A Continuous-Time Asset Pricing Model with Boundedly Rational Heterogeneous Agents

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Abstract

Asset prices are forward looking. This evidence implies that prices of financial assets are essentially determined by the traders expectations about future prices. Another evidence about asset prices is that these do not seem to follow a predictable pattern over time; we observe periods of high volatility, periods of large negative returns and periods of observation clustering, without noticing any kind of regular pattern. How can one conciliate the formation of expectations with unpredictable erratic behavior?

The 'routes to randomness' strand of literature has tried to answer the previous question in the last few years. Two conditions are essential to explain asset price unpredictability. (1) agents have different beliefs about future prices, (2) agents follow a rule of bounded rationality, under which they can change the way they form expectations, but such change does not occur instantly and permanently. In this paper the bounded rationality heterogeneous agents setup concerning asset prices is adapted to a continuous-time framework and general conditions conducting to erratic price behavior are presented and discussed.

Keywords: Heterogeneity; Bounded rationality; Asset pricing; Expectations; Stochastic Optimal Control.

JEL classification: C61, G12

I. INTRODUCTION

The 'routes to randomness' literature, that has been developed having as a reference the pioneer work by Brock and Hommes (1997, 1998), attempts to conciliate the two main views concerning economic fluctuations – new classical and Keynesian views – and the underlying rules relating to the formation of expectations.

Many observable economic variables, as asset prices or consumption levels do not display a steady long run behavior, that is, neither they assume constant values or grow at constant rates. There are fluctuations, that under a traditional economic analysis perspective should be modelled as random shocks to fundamentals; in this sense, there is a fundamental long run value for each economic variable to which the economic system converges over time. An alternative view, of Keynesian inspiration, sustains that even in the absence of external disturbances, the motion of economic variables is driven by nonlinear rules. The first approach relies on the central concept of rational expectations while the second is in part an animal spirits interpretation of human behavior.

Rational expectations are present in most of the explanation of economic behavior, for instance relating macroeconomic policy or asset pricing theory. Although they are a benchmark, rational expectations are an oversimplified explanation of human behavior. As a result, research along the past few years has focused on exploring alternative expectation formation rules, as documented in Grandmont (1998), Sargent (1999) and Evans and Honkapohja (2001). Two concepts play a central role on a more sophisticated look upon expectations: behavior heterogeneity and bounded rationality.

Certainly agents have different degrees of information, different life experiences and different attitudes towards risk. These and other differences have always been in the mind of economists, but under rational expectations one takes as granted that heterogeneous beliefs that surely exist average out due to aggregation. To hold in the long run, initial heterogeneous beliefs or expectations must be associated with our second feature: the absence of a fully rational behavior. In the real world, individuals do not make automatically the choice for the best strategy; they have to evaluate and learn about different options. Brock and Hommes (1998) and subsequent literature use the term 'Adaptive Belief System' to acknowledge the fact that individuals may have some initial belief that can be changed over time according to accomplished results, but this change probably takes time, i.e., individuals may not react immediately to a worse result in the choice of strategy. Adaptive beliefs produce a less straightforward path to the fundamental long term state than rational expectations, and such state may not even be reached in the sense that a non linear long run result can subsist.

Routes to chaos are present in numerous models in various fields of economic analysis. For example, Benhabib, Schmitt-Grohé and Uribe (2001*a*, 2001*b*, 2001*c*) analyze strange dynamics associated with monetary policy and the so called Taylor (1993) rules relating interest rates; Tuinstra and Wagener (2003) focus the analysis of chaotic motion in an overlapping generations model under which households predict future inflation rates; Azariadis and Kaas (2002), in turn, find equilibria with unusual properties for a standard intertemporal consumption utility maximization model where the optimization problem is constrained by a dynamical budget equation. Though, perhaps the field with a wider and more systematic approach to non linear dynamics is asset price modelling. Hommes (2001), Brock, Hommes and Wagener (2001), Brock and Hommes (2002), Hommes, Sonnemans, Tuinstra and van de Velden (2002), Gaunersdorfer, Hommes and Wagener (2003), Chiarella and He (2001, 2002*a*) and LeBaron (2000), are good examples of the work under progress relating to how a simple standard pricing model is able to lead to complex dynamics that make it extremely hard to predict the evolution of prices in asset markets.

