Multifractal Spectral Analysis of the 1987 Stock Market Crash

Cornelis A. Los* and Rossitsa Yalamova**
*Kent State University, Department of Finance, BSA416, Kent, OH, 44242-0001, Email: clos500@cs.com
**University of Lethbridge, Faculty of Management, Finance, E520 University Hall, Canada T1K 3M4
Email: rossitsa.yalamova@uleth.ca

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Abstract

The multifractal model of asset returns captures the volatility persistence of many financial time series. Its multifractal spectrum computed from wavelet modulus maxima lines provides the spectrum of irregularities in the distribution of market returns over time and thereby of the kind of uncertainty or "randomness" in a particular market. Changes in this multifractal spectrum display distinctive patterns around substantial market crashes or "drawdowns." In other words, the kinds of singularities and the kinds of irregularity change in a distinct fashion in the periods immediately preceding and following major market drawdowns. This paper focuses on these identifiable multifractal spectral patterns surrounding the stock market crash of 1987. Although we are not able to find a uniquely identifiable irregularity pattern within the same market preceding different crashes at different times, we do find the same uniquely identifiable pattern in various stock markets experiencing the same crash at the same time. Moreover, our results suggest that all such crashes are preceded by a gradual increase in the weighted average of the values of the Lipschitz regularity exponents, under low dispersion of the multifractal spectrum. At a crash, this weighted average irregularity value drops to a much lower value, while the dispersion of the spectrum of Lipschitz exponents jumps up to a much higher level after the crash. Our most striking result, however, is that the multifractal spectra of stock market returns are not stationary. Also, while the stock market returns show a global Hurst exponent of slight persistence $0.5 < H < 0.7$, these spectra tend to be skwed towards anti-persistence in the returns.

1 Introduction

The accuracy of measured financial risk models crucially depends on the assumptions about the time-frequency properties of asset prices. These time-frequency properties of market data during financial crises are very different from the propert'ies of the data obtained from normally functioning markets. This corroborated observation may require considerable adjustment of the existent market risk monitoring techniques, which were developed for non-extreme financial market behavior.[1]

The classical assumption of independent and stationary (i.i.d.) market return innovations implies that the degree of fatness of the distributional tails remains the same as the investment horizon extends. But under extreme circumstances this can drastically change. For example, Johansen and Sornette (2000) find that such asset return innovations exhibit strong positive correlations exactly at the time of extreme events. Dacorogna *et al.* (1993) find the same for FX returns. Muzy *et al.* (2001) and Breymann et al. (2000) show that the return volatility displays different long-term correlations from large to small time scales. Therefore, using a fixed time scale is found to be unsuitable for an analysis of the dynamics of such extreme market price moves. Low-order statistics with adjustment to the varying time scales of the market may provide more efficient descriptions.

Dacorogna *et al.* (1996) propose to expand the time periods of high volatility and contract those of low volatility to eliminate this so-called *time-warping.* However, Los (2003) conjectures that these periods of *condensation* and *rarefaction* of market returns are essential for a proper functioning of the financial markets. In periods of condensation, *i.e.*, in peruiods of faster trading, more liquidity is provided to the market within the same real time unit than in periods of rarefaction, *i.e.*, in periods of slower trading. Since fixed time scales are thus not adequate for adequately describing all volatility or risk level changes in the market, a better insight into the dynamics of financial markets can be achieved with a time-adaptive framework that simultaneously takes all time-scales of the statistical distributions of the return innovations into account, *i.e.*, with a complete time-frequency analysis.

Mallat's (1989) wavelet multiresolution analysis (MRA) can analyze all these varying modeling descriptions. It produces a complete time-scale, or time-frequency, representation of the statistical properties of a financial time series and it can successfully be extended to *pattern recognition* and *crash (= discontinuity,* or *singularity) detection* (Mallat and Hwang, 1992). Furthermore, wavelets allow for multifractal analysis of asset returns with proven advantages to the usual structural function approach. The multifractal model of asset returns can describe important empirical regularities observed in financial time

[1] This paper is based on the succesfully defended doctoral thesis of Rossitsa Yalamova, PhD, Kent State University, Kent, OH, USA, December 2003. Cornelis Los , PhD was her Supervisor and Chairman of her Doctoral Thesis Defense Committee. The constuctive citique and editorial suggestions of the other two Committee members, Kazim Khan, PhD, and Federick Schroath, PhD, are gratefully acknowledged.

series, including fat distributional tails (= non-normal occurrence of extreme events) and long memory, and has now a firm theoretical mathematical finance foundation (Elliott and van der Hoek, 2003). The local variability of the irregularities of a multifractal process is highly heterogeneous. It should therefore be characterized by a spectrum of local Hölder exponents, instead of by one monofractal Hölder, *i.e.*, one Hurst exponent.

Johansen and Sornette (1998) apply conventional statistical analysis to market indices and find that the largest crashes may be extreme outliers triggered by amplifying factors. Stock markets in normal times exhibit simple self-organization with leptokurtic distributions of their return innovations, implying that normal "noise" trading combines with fairly regular, but more-frequent-than-nornmal occurrence of extreme events. The multifractal model of a asset returns is suitable to model the moment scaling, the volatility persistence, and the abnormally occurring outliers observed in financial time series (Calvet and Fisher 1999).

The conventional statistical financial models, which are based on the complementary assumptions of ergodicity and stationarity cannot properly analyze crashes, precisely because the statistical properties of such highly nonstationary data are very different from the statistical properties of stationary data. Therefore, a model based on data obtained in stationary and stable times is not of much use in times of crises. Indeed, Morris and Shin (1999), Danielson and Zigrand (2001), Danielson *et al.* (2001), and Los (2003) suggest that most existent statistical modeling in finance is based on a misunderstanding of the properties of financial risk (= volatility = energy = power), be it in normal trading times or in critical times.

Stock market criticality suggests that stock market crashes are preceded by increased susceptibility and pre-cursory signals similar to critical instability in physical fluid dynamics. Therefore, a *multifractal spectrum* (MFS) analysis around such significant drawdowns may reveal the existence of an identifiable pre-cursory pattern.

Thus the objective of this paper is to extract empirically descriptive information from the data about the dynamics of stock market returns by studying the 1987 stock market crash, when the stock ,arket was drawn down by 26% in less than two days. According to Sornette and Johansen (1998) stock market behavior before a crash is related to the *transient behavior* preceding a set of steady state equilibria. This observation enables, perhaps, early detection (and warning) of stock market crashes. It is already known that a normally functioning financial market exhibits properties of a complex dynamical system.

Criticality of the market pricing process may imply particular scale invariances in the rate of return process. Recently, the analogy of financial crashes to critical energy (= risk) diffusion points in fluid dynamics and in statistical mechanics has begun to be researched (*cf.* Los, 2003, for an in-depth review). For example, Johansen *et al.* (2000) claim that log-periodic oscillations appear in the price of the asset just before the critical date of a crash. Such oscillations can be detected by the scalograms of the MRA.

2 Methodology

The financial markets have been shown to be similar to complex dynamical systems (Johansen *et al.*, 2000). The idea to research stock market data during crashes is based on scientific evidence in physics that such complex dynamical systems reveal their properties better under stress than in normal conditions.

Wavelet analysis allows to test for nonlinear long term dependence even in the presence of trends. In particular, wavelet MRA allows the identification of MFS, or Hausdorff fractal dimension, of financial time series and of the dimensionally most prevalent monofractal Hurst exponent.

This monofractal Hurst exponent H of daily market index data is calculated to determine the *global* or average statistical self-similarity of the market return series. Such statistical self-similarity is manifest in the power law spectrum of the series. First, we selected daily index price series of DJIA, NASDAQ and S&P500 of approximately 2000 observations each and computed the monofractal H exponent using two methods. In this way we were able to assess somehwat the accuracy of such calculations and the stability of the computed H exponent, by comparing these two replication results. We found many differences in the values of the obtained Hurst exponents for the same series over various time intervals.[2] Differences in the value of the scaling Hurst exponent H over time are, of course, possible indications of the existence of multifractality, as Mandelbrot (1997) pointed out.

Next, we computed multifractal spectra. Regular time series can be fitted by a polynomial dynamic process, or Taylor expansion. The degree of irregularity of the residuals of such a Taylor polynomial is measured by the Lipschitz regularity exponent α_L, which is the exponents of the residual term of the Taylor series expansion. The Lipshitz α_L measures the degree of irregularity, or "randomness," of a time series. For each market index series, a singularity spectrum of these Lipschitz α_L's is computed following a five-step-procedure, based on the exact results of Bacry, Muzy and Arnéodo (1993; *cf.* Los, 2003, Chapter 8 for a detailed description).

The data consists of daily returns of stock market indices before and after the market crash of October 1987. The multifractal spectra of these observations are calculated on a 512-day moving window. The changes in the multifractal spectra over time are then described mainly by the weighted average of the resulting spectra of the identified Lipschitz α_Ls. The weighted moving average is computed using the spectra's Hausdorff dimension values $D(\alpha_L)$ as weights. We also document the minimum α_L and maximum α_L at each point in time for a better characterization of the changes in the multifractal spectra, like the changes in their skewness.

[2] For some other recent empirical examples of the identification of these monofractal exponents for various markets in Latin America and in Europe, *cf.* Kyaw and Szong, 2003 and Lipka, 2003.

3 Functional Irregularity or Uncertainty

3.1 Geometrical Measurements - Dimensions

The (ir-)regularity of a function can be measured geometrically by the Hausdorff dimension. The Hausdorff dimensions $D(a)$ of a bounded set in n - dimensional real space is measured by the minimum number N of balls with radius a, needed to cover the set when $a \rightarrow 0$.

$$D(a) = \lim_{a \rightarrow 0} \frac{\log N(a)}{\log a^{-1}} \tag{1}$$

When $D(a)$ is an integer, the Hausdorff dimension coincides with the classical Euclidean dimension. The Hausdorff dimension can be computed from the Legendre transform of the scaling exponents $\tau(q)$ of the partition function $Z(q, a)$, as explained below, where q is the moment order and a is the scale.

This wavelet transform modulus maxima (WTMM) method overcomes the limitations of the structure function approach of Parisi and Frisch (1986), who computed the Hausdorff dimension, or singularity spectrum directly from the scaling exponent of a p-order structure function. Muzy *et al.*(1993) prove that the WTMM based method is more appropriate for multifractal description of self-affine distributions.

