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MEASURING FINANCIAL CASH FLOW AND TERM STRUCTURE DYNAMICS

Abstract

Financial turbulence is a phenomenon occurring in anti - persistent markets. In contrast, financial crises occur in persistent markets. A relationship can be established between these two extreme phenomena of long term market dependence and the older financial concept of financial (il-)liquidity. The measurement of the degree of market persistence and the measurement of the degree of market liquidity are related. To accomplish the two research objectives of measurement and simulation of different degrees of financial liquidity, I propose to boldly reformulate and reinterpret the classical laws of fluid mechanics into cash flow mechanics. At first this approach may appear contrived and artificial, but the end results of these reformulations and reinterpretations are useful quantifiable financial quantities, which will assist us with the measurement, analysis and proper characterization of modern dynamic financial markets in ways that classical comparative static financial - economic analyses do not allow.

Cornelis A. Los, Ph.D.
Kent State University, College of Business Administration BSA416, & Graduate School of Management, Kent, OH 44242, U
tel: 1-330-672-1207; fax: 1-330-672-9806; e-mail: clos@bsa3.kent.edu
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1 Introduction

Peters (1989, 1994) of PanAgora Management suggests that, to understand financial turbulence, the dynamics of cash flows between the various market participants, within and between different asset markets, should be analyzed and measured more carefully. Although there exists not yet a complete theory of physical turbulence, let alone a theory of financial turbulence, many parallels between the two phenomena have been noted by, for example, Mandelbrot (1982, 1998).

Simultaneously, the accurate measurement of financial illiquidity and of financial illiquidity risk has gained in importance, as the example of the tax - payer financed bail - out of the collapsed Long Term Capital Management (LTCM) hedge fund in 1998 demonstrates.[1] This hedge fund applied a trading strategy known as *convergence arbitrage,* which is based on the idea that if two securities have the same theoretical price, because they have the same return - risk profile, their market prices should eventually be the same. But this convergence strategy ignores the observation that financial risk is a time - dependent phenomenon and not a time - independent phenomenon to which the usual central limit theory based on i.i.d. assumptions applies. Indeed, in the summer of 1998 LTCM made a huge $4 billion loss. This was triggered when Russia defaulted on its debt, which caused a flight to quality in the German bond market.

LTCM itself did not have a large exposure to Russian debt, but it tended to be long illiquid German (off - the - run) bonds and short the corresponding liquid German (on - the - run) bonds. The spreads between the prices of the illiquid bonds and the corresponding liquid bonds widened sharply after the Russian default. Credit spreads also increased and LTCM was highly leveraged, with a debt/equity ratio of close to 300, so that the loss of profits on its hedges immediately translated in a very rapid vanishing of its market valued equity and threatened bankruptcy. When it was unable to make its projected "risk - free" arbitrage profits, it experienced huge losses and there were margin calls on its positions that it was unable to meet.

[1] This example is borrowed from Hull, 2001, p. 428 - 429. *Cf.* also Dunbar (2000) and Jorion (1999).

But LTCM did not operate in a vacuum. It operated in a particular financial environment. LTCM's position was made more difficult by the fact that many other hedge funds followed similar convergence arbitrage strategies. When LTCM tried to liquidate part of its portfolio to meet its margin calls, by selling its illiquid off - the - run bonds and by buying its liquid on - the run bonds, other hedge funds faced with similar problems tried to do similar trades. The price of the on - the - run bonds rose relative to the price of the off - the -run bonds. This caused the illiquidity spreads to widen even further and to reinforce the flight to quality. Thus the illiquidity problem was exacerbated and not alleviated. This was a clear example of long - term illiquidity risk dependence that was not foreseen by the time independence assumptions of geometric Brownian motion arbitrage models used by LTCM's managers and partners.[2]

Despite its obvious importance, the proper empirical measurement and analysis of liquidity and of illiquidity risk is still in its infancy. For example, there is still no agreement on how financial market liquidity should be measured. More precisely stated: there does not yet exist a measurement standard for the various *degrees of financial liquidity*. Which levels of financial illiquidity are prone to generate financial catastrophes, which levels of financial illiquidity allow regular, liquid trading activity, and which levels of ultra financial liquidity are prone to generate financial turbulence?

Therefore, in this paper we discuss the empirical measurement of the dynamics of market illiquidity, in particular of the illiquidity of cash flows, which is directly related to the dynamics of the term structure of rates of return on cash investments and to the concept of (bond and equity) duration.

For a recent survey of current theoretical work on term structure dynamics, see the recent article by Hu (2001). For a thorough discussion of a few current approaches to the empirical

[2] Some of these partners which were the same Nobel Prize winners, who had proposed this theory in 1973. At least, they had courageously put their money where there mouthes were. But the empirical experiment of 1998 drastically refuted the non-empirical assumptions of their theory......"A Trillion Dollar Bet." is an excellent pedagogic video documentary of the rise and fall of LTCM, of the crucial role the Nobel prize winning Black-Scholes solution model in this financial market affair, and of the regulatory aftermath of LTCM's bail-out by the Fed.

model identification of term structure dynamics, in particular of its principal component analysis, see the companion article by Chapman and Pearson (2001). In 1989 we thoroughly criticised and rejected principal component analysis, since it was non - scientifically subjective (Los, 1989).

To accomplish the two research objectives of measurement and simulation of various degrees of financial liquidity illiquidity, I propose to boldly reformulate and reinterpret the classical laws of fluid mechanics into financial cash flow mechanics, in particular, the laws of conservation of investment capital (= "mass"), of the financial cash flow rate (= "momentum") and of the rate of financial cash return (= $2\times$ "energy"). My newly proposed financial cash flow laws are unashamedly modeled on the basic conservation laws of hydraulics and aerodynamics.[3]

For example, we'll study a "scale - by - scale cash flow rate of return budget equation," which allows us to interpret the transfer of cash returns among different scales of investment cash flow (*cf.* Los, 1999). For more detailed background on the formulation of these laws in the mechanics of incompressible fluids, see Batchelor (1970), Landau and Lifshitz (1987) and Tritton (1988). Van Dyke 's (1982) album provided me with inspiring visualizations of physical fluid dynamics and turbulence.