The recent literature on asset pricing, building upon heterogeneity and bounded rationality, aims to describe asset prices observed fluctuations over time even in the absence of stochastic disturbances. Under this framework, a purely deterministic model appears as a reasonable setup to explain the main financial market features, as the ones presented in Franses and van Dijk (2000), which include the evidence of periods of high volatility, periods of large negative returns, clustering of observations and other kinds of erratic behavior.

The asset pricing adaptive beliefs theory focus the idea of heterogeneity on the observation that typically two kinds of traders exist in financial markets. The first, fundamentalists, believe that asset prices converge to the long term fundamental rational expectations value, which is given by the expected discounted sum of future dividends. The second group is composed by technical analysts, chartists or trend extrapolators, who base their expectations on recently observed past prices, thus extrapolating historical patterns. A classical interpretation of such a system implies that individuals rapidly learn with their mistakes and consequently only fundamentalist rational expectations traders tend to prevail in a long run market scenario. Under bounded rationality, there will be an irregular switching between phases where fundamentalist beliefs dominate the market and phases where prices deviate from the fundamental value as technical analysts sometimes perform better than the other group. The absence of linear dynamics arises in this way as an endogenous phenomenon that is caused (or at least amplified) by the behavior of traders, given that these have different expectations and that the expectations change under an adaptive or evolutionary learning process.

The financial asset market will be described in the following sections, using a continuous time model, where a typical consumption utility optimal control problem is developed. The aim is to investigate how heterogeneous behavior in asset markets together with the previously cited evolutionary learning mechanism, based on a bounded rationality assumption, has impact over the long term equilibrium concerning the macroeconomic choice among consumption and savings. A stochastic intertemporal optimization model is considered, following Merton (1969); this model allows for a deterministic relation over time between consumption and wealth accumulation, which in turn can be associated to asset market behavior. A fully deterministic differential equation system is constructed, allowing to jointly analyze time paths of the economy's consumption-wealth ratio, different traders consumption-wealth ratio, the economy's shares of investment in risky and risk free assets, the different trader types shares of investment in risky and risk free assets and the evolution of asset prices over time.

In short, our task consists on combining the asset pricing features relating bounded rationality and heterogeneous agents with a trivial stochastic utility maximization problem.

The remainder of this paper has the following contents. Section II presents the basic structure of a continuous time asset pricing model under the adaptive beliefs setup. Section III solves the utility maximization intertemporal problem using Pontryagin's principle. Section IV discusses the conditions under which the built model implies an erratic and unpredictable time path for the price of some asset. Finally, section V displays the most relevant conclusions.

II. Asset Pricing, Beliefs Heterogeneity and Bounded Rationality

Consider a financial market where two investment choices are available. Investors split their savings between a risk free asset and a risky asset. Associated to the sure asset is a fixed rate of return, $r>0$, while the expected return on the risky asset corresponds to a variable rate, $a(t)>r$, dependent on the price per share, $p(t)$, and on dividend at time t, $y(t)$. The return on the risky asset is presented as the sum of a dividend yield with the growth rate of the asset price, that is,

$$a(t) = \frac{y(t)}{p(t)} + \frac{\dot{p}(t)}{p(t)} \tag{1}$$

The two rates of return will be recovered in the next section, where a consumption-wealth dynamic model is developed. In that model, an intertemporal wealth constraint reveals the choices concerning the investment on risky and risk free assets. For now, we concentrate on the dynamics of the price per share variable, adapting the Brock and Hommes (1997, 1998) setup. As highlighted in the introduction, this setup is based upon two main features: heterogeneous agents and bounded rationality.

Our heterogeneity assumption is that individuals have different beliefs about future asset prices.[1] This goes against the basis of the rational expectations hypothesis, under which prices move to the fundamental long term price that is constant and that will be defined, according to (1), as the ratio between the dividend and the return rate. Asset price changes in a long run perspective would be solely determined by unexpected changes in dividends or some kind of random shock. The heterogeneous beliefs literature reviewed in the introduction points to another direction. Well informed fundamentalist traders co-exist with trend extrapolators that base their market forecasts on past prices, or past price variation. In contradiction with Friedman (1953) belief that rational traders will drive irrational trend followers from the markets, the Brock and Hommes (1997, 1998) strand of thought makes the persistence of behavior heterogeneity a key argument to the explanation of variable $p(t)$ evolution over time.