3.2 Analytical Measurements - Functional Spaces

The analytical way of measuring the (ir-)regularity of a time series is to consider a family of dynamic functional spaces in their local, or pointwise version. The local information about such spaces is given by the Hölder exponent at each point, while all such information is captured by the Hausdorff dimensions or multifractal singularity spectrum $D(\alpha_L)$. Therefore, the Hausdorff dimensions of a set of Lipschitz-Hölder exponents α_L form a MFS that characterizes the set of singularities of that particular time series. If there is only one Hausdorff dimension in the spectrum for one particular Lipschiotz α_L, the time series exhibits only one kind of fractional irregularity or "randomness" and is thus monofractal. But more often it is found that several different Lipschitz α_Ls exist in a time series with singularities and then the time series is multifractal.

The continuous wavelet transform is useful for obtaining information about the local scaling behavior of functions by computing the maxima lines in the wavelet scalograms, i.e. the matrices of squared wavelet resonance coefficients (= correlation coefficients beteen the time series and the various orthogonal wavelet bases). Maxima lines follow the local maxima of the wavelet resonance cofficients in the scalogram over all available scales. The wavelet resonace coefficients of the scalorgram measure the degree of local correlation of a time series with the wavelet basis of a particular resolution scale. Thus maxima lines can carry information about singularities across the various resolution scales. These time resolution scales are proportional to the inverses of the radian frequencies: $a \sim \frac{1}{\omega}$.

If the empirical time series contains only cusp-type singularities, the wavelet transform modulus maxima (WTMM) provide very reliable estimation of the singularity spectrum $D(\alpha_L)$. However, in the presence of oscillating singularities, an additional oscillating exponent besides the Hausdorf dimension $D(\alpha_L)$ is needed to describe the behavior of the time series. In this paper we ignore this latest theoretical advance in the measurement of singularities, although we acknowledge that it may be important for our future research. The reason for abandoning that project is that there is not yet a reliable computational algorithm to compute the canonical characterization of singularities in both their real and imganinary dimensional components, only an algorithm for the real part of the dimension $D(\alpha_L)$. Consequently, we were unable to detect any oscillating singularities and were able to characterize only the non-oscillating singularities, *i.e.*, the non-periodic discontinuities of the stock market return series and not the periodic ones.

Pointwise Hölder Exponent Our formal approach is as follows. The pointwise Hölder exponent of a function $f(x)$ at an observed point x_0 is the supremum of the Lipschitz-α_L of the series x such that:

$$|f(x) - P_n(x - x_0)| < C\,|x - x_0|^{\alpha_L}\,, \tag{2}$$

where C is a constant and P_n is a polynomial of degree n. If $f(x)$ is Lipschitz α_L, for $\alpha_L > n$, then $f(x)$ is n times continuously differentiable in x_0 and the polynomial $P_n(x)$ is the first $n+1$ terms of the Taylor series of $f(x)$ in x_0.We say that $f(x)$ is uniformly Lipschitz α_L on an interval $[a, b]$ if the difference between $f(x)$ and the first n terms of the Taylor series defined with respect to the local point x_0, satisfy the above equation for any $(x, x_0) \in [a, b]^2$.

The pointwise Hölder exponent of f is defined by the supremum:

$$\alpha_L(x) = \sup\{\alpha_L : f(x) \in C_x^{\alpha_L}\} \tag{3}$$

3.3 Measurement Methodology

Global Self-Similarity Statistic Recent empirical studies suggest that market price changes exhibit certain properties: the price increments, or return innovations, are not correlated, but their volatilities are power-law-dependent, and the shape of the probability density function (pdf) of the price increments depends on the time scale: from Gaussian at large time scales of a month and more, to stable leptokurtic distributions with fat tails at fine scales of intra-day data, as shown in, for example, Bouchaud and Potters (2003).

A random process $x(t)$ is said to be self-similar with the Hurst exponent H if for any scale $a > 0$ it obeys the scaling relation

$$x(t) \triangleq a^{-H} x(at) \tag{4}$$

Thus a self-similar process is monofractal when its singularity spectrum $D(\alpha_L) = D(H)$ is reduced to a single point and the Lipschitz-Hölder α_L exponent is the same H at any point in time. Its increments display long range dependence and their autocovariance function decays as

$$\gamma(\tau) = \sigma^2 \tau^{2H-1} \tag{5}$$

as the horizon $\tau \to +\infty$.

The Hurst exponent provides us with a means to analyze the global dependence characteristics of a financial time series and to determine if the statistical observations are persistently, neutrally or anti-persistently dependent. When the Hurst exponent is $0 < H < 0.5$, the time series is called anti-persistent. Fractional Brownian Motion (FBM) increments with anti-persistence diffuse more quickly than Geometric Brownian (GBM) increments with the neutral $H = 0.5$. Such an anti-persistent FBM returns continuously to the initial point. The GBM with $H = 0.5$ has independent increments, without any particular tendency, and its autocovariance is a constant $\gamma(\tau) = \sigma^2$, no matter what the time horizon. When $0.5 < H < 1$, the time series is persistent and its time adjusted volatility lasts forever.

Fig. 1 provides three graphs of Fractional Brownian Motion (right) with $H = 0.2, 0.5, 0.8$ (top to bottom on right), with their respective increment processes on the left. Notice that the anti-persistent increments are "denser" and reverse more quickly than the neutral increments, giving the GBM process a more "ragged" appearance. The persistent increments are less "dense" than the neutral increments, giving the GBM process a more "smooth" appearance.

The risk (= volatility = energy = power) spectrum $P(\omega)$ can be used to determine the kind of self-similarity or degree of persistence of a time series, since it can easily be established, by taking the Fourier Transform of the autocovariance function, that:

$$P(\omega) \propto \omega^{-\gamma} \tag{6}$$

with the power exponent $\gamma = 2H = 1$ and the frequency proportional to the inverse of the time scale: $\omega \sim \alpha^{-1}$, such that the monofractal Hölder exponent H can be directly identified by plotting $\ln P(\omega)$ against the radian frequency ω. Such a risk spectrum $P(\omega)$ can be most accurately measured by a scalegram, or averaged scalogram, based on the complete wavelet multiresolution analysis (MRA), based on orthogonal wavelet bases, which have finite support, instead of the orthogonal sinuses and cosinuses of Fourier analysis, which have infinite support and thus lead to overlapping and to statistical "double-counting." Wavelet multiresolution based on ortgonal wavelet bases is exhaustive and complete and does not lead to statistical "double-counting."

Fleming *et al.* (2001) extract the set of wavelet detail coefficients $\{d_{j,k}\}$ using a wavelet, $\psi(t)$ with N vanishing moments. One can identify the Hurst exponent of the series from the variances of wavelet detail coefficients based on dyadic scaling, as follows

$$Var(d_{j,k}) \propto 2^{-j\gamma} \tag{7}$$

This is similar to a conventional power spectrum, whereby the financial risk = power of the time series is measured by its variances. If $0 < \gamma < 2N$, one can plot the $\log_2 Var(d_{j,k})$ versus the level of decomposition j to produce a straight line of slope $-\gamma$ for the given time series. Deviations in these log-variance or power plots from a straight line have been shown (Los, 2003) to reveal coherent risk structures present at a given scale with an increase in the variance. Such periodic risk structures are often induced by institutional constraints, like the observed diurnal cycle of trading, which is caused by the absence of stock trading floors in the Pacific Ocean. During the half to three-quarter day of trading the electronic trading book gets passed around the world, except in the Pacific Ocean.

Wavelet analysis is thus the basic tool to study the scale dependent properties of data directly via the coefficients of time-scale wavelet decomposition, since almost nothing needs to be assumed about the characteristics of the underlying trading processes. For example, the time series process of market returns does not need to be bounded, so that sharp discontinuities and drawdowns in the pricing process can easily be analyzed, without any assumption of possible jump processes.

Classical Multifractal Formalism - Series Estimation The concept of multifractality originated from a general class of multiplicative cascade models introduced by Mandelbrot (1974). This multifractal formalism was originally established to account for the statistical scaling properties of singular measures determined by their singularity spectrum $D(\alpha_L)$. Fractals appear not only as singularity measures, but also as time series of singularities. Parisi and Frisch (1985) proposed extracting the MFS of the velocity field from the inertial scaling properties of multidimensional structure functions

$$S_q(\tau) = \langle (\delta x \tau)^q \rangle \sim \tau^{\xi_q}, \tag{8}$$

where δx_τ is a longitudinal velocity (or rate of return) increment over a time horizon or distance τ, integer order coefficient $q > 0$ and ξ_q is the scaling exponent for that particular order moment. The inner product of the velocity incerements $\langle (\delta x_\tau)^q \rangle$ computes a moment of order q, which scales over time horizon τ with scaling exponent ξ_q. This scaling exponent can be moment-dependent.

This structure function approach has serious drawbacks as shown by Muzy *et al.* (1993). This approach fails to fully characterize the singularity spectrum due to some fundamental limitations in the range of accessible irregularity exponents: it fails to detect the part of the $D(\alpha_L)$ spectrum which lies beyond the value $\alpha_L \geq 1$, for example. This may not be such a large deficiency since a Lipschitz $\alpha_L \geq 1$ implies that at least one extra unit order of differentiation should be included in the Taylor expansion, instead of computing just (relative) first differences.

Multifractal formalism based on wavelet transform modulus maxima (WTMM) allows us to determine the whole singularity spectrum $D(\alpha_L)$ directly from any experimental signal or time series (Muzy *et al.*, 1991). It works in most situations and provides a unified multifractal description of self-affine distributions. The following procedure of computing the multifractal singularity spectrum based on WTMM is described in Los (2003, Chapter 8). For this paper we developed computational algorithms in MATLAB and used fast-computing C/C++ based programs.

First, a decay scaling exponent $\tau(q)$ is computed as the slope of the dyadic logarithm of Gibb's power partition function:

$$\log_2 Z(q,a) \approx \tau(q)\log_2 a + C(q), \tag{9}$$

where q is the power of the moments and a represents the scale. The partition function $Z(q,a)$ can be computed from the sum of the modulus maxima raised to the power of q. The MFS $D(\alpha_L)$ can then be found as the inverse Legendre transform of the scaling exponent $\tau(q)$.

Because this procedure is based on Gibb's statistical averaging or partition function $Z(q,a)$, there is no explicit local information present in the resulting scaling estimates. Therefore, the usefulness of the partition function method resides in the fact that it obtains information on global average moments, which tend to be more stable than pure local information. Its disadvantage is that, for the same reason, it tends to obscure this local information. Therefore, in order to capture possible changes in such global average exponents over time, we calculate the MFS on a 512-day moving window with daily increments.