At first this approach may appear contrived and artificial, but the end results of these reformulations and reinterpretations are various useful quantifiable financial quantities, which will assist us with the measurement, analysis and characterization of the modern dynamic financial

[3] A flow dynamics approach to the cash flows in financial markets is not new in economics. Compare, for example, the physical income flow dynamics modeling of an economy in the 1950s at the London School of Economics and Political Science (LSE) by the engineer and economist A. W. Phillips, familiar from the (expectations - augmented) Phillip's Curve in undergraduate textbooks on macro-economics. Unfortunately, the financial economic research of markets has focused on the static "equilibrium theory " of markets of Nobel Memorial prize winners Gerard Debreu and Kenneth Arrow, while an dynamic "transport theory " has mostly been neglected. Consequently, the dynamics of cash flows has not been further developed since Phillips' heuristic macro - economics research. But fluid dynamics has a long mathematical history going back to Dutch mathematician and engineer Simon Stevin 's *Hydrostatics* (1605), Italian Evangelista Torricelli's law of e- ux (1644), and French Blaise Pascal's uniform pressure law of hydrostatics (1653). Sir Isaac Newton devoted over one quarter of his famous *Principia* book to the analysis of fluids (Book II, 1713) and added some original ideas, like his hypothesis of viscosity. This was followed by the mathematical explanations of the mechanics of ideal, frictionless fluid mechanics by Daniel and Jacques Bernouilli (1738) and Leonhard Euler (1755). In 1827, Claude Navier derived the equations of viscous flow, which were published by Sir George Gabriel Stokes in 1845, the celebrated Navier - Stokes equations. Work by Joseph Boussinesq (1877) and Osborne Reynolds (1883) on laminar (streamlined) flow and turbulent (erratic) flow extended these Navier - Stokes equations to turbulent flow, by including the Reynolds stresses and measurements, to be discussed in this paper.

markets. They will do that in ways that could not be achieved by the concepts of finance and economics, which were developed in Marshallian comparative static equilibrium models and not in the currently fashionable dynamic partial differentiation equations, which are, for example, used in stochastic dynamic asset valuation. Despite some difficulties we encounter with such an unashamed copy - cat approach, the current underdeveloped status of the liquidity analysis of the financial markets forces us us to break new grounds so that we can make slow but steady progress in this difficult area of risk measurement and analysis of dynamic cash flow markets.

The fundamental cash growth rate accounting framework has already been developed and its empirical applicability has been established by Karnosky and Singer (1994, 1995) and by Los (1998a & b), who extended it to Markowitz' classical mean - variance portfolio optimization and risk attribution framework (Los, 2001). It is easy to extend it further beyond mean - variance analysis, taking account of the latest developments in stable distribution and extreme value theories. In this paper, we'll focus on the term structure of so - called *laminar* and *turbulent* financial cash flows. By way of several examples we'll demonstrate the empirical applicability of these new ideas and compute, for example, the coefficient of illiquidity (= "kinetic viscosity of cash flows") for a term structure based on the S&P500 stock market index. This is a direct follow - up of our earlier, more philosophical remarks about the importance of time - frequency analysis of financial time series for financial management (Los, 2000a).

In another paper, we discuss the theory of financial turbulence, starting with Kolmogorov's 1941 explanation for homogeneous fluid turbulence. In that companion paper, we'll draw upon examples from meteorology, as originally suggested in Los (1991) and we'll describe the Galerkin procedure, based on the wavelet multiresolution analysis (MRA) to solve the Navier - Stokes equations of financial turbulence, which may become central to the study of turbulence in the global financial markets.

One important point we should emphasize: financial turbulence is not a "negative" phenomenon, in contrast to financial catastrophes or crises. Turbulence is an *efficiency enhancing*, deterministic, dynamic phenomenon, that we have to understand better for a full appreciation of the ongoing proper functioning of the financial markets. Financial turbulence is a limited phenomenon that can be observed under particular conditions of well - functioning financial markets, such as in the foreign exchange markets of the Deutschemark, the Euro or the Japanese Yen, when these markets are coping with *differentials in liquidity.*

In other words, financial *turbulence*, which occurs in anti - persistent financial markets, must be sharply differentiated from financial *crises,* which are unpredictable discontinuities and truly catastrophic crisis phenomena occurring in highly persistent financial markets. Such financial crises we either need to prevent by changing the institutional structure of the existing financial markets to make them less persistent and more liquid, or against which we need to insure and hedge by trading financial catastrophe bonds or derivatives, to prevent their truly catastrophic consequences, as suggested by Chichilnisky and Heal (1992) and Chichilnisky (1996).

Can a financial crises, which emerge in persistent fianancial markets, like real estate, stock, bond and commercial loan markets, generate financial turbulence in interlinked anti - persistent financial markets, like the foreign exchange and cash markets? Yes, it can and it does, as we've observed in several Asian financial markets in 1997 - 1998.

2 Dynamic Financial Cash Flow Theory

When financial cash is in motion, its properties are described at each point in time by the properties of its cash flow rate or volume flux.

Definition 1 *The financial* ***cash flow rate****, or* ***cash volume flux****,* $\Delta X(t)$*, is the product of the cash invested (= cash position)* $X(t-1)$ *at the beginning of time period* t *and its cash rate of return (= cash velocity)* $x(t)$*:*

$$\Delta X(t) = x(t)X(t-1) \tag{1}$$

since the value of an asset grows as

$$X(t) = X(t-1).e^{x(t)} \tag{2}$$

so that (approximately)

$$\ln \frac{X(t)}{X(t-1)} = x(t) = \frac{\Delta X(t)}{X(t-1)} \tag{3}$$

Example 2 *According to the exact cash accounting framework, at time t an investor has three possible investment instruments: (1) investment in an asset k in country i with rate of return $r_{ik}(t)$, (2) a cash swap with rate of return $c_j(t) - c_i(t)$, with $c_j(t)$ being the risk free cash rate in country j into which the nominal is swapped, and $c_i(t)$ being the risk free cash rate in country i out of which the nominal is swapped and (3) the foreign currency appreciation rate $f_j(t)$ of country j.*[4] *Thus, one particular bilateral investment strategy at time t is represented by the following strategic rate of return*

$$x_{ijk}(t) = r_{ik}(t) + [c_j(t) - c_i(t)] + f_j(t) \tag{4}$$

with $i, j \in \mathbb{Z}_1^2$ being the indices for bilateral cash flows between the various countries and $k \in \mathbb{Z}_2$ being the index for the various assets (stocks, bonds, real estate, commodities, etc.). Notice that such an international investment strategy is, according to the Capital Asset Pricing Model, equivalent to the sum of the asset risk premium in local market i $[r_{ik}(t) - c_i(t)]$ and the cash return on currency j, $[c_j(t) + f_j(t)]$ (Los, 1998, 1999; Leow & Los, 1999).

This leads to the following definitions of continuous and steady cash flows.

Definition 3 ***Continuous*** *financial cash flow occurs in a particular cash flow channel, when its cash flow rate $\Delta X(t)$ is constant*

$$\Delta X(t) = X(t-1)x(t) = \text{constant (= independent of time)} \tag{5}$$

Remark 4 *This equation is also called the* ***cash continuity equation****.*

Definition 5 ***Steady*** *financial cash flow occurs in a particular investment channel, when its cash flow return rate $x(t)$ = constant (= independent of time).*

One would expect, under conditions of continuous cash flows, that the cash flow return rate $x(t)$ to be high when the cash flow channel is constricted, *i.e.*, where the investment position $X(t-1)$ is small, and the cash flow return rate $x(t)$ to be low where the channel is wide, *i.e.*, where the investment position $X(t-1)$ is large.