[1] We could consider that individuals also have different beliefs about future dividends. To simplify, we shall assume that there is belief homogeneity concerning dividends and that the stochastic dividend process underlying variable $y(t)$ is an IID process with constant mean $E[y(t)] = \bar{y}$. If $a(t)$ is defined as the expected value of the risky asset return, variable $y(t)$ in equation (1) must, then, be replaced by $\bar{y}$.

One may adopt the simplistic distinction between fundamentalists and technical analysts or chartists, or, otherwise, consider a continuum of distinct trading strategies on a $(0,H]$ set. Let us assume that an unknown but large number of traders have different beliefs about which will be the long term asset price. That is, let $E_h(p)$, $h\in(0,H]$, to be the expected price of the asset for each type of agent, relating some future moment.

By itself, agent heterogeneity is unable to explain asset pricing behavior, because it is not realistic to assume that traders stick with a price belief if this performs badly in what concerns returned results. It makes sense to recognize that individuals act rationally and thus, in the presence of better forecasting rules than the adopted one, agents will change their behavior. The key point in our argument is that individuals are not seen as fully rational but instead they follow some kind of bounded rationality rule.

An adaptive belief system based on the discrete choice model developed in McFadden (1973), Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993) puts together the heterogeneity concept presented above and a rule for changing trading strategies. A discrete choice model allows to quantify how the fraction of individuals attached to each trading strategy or forecasting rule changes over time.

Let $n_h(t)$ be the percentage of traders that follow rule $h\in(0,H]$. These fractions are updated through the assumption of a performance measure or fitness function $\tilde{U}_h(t)=U_h(t)+\varepsilon_h(t)$, where $U_h(t)$ is the deterministic part of the function and $\varepsilon_h(t)$ is a noise term that is IID across strategies. Discrete choice theory points to the following fraction as the probability that an agent chooses strategy h (in the limit, as the number of agents tends to infinity),[2]

$$n_h(t)=\frac{e^{\beta.U_h(t)}}{\int_0^H e^{\beta.U_h(t)}.dh} \tag{2}$$

Parameter β plays a fundamental role. It is called the intensity of choice and it measures the sensitivity of traders to the differences in performance of the various trading strategies. This parameter is related to the noise term $\varepsilon_h(t)$ in the sense that the larger the noise the lower will be the value of β, meaning a smaller probability of

[2] To be precise, discrete choice theory works with a version of (2) where h is not defined in an interval $(0,H]$ but in discrete terms: h=1, …, H. The continuous version analogue in (2) is explored in detail by Dagsvik (1994, 2002) and Diks and van der Weide (2003).

traders to change their strategy even if this performs not as well as other strategies (what is measured by the fitness function U_h). In the extreme case β=0, fractions $n_h(t)$ are fixed over time, meaning that there is an infinite noise variance that do not allow for any adjustment process; the other limit case, $\beta\to\infty$, translates the situation without any noise, where all individuals may immediately choose the optimal forecast, what implies homogeneous rational expectations. Assuming that β is some finite and different from zero value, agents will be boundedly rational in the sense that some noise prevents the immediate choice of the best performance strategy. Accordingly, an increase in the choice parameter value represents an increase in rationality or, in other words, $1/\beta$ may be interpreted as the propensity of agents to fail in choosing the best trading strategy.

Hirshleifer (2001) makes a detailed presentation of what means different values of the intensity of choice parameter. Basically, this author distinguishes between, on one hand, conservatism behavior (β small) which can be explained on the grounds of overconfidence or difficulty to react under costly information processing and, on the other hand, overreaction or herd behavior (β high) which may involve as well rationality deficiencies in the sense that frequently it is easier to 'follow the mob' than to wait and see if the momentum successful trading strategy is a long run well succeeded strategy.

The fitness function represents how well the chosen *h* strategy has performed. Performance is associated with accumulated realized profits or realized wealth return. To present this function expression, we rely on the Chiarella and He (2002*b*) presentation around this subject. An expression similar to the following is there presented,

$$U_h(t) = \phi(t) + \eta . U_h(t - \Delta t) \tag{3}$$

In (3), $0 \leq \eta \leq 1$ is a memory parameter that measures how past realized fitness is discounted for the selection of a strategy. The higher the value of η, the lower the impact of past fitness over present choices. In continuous time we define $U_h(t - \Delta t) \equiv U_h(t) - \dot{U}_h(t)$. Function $\phi(t)$ is the expected wealth return on the risky asset at time *t*, that corresponds to

$$\phi(t) = z_h(t).E\left[\frac{dw_h(t)}{w_h(t)}\right] \qquad (4)$$

where $z_h(t)$ is the share of wealth of trader type h invested in the risky asset at each time moment [correspondently, 1- $z_h(t)$ will be the share of wealth allocated to the sure asset] and $w_h(t)$ represents the wealth of an agent of type h in each t moment.