We implement the following steps:

3.3.1 Step 1: Compute the wavelet transform $W(\tau, a)$ for all translations and dilations:

A wavelet is simply a finite energy function with a zero mean. The wavelet transform is defined by the continuous time correlation between the time series and the particular wavelet of horizon τ and scale a:

$$W(\tau,a) = \int_{-\infty}^{+\infty} f(t)\psi_{\tau,a}(t)dt, \tag{10}$$

where the base atom $\psi_{\tau,a}(t)$ is a zero average function, centered around zero with finite energy, volatility or risk. The family of wavelet vectors is obtained by translations and dilatations of the basic ("mother") wavelet atom:

$$\psi_{\tau,a}(t) = \frac{1}{\sqrt{a}}\psi(\frac{t-\tau}{a}) \tag{11}$$

This wavelet is centered around τ, like the (windowed) Fourier atom. If η denotes the frequency center of the base wavelet, then the frequency center of a dilated wavelet is $\xi = \eta/a$. The wavelet transform has a time frequency resolution which depends on the scale a. It is a complete, stable and potentially redundant representation of the signal or time series. Orthogonal wavelets produce complete, exhaustive and non-redundant analysis. The analyzing wavelet ψ is viewed as a (Heisenberg) box of particular shape and the scale a is its relative size.

The scalogram of a signal is defined by the array of squared wavelet resonance coefficients (= "coefficients of determination") according to time dilation τ and scale a:

$$P_W(\tau, \xi) = |W(\tau, a)|^2 = \left| W(\tau, \frac{\eta}{\xi}) \right|^2 \tag{12}$$

The scalogram is therefore a 2-dimensional array which gives us normalized risks or volatilities across time and scale (= inverse of frequency). It analyzes over time and scale how well the time series conforms to the wavelets at each time horizon and at each scale. For this procedure we actually use a Gaussian wavelet basis, following Mallat's (1998) procedure. First, the wavelet transform $W(\tau, a)$ for all translations τ and all scaled dilations a is obtained. The largest dyadic scale $a = 2^j$ depends on the number of available time series observations.

3.3.2 Step 2: Find wavelet transform modulus maxima (WTMM) for each scale a

The modulus maxima (largest wavelet transform coefficients) are found at each scale a as the suprema of the computed wavelet transforms such that:

$$\frac{\partial W(\tau, a)}{\partial \tau} = 0. \tag{13}$$

The advanbtage of the Gaussian wavelet is that it can be shown that the analyzing wavelet ψ is the N-th derivative of the Gaussian function, where the order of the differentiation is related to the scale a:

$$\psi^{(N)}(x) = d^N(e^{-\frac{x^2}{2}})/dx^N. \tag{14}$$

as is shown in Fig. 2.

In order to estimate the Liptschitz exponents up to a maximum value N, we need a wavelet with at least N vanishing moments. The Gaussian wavelet has compact support and it is also N times continuously differentiable, therefore it is appropriate for calculation of local maxima. Using wavelets with more vanishing moments has the advantage of being able to measure the Lipschitz regularity up to a higher order and increases the number of maxima lines. Mallat and Hwang (1992) recommend the Gaussian wavelet for singularity detection. For most types of singularities, the number of maxima lines converging to the singularity depends upon the number of local extrema of the wavelet itself.

These WTMM are positioned on connected curves, or maxima lines, like the top ridges of mountain ranges. When the analyzed signal has a local Hölder exponent $\alpha_L(x_0) < N$ at point x_0, there is a maxima line pointing at x_0. Thus each maxima line displays the hierarchical organization of the various singularities.

3.3.3 Step 3: Compute Gibb's partition function based on wavelets

The originality of the WTMM method is in the calculation of the partition function $Z(q, a)$ from these maxima lines. The time scale partitioning given by the wavelet tiling defines the particular Gibb's partition function. A matrix containing the maxima lines (maxmap) from the previous step allows the computation of Gibbs' partition function, where a is the scale, e.g., $a = 2^j$, $j = 1, 2, ..$

$$Z(q, a) = \sum_{\tau, a} \sup_a |W(\tau, a)|^q \tag{15}$$

This partition function effectively computes the moments of the absolute values of the wavelet resonance coefficients $W(\tau, a)$. There is an analogy between the classical partitions defined for measures and the one provided by the wavelet transform used for functions. The supremum allows us to define a scale-adaptive partition preventing divergencies for negative values of the moment order q.

3.3.4 Step 4: Compute the decay scaling exponent $\tau(q)$

The slope in the double-logarithmic plot

$$\log_2 Z(q, a) \approx \tau(q) \log_2 a + C(q) \tag{16}$$

allows the computation of the decay scaling exponent $\tau(q)$. The scaling exponent $\tau(q)$ is the Legendre Transform of the MFS $D(\alpha_L)$ for self-similar time series and relates the fractal dimensions to the order q of the partition function $Z(q, a)$.

The general idea is best explained in one dimension. For each function $f(x)$ we define a new function $\mathcal{L}f(z)$ called the Legendre transform. We do this as follows:

Define $z = \frac{df}{dx}$ which relates the new variable z to the old variable x. The condition $\frac{\partial^2 f}{\partial x^2} \neq 0$ guarantees that we can find the inverse function $x(z)$. Hence, we have a unique relation between x and z. Now consider the mapping

$$z \to x : x\frac{df}{dx}(x) - f(x) = xz - f(x) \equiv \mathcal{L}f(z) \tag{17}$$

This defines the Legendre transform of the function f. It is a very special change of variables, and can also be written as

$$\mathcal{L}f(z) \equiv zx(z) - f(x(z)) \tag{18}$$

Note that simultaneously the variable x is changed to the derivative and modify the function modified.

The decay scaling exponent $\tau(q)$ is defined by the power-law behavior of Gibb's partition function in the limit when the scale $a \to 0$. Using the property of self-similarity it is easy to find that the partition function is proportional to the scale with the scaling exponent $\tau(q)$ for $a \to 0$ (Los, 2003):

$$Z(q, a) \sim a^{\tau(q)}$$

Thus, this exponent measures the asymptotic decay of the partition function at fine scales. In other words, the partition function is scale dependent and it is this scale depenence that is exploited to find the MFS $D(\alpha_L)$.

3.3.5 Step 5: Compute the MFS $D(\alpha_L)$

By using both the scaling behavior of the wavelet transform $W(\tau, a)$ along the maxima lines and the definition of the singularity spectrum, we can compute the MFS $D(\alpha_L)$ as follows:

$$D(\alpha_L) = \min_q [qH - \tau(q)] \tag{19}$$

The moments of a self-similar process $X(t)$ satisfy the expectational equation:

$$E\,|X(t)|^q = E\,|X(t)|^q \,.\, |t|^{qH} \,. \tag{20}$$

Therefore, the relation

$$E\sigma_j^{(q)} = C_q 2^{j(\tau(q)+q/2)}, \tag{21}$$

suggests that self-similarity can be detected by testing the linearity of $\tau(q)$ relative to the order q. From the properties of the Legendre transform we can deduct that monofractal functions are characterized by a linear $\tau(q)$ spectrum, with a unique slope $H = \tau(q)/q$. In contrast, a non-linear $\tau(q)$ curve is a signature of multifractal functions that display multifractal properties: there are many slope coefficients $\alpha_L = \partial\tau(q)/\partial q$, depending on where the derivative is calculated on the $\tau(q)$ curve. Measuring the deviation from a simple linear relationship is thus the crucial issue to determine if a fractal process is monofractal or multifractal.

The WTMM approach is now the foundation of a unified multifractal description of self-affined distributions, as shown by Muzy *et al.* (1993). There are two obvious advantages of the WTMM method to the structure function approach: (1) the scale-adaptive partition (defined by the sup) which prevents divergencies from showing up in the calculation of $Z(q, a)$ for negative values of q and (2) the accessibility of the entire range of singularities made possible by the choice of the number of vanishing moments, thus allowing for negative spectrum values $D(\alpha_L)$.

Mandelbrot (1990) defines such negative dimensions as measuring the emptiness of empty sets. The positive $D(\alpha_L)$ are shown to define a 'typical' distribution, while the negative $-2 \leq D(\alpha_L) < 0$ characterize the sampling variability. Mandelbrot also shows that negative $D(\alpha_L)$ are essential for revealing the generating process, generalizing it to random multifractals.

4 Empirical Measurement Results

Two broad categories of departures from a pure scaling model are possible. First,a scaling exponent might be well defined, although existing nonstationarities may not be of a scaling nature, which leads to a problem of correct measurement of the constant underlying scaling parameter $\tau(q)$. Second, there is the possibility that the scaling changes with time, *e.g.*, the parameters of constant in-time scaling are not invariant. Special properties of the wavelet approach allow us to overcome these two challenges.

The problem of variance of the scaling exponent problem (time-varying scaling) can be reduced to a simple model inference problem because of the quasi-decorrelation in the time-scale or time-frequency plane. Wavelet coefficients can be treated as almost independent and normally distributed with known variances.

The log-scale diagram can also be generalized to the study of statistics other than those of the second order. The resulting q-th order log-scale diagram contains relevant information to the analysis of scaling beyond the reach of second-order statistics.

4.1 Inconsistent Hurst Exponent Measurements

To measure the Hurst exponents for the time series of index prices, we implemented two methods: the method of Fleming *et al.* (2001) and the method based on the built-in algorithm in Fraclab developed by the French Institute National de Recherche en Informatique et en Automatique (National Institute for Information and Automation Research). Their method is based on discrete wavelet coefficients. For the first method we developed an algorithm in MATLAB based on the continuous wavelet transform (CWT) using the Gaussian wavelet with only two vanishing moments. The results we obtained were not unique. In other words, each of the two methods did not provide consistent Hurst exponents for the same data sets, as can be seen in Fig. 3 for the NASDAQ, in Fig. 4 for the DJIA, and in Fig. 5 for the S&P500 stock market returns using moving windows, computing Hurst exponents for each window. It appears that the CWT provides smoother and less variable Hurst exponent values than the Fraclab procedure.

4.2 Multifractal Spectra of Stock Markets

The non-uniqueness of the computed Hurst exponents provided additional motivation to attempt to produce MFS calculations. The two methodologies we used for computing the global Hurst exponents also assign different cutting off points for finer scales, adding to the non-unique results. It demonstrates that averaging the risk or irregularity over a longer time period may lead to inaccurate estimates and imprecise fractal models. There is thus an urgent need for examining the MFS and its dynamics over time, and not to rely on the measurement of a possible monofractal Hurst exponent only, which tends to show less or more variability over time, depending on which measurement method is chosen.

Indeed, the S&P500 stock market price index series exhibit deviations from linearity of the scaling exponent $\tau(q)$ in Fig. 6 suggesting the existence of a MFS, instead of one monofractal Hurst exponent. In finance, Corrazza and Malliaris (2002) have already drawn a similar conclusion for a number of foreign currency markets.

The MFS is calculated following the aforementioned five-step procedure described in more detail by Los (2003). A MFS can be visualized as an interwoven ensemble of independent monofractals each with their own dimension $D(\alpha_L)$ as seen in Fig. 7. We examined the changes in these multifractal spectra of each stock market index over time and to detect if there are characteristic levels or patterns before significant drawdowns. In other words, did the spectrum of Lipschitz α_Ls, which characterizes the kind of singularities occurring in a market, change in the period preceding a major market drawdown?

