of financial market risk, but also for the correct and accurate valuation and pricing of both fundamental and derivative financial instruments. In particular, it is already an important research topic in the leading derivatives markets, which exist specifically to value and price risk. The following discussion, which illustrates this point in abundance, borrows from Hull (2001, pp. 27 - 28 and 64 - 66) and Johnson (1960).[6]

[6] Figlewski (1986) is a good general source for theoretical and empirical issues of hedging with futures.

4.1 Speculators' Forecasts and Normal Backwardation in Futures Markets

Referring to his ow personal negative experience, the financier Cowles (1933) asked if forecasters could predict the stock market.[7] However, it took until Houthakker (1957), before the individual commodity markets were actually empirically analyzed. Houthakker looked at empirical futures prices for wheat, cotton and corn in the period $1937 - 1957$ and he found significant market inefficiencies. He showed that significant profits could be earned by taking long futures positions, suggesting that an investment in an asset has positive systematic risk and consequently the present futures price F_0 systematically under-predicts the expected future spot price of the asset $E\{S_T\}$ at the time of maturity T, as follows.

Finance theory tells us that the present value of the expected cash flows of a risky investment to the speculator, who takes a long position in the futures market, consists of the sum of a certain cash outflow and a, certainty equivalent, expected cash inflow:

$$-F_0 e^{-rT} + E\{S_T\} e^{-xt} \tag{34}$$

where r is the risk free rate of return and x is the discount rate appropriate for the investment, *i.e.*, the expected return required by investors on the risky investment, which depends on the systematic risk of the investment.[8] Assuming that all investment opportunities in an free, competitive securities markets have zero net present value, we have

$$\begin{aligned} -F_0 e^{-rT} + E\{S_T\} e^{-xt} = 0, \text{ or} \\ F_0 = E\{S_T\} e^{(r-x)T} \\ < E\{S_T\} \text{ when } k > r \end{aligned} \tag{35}$$

[7] In 1931 Cowles funded the foundation of The Econometric Society to answer his own question.

[8] According to the celebrated Capital Asset Pricing Model (CAPM). Los (1999) and (2001), Chapter 7, pp. 113 - 122, discusses the systematic under - measurement of the CAPM's beta by the conventional downward least squares projections, which ignore the epistemic uncertainty inherent in the positivity of the covariance matrices based on empirical data.

Both economists Keynes (1930) and Hicks (1939) had already discussed this situation in theoretical terms under the term of *normal backwardation*, when the speculators tend to hold long positions and the hedgers tend to hold short positions. This occurs because speculators require compensation for the systematic market risks they are bearing. While the hedgers lose money on average, they accept this situation because the futures contract reduces their risks.

Of course, theoretically, there are two other cases possible. First, the case of unbiased prediction by futures prices, which would indicated that the futures markets are efficient. If the spot price S_T would be uncorrelated with the level of the stock market, the investment would have zero systematic risk, in which case $x = r$ and $F_0 = E\{S_T\}$. Second, the case where S_T would be negatively correlated with the level of the stock market and the investment would have negative systematic risk. In this case $x < r$, and $F_0 > E\{S_T\}$. In this case, called *cotango* by both Keynes and Hicks, it would be profitable for hedgers to hold long positions and speculators to hold short positions.

Telser (1958) appeared to have empirically found the first case of unbiased prediction. He studied the period from 1926 to 1950 for cotton and from 1927 to 1954 for wheat and found no significant profits for traders taking either long or short positions. His results are corroborated by Gray (1961), who looked at corn futures prices during the period $1921 - 1959$, and by Dusak (1973), who studied more recent data on corn, wheat and soybeans in the period 1952 - 1967.

Dusak calculated directly the correlation of the movements in the commodity prices with movements in the S&P500 and found little or no systematic risk, lending support to the hypothesis of unbiasedness or efficiency. However, from the current perspective of this paper, Dusak's study suffers from the prejudice to look only at linear correlations and to ignore nonlinear dependencies. Moreover, his unidirectional projection method of regression, strongly biased his correlation results downwards, since the level of uncertainty in his data is high.[9] More recent work by Chang (1985),

[9] For a strong scientific critique of the prejudices of unidirectional least squares projection and the resulting systematic downward bias of the correlation results, *cf.*, Los, 2001, Chapters 4 and 5, and Los, 2004.

using data on the same commodities but with more advanced statistical techniques supports the normal backwardation hypothesis, $F_0 < E\{S_T\}$.

Other shortcomings of most of these earlier studies is that they assume that the market return distributions are Gaussian, *i.e.*, symmetric and time independent, and that they are stationary, so that linear correlation studies can be applied. The empirical market reality differs from both these assumptions and these conventional correlation studies can be shown to be scientifically deficient.

5 CONCLUSION: MEASURE THE DEGREE OF MARKET EFFICIENCY

Overall, our own conclusion from our survey of the recent literature is that since empirically more asset prices are positively correlated with the levels of the market indices than negatively correlated, normal backwardation should be expected to be prevalent in the futures markets. The empirical financial markets are just not as efficient as is theoretically assumed in the financial textbooks. The financial markets are not black or white. They are not inefficient or efficient. The financial markets exhibit a continuum of different degrees of efficiency. Since this is the empirically observed case, financial research has to concentrate on different issues.

The current research questions try to refocus financial analysis: what is the character and degree of dependence in the markets - in other words, what is the *degree of efficiency* in the various markets - and what is the degree of the resulting prediction bias in the derivatives markets? These research issues are important for the derivatives markets, as testified by the research of, for example, Rendleman and Carabini (1979), Bhattacharya (1983), Klemkosky and Lasser (1985) and Harvey and Whaley (1992).

But is also clear, at least to this author, that a lot of these classical empirical studies should be redone with current up - to - date measurement concepts and technology to correct the errors now conventionally adopted in finance textbooks, in particular the errors of unidirectional least squares projection technology.[10]

We make the following suggestions for improvement in the scientific investigation of these important financial market issues. First, issue of how to identify the degree of market efficiency. The frequency distributions of the various asset classes must be made more realistic than the usual Gaussian assumption, *i.e.*, approximating closer to empirical reality, by introducing non - Gaussian, in particular skewed and heavy - tailed stable distributions. However, it has already been observed that this elegant statistical approximation approach has substantial scientific shortcomings, because the degree of approximation becomes a subjective choice. That's unacceptable in science.

Second, financial analysts must become more aware of the various investment horizon dependencies of these frequency distributions and how we can possibly analyze them in a scientific, non - approximation fashion. The conventional finance theory takes that every investor in the economy has the same investment horizon, but this is inconsistent with the economic life- cycle theory of consumption.

Third, financial analysis must move from the (assumed) stationary investment return processes of, say, classical option pricing, via non - stationary rate of return processes, to the transient rate of return processes and ultimately, perhaps to the theory of series of singularities, which are not necessarily evenly spaced throughout time and which represent essentially unpredictable, but still "characterizable" financial market risks. Some of these possible return innovation series can be characterized as the Cantor "dust" of Mandelbrot (1982).

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[10] Plus all its algebraically equivalent offshoots, such as maximum likelihood estimation, principal component analysis, etc.

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