Now we rewrite equation (3) as a differential equation describing the motion of the performance measure,

$$\dot{U}_h(t) = \frac{1}{\eta}.z_h(t).E\left[\frac{dw_h(t)}{w_h(t)}\right] - \frac{1-\eta}{\eta}.U_h(t) \qquad (5)$$

Two points are worth noting with respect to (5). First, the dynamics of wealth, necessary to solve (5), are discussed along section III; second, the negative sign associated to the second term on the right side of the equation means a convergence process where high fitness values contribute to the decay of U_h to the steady state value.

At last, we have the necessary information to understand how the aggregate stock price evolves over time. Given the existence of various possible strategies in the interval (0,H] and given that individuals are allowed to change how they predict prices over time in a boundedly rational way, the time path of the stock price is the result of the following differential equation

$$\dot{}(\ ) = \left[\int_0 \quad (\ ). \quad (\ ). \quad \right]. \ (\ ) \quad - \qquad (6)$$

with $a_h(t) = \frac{\bar{y}}{E_h(p)} + \frac{[\dot{E_h}(p)]}{E_h(p)}$. Expression (6) is similar to (1), with the expected return given as a weighted average of the expected return of each trader type h.

A rigorous analysis of the dynamics underlying equation (6) is not easy to undertake. Only through numerical simulation we could sketch some of the main features concerning the model dynamics. This is not attempted here; we will just present the conditions under which the persistence of an unpredictable price time path is likely to occur. In section IV, this discussion is undertaken. Previously to that, we have to look to wealth dynamics, the central topic in section III. Assuming, for each trader type *h*, a usual intertemporal consumption utility optimization model, we will be able to endogenously determine the wealth shares invested on the risky and on the risk free asset and to explicitly introduce volatility concerning the investment on the risky asset. Basically, we shall revisit the Merton (1969) stochastic consumption-wealth model.

III. THE MERTON MODEL AND THE LONG-RUN CONSUMPTION-WEALTH RATIO

We now develop the Merton (1969) stochastic consumption utility optimal control model, following the presentation in Kamien and Schwartz (1991). Consider a CRRA utility function where $c_h(t)$ represents the consumption level of an individual that has chosen the asset pricing strategy *h*. Parameter $\theta>1$ is a risk aversion parameter.

$$U[c_h(t)] = \frac{c_h(t)^{1-\theta} - 1}{1-\theta} \tag{7}$$

An agent of type *h* maximizes the expected value of an infinite flow of utility functions, assuming a $\rho>0$ discount rate,

$$Max\ \ E\left\{\int_0^{+\infty} U[c_h(t)].e^{-\rho.t}.dt\right\} \tag{8}$$

The optimization problem (8) is constrained by an equation of motion that represents the accumulation of wealth. As stated in the previous section, individuals invest part of their wealth in a risky asset and the other part in a risk free asset. As before, let $z_h(t)$ represent the fraction of wealth reinvested in the risky asset, $a_h(t)$ the expected return on the risky asset, as defined along with equation (6), and *r* the rate of return of the sure asset. The equation of motion representing the accumulation of wealth is

$$\begin{aligned} &dw_h(t) = [r.(1-z_h(t)).w_h(t) + a_h(t).z_h(t).w_h(t) - c_h(t)].dt + z_h(t).w_h(t).\sigma.dB(t), \\ &w_h(0) = w_{h0} \text{ given.} \end{aligned} \tag{9}$$

Equation (9) displays two components. The deterministic component shows that wealth grows with the investment in risk free and risky assets and that wealth declines with the consumption level. The second component is a stochastic one and reflects the volatility associated with the investment in the risky asset. Parameter $\sigma>0$ is a standard deviation parameter and variable $B(t)$ is a Wiener process or Brownian motion, that is, a stochastic process on some probability space with the following properties: $B(t)$ is a continuous function in t, the increments of the process are independent random variables and these same increments are normally distributed with zero mean and a constant variance.