Monofractal time series display only one regularity exponent $\alpha_L = H$ with dimension $D(\alpha) = 1$, defining the Lévy scaling of the underlying stable distribution If the process is stationary that exponent tends to 0, because the underlying time polynomial is stationary, has constant frequencies and no irregurlarity, uncertainty, or "randomness". In that case, no degree of irregularity as measured by the Hurst exponent remains. A multifractal process displays more than one Lipschitz-α_L, defined by the order of the partition function $Z(q, \alpha)$ and the empirical existence of irregularities of that order q. Therefore, the existence of positive higher-than-second-order-moments of the distribution of the stock market time series will be reflected by Lipschitz α_L's smaller than the dominant or global Hurst exponent and by negative higher moments with larger α_L's.

Fig. 7 shows an empirical MFS the Lipschitz α_L's. Compare this with the theoretical MFS spectrum in Fig. 8. Where $q = 0$ we measure dimension $D(\alpha_L) = 1$ for the Hurst exponent. Increasing the absolute value of q displays lower dimensions. For moment order $q > 0$ we measure the dimensions of the Lipshitz α_L smaller than the Hurst exponent, while for moment order $q < 0$ we measure the doiemnsions of the Lipschitz α_L larger than the Hurst exponent. Thus in Fig. 7 we see that the spectrum is skewed to the right towards the more persistent values of the α_L's and has a cutoff point at the values of the α_Ls that indicates anti-persistence at the turbulence level close to $\alpha_L = \frac{1}{3}$. In our program we use partition functions between the order of -5 and $+5$ with .5 increments. Therefore, the maximum number of the Lipschitz α_L's computed for each multifractal spevctrum is 21. The following picture relates the moment orders q to the MFS $D(\alpha_L)$.

The empirical MFS in Fig. 7 is characterized by a number of Lipschitz α_L regularity exponents and their respective dimensions. This spectrum allows to compute a weighted average and a standard deviation of the distribution of the regularity exponents. The exponent with $D(\alpha_L) = 1$ (= the mode of the spectrum) is the global Hurst exponent. However, we found that the weighted average gives more information on the higher moments of the empirical distribution of the index prices, which obviously deviates from normal.

Figures 9, 10 and 11 provide information on these weighted averages and modes of the NASDAQ, DJIA and S&P500 stock market returns, respectively. Notice that in all cases the mean is smaller than the mode, suggesting that although globally these markets are mildly persistent as measured by their Hurst exponent (MFS mode) of $0.5 < H < 0.7$, their multifractal spectra are skewed towards anti-persistence: the NASDAQ and DJIA stock market returns have weighted mean Lipschitz $\alpha_L \approx 0.4$.

We also report the minimum and maximum α_Ls as well as their range = maximum - minimum in the following figures. The empirical MFS is computed on a 512-day sliding window with one-day increments. The length of the window had to be judiciously chosen to allow for the use of finer scales in the wavelet analysis, while at the same time detecting any changes in the MFS in the period before the crash. In the literature patterns have been reported for two to four years before crashes.

4.3 Multifractal Patterns Around Stock Market Crashes

By plotting these multifractal dimension statistics we expected to identify patterns characterizing the stock price index returns preceding significant market drawdowns. If such a pattern could be identified, it would contribute to the empirical evidence of the possible predictability of market crashes. Johansen *et al.* (1999) proposed that such hypothesized patterns preceding market crashes are caused by a slow build-up of long-range time dependencies reflecting interactions among traders. Indeed, Hirshleifer and Teoh (2003) anecdotally suggested the same hypothesis. Thus, we made the identifiability of a pattern preceding stock market crashes our null hypothesis.

We observe in Fig. 12 for the DJIA that the spread between minimum and maximum Lipshitz α_L's increases significantly when the day of the crash (e.g. October 19, 1987) is included in the calculations of the MFS. It widens both the range and the dispersion of the MFS. The minimum α_L drops significantly and the maximum α_L exhibits several peaks in the days after the crash. Careful examination of the min/max, as well as dispersion of α_L plots for different stock market indices allows us to detect ex-post the exact time of the crash, but not a uniquely identifiable preceding pattern, apparently rejecting our null hypothesis.

Mandelbrot (1991) describes the possibility of the occurrence of "anomalous" negative α_Ls that may be a flaw of the methodology based on the partition function. However, all extracted minimum α_L from our computations show that there is no such anomaly in our empirical results, since we did not compute any negative α_L, although we do find minimum α_Ls close to zero. Notice that there are several large maximum α_Ls in the $(0, 2)$ range, and that thesze larger α_Ls mostly occur after the crash.

By dividing the sum of the dimensions by the average dimension we recover the number of $q^{'}$s at each point. This provides additional information about the changes of the MFS and allows a possible normalization of the multifractal spectra statistics. The mean/standard deviation of the MFS at each point was divided by the average dimension.

The shape of the MFS changes over time, therefore we decided to examine the changes in the skewness of the spectrum by calculating the asymmetry of the MFS. The asymmetry of the MFS at each point is calculated as the difference between the weighted average Lipschitz-α_L and the minimum Lipschitz-α_L

divided by the range:

$$asym = \frac{\sum [D(\alpha_L).\alpha_L] - \min(\alpha_L)}{\max(\alpha_L) - \min(\alpha_L)}. \tag{22}$$

A coefficient close to 0.5 implies a symmetric spectrum. Plots of these asymmetry coefficients did also not reveal any particular reliable pattern preceding significant drawdowns.

Changes of the weighted mean and standard deviation of the spectrum of Lipschitz-α_L exponents were plotted in particular for patterns before the crash of October 19, 1987. Surprisingly such simple statistics of the MFS of NASDAQ, DJIA and S&P500 index returns do exhibit a very similar and clearly identifiable pattern, as seen in Figs 13, 14 and 15, respectively. We find very similar extended periods of substantial weighted averages of the regularity exponents combined with low standard deviations of the singularity spectrum. Moreover, these statistics do not exhibit the spikes observed in other periods and may reflect a higher regularity of the pricing processes.

We conjecture that the increased regularity in these market returns is the result of coordination in the behavior of a large number of agents in the market that leads to increased order. This is in line with Sornette's (1998) hypothesis that a normally functioning market shows a high weighted average value of regularity exponents, combined with a narrow spectrum, indicating fairly accurate global persistence.

A market crash then drops the weighted average value of the regularity exponents and combines that with a jump in the dispersion of the spectrum of these exponents. Thus preceding a crash the average persistence gradually increases and there is a fairly narrow range of irregularity. This slowly increasing persistence and more ordered structure - with less dispersion of the "randomness' in the market pricing process - tends then to lead to a sharp "drawdown." The crash itself sharply reduces the degree of persistence of a market and simultaneously increases the dispersion or diversity of the irregularity, uncertainty or "randomness" in the stock market, thereby returning it to its "normal" functioning.

This suggests that in the periods immediately preceding stock market crashes, the degrees of horizon uncertainty or risk experienced by the various market participants become too similar, possibly indicating that the participants all tend to converge to similar time horizons and thus to similar shapes of stable market returns distributions. A market crash reduces this too high level of regularity and persistence in the market. It increases the diversity of risks and the diversity of time horizons of the market participants, so that the market can normally function with less persistence and greater "fluidity."

Gutzwiller and Mandelbrot (1988) encountered unusual multifractal measures $\alpha_{\min} = 0$, and/or $\alpha_{\max} = \infty$ in classical Hamiltonian systems that display "hard chaos." This may explain the spikes of the MFS statistics in a "normally" functioning stock market, that might otherwise seem unusual, as these exceptionally regularity exponents considerably deviate from the usual $(0, 1)$ range.

Furthermore similar patterns preceding the 1987 crash are observed in other world indices like the London's FTSE, Tokyo's Nikkei, and Hong Kong's Hang Seng indices in Figs. 15, 17 and 18, respectively. This may indicate not only the presence of similar MFS patterns in each of these financial markets, but also the existing coordination between the trading in these international markets. Thus when we compare the patterns of different stock markets all over the world in the period preceding the same stock market crash of 1987, we do find similar identifiable patterns. In other words, not every crash within the same market may show the same identifiable pattern, because the level and dispersion of irregularity or "randomness" preceding different crashes vary, since the market participants have different time horizons and are confronted by different kinds of information sets.

But the same crash at the same time in different markets does show similar identifiable behavior preceding the date of the joint crash - a gradual increase in the weighted average value of the spectrum of Lipschitz α_L irregularity exponents - since the market participants have similar time horizons and are confronted by the same information set. This can is even confirmed by summary statistics of the windowed multifractal spectra of the Canadian, Australian All Ordinaries, and Singapore Straits Times index returns. In all these international stock markets the average Lipschitz α_L hovers around market neutrality, but shows a clear tendency to move from the anti-persistence to the persistence range in the period preceding a market crash. In all cases the 1987 crash sharply pushes the market back to considerable anti-persistent behavior together with a simultaneous increase in the spectrum of irregularity of the markets.

5 Conclusions

Johansen *et al.* (2000) and Johansen (1997) show that the reported patterns can never occur in $\approx 10^5$ years using conventional time-series models like Engle's GARCH$(1,1)$ model. The frequency of occurrence of drawdowns larger than 15% implies nonrandomness, as is shown by Johansen and Sornette (2001) and as we have consistentlly observed. Therefore, we decided to attempt to contribute additional evidence for the existence of nonlinear patterns in the market price diffusion process with a methodology that allows for multifractal processes, scale consistency, and long memory in volatility.

The predictability of prices is not theoretically specified, since the multifractal model has enough flexibility to satisfy the short term martingale property in some cases and long memory in its price increments otherwise (Elliott and vand der Hoek, 2003). The multifractal model of asset returns incorporates fat tails and several economists now agree that there is no *a apriori* justification for rejecting infinite second moments in the markets. There have been too often discontinuities in the world's stock markets to justify constant or even finite variances. In fact it is found that the monofractal Hurst exponent of the S&P500 $H = \frac{2}{3}$, implying a stability exponent of $\alpha_Z = 1/H = \frac{3}{2}$. This stability value indicates that even the S&P500 portfolio rates of return have no convergent or "existent" volatility value. The volatility "wanders" over time, thereby making it a very questionable underlying asset for consistent option pricing.

The multifractal model exhibits long memory. The FBM is useful for modeling the tendency of price changes to be followed by changes in the same (or opposite) direction. Mandelbrot *et al.* (1997) assert that the FBM captures neither fat tails nor fluctuations in volatility that are unrelated to the predictability of future returns. This methodology is a promising alternative to (G)ARCH-type models, because such models have finite second and fourth moments that can't model the long memory phenomenon.