Example 6 *Day traders, who have very short investment horizons, trade very quickly with small investment positions, in the order of a \$1,000 and reap highly fluctuating returns $x(t)$ with very large amplitudes $|x(t)|$. In contrast, institutional investors, such as pension funds, insurance funds, or large mutual funds, who usually have long term investment horizons, trade much slower with very large investment positions in the order of \$100 million, and who invest for steady rates of return $x(t)$ with moderate amplitudes $|x(t)|$. The trillion dollar research question is: can such*

[4] *In such cash accounting frameworks, one usually adopts the US dollar as the base currency, or numéraire.*

different types of traders, with investment positions of different size and investment horizons of different lengths, coexist in the same global markets without financial turbulence, or is that a necessary and unavoidable consequence? Since day traders add liquidity and reduce the persistence of a financial market, it is likely that they reduce the possible occurrence of financial crises, which is a phenomenon occurring in illiquid, persistent financial markets, when suddenly large investment positions are set up or unwound.

2.1 Perfectly Efficient Dynamic Financial Markets

The motion of empirical cash flows in globally interconnected financial markets is very complex and still not fully understood, because the flows start and stop at various times and go back and forth between millions of bilateral positions. Until very recently, financial economists have proceeded by making several heroic, simplifying assumptions for stationary, and often static, markets. Following the classical definitions of ideal, perfectly efficient physical flows, the following is my suggestion for a new theoretical definition of perfectly efficient dynamic financial markets, which can provide a standard for the comparative measurement of dynamic cash flows in non - stationary markets.

Definition 7 ***Perfectly efficient dynamic financial markets*** *have cash flows, which are:*

(1) ***perfectly liquid*** *(or* ***nonviscous****): there is no internal (institutional) friction in the markets and all assets of different maturities in all markets can be instantaneously purchased and liquidated;*

(2) ***steady****: the rate of return $x(t)$ of each cash flow remains constant;*

(3) ***incompressible****: the density of the cash transactions remains constant in time, i.e., the volume of market buy and sell transactions remains uniform over time;*

(4) ***irrotational****: there is no angular momentum of the cash flows, i.e., all cash flows in a particular investment channel at a time t are in the same direction. All decision makers with different time horizons for investment receive the same information at the same time and interpret that information correctly and simultaneously (= the market's rational "herd instinct").*

Thus, in the ideal situation of a perfectly efficient dynamic financial market, the cash flows in the various investment channels are all streamlined in the same direction (there are no cross - over trades) and move at the same constant flow rates $x(t)$. There are no "bulls" (positive investors) and "bears" (negative investors) trading at the same maturity levels. This means, for example, that the term structure of interest rates can only move parallel to itself and is not allowed to rotate. This is, of course, an abstraction from empirical financial market reality, where rotations in the term structures of interest rates are a, difficult to model, empirical phenomenon (*cf.* Chapman and Pearson, 2001).

2.2 Financial Pressure and Cash Flow Power

Under such abstract, perfectly efficient dynamic financial market conditions, one can formulate a Bernoulli equation for cash flows which quantifies the concept of financial pressure.[5]

Definition 8 *(**Bernoulli equation for cash flows**) In perfectly efficient dynamic financial markets*

$$FP(t) + \frac{1}{2}\rho x^2(t) = constant \tag{6}$$

*where $FP(t)$ is the **financial pressure** at time t and $\rho = \frac{1}{\$^2}$ is the uniform cash density.*

Remark 9 *The Bernouilli cash flow equation is also called the **cash energy equation**.*

This Bernoulli equation for cash flows states that the sum of financial pressure $FP(t)$ and the kinetic energy of the cash flow per dollar, $\frac{1}{2}\rho x^2(t)$, is constant along a cash flow investment channel (streamline). This Bernoulli equation for cash flows describes a *Venturi cash flow channel* to measure the difference in financial pressure between two different places along an investment cash flow channel, as follows. The Bernoulli equation implies that

$$FP_i(t) + \frac{1}{2}\rho x_i^2(t) = FP_j(t) + \frac{1}{2}\rho x_j^2(t), \text{ where } (i,j) \in \mathbb{Z}^2 \tag{7}$$

so that the *difference in financial pressure* $FP_i(t) - FP_j(t)$, measured simultaneously at two different points i and j in a particular investment cash flow channel at time t, is the difference between the kinetic energies of the uniform cash flow at these two points:

$$\begin{aligned} \Delta_{ij} FP(t) &= FP_i(t) - FP_j(t) \\ &= \frac{1}{2}\rho \left[x_j^2(t) - x_i^2(t)\right] \\ &= -\frac{1}{2}\rho \Delta_{ij} x^2(t) \end{aligned} \tag{8}$$

When the financial pressure at "upstream" point i is higher than at "downstream" point j, the cash flow velocity at "downstream" point j is higher than at "upstream" point i, and vice versa.

[5] Daniel Bernoulli's most famous work, *Hydrodynamics* (1738) is both a theoretical and practical study of equilibrium, pressure and velocity of fluids. He demonstrated that as the velocity of fluid flow increases, its pressure decreases.

Financial pressure is relatively high where investment cash flow moves slowly, and is relatively low where it moves fast.

This point about the cash flow pressure differential becomes clearer, when we take account of the size of the investment positions. When the cash flow is continuous, as we thus far have assumed, thus, when

$$x_i(t)X_i(t-1) = x_j(t)X_j(t-1) = \text{constant} \tag{9}$$

so that

$$x_j(t) = \frac{X_i(t-1)}{X_j(t-1)}x_i(t) \tag{10}$$

then

$$\begin{aligned}
\Delta_{ij}FP(t) &= -\frac{1}{2}\rho\Delta_{ij}x^2(t) \\
&= \frac{1}{2}\rho\left[x_j^2(t) - x_i^2(t)\right] \\
&= \frac{1}{2}\rho\left[\left(\frac{X_i(t-1)}{X_j(t-1)}x_i(t)\right)^2 - x_i^2(t)\right] \\
&= \frac{1}{2}\rho\left[\left\{\left(\frac{X_i(t-1)}{X_j(t-1)}\right)^2 - 1\right\}x_i^2(t)\right] \\
&= \frac{1}{2}\rho\left[\left\{\frac{X_i^2(t-1) - X_j^2(t-1)}{X_j^2(t-1)}\right\}x_i^2(t)\right]
\end{aligned} \tag{11}$$

Note that $\Delta_{ij}FP(t) > 0$ if and only if $X_i^2(t-1) > X_j^2(t-1)$. This financial market cash flow system operates as follows. Consider a particular very crowded asset investment market, with many market participants, where a lot of cash is squeezed together. As soon as an extra investment channel is opened and cash begins to exit that particular asset market, the squeezing, or pressure FP, is least near this small exit, where the motion (= cash flow rate of return $x(t)$) is greatest. Thus, under the assumption of perfectly efficient dynamic financial markets, the lowest financial pressure occurs where the investment channel is smallest and the cash return rate is highest, *e.g.*, in emerging financial markets, where the marginal productivity and the risk of capital investment is highest. The highest financial pressure occurs where the investment channel is largest and the cash return rate is lowest, *e.g.*, in developed financial markets, where the marginal productivity and the risk of capital investment is lowest.