To solve maximization problem (8) subject to budget constraint (9) we apply a stochastic version of the Pontryagin principle.[3] A current value Hamiltonian function takes the form

$$\aleph\left[w_h(t), c_h(t), z_h(t)\right] = U\left[c_h(t)\right] + p\left[w_h(t)\right].f_h + \frac{1}{2}.p_w.g_h{}^2 \tag{10}$$

In (10), $p[w_h(t)]$ is a co-state variable or shadow-price of $w_h(t)$ and p_w represents the derivative of this function in order to the wealth argument. The f_h and g_h functions correspond to the deterministic and stochastic parts of (9), respectively,

$$f_h = r.(1 - z_h(t)).w_h(t) + a_h(t).z_h(t).w_h(t) - c_h(t) \tag{11}$$

$$g_h = z_h(t).w_h(t).\sigma \tag{12}$$

The necessary optimality conditions will allow to find an equation of motion for consumption. Recall that $c_h(t)$ and $z_h(t)$ are control variables in the sense that the correspondent values are determined in order to maximize utility and, thus, first order conditions include

$$\aleph_c = 0 \Rightarrow c_h(t) = p\left[w_h(t)\right]^{-1/\theta} \tag{13}$$

[3] See Malliaris and Brock (1985), page 112, for a statement of the Pontryagin stochastic maximum principle.

$$\aleph_z = 0 \Rightarrow z_h(t) = -\frac{a_h(t) - r}{\sigma^2 . w_h(t)} . \frac{p[w_h(t)]}{p_w} \tag{14}$$

Note from (14) that the derivative of the shadow-price in order to wealth must be a negative value in order to guarantee a positive value for the fraction $z_h(t)$. Under the Pontryagin principle the co-state variable satisfies the following stochastic differential equation,

$$\begin{aligned} & dp_h(t) = [\rho . p_h(t) - \aleph_w].dt + g_h . p_w . dB(t) \Rightarrow \\ & \Rightarrow dp_h(t) = (\rho - r).p_h(t).dt - \frac{a_h(t) - r}{\sigma}.p_h(t).dB(t) \end{aligned} \tag{15}$$

The transversality condition $\lim_{t \to +\infty} p_h(t).e^{-\rho.t}.w_h(t) = 0$ also applies.

Given (13) and (15), to obtain a stochastic differential equation for consumption is a straightforward task through the application of Itô's lemma,

$$dc_h(t) = \frac{1}{\theta}.\left[r - \rho + \frac{1}{2}.\frac{1+\theta}{\theta}.\frac{[a_h(t) - r]^2}{\sigma^2} \right].c_h(t).dt + \frac{1}{\theta}.\frac{a_h(t) - r}{\sigma}.c_h(t).dB(t) \tag{16}$$

To further develop the model we have to define the steady state; this can be presented as the long run locus with the following properties:

- fraction $z_h(t)$ assumes a constant value;
- variables $c_h(t)$ and $w_h(t)$ grow at constant rates;
- the ratio $\psi_h(t) \equiv c_h(t) / w_h(t)$ is a constant value.

The ratio variable, $\psi_h(t)$, allows to present a specific functional form for the shadow-price function [given (13)],

$$p[w_h(t)] = [\psi_h(t).w_h(t)]^{-\theta} \tag{17}$$

The first-order derivative of (17) is

$$p_w = -\theta . \psi_h(t)^{-\theta} . w_h(t)^{-\theta - 1} \tag{18}$$

Expressions (17) and (18) constitute the necessary information to present an explicit value for fraction (14),

$$z_h(t) = \frac{a_h(t) - r}{\sigma^2.\theta} \tag{19}$$

From (19) it is straightforward to understand that in the absence of disturbances on the rate of return of risk free assets, on the variance parameter or on the risk aversion parameter, the fraction of wealth allocated to the stock market varies with the correspondent expected return. Note that $0 \le z_h(t) \le 1$ and thus $r \le a_h(t) \le \sigma^2.\theta + r$. With (19) we simplify the budget constraint expression, that reduces to

$$dw_h(t) = \left[\left(r + \frac{[a_h(t) - r]^2}{\sigma^2.\theta}\right).w_h(t) - c_h(t)\right].dt + \frac{a_h(t) - r}{\sigma.\theta}.w_h(t).dB(t) \tag{20}$$