We started our search for identifiable patterns in the period immediately preceding the market crash of October 19, 1987, as complex dynamical systems tend to reveal their structure and organization better in extreme conditions.[3] We developed an algorithm for calculating the MFS of financial market time series and examined the changes in the financial market price diffusion process using wavelet multiresolution analysis, as suggested by Los (2003), since wavelet MRA is used with great success by signal processing engineers.

Wavelet multiresolution analysis of time series of financial market returns yields a time-frequency analysis with contracted and dilated versions of a chosen prototype wavelet basis (Fleming *et al.*, 2001). The application of wavelet MRA enables us, first, to perform time - scale or time - frequency analysis and, second, to achieve de-noising of the data for better detection of patterns. Wavelet MRA is most useful, when the time series exhibits sharp spikes and jumps, a situation typical for stock market crashes. Wavelets can be used not only to identify the localizations, magnitudes and Lipschitz characterizations of these singularities, but also to analyze the actual recovery of the market price diffusion process. In terms of financial time series, wavelets appear to be more robust in analyzing market indices around crashes compared with the conventional numerical analysis with several free parameters, as has been used by classical physicists.

Our research identified coherent patterns of change in the MFS of stock market indices before significant drawdowns. The exact mathematical representation or model of these patterns has not yet been identified, because they thend to differ from crash to crash, although they are cross-sectionally similar at the same time. The differences in the multifractal spectra between periods around market crashes and periods of 'normal' market behavior might be due to the differences in the types of singularities and their respective dimensions. The classical multifractal formalism applied in this study is accurate for cusp type singularities we observe in the stock market, while the so called time-varying chirps of slowly varying volatilities tend to be underestimated.

If log-periodic oscillations appear in the irregularities before market crashes they can possibly be detected with grand canonical multifractal formalism based on complex wavelets, *i.e.*, on wavelets with real and imaginary parts. Such an analytic approach requires a two dimensional partition function and results in

[3] We also looked at many other stock market crashes, defined as 15% or larger drawdowns within a short period of a few days, and found similar identifiable patterns.

a three dimensional MFS, with a real value axis, an imaginary value axis, and an Hausdorff dimension axis. The plane projection of this complex MFS in the real plane that we have used in this paper is shown to be of the same shape, but slightly to the right of its classical counterpart. Arnéodo *et al.* (1997) show that the projection of the grand canonical MFS is shifted to the right relative to the classical multifractal formalism.

In other words, the real multifractal spectra computed by us tend to show somewhat less persistence in the stock markets than there actually exists, especially in the presence of oscillating singularities. There is more persistence in the stock markets than we were able to identify with our simple two-dimensional multifractal spectra (MFS). Thus the recently proposed complex MFS approach is expected to even further increase the average Lipschitz-α_L irregularity exponent valueds, accentuating the patterns identified in this paper. The results obtained with our classical multifractal formalism therefore do not exclude the appearance of log-periodic oscillations before market crashes (which is still an open empirical research question), but only point to the differences in the cusp singularity composition between periods around crashes and periods of normally functioning markets.

As far as we have been able to detect in the literature, we are the first to apply our real-plane MFS measurements to stock market data. Our method is aimed at detecting irregularity patterns of index price dynamics preceding market crashes. We attempt a variety of measurements to track the changes in the computed multifractal spectra over time. Difficulties for comparison arise from changes in the distance from the α_L-axis, the spread, the concavity of the Lipschitz-α_L's MFS and the number of observed irregularity exponents for each window.

As we computed the weighted average and standard deviation for those multifractal spectra in 'classical statistical sense,' the negative fractality measured by the maximum Lipschitz-α_L's showed on the plots as frequent spikes observed predominantly in "normal' markets. The presence of negative dimensions reveals the randomness of the multifractals as explained by Mandelbrot (1990). The absence of such jumps in the periods before crashes points on some type of hierarchical structure and less "randomness."

The methodological approach developed in this paper provides information for analysis and comparison of the episodes of market crashes and the subsequent recovery patterns in different stock markets. The patterns identified by us reveal relationships between the magnitude of the crash, degree of persistence of the stock market, and the impact of the crash on the financial system. The observed coherent patterns suggest that there is a measurable build-up of market pressure - an increase of the market's degree of persistence and reduction in its "randomness' - leading to a crash. They also show a worldwide coordination or sharing of these kinds of pressures among the markets before the crash of October 1987, because all stock markets we studied show the same pattern preceding this particular crash.

Our hypothesis is that the increase of regularity and the low diversity of financial multifractality, indicated by a higher mean and a low standard deviation of the MFS, as well as by precursory absence of multifractal randomness, reflects an increased dependence between the actions of the various market participants. In contrast, the crash drastically reduces the average degree of persistence of a stock market, and it increases the prevalence and the diversity of its irregularity or "randomness" necessary for the normal functioning of a stock market. Apparently a stock market cannot exhibit too much order and structure, persistence and predictability, because then it will have a tendency to crash. Reduced persistence and greatest diversity of irregular behavior appears to provide the smoothest functoning of a stock market. Thus the introduction of more rules, regulations and institutional restrictions in a stock market, which make it more persistent, only enhances its tendency towards crashing (Danielson and Zigrand, 2001).

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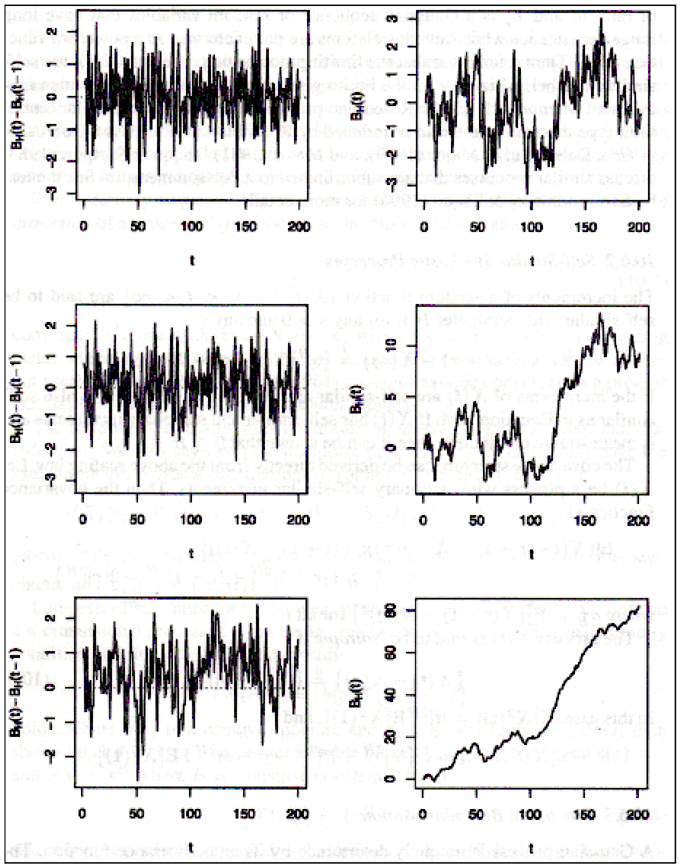

Figure 1: Graphs of Fractional Brownian Motions (right) with long memory Hurst exponents $H = 0.2, 0.5, 0.8$ (top to bottom on right), with the increment processes on the left. The series have been simulated using the method of Davies and Harte (1987).

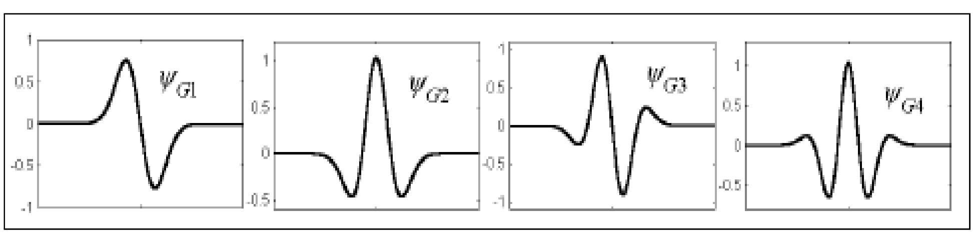

Figure 2: Different Gaussian wavelets obtained from derivatives of the Gaussian mother function

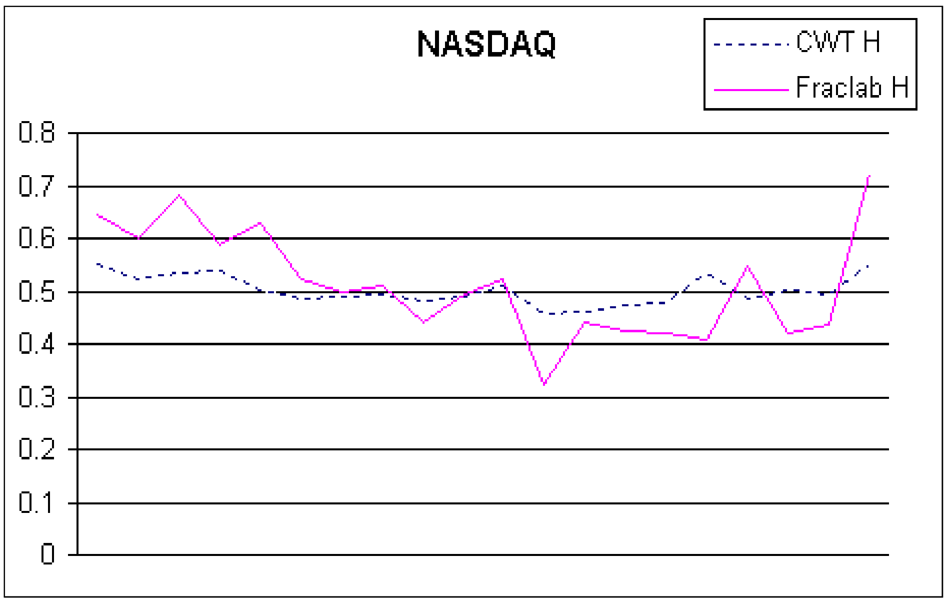

Figure 3: The time series of the NASDAQ index price (1980-2003) is divided in 20 subsamples. The Hurst exponent (on the $y-axis$) was calculated for windows of approximately 2000 observations (sequence of subsamples on the $x-axis$). Results show mild variability of the Hurst exponent, using the MATLAB wavelet MRA and even more variability using the second methodology (developed by INRIA). Most prblematic is the apparent divergence between the two computed value series.