Remark 10 *One should be on the alert for financial turbulence when the difference in financial pressure between the upstream developed and downstream emerging financial markets, $\Delta_{ij}FP(t)$, is large, because the difference in the cash investments in each of the two connected markets $X_i^2(t-1)-X_j^2(t-1)$ is very large, so that the kinetic energy $x_j^2(t) = \left(\frac{X_i(t-1)}{X_j(t-1)}x_i(t)\right)^2$ in the underdeveloped downstream market is very large, since the uniform kinetic energy term $\frac{1}{2}\rho x^2(t)$ figures prominently in the financial Reynolds number (see below). A high Reynolds number empirically measures the onset of financial turbulence.*

Multiplying the kinetic energy of the uniform cash flow by its flow rate provides the rate at which the energy is transferred, *i.e.*, it provides its power.

Definition 11 *The **power of a cash flow** at time t is defined by*

$$\begin{aligned} Power(t) &= \left[\frac{1}{2}\rho x^2(t)\right][x(t)X(t-1)] \\ &= \frac{1}{2}\rho x^3(t)X(t-1) \end{aligned} \tag{12}$$

Therefore, the *available cash flow power per dollar invested* is measured by

$$\frac{Power(t)}{X(t-1)} = \frac{1}{2}\rho x^3(t) \tag{13}$$

and the *average available cash flow power per dollar invested* is

$$\frac{1}{2}\rho E\left\{x^3(t)\right\} \tag{14}$$

Notice that these expressions involve the third moment (skewness) $m_3 = E\left\{x^3(t)\right\}$ of the distribution of the cash flow rates of return in the term structure of spot rates. For (symmetric) Gaussian investment rates returns $m_3 = 0$. For asymmetric distributions of investment rates of return $m_3 \neq 0$. Thus, the average available cash flow power per dollar invested is nil, when the distribution of the rates of return is symmetric around zero, *e.g.*, when it is as likely to make money as it is to lose it. The average cash flow power is positive when the distribution of the rates of return is positively skewed, and negative when their distribution is negatively skewed.

3 Illiquidity and Financial Turbulence

In the imperfect empirical international financial markets, in which asset investors with different cash investment horizons - for example, day traders, bank treasurers and pension fund managers - trade, friction can occur between the various investment cash flows, which show a great diversity of changing strategic cash flow rates of return $x_{ijk}(t)$. Such friction can cause either financial turbulence, or financial crises, depending on the degree of persistence of the financial markets as measured by the persistence of these time series of the strategic rates of return. Now that we have defined continuous and steady cash flows, let's, initially rather informally, define turbulent cash flow.

Definition 12 ***Turbulent*** *financial cash flow is irregular cash flow characterized by small whirlpool - like regions, vortices, "eddies "*

$$\Delta X(t) = x(t)X(t-1) = \textit{irregular, with possible vortices} \tag{15}$$

Remark 13 *This effectively means that the cash flow rates of return series $x(t)$ is irregular. We've already characterized and measured various degrees of irregularity by computing the fractal spectrum of financial cash flow rates based on the Lipschitz-α_L and the corresponding fractal dimensions (Karuppiah and Los, 2000).*

But what can we imagine a cash flow vortex or "eddy" to be? We suggest the following definition.

Definition 14 *A financial cash flow* ***vortex*** *occurs, when the fractal differences $\Delta^d x(t)$ first rapidly increase in density (become quickly more rapid) and then rapidly decreases in density (become quickly less rapid). In other words, a cash vortex occurs,, when the fractional differentiation (in particular the second order differentiation, or convexity) of $x(t)$ is non - homogeneous and there exists a fractal spectrum of irregularities.*

Thus cash flow vortices are regions of non - homogeneous fractional differentiation of the spot rates of return $x(t)$, which can lead to period - doubling and intermittent behavior. Period doubling and intermittency will be discussed in the next chapter. But, it shows up in the records of financial time series as *clustering* of the increments of the Fractional Brownian Motion (FBM). For an example, notice the vortex in the Philippine Pesos after the Thai baht crisis on July 2nd, 2001 in Fig. 1.

3.1 Illiquidity: Cash Flow Viscosity

Why would such financial turbulence, consisting of a series of financial vortices, occur? The physical *theory of flow dynamics* may provide us with some clues, or, at least, with some quantifiable entities, which can measure when turbulence occurs, even though no completely acceptable model or theory of financial turbulence exists. The term *viscosity* is used in flow dynamics to characterize the degree of internal friction, or illiquidity, in a fluid. Equivalently, we can define in finance:

Definition 15

$$\begin{aligned} \textbf{\textit{financial cash flow viscosity}} &= \textbf{\textit{fianncial cash flow illiquidity}} \\ &= \textit{degree of illiquidity of adjacent financial cash flows} \end{aligned} \tag{16}$$

This internal cash flow friction is associated with the resistance of two adjacent layers of cash flows to move relative to each other, due to *Newton's Second Law* for flows

Proposition 16 ***(Newton's Second Law)*** *Any force applied to a flow results in an equal but opposite reaction, which in turn causes a rate of change of momentum in the flow.*

Because of the illiquidity of investment cash flows, part of their steady kinetic energy, represented by a finite, integrated amount of energy $\int |x(t)|^2 \, dt \ < \infty$ is converted to thermal (= "random") energy, represented by an infinite integrated amount of energy $\int |x(t)|^2 \, dt \ \to \infty$, which can cause cash vortices ("eddies") on the edges of the adjacent investment cash flow channels.

Again, analogously to definitions in physical flow dynamics, we can define a *coefficient of cash flow illiquidity*, which would allow us to measure the degree of illiquidity in the financial markets, as follows.

Definition 17 *The* ***ex post term structure*** $x_\tau(t)$ *of any asset at time* t *is defined by the expression*

$$\begin{aligned} x_\tau(t) &= \left[\frac{X(t)}{X(t-\tau)} \right]^{\frac{1}{\tau}} - 1 \\ &= \frac{\ln X(t) - \ln X(t-\tau)}{\tau} \end{aligned} \tag{17}$$

This is easy to see, since

$$[1 + x_\tau(t)]^\tau = \frac{X(t)}{X(t-\tau)} \tag{18}$$

so that, by approximation,

$$\begin{aligned} \tau \ln [1 + x_\tau(t)] &= \tau x_\tau(t) \\ &= \ln X(t) - \ln X(t-\tau) \end{aligned} \tag{19}$$

It is important to understand that the ex post term structure $x_\tau(t)$ of an asset is nonlinearly related to the corresponding cash flows $X(t)$ and $X(t-\tau)$, defining the asset and its value. This becomes even clearer in the following examples of the fundamental and derivative instruments used in the financial markets and the way they are accounted for by the double - entry bookkeeping system of market values.