A generalized Itô's lemma can be applied over variable $\psi_h(t)$, given (16) and (20), to obtain an equation of motion for this variable with a constant steady state value. The remarkable point underlying such equation is that two stochastic differential equations give rise to a fully deterministic relation. Some computation leads to

$$\dot{\psi}_h(t) = \left\{\psi_h(t) - \frac{\theta - 1}{\theta}.r - \frac{1}{\theta}.\rho - \frac{1}{2}.(\theta - 1).\frac{[a_h(t) - r]^2}{\sigma^2.\theta^2}\right\}.\psi_h(t) \tag{21}$$

As it is clear from (21) a long term constant consumption-wealth ratio for trader type *h* is achieved for a steady state constant risky asset return.

We are in conditions to put together the consumption-wealth dynamics sketched in this section for trader type *h* and the bounded rationality scenario of section II. The link between the two is essentially found in equation (5), where the performance measure evolution over time appears as depending on the expected growth rate of wealth. Combining (20) with (5), the fitness function differential equation reduces to

$$\dot{U}_h(t) = \frac{1}{\eta}\cdot\frac{a_h(t)-r}{\sigma^2.\theta}\cdot\left[r + \frac{[a_h(t)-r]^2}{\sigma^2.\theta^2} - \psi_h(t)\right] - \frac{1-\eta}{\eta}.U_h(t) \tag{22}$$

A system of equations, that includes equation (6), equations for the change in trader type h price prediction, equations (21) and equations (22), in these three last cases for $h \in (0,H]$, will lead to the determination of the time paths of the several variables, namely $p(t)$, $E_h(p)$, $\psi_h(t)$ and $U_h(t)$, $h \in (0,H]$. Recall also that $a_h(t)$ and $n_h(t)$ are defined along with (6) and in (2), respectively. In section II it was made clear that individuals change asset trading strategies and such a fact probably implies strange dynamic behavior. The next section discusses the conditions under which such strange dynamics arises.

IV. CONDITIONS FOR INSTABILITY

The system that was constructed along sections II and III includes a large set of variables that makes it difficult to proceed with a clear analysis about its dynamics and particularly to understand what kind of dynamics underlies the evolution of the asset price variables, $p(t)$. Furthermore, we have not presented explicit rules concerning future price expectations, $E_h(p)$. Therefore, we will not solve the model in search for strange dynamics. Alternatively, we will intuitively discuss the properties to which expectations $E_h(p)$ should obey for an erratic behavior of the time path of $p(t)$ to be encountered.

Our choice model states that in a universe of many asset traders, any of the traders has a choice about how it expects the asset price to evolve. If one of such expectations reveals to be always the best choice in terms of wealth growth, then the individuals have the possibility to change the way they form expectations and everyone will end up by adopting the expectation rule that best performs. In this way heterogeneity is eliminated because $n_h(t)$ will be equal to one for the best h strategy and equal to zero in every other case. Under these circumstances, the price of an asset would converge to a long run constant fundamental price.

To find an unpredictable price movement one has first of all to find an unpredictable movement for each $n_h(t)$ share. This share will exhibit an unpredictable time evolution if the several trading strategies perform in some moments (but not in all) better than the others. This is true for a market where fundamentalists and trend extrapolators co-exist. In some periods of time the price approaches its fundamentals (the present value of future dividends), and in this situation individual traders will left the chartist group (in a discrete choice boundedly rational kind of way) to become fundamentalists. But in other periods, speculative bubbles may arise making a result that deviates from the fundamental one to depart further more. In this case, individuals will change from the fundamentalists group to the trendists group.

The referred dynamics tends to reflect with some accuracy what really is found in financial markets – individuals tend to form expectations according to perceived results and they will change their expectations, according to bounded rationality rules, if strategies followed by others tend to perform better. Traders may believe that the long run price of an asset is a fundamental one, but if the other kind of expectation proves to lead to higher income, then there is a change. This willingness to change (in a sluggish not completely rational way) is combined with a market behavior that indeed alternates between situations where a higher than the fundamental price leads to an expectation of fall with situations where the opposite succeeds.