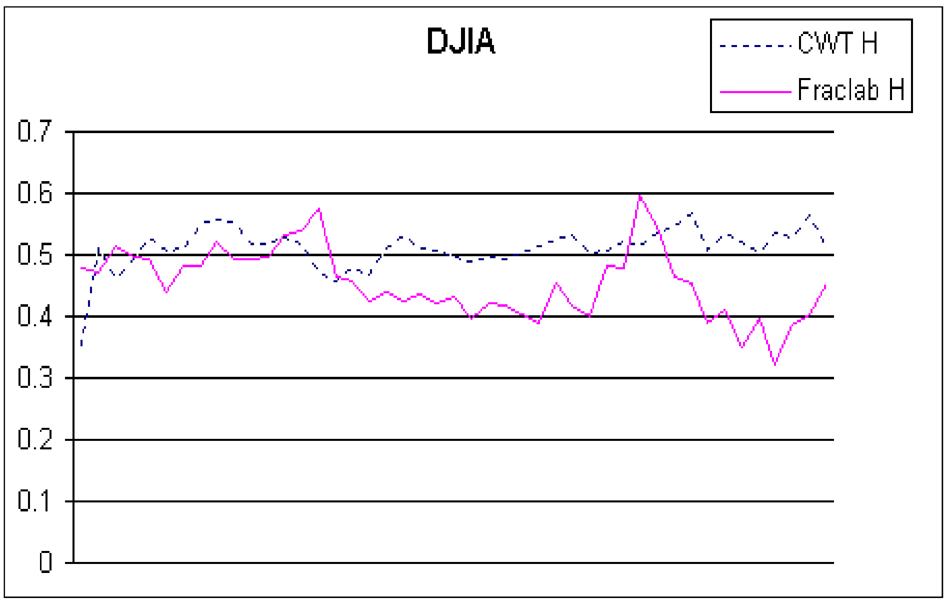

Figure 4: The time series of the DJIA index price (1930-2003) is divided in 45 subsamples. The Hurst exponent was calculated for windows of approximately 2000 observations. Results show the fairly substantial variability of the computed Hurst exponents, using the MATLAB wavelet MRA and even higher variability using the second methodology (developed by INRIA). Most prblematic is the divergence of the two value series.

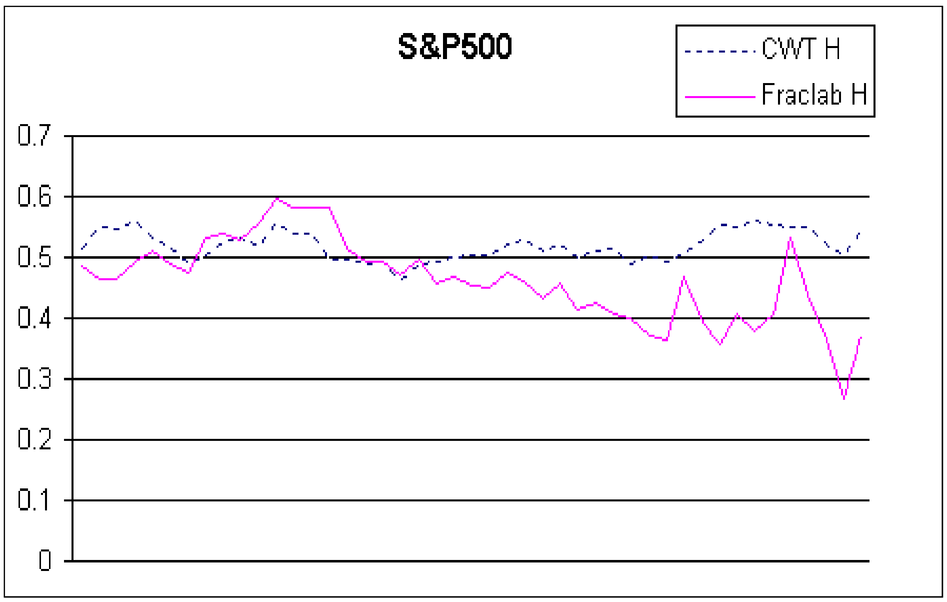

Figure 5: The time series of the S&P500 index price (1950-2003) is divided in 45 subsamples. The Hurst exponent was calculated for windows of approximately 2000 observations. Results show instability of the Hurst exponent, using the MATLAB wavelet MRA and even higher variability using the second methodology (developed by INRIA). The continuous wavelet transform shows greater stability around $H = 0.5$. The Fraclab H deviates considerably and tends to show that the S&P500 went from neutral to anti-persistent trading.

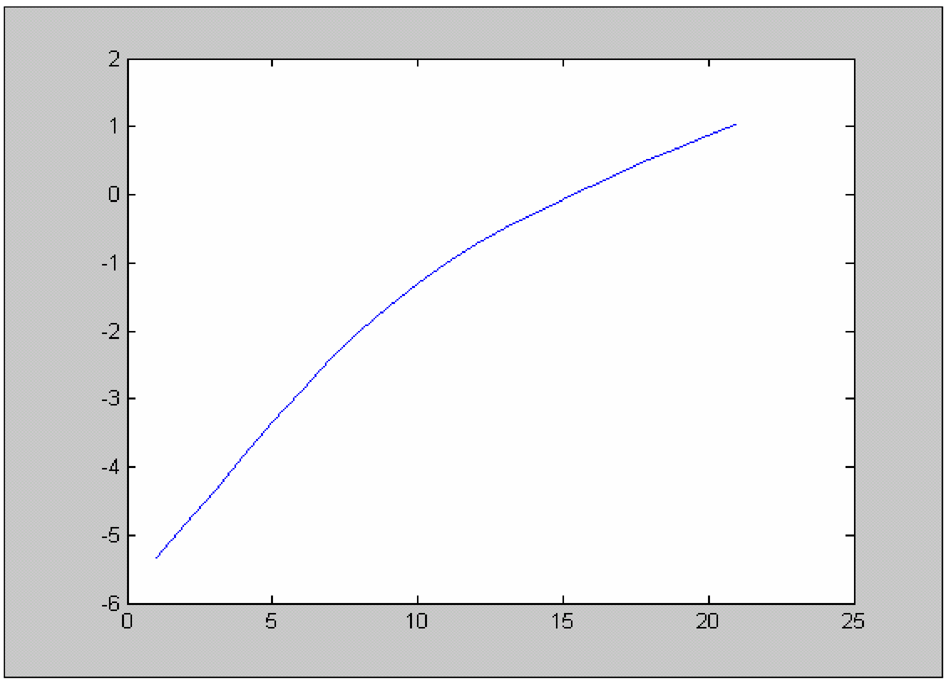

Figure 6: Plot of the decay scaling exponent $\tau(q)$ of the Gibb's partition function (y-axis) versus the moment order q (x-axis). This plot displays nonlinearity in the slope $\frac{\partial \tau(q)}{\partial q}$, indicating the existence of multifractality.

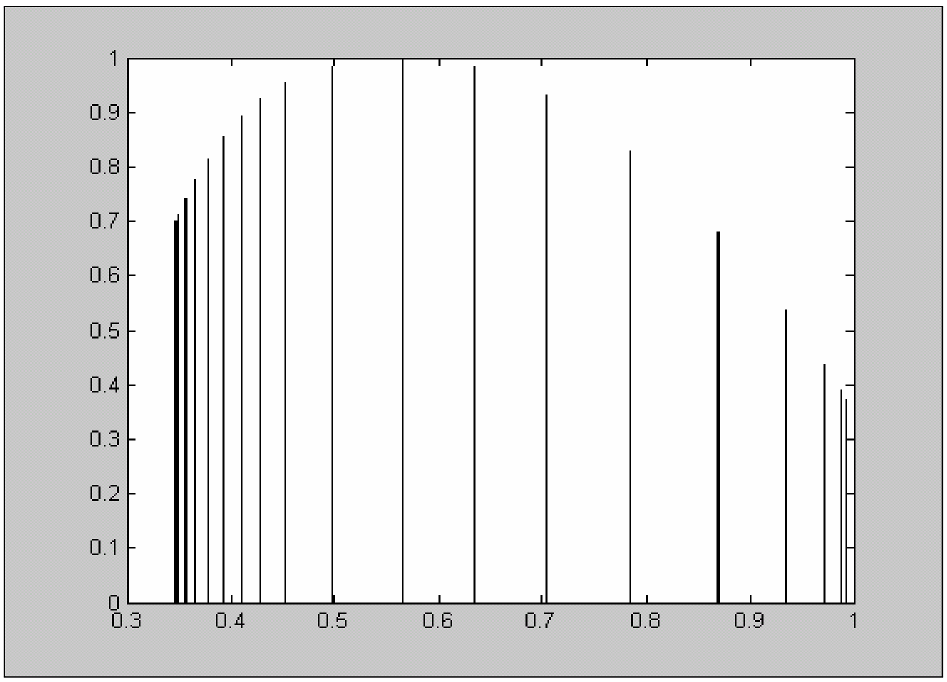

Figure 7: "Classical" multifractal spectrum (MFS) with Hölder regularity exponents on the x-axis and the fractal dimension $D(\alpha_L)$ on the y-axis. The Hurst exponent, with Hausdorff dimension of 1, is about 0.56, indicating slight persistence.

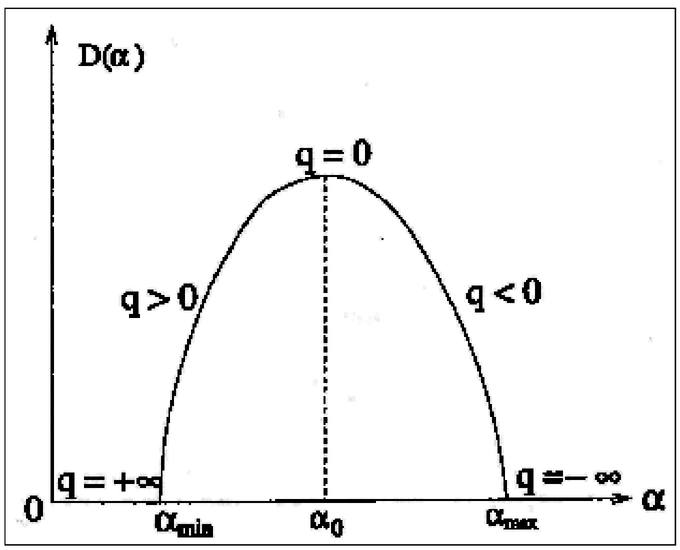

Figure 8: Theoretical multifractal spectrum, with the complete range of Lipschitz α_L, for the complete range of moment orders q of the market returns.