Example 18 *The fundamental security of a simple (Treasury)* ***bond*** *at time t can be decomposed into a series of zero coupon bonds. A zero coupon bond, or pure discount bond, with maturity τ and principal payment $B_\tau(t)$, has zero coupons, so that its (discounted) present value is*

$$PB_0(t) = \frac{B_\tau(t)}{[1 + x_\tau(t)]^\tau} \tag{20}$$

Thus, we have, ex post,

$$\begin{aligned} [1 + x_\tau(t)]^\tau &= \frac{B_\tau(t)}{PB_0(t)} \\ &= \frac{X(t)}{X(t-\tau)} \end{aligned} \tag{21}$$

A $\tau-$year ***spot interest rate*** *at time t is the interest rate of a zero bond*

$$x_\tau(t) = \left[\frac{B_\tau(t)}{PB_0(t)}\right]^{\frac{1}{\tau}} - 1 \tag{22}$$

This ***term structure of interest rates*** *is the nonlinear relationship between the spot interest rate $x_\tau(t)$ and the maturity τ for a particular grade of credit quality of obligations (e.g., bills, notes, bonds) (Vasicek, 1977; Cox, Ingersöll and Ross, 1985; Heath, Jarrow and Morton, 1992; Los, 2001, p. 52).*

Example 19 *In Fig. 2 we've plotted the term structure of interest rates, as follows. We define the following three coordinates: $x \equiv x_\tau(t)$ the term structure, $y \equiv \frac{B_\tau(t)}{PB_0(t)} = \frac{X(t)}{X(t-\tau)}$ the relative value of the bond investment (when $0 < y < 1$ it is a premium bond; when $y = 1$ a par bond; and when $1 < y$ a discount bond), and $z \equiv \tau$, the maturity in years, so that*

$$x - y^{\frac{1}{z}} = -1, 0 < y, 0 < z < 30 \tag{23}$$

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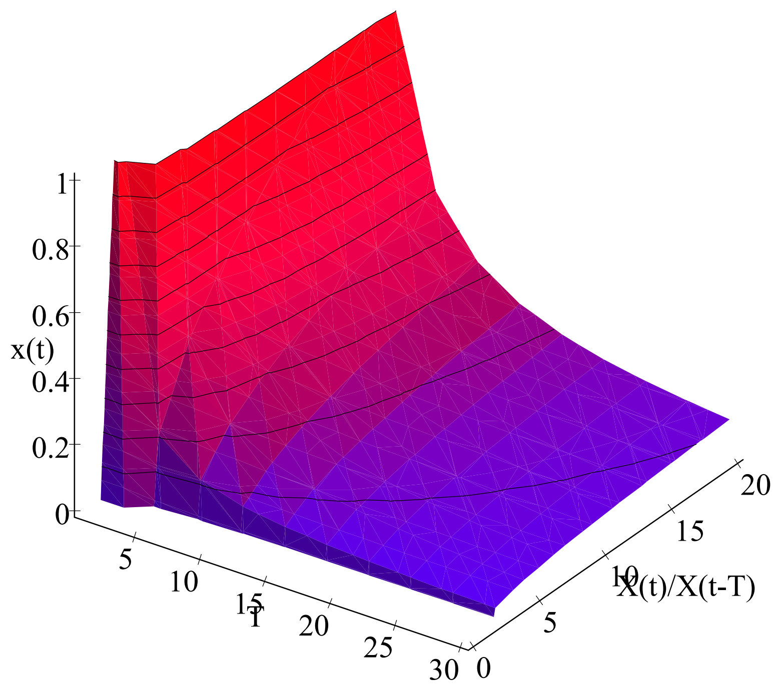

Term structure of interest rates $x_\tau(t)$as function of maturity $\tau$and gross investment cash flow change $X(t)/X(t-\tau)$

Example 20 *For the second fundamental security of a* ***stock*** *we have the following valuation. For (not necessarily constant) dividend payments $D_\tau(t)$ and a cost of capital $x_\tau(t)$ at time t, the dividend discount model (DDM) of stock valuation for an ongoing concern (= a firm with an infinite life time) computes the stock's present value at time t to be equivalent to an infinite series of "zero coupon bonds" with maturities $\tau = 1, 2, ..., \infty$*

$$PS_0(t) = \sum_{\tau=1}^{\infty} \frac{D_\tau(t)}{[1 + x_\tau(t)]^\tau} \tag{24}$$

Thus for each of the dividend payments we have the cash flow relationship

$$\begin{aligned} [1 + x_\tau(t)]^\tau &= \frac{D_\tau(t)}{PS_0(t)} \\ &= \frac{X(t)}{X(t-\tau)} \end{aligned} \tag{25}$$

so that the $\tau-$year ***spot dividend yield*** *is*

$$x_\tau(t) = \left[\frac{D_\tau(t)}{PS_0(t)}\right]^{\frac{1}{\tau}} - 1 \tag{26}$$

Example 21 *The rational law of* ***pricing by arbitrage*** *implies that all other financial instruments, like derivatives, can be derived from these two fundamental asset valuations by simple* *linear transformations. Call and put options can be synthesized from a linear portfolio of bonds and stocks (cf. Los, 2001, pp. 164 - 166). Forwards and futures can be represented as time-shifted bonds or stocks (cf. Los, 2001, pp. 225 - 227). Swaps can be represented by a combination of a long and a short bond, or, equivalently, by a series of zero coupon bonds, or a series of long and short forwards (cf. Los, 2001, pp. 239 - 243).*

Example 22 *The double-entry bookkeeping model of accounting shows that the value of any concern can be represented as a linear portfolio of fundamental and derivative financial instruments (cf. Los, 2001, p. 24). The **Accounting Identity** of the **balance sheet** with current market values is:*

$$\begin{aligned} Assets(t) &= Liabilities(t) + Equity(t) \\ \text{or } A(t) &= L(t) + E(t) \\ \text{or, rewritten, } E(t) &= A(t) - L(t) \end{aligned} \tag{27}$$

*This Accounting Identity is the basic model for exact financial modeling, since financial instruments can be viewed as combinations of long and short positions of the fundamental securities. Changes in the net equity position $\Delta_\tau E(t)$ over the accounting period τ, which is usually one quarter of a year or a year, are produced by the corresponding changes in the assets and liabilities, as reflected in the **income statement**[6]*

$$\begin{aligned} \Delta E_\tau(t) &= \Delta A_\tau(t) - \Delta L_\tau(t) \\ \text{or } x_\tau^E(t)E(t-\tau) &= x_\tau^A(t)A(t-\tau) - x_\tau^L(t)L(t-\tau) \\ \text{or } Net\ Income_\tau(t) &= Revenues_\tau(t) - Expenses_\tau(t) \end{aligned} \tag{28}$$

on market value basis, where $x_\tau^A(t)$ is the market valued Rate of Return on Assets (ROA) over the period τ, $x_\tau^L(t)$ is the market valued rate of debt liability expense (= rate of interest on all debt) over the period τ, and $x_\tau^E(t)$ is the market valued Rate of Return on Equity (ROE) over the period τ. Thus the Accounting Identity combines the bundles of discounted cash flows in a linear fashion and these discounted cash flows are nonlinearly related to the rates of return.