Financial markets are in this way markets in which the best price expectation in one moment, $E_h(p)$, does not tend to prevail forever as the best price expectation. Thus, fitness functions $U_h(t)$ will intercept frequently over time, as one of the strategies tends to outperform the others in a specific period. The only way in which $n_h(t)$, $h\in(0,H]$, will exhibit an erratic behavior is in the circumstance where functions $U_h(t)$ will translate in different time moments different relative performance of each strategy.

We have argumented that the coexistence of periods where high (or low) asset prices lead to a dominance of expectations about an increase in prices with periods where the opposite happens imply that fitness functions will not rank equally in time and consequently each share n_h is systematically changing over time. To reach the asset price variability result over time just one more step needs to be added. Look at expression (6). The variation of the price of a financial asset is determined by the weighted sum of the expected return concerning each trading strategy. If the weights, the $n_h(t)$ shares, vary constantly over time then there is not a deterministic path for the time evolution of $p(t)$.

The unpredictability of $p(t)$ has, according to the previous reasoning, as remote cause the fact that no one knows in each time moment which expectation about future prices will perform best. The prevalence of a chartist result in some initial time period may lead to a price dynamics completely diverse from the case where the fundamental result expectation implies a higher wealth return; in this way, one may talk about a chaotic behavior for $p(t)$ – the initial conditions conditionate all the future pattern of price evolution. Furthermore, if one of the two (or more) expectation rules performs better than the other in periods with unsystematic time lengths we will certainly find periods of high volatility and periods of low volatility in a same asset price time series.

The undertaken discussion is synthesized in the following argument:

- In the presence of heterogeneous expectations and bounded rationality concerning expectation changes, the absence of a behavior rule that systematically performs better than all the other implies that the share of individuals linked to each rule will never be a constant value. Consequently, the price of a financial asset, that depends on expectations about future prices, will follow a time path that is erratic and impossible to predict.

V. Conclusions

Three facts about financial markets motivate the 'routes to randomness' literature. First, asset prices vary in an erratic, unpredictable way over time. Second, individuals have different beliefs about how asset prices will evolve. Third, asset prices vary due to changes in expectations relating future prices. The second and third facts should explain the first.

In the presence of homogeneous beliefs or heterogeneity that do not tend to be perpetuated over time, the price time path would be a random walk, that is, only random shocks would make the price to deviate from its fundamental and no deviation tends to persist over time. Therefore, building a framework where belief heterogeneity does not erode with time may be the answer to explain the erratic pattern of asset prices evolution. A market where individuals can change their expectations about future prices but there is no tendency to one of the expectation rules to be in all moments the best one implies long run heterogeneity. Long run expectations heterogeneity, combined with systematic changes in the shares of agents following one of various expectation rules, is in this case the key to justify that asset prices do not follow a predictable pattern.

In this paper, the important Brock and Hommes (1997, 1998) interpretation of asset price movements was adapted to a continuous-time framework. The continuous-time setup allowed for assembling the discrete choice mechanism regarding bounded rationality with a simple model of stochastic utility maximization. The optimization problem was solved in order to find the dynamics of wealth and consumption. Looking at equations (20) and (16) one observes that the existence of a risky asset in the agents portfolio introduces a stochastic term in each of the equations; furthermore, because individuals have distinct expectations about asset prices, the return that they obtain from their savings applications differ, what has implications over wealth and consumption dynamics. For each type of expectation / belief we have also concluded that a fully deterministic dynamic equation explains the time evolution of the consumption-wealth ratio.

Given the optimization framework with expectation heterogeneity, individuals have different levels of wealth and different levels of consumption just because they choose distinct rules to apply their savings in financial markets. If they want to maximize their consumption and wealth values they must change to the best investment strategy, and in fact the described model predict that they do this. Trying to stay close to reality, we state that individuals do not change their behavior immediately, because this change has certainly some costs. A fitness or performance function will accumulate previous results in order to help individuals form their expectations. If one trading rule performs systematically better, an equation like (22) will give a reason for the change to that strategy, and thus we might say that individuals are rational but simultaneously suspicious about momentaneous good results.

The key argument for an unpredictable result for the asset price time trajectory (and also for the trajectory of the individuals consumption and wealth) is that the number of individuals choosing each strategy in each moment is never equal. In this way, an explanation for the unpredictability of price movements must have as ingredients not only the heterogeneity of agents but also the constant variation in the number of agents within each behavioral group.

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