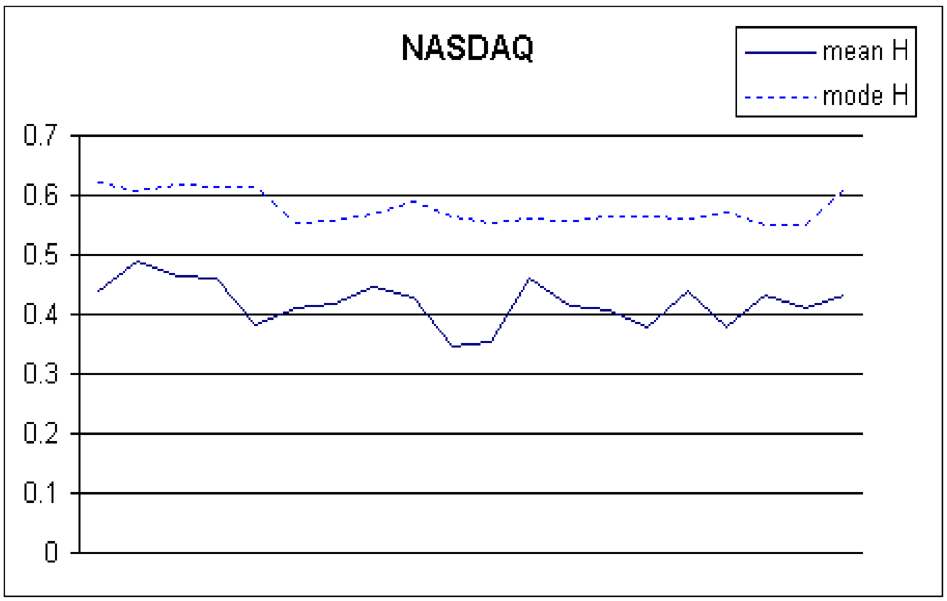

Figure 9: The NASDAQ index price series is first divided into 20 files having each an approximate length of 2000. The MFS of each time series window is then calculated. The graph displays the mode and the mean of the resulting spectra. Since the mode is above the mean in this series of windows, the underlying MFS is skewed to the left.

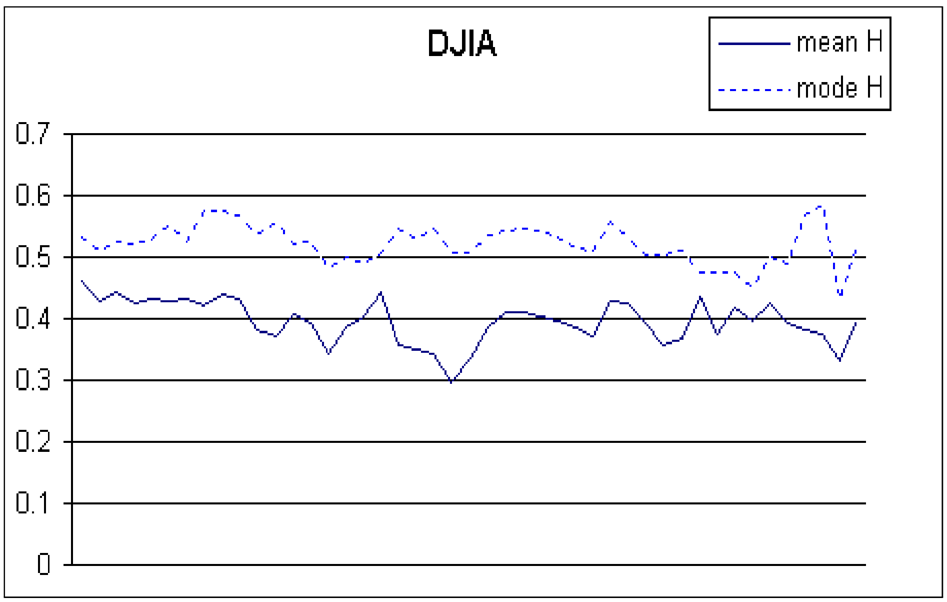

Figure 10: The DJIA index price series is first divided into 45 files having each an approximate length of 2000. The MFS of each time series window is then calculated. The graph displays the mode and the mean of the resulting spectra. The graph displays the mode and the mean of the resulting spectra. Since the mode is above the mean in this series of windows, the underlying MFS is weighted to the left.

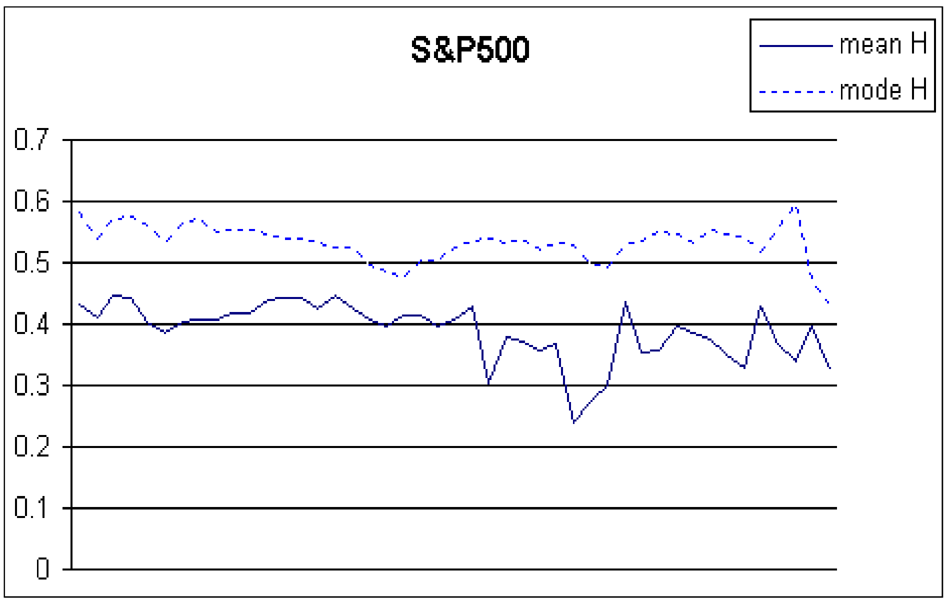

Figure 11: The S&P500 index price series is first divided into 42 files each having an approximate length of 2000. The MFS of each time series is calculated. The graph displays the mode and the mean of the resulting spectra. The graph displays the mode and the mean of the resulting spectra. Since the mode is above the mean in this series of windows, the underlying MFS is weighted to the left.

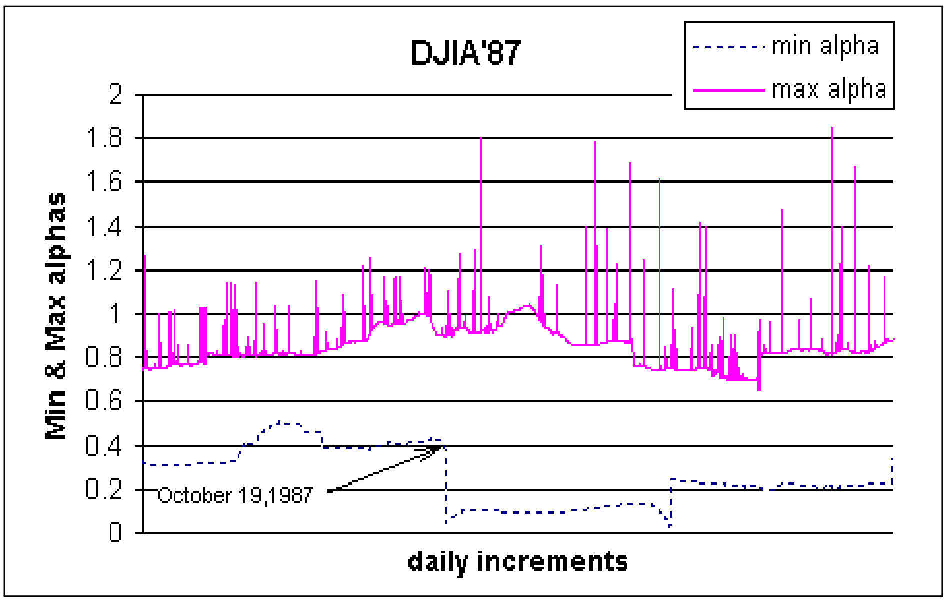

Figure 12: Plot of minimum and maximum Lipschitz-α_L's of the MFS (on the $y-axis$) using 512 days windows with daily increments (on the $x-axis$) displays sharp decrease of the minimum α_L when October 19, 1987 is first included in the calculations. The maximum α_L exhibits large spikes in the period after the crash.

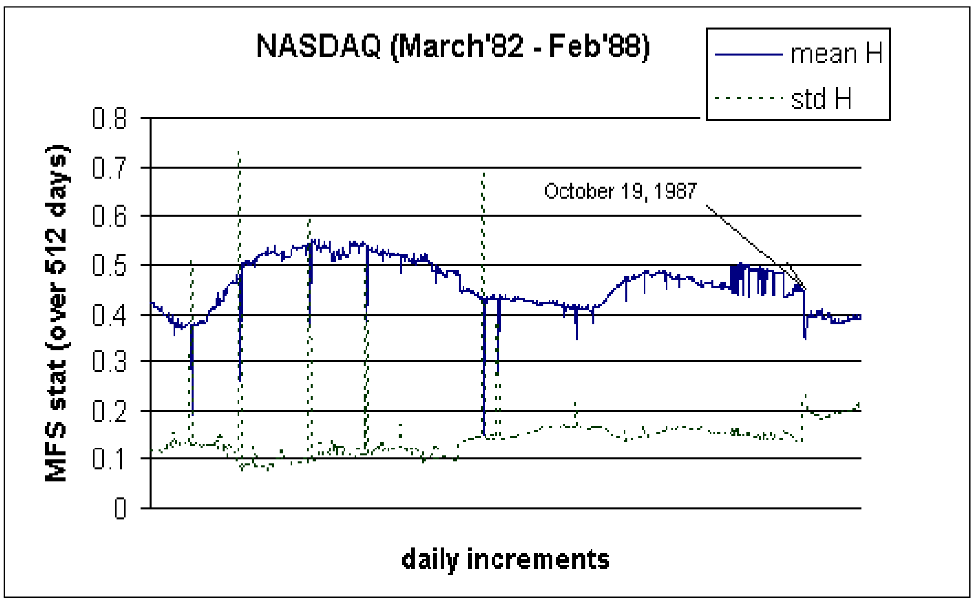

Figure 13: $Y - axis$: mean and standard deviation of the MFS over 512 days. $X - axis$: daily incremental shifts, each point is calculated from the MFS of 512 daily index prices. The mean and the standard deviation of the Hölder exponent sharply decreases and increases, respectively, when October 19,1987 is included. The period immediately prior to this crash is characterized by a sustained high mean and a low standard deviation in the MFS.