In physics, *shear stresses* are the internal friction forces opposing flowing of one part of a substance past the adjacent parts. In finance, we are concerned about the shear stresses of investment cash flows with different degrees of illiquidity adjacent in the complete term structure, flowing past each other within and between the various financial term markets and their maturity "buckets."

Example 23 *One can form a (not necessarily square) matrix of bilateral cash flow channels between two countries, so that every cash flow investment channel associated with a particular term rate of return $x_{\tau_i}(t)$ is adjacent to every other cash flow investment channel, resulting in a net rate of return for both cash channels together $x_{\tau_i,\tau_j}(t) = x_{\tau_i}(t) - x_{\tau_j}(t) = x_{ij}(t)$:*

$$\mathbf{x}(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & ... & x_{1,T_2}(t) \\ x_{21}(t) & x_{22}(t) & ... & ... \\ ... & ... & ... & ... \\ x_{\tau_1,1}(t) & ... & ... & x_{\tau_1,\tau_2}(t) \end{bmatrix} \tag{29}$$

[6] *It is easy to view liabilities as "negative assets."*

Each side of the matrix can represent, say, the term structure of assets in a country, whereby one country may have more investment channels and different asset term structures than another (as is empirically the case), so that this matrix is not necessarily symmetric, or square. For example, the U.S.A. has a very fine term structure with $T_1 = 30$ year bonds, while other countries have only $T_2 = 10$ year bonds. Using tensor algebra, Los (1998a & b) provides empirical examples of such bilateral cash rates of return matrices based on the cash rates and stock market return rates in 10 *Asian countries plus Germany.*

But how does cash flow shear stress and strain occur and what do these terms mean?

Definition 24 ***Cash flow shear stress*** *is the change in the ex post asset term structure relative to the original asset prices, $\Delta x_\tau(t)/X(t-\tau)$. It is the quantity that is proportional to the cash flow supply of earnings on an investment made τ periods ago, causing deformation of the term structure of the asset return rates.*

Definition 25 ***Cash flow shear strain*** *is the change in an asset price relative to its maturity $\Delta X_\tau(t)/\tau$. It is the quantity that is proportional to the cash flow demand.*

When we express the cash strain per unit of time $\Delta t = 1$ we have the cash rate of return (velocity) gradient.

Definition 26 *The* ***cash rate of return (velocity) gradient*** *is*

$$\begin{aligned} \frac{\textit{Cash flow shear strain}}{\Delta t} &= \frac{\Delta X_\tau(t)/\tau}{\Delta t} \\ &= \frac{x_\tau(t)}{\tau} \end{aligned} \tag{30}$$

This gradient is the cash flow rate of return of an asset with maturity τ relative to its term, as a measure of the degree of term structure deformation caused by cash flow shearing, *i.e.*, caused by the differences between the cash flow rates of return with different time horizons for investment. This gradient is conceptually similar to the laminar velocity profile of the hydraulic flow in pipes.

For small cash flow stresses, stress is proportional to strain and we can define a coefficient of cash flow illiquidity, or kinetic cash viscosity, based on this simple linear stress - deformation relationship.

Definition 27 *The* ***coefficient of cash flow illiquidity*** *or* ***kinetic cash flow viscosity*** *at time t is*

$$\begin{aligned} \eta_\tau(t) &= \left| \frac{\textit{cash flow shear stress}}{\textit{cash flow shear strain}} \right| \\ &= \left| \frac{\Delta x_\tau(t)/X(t-\tau)}{x_\tau(t)/\tau} \right| \\ &= \left| \frac{\Delta x_\tau(t)\tau}{x_\tau(t)X(t-\tau)} \right| \end{aligned} \tag{31}$$

This relative illiquidity coefficient measures the change in the term structure $\Delta x_\tau(t)$ multiplied by its term (maturity) τ relative to the cash flow rate $x_\tau(t)X(t-\tau)$. When the investment cash flow is continuous, *i.e.*, when the investment cash flow rate $x_\tau(t)X(t-\tau)$ is constant, all illiquidity is measured by the empirical changes in the term structure $\Delta x_\tau(t)$ for each term τ.

When $\Delta x_\tau(t)$ varies inversely with the term τ, the illiquidity coefficient is constant, $\eta_\tau(t) = \eta_\tau$. Another way of expressing this, is to state

$$\Delta x_\tau(t)\tau = \eta_\tau x_\tau(t)X(t-\tau) \tag{32}$$

Thus, for *laminar cash flow*, the "force of illiquidity resistance" is

$$\Delta x_\tau(t)\tau \propto x_\tau(t)X(t-\tau) \tag{33}$$

i.e., is proportional to the rate of cash return $x_\tau(t)$ for maturity τ and the size of the original cash investment $X(t-\tau)$. Still a different way of interpreting this relative illiquidity coefficient is to state that it measures the absolute relative change in the interest rates of a particular maturity $\frac{\Delta x_\tau(t)}{x_\tau(t)}$, relative to the size of the average invested cash flow $\frac{X(t-\tau)}{\tau}$ in particular maturity market:

$$\eta_\tau(t) = \left| \frac{\frac{\Delta x_\tau(t)}{x_\tau(t)}}{\frac{X(t-\tau)}{\tau}} \right| \tag{34}$$

After these introductory definitions - which can all be calculated from the basic data - we are in the position to define perfect cash flow liquidity.

Definition 28 ***Perfect cash flow liquidity*** *exists when $\eta_\tau(t) = 0$. That occurs either (1) when $\Delta x_\tau(t) = 0$ for finite invested cash flows $X(t-\tau)$ and finite investment horizon terms τ, implying that the finite term rates of return structure remains unchanged under financial stress, thus $x_\tau(t) =$ constant; or (2) when, unrealistically, the term rates of return are infinite $x_\tau(t) = \infty$ for all finite cash flows and finite terms; or when the cash flow for a finite term τ and a finite term structure $x\tau(t)$ is infinite $X(t-\tau) = \infty$; or when the term equals zero, $\tau = 0$, for a finite investment cash flow $X(t-\tau)$ and a finite spot rate of return $x_\tau(t)$.*

Remark 29 *Most real world cash flows are non - perfectly liquid with $\eta_\tau(t) > 0$, because most real world cash flows are term dependent. There is not much instantaneous cash in comparison. Also, infinite term rates of return and invested cash flows empirically don't exist, nor do zero investment horizons. Thus the only realistic outcome is that perfect cash flow liquidity exists when $\Delta x_\tau(t) = 0$ for finite invested cash flows $X(t-\tau)$ and finite investment horizon terms τ.*