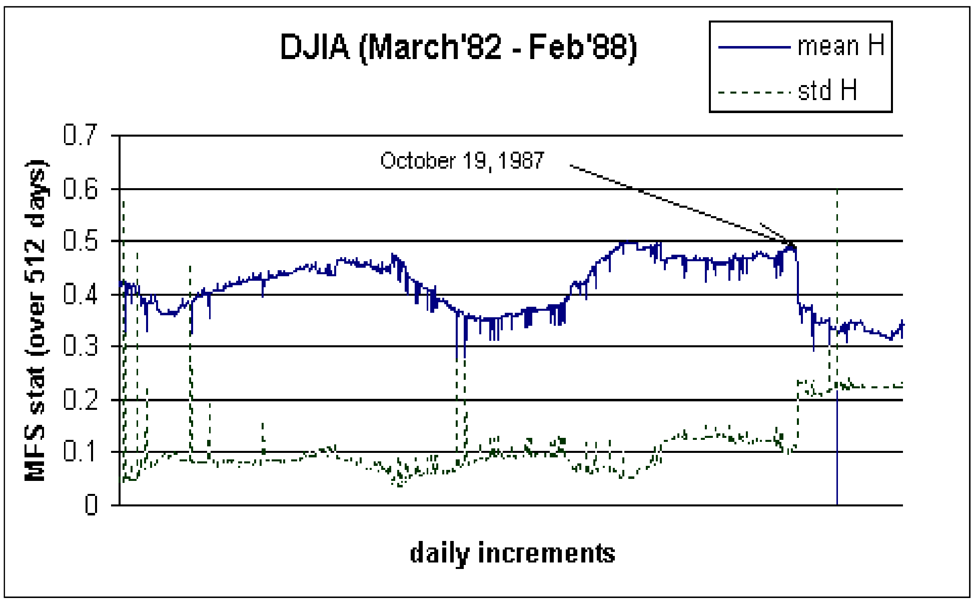

Figure 14: $Y - axis$: mean and standard deviation of the MFS over 512 days. $X - axis$: daily incremental shifts, each point is calculated from the MFS of 512 daily index prices. A period MFS with high mean and low standard deviation precedes the October 19, 1987 crash. The value of these first two MFS moments change sharply after the crash.

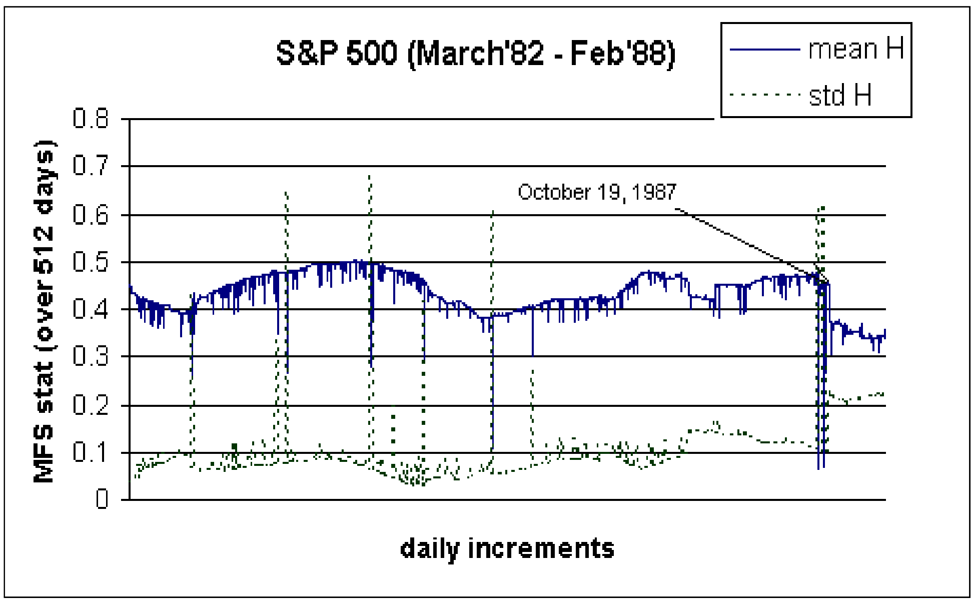

Figure 15: $Y - axis$: mean and standard deviation of the MFS over 512 days. $X - axis$: daily incremental shifts, each point is calculated from the MFS of 512 daily index prices. S&P500 Hölder exponent statistics before the crash of October 19, 1987 exhibit a similar pattern to that of the DJIA and the NASDAQ.

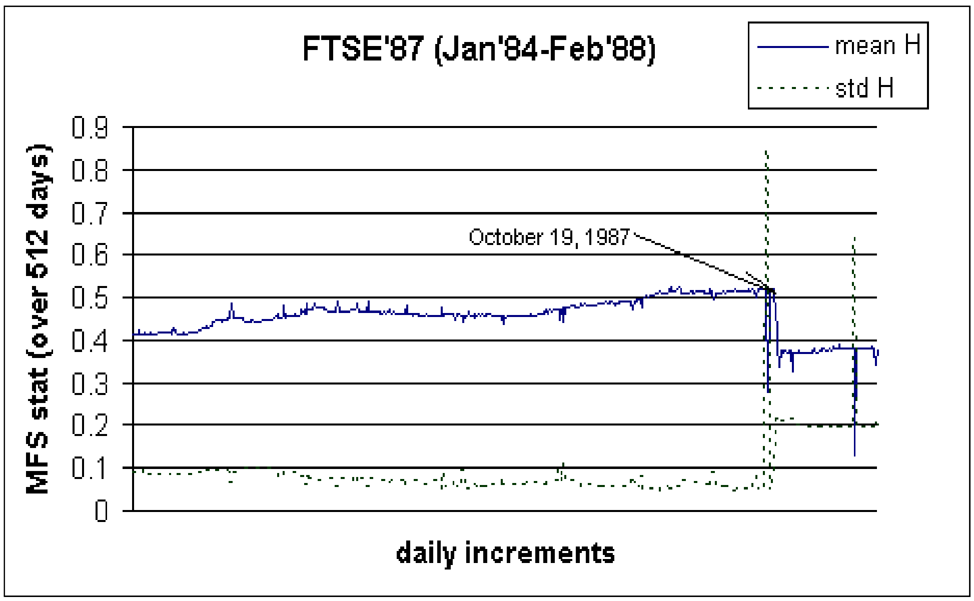

Figure 16: $Y-axis$: mean and standard deviation of the MFS over 512 days. $X-axis$: daily incremental shifts, each point is calculated from the MFS of 512 daily index prices. FTSE displays a steady increase of the average MFS regularity exponent and a very gradual decrease in its MFS standard deviation.

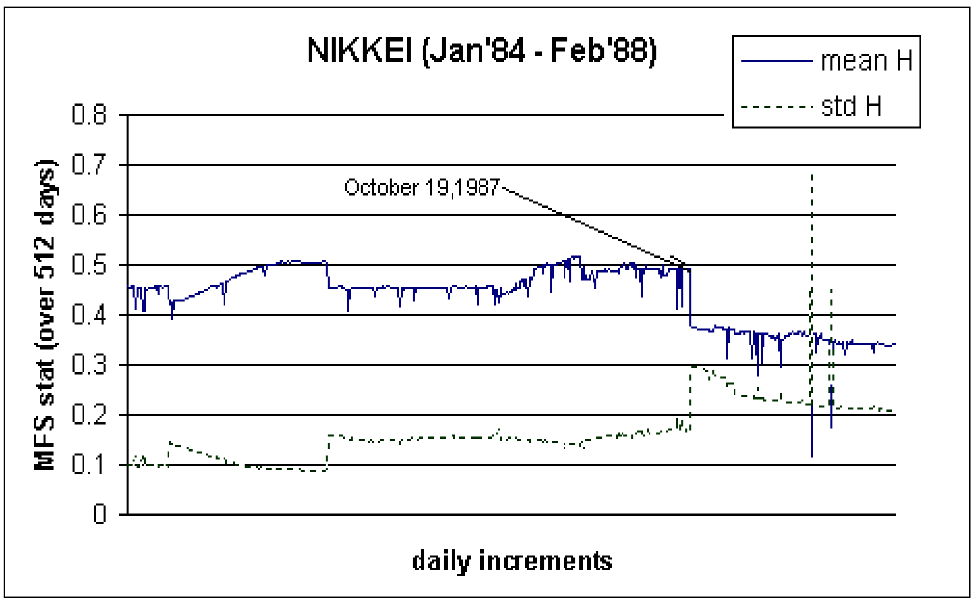

Figure 17: $Y - axis$: mean and standard deviation of the MFS over 512 days. $X - axis$: daily incremental shifts, each point is calculated from the MFS of 512 daily index prices. The Nikkei market index exhibits a similar pattern before the crash of October 19, 1987 as the other major stock market indices.

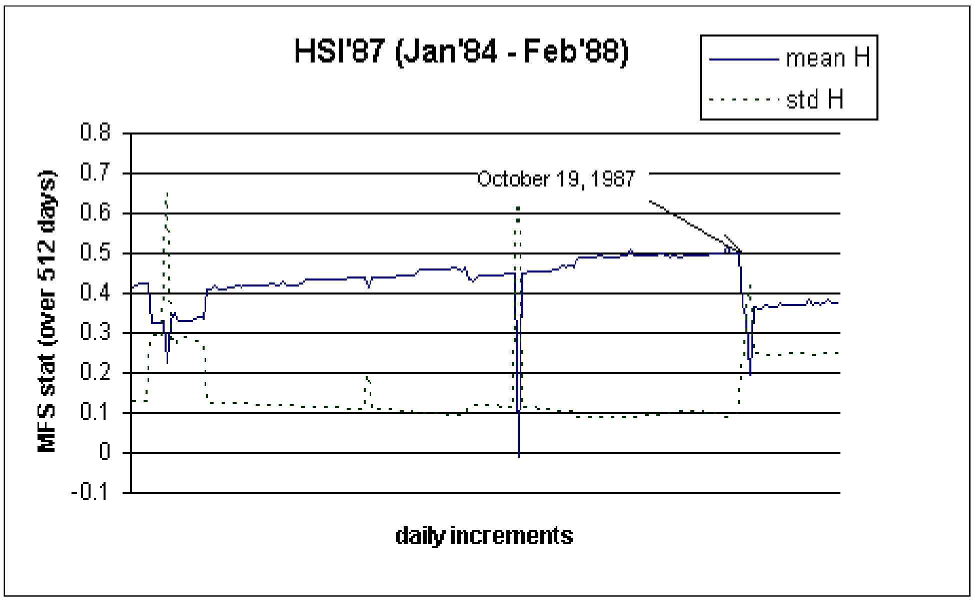

Figure 18: The dynamics of the Hang Seng index displays jumps in the MFS statistics of the regularity exponents. Before the crash of October 19, 1987 apparently a certain order builds up a pressure in the market. Notice again the increasing "gap" between the mean (upper line) and the standard deviation (lower line) of the MFS of this index before the October 19, 1987 crash.