This particular expression for the liquidity coefficient is only really valid when the term structure $x_\tau(t)$ varies linearly with the term τ, *i.e.*, when the rate of cash return gradient $x_\tau(t)/\tau$ is uniformly constant. When this is the case, one may speak of *Newtonian cash flows*, in which stress, illiquidity, and rate of strain are linearly related. This is, of course, seldom the case with the *dynamic term structures* in the international financial markets. Empirically, the term structures of the cash investment markets vary nonlinearly, in such a way that the short term cash rates of return vary relatively more (= have larger amplitudes), and vary more frequently, than the long term cash rates of return.[7]

Remark 30 ***Term structure modelers*** *have attempted to capture this nonlinear dynamics by, first, intercorrelating the various terms $x_\tau(t)$ for a limited number of terms $1 < \tau < T$, and, then, by analyzing the resulting covariance matrix using a static principal component, or factor analysis (cf. the survey by Chapman and Pearson, 2001). The objective of such spectral analysis is to capture most of the variation of the term structure by retaining a small number, say three, principal component factor loadings, like Factor 1 = Level, Factor 2 = slope, and Factor 3 = Curvature of the term structure. But such inexact identification scheme of a covariance matrix is inherently subjective, because which eigenvalues are to be considered significant and to be retained and which should be considered representing noise? (For a detailed discussion of such "prejudices" of principal components, cf. Los, 1989).*

When the rate of return gradient is not uniform, as is usually the case, we must express the illiquidity coefficient in the general marginal form:

$$\begin{aligned}\eta_\tau(t) &= \left|\frac{\Delta x_\tau(t)/X(t-\tau)}{\partial x_\tau(t)/\partial\tau}\right| \\ &= \left|\frac{\Delta x_\tau(t)\partial\tau}{\partial x_\tau(t)X(t-\tau)}\right| \end{aligned} \quad (35)$$

Thus, we compare the empirical change in the required cash flow over an infinitesimal small change in the term, $\Delta x_\tau(t)\partial\tau$, with the infinitesimal change in the available cash flow rate $\partial x_\tau(t)X(t-\tau)$. That is a comparison between the marginal investment cash flow shear stress and the marginal investment cash flow shear strain.[8]

[7] Professor Craig Holden of Indiana University has made is a simple historical demonstration of this nonlinearly varying term structure: "Craig Holden's Excel-based Interactive "Movie" of Term Structure Dynamics " (1996), which can be downloaded from his web page.

[8] We use the partial derivatives ∂ to indicate that we're interested in variation on the term structure $x_\tau(t)$ caused by a marginal change in the term τ at a fixed time t.

3.2 Laminar and Turbulent Financial Cash Flows

We will now define the important concepts of laminar and turbulent cash flows.

Definition 31 *If the adjacent layers of illiquid term cash flow smoothly alongside each other in a financial market, and the term cash flow rates $x_\tau(t)$ are constant, the stable streamline flow of investment cash through a particular financial market (of a particular term and risk profile), is called* ***laminar****.*

Definition 32 *If the streamlined term cash flows in a financial market, at sufficiently high term rates of return $x_\tau(t)$, change from a laminar cash flow to a highly irregular cash flow with highly irregular cash flow (velocity) rates $x_\tau(t)$, it is called* ***turbulent****.*

Both, laminar and turbulent flows are easily illustrated by the hot cigarette smoke, which initially is laminar, but then turns turbulent due to the nonlinear constraints imposed by the interactive heat exchange with the cooler environment. (Fig. 16 in Schroeder, 1991, p.26). One can measure such cash flow velocity rate changes by a spectrogram or a scalogram.

My current conjecture is that the nonlinear expectation constraints inherent in the logical term structure $x_\tau(t)$ cause such chaos in the cash flows in financial markets to emerge, when particular parametric thresholds in the interlocking term structures are surpassed. In other words, particular shapes of interlocking term structures, at sufficiently high term rates, can cause irregularities to emerge.

Remark 33 *Small, perfectly liquid, laminar cash flow motions are called* ***acoustic*** *cash flow motions and (linear) Fourier Transform analysis can be used to decompose such steady and continuous global cash flow motions in subcomponent flows. However, when any of these conditions do not apply, and we deal with large - scale, so - called* ***non-acoustical*** *cash flow motions, like cash flow turbulence, we need (nonlinear) Wavelet Transform analysis to analyze and decompose them.*

At this moment, analogously to physical turbulence measurements, I also conjecture that the onset of cash flow turbulence can be identified by a dimensionless parameter, called the Reynolds number for uniform cash flow. In hydrodynamics and aerodynamics, the *Reynolds number* is a dimensionless ratio related to the velocity at which smooth flow shifts to turbulent flow.[9] There is no fundamental reason why we should not be able to measure a similar Reynolds numbers for financial cash flows.

[9] Osborne Reynolds (1842 -1912), was an English scientist whose papers on fluid dynamics and turbulence remain basic to turbine operation, lubrication, fluid flow and streamlining, cavitation, and the effects of tides.

Definition 34 *The financial **Reynolds number** $Re_\tau(t)$ for a uniform cash flow of maturity term τ is given by*

$$\begin{aligned} Re_\tau(t) &= \left| \frac{a\rho x_\tau(t)}{\eta_\tau(t)} \right| \\ &= \left| \frac{\rho x_\tau(t) X(t-\tau)}{\frac{\Delta x_\tau(t)\tau}{x_\tau(t)X(t-\tau)}} \right| \\ &= \left| \frac{\rho x_\tau^2(t) X^2(t-\tau)}{\Delta x_\tau(t)\tau} \right| \end{aligned} \tag{36}$$

where $\rho = \frac{1}{\$^2}$ is again the uniform cash flow density, which renders the Reynolds number dimensionless, and $a = X(t-\tau)$ is the scale of the investment cash flow channel.

The beauty of the Reynolds number is that it measures dynamic similarity: flows with the same Reynolds number look the same, whereas flows with different Reynolds numbers look quite different. It is a measure of the amount of cash flow present at a particular time in a particular channel. When the cash flow Reynolds number is constant, $Re_\tau(t) = Re_\tau$, for turbulent investment cash flows show

$$\Delta x_\tau(t)\tau = \frac{\rho x_\tau^2(t) X^2(t-\tau)}{Re_\tau} \tag{37}$$

Thus, for turbulent cash flow, the "force of illiquidity resistance" is a quadratic term. It is proportional to the squares of the rate of cash return and the size of the cash investment,

$$\Delta x_\tau(t)\tau \propto x_\tau^2(t) X^2(t-\tau) \tag{38}$$

This quadratic expression has a much higher value than the "force of illiquidity resistance" or viscosity, for laminar cash flow, which we know to be linearly proportional

$$\Delta x_\tau(t)\tau \propto x_\tau(t) X(t-\tau) \tag{39}$$

Remark 35 *The cash flow Reynolds number is proportional to the uniform cash flow energy $\frac{1}{2}\rho x_\tau^2(t)$ at the ("downstream ") measurement point. When the financial pressure "upstream " is high, this "downstream " energy increases rapidly, producing a large Reynolds number. Turbulent cash flows have violent and erratic fluctuations in velocity and pressure, which are not associated with any corresponding fluctuations in the exogenous forces driving the flows.*

Physical turbulence is generally considered to be a manifestation of the nonlinear nature of the underlying fundamental equations. The "force of illiquidity resistance" is a quadratic, therefore highly nonlinear, term. We will see in Chapter 10, that this quadratic term is the essential constraint "causing " deterministic chaos or turbulence to occur at particular values of the growth rate parameters of the underlying simple dynamic processes. Feedback loops of the simple dynamic investment cash flow processes quickly increases the complexity of the aggregated dynamic investment cash flow process of several interconnected financial markets.

Remark 36 *The physical flows in round pipes are laminar for* $Re_\tau(t) < 2000$*, but physical turbulence begins to occur for Reynolds numbers,* $Re_\tau(t) > 3000$*. On the other hand, in physical turbulent flow over flat surfaces* $Re_\tau > 500,000$*. We don't know yet what the critical values are for the investment cash flow Reynolds numbers. This is an important issue for the empirical research into financial turbulence we're currently conducting at Kent State University.*

Since the scale of the investment cash flow channel is measured by a uniform dollar $a = 1\$$, we have $a\rho = \frac{1}{\$}$, so that the Reynolds number for uniform cash flows can be interpreted as a term based *return/liquidity risk ratio:*

$$Re_\tau(t) = \left| \frac{x_\tau(t)}{\eta_\tau(t)} \right| \tag{40}$$

The higher the cash rate of return on an investment asset with maturity τ, and the lower the cash illiquidity (= the higher the cash liquidity) for that same maturity, the higher the Reynolds number and, thus, the opportunity for financial turbulence.[10]

In other words, the financial Reynolds number measures the cash rate of return on an investment relative to its *illiquidity risk*, which is a quite different, and, perhaps, a much more important measure than the usual market *volatility risk* $\sigma_\tau(t)$.

It's a matter of current empirical financial research to determine at which magnitudes of this cash flow Reynolds number financial turbulence begins to occur and if it is a real empirical problem. But this expression shows that the Reynolds number for finite term rates of return $x_\tau(t)$ only becomes very large when the illiquidity risk approaches zero, $\eta_\tau(t) \to 0$. Thus financial turbulence is a phenomenon of very liquid financial markets and not of illiquid markets. In fact,

[10] The value of an asset is computed as the sum of the discounted (future) cash flows associated with that asset. Each asset has a particular maturity, even though it is sometimes assumed to have an infinite maturity, as in the case of, for example, the assets of a firm, based on the accounting principle of an "ongoing concern." However, in reality, any firm has finite assets, *i.e.*, projects with finite maturities.

the efficiency - enhancing financial turbulence contrasts sharply with the high risk discontinuity phenomena of the very illiquid and efficient markets (which show very low Reynolds numbers).

Remark 37 *Indeed, the current empirical evidence in Chapter 11 shows that turbulence is likely to occur in the anti - persistent anchor currency markets of the D - Mark and the Yen, relative to the US dollar.*

3.3 Financial Wavelet Reynolds Numbers and Financial Intermittency

How can we translate this financial Reynolds number in terms of the wavelet Multiresolution Analysis (MRA) of Chapters 7 and 8? Recall from Chapter 7 the definition of the scalogram of the cash rate of investment return $x(t)$. Then, analogously to Farge et al. (1996), we have the following definition:

Definition 38 *The local **financial wavelet spectrum** $S_W(\tau, a)$ of $x(t)$ is defined as its scale - standardized scalogram*

$$S_W(\tau, a) = \frac{P_W(\tau, a)}{a} = \frac{|W(\tau, a)|^2}{a} \tag{41}$$

Remark 39 *With the suggested financial uniform scale $a = 1\$$ that standardization is very simple and*

$$S_W(\tau, a) = |W(\tau, a)|^2 /\$ \tag{42}$$

A characterization of the local "activity " of $x(t)$ is given by its *wavelet intermittency*, which measures local deviations from the mean spectrum of $x(t)$ at every time lag τ and scale a. It is a local measure of *financial risk*, *i.e.*, a measure of risk localized in the scale (frequency) - time domain. Thus we no longer need the idealized statistical concept of ergodicity of Chapter 1 to measure financial risk. We can locally measure financial risk. While we provide already here a way of quantitatively measuring intermittency, the concept of intermittency has been explained in greater detail in Los (2000b), when we discuss the various phases of degeneration into chaos of nonlinear financial pricing processes.

Definition 40 *The **financial wavelet intermittency** of $x(t)$ is defined as*

$$I_W(\tau, a) = \frac{|W(\tau, a)|^2}{\int_{\mathbb{R}} |W(\tau, a)|^2 \, d\tau} \tag{43}$$

Remark 41 *One advantage of this wavelet intermittency measure is that it is dimension - free.*

Next, we implement the suggestion by Farge et al. (1996, p. 651) to express the Reynolds number in terms of wavelets, as follows.

Definition 42 *The* ***financial wavelet Reynolds number*** *is defined as*

$$Re_W(\tau, a) = \left| \frac{a\rho\,|W(\tau, a)|}{\eta_\tau(t)} \right| = \left| \frac{a\,|W(\tau, a)|}{\eta_\tau(t) V_\psi^{0.5}} \right| \tag{44}$$

so that $\rho = 1/V_\psi^{0.5}$, *where the constant*

$$V_\psi = \int_{\mathbb{R}^2} |\psi_{\tau,a}(t)|^2 \, d\tau da \tag{45}$$

is the uniform variance of the analyzing wavelet $\psi_{\tau,a}(t)$ *of the scalogram* $W(\tau, a)$ *over all time horizons* τ *and scales* a.

Remark 43 *Our expectation is that at large (inverted frequency) scales* a, *the wavelet financial Reynolds number coincides with the usual large - scale Reynolds number* $Re_\tau(t)$. *In the smallest scales (say* $a = \sigma_\varepsilon$. *where* σ_ε *is the Kolmogorov scale of turbulent cash flow, as in Chapter 11), one expects this wavelet Reynolds number to be close to unity, when averaged over time.*

Analogously to Farge et al. (1996), the current empirical research question regarding the analysis of financial turbulence is the following. With such a wavelet Reynolds number defined for time lag τ and scale a, are there time lags τ where such a Reynolds number at some very small scale is much larger than at others, and how do such time periods correlate with time periods of small - scale activity within the cash flow? If so, then $Re_W(\tau, a)$ could give an unambiguous answer of the activity at small scales (or at any desired scale a). Such time periods of high $Re_W(\tau, a)$ can then be interpreted as periods of *strong nonlinearity*, *i.e.*, periods of financial turbulence.

In other words, if the interpretation of Farge et al. (1996) is correct, we can just look at "scalograms of financial Reynolds numbers" to detect periods of strong nonlinearities and thus of financial turbulence.[11]

[11] At Kent State University, we are currently creating such financial scalograms.

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