When Does Extra Risk Strictly Increase an Option's Value?

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Abstract

It is well known that risk increases the value of options. This paper makes that precise in a new way. The conventional theorem says that the value of an option does not fall if the underlying option becomes riskier in the conventional sense of the mean-preserving spread. This paper uses two new definitions of "riskier" to show that the value of an option strictly increases (a) if the underlying asset becomes "pointwise riskier," and (b) only if the underlying asset becomes "extremum riskier."

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I. Introduction

A call option is the right to buy the asset at a strike price, P. It has been well known at least since Robert Merton (1973) that the value of a call option increases with the riskiness of the underlying asset. If extra risk increases the probability that the market price exceeds P, then the value of the option increases. A standard finance text says

> "The holder of a call option will prefer more variance in the price of the stock to less. The greater the variance, the greater the probability that the stock price will exceed the exercise price, and this is of value to the call holder." (Copeland & Weston, 3rd edition, p. 243)

But this is not quite correct, despite being the sort of thing that even experts say in conversation and in textbooks. As I am sure Thomas Copeland and Fred Weston knew as they wrote this passage, it quite possible for the risk and variance of the underlying asset to increase while the value of the option remains does not increase. The value will not fall, but it might remain unchanged. Suppose the strike price is \$50 and the current price of the asset is \$40. If the probability of the price being between \$45 and \$49 increases, while the probability it is between \$38 and \$42 falls, the asset has become riskier, but value of the option is unchanged because the probabilitiesof asset values above the strike price of \$50 are unchanged.

This, too, is well-known, but it leaves open the question of what kind of risk does actually increase the value of options. It is false, strictly speaking, to say that additional risk increases the value. On the other hand it is true but uninteresting to say that additional risk does not reduce the value. A great many variables do not reduce the value of an option, usually because they never affect the value either way. For introductory textbooks no great harm is done in stating a risk-value proposition loosely, but it is worth thinking about how we can come up with a proposition for this basic intuition that is both interesting and true.

One way out is to surrender generality in the kinds of asset distributions that we describe. Bliss (2001), noting the problem of coming up with a rigorous proposition, points out that a sufficient condition for option value to increase with risk is that the underlying asset value have a two-parameter distribution such as the normal or lognormal. The relationships between option value and risk, however, clearly holds for much more general distributions. (Bliss's attention in that article is about a different problem, also trivially solved by restricting attention to normal distributions– what to do when the underlying assets cannot be ordered using the standard definition of risk.)

The options literature has travelled down the route of studying particular stochastic processes for asset returns– diffusion or jump processes– rather than looking at general distributions for end-states as Merton (1973) did. This began with the log-normal diffusion processes of Fischer Black and Myron Scholes (1973) and continued with such generalizations as John Cox & Stephen Ross (1976) and Merton (1976) More recent entries in the literature include Yaacov Bergman, Bruce Grundy & Zvi Wiener (1996) and Masaaki Kijima (2002). Other papers look at other considerations absent in the simplest model of one underlying asset, risk-neutral investors, and zero transaction costs. Ravi Jagannathan (1984), for example, looks at values when investors are not risk neutral, and value wealth more in particular states of the world. In such a situation, a riskier asset might not have a higher option value because the option might yield its highest returns in a state of the world when investors are wealthier anyway and hence value the return less.

In this article I will return to the original problem of how risk affects option value, but from a different direction. First, I will note that if the underlying asset becomes riskier, then we can at least say that for *some* strike prices a call option will become more valuable– a very simple result, but worth noting. (I will use call options rather than put options throughout, but it will be clear that the proofs easily extend to puts.) Second, I will show that only if the underlying asset becomes riskier in the special way I call "extremum riskier" will every call option will rise in value regardless of the strike price- a necessary condition for a rise in value. Third, I will show that if the underlying asset becomes riskier in the special way I call "pointwise riskier" then every call option will rise in value regardless of the strike price- a sufficient condition for a rise in value.

II. The Model

Let there be an asset which has terminal value x_i with probability $f(x_i)$, where the values of x_i with positive probability are $x_1 < x_2 < ... < x_m$. Denote by $V_{call}(f,p)$ the current value of a call option on that asset with strike price p such that $x_1 < p < x_m$. This rules out strike prices of x_1 or below and x_m and above, because they would lead to riskless options which would be exercised always or never. It does allow a strike price that does not happen to equal any of the x_i. The call option entitles its owner to buy the asset at price p at the terminal time if he wishes. We will assume the discount rate is zero and use only two dates, the current date and the terminal date, to avoid distraction by the many issues that would otherwise arise (the date of exercise, diffusion versus jump processes, the time value of money, dividend payments, and so forth). Instead, our focus is on seeing how the option value would change if the underlying asset followed a different distribution $g(x)$ which has the same mean as $f(x)$, so

$$Ex = \sum_{i=1}^{m} f(x_i)x_i = \sum_{i=1}^{m} g(x_i)x_i + \sum_{i=m+1}^{n} g(x_i)x_i, \tag{1}$$

where $x_{m+1} < x_{m+2} < ... < x_n$ are points in the support of g but not f. This allows, for example, $x_{m+1} < x_1$, which in words means that g can have positive probability on x values less than or greater than the support of $f(x)$, or on values between x's in $f(x)$'s support. Let us denote the cumulative distributions by $F(x)$ and $G(x)$.

The value of a call option with strike price p is

$$\begin{aligned} V_{call}(f,p) &= \sum_{i=1}^{m} Max\{0, f(x_i)(x_i - p)\} \\ &= \sum_{i=j}^{m} f(x_i)(x_i - p) \; where \; j : x_{j-1} < p < x_j \end{aligned} \tag{2}$$

Defining Risk

The standard definition of risk is based on the idea of a "mean- preserving spread," which in the present context we can define as follows.

Definition 1a: *A* **mean-preserving spread** *consists of three numbers $s(y_1)$, $s(y_2)$, and $s(y_3)$ for $y_1 < y_2 < y_3$ such that*

$$s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0, \text{ (the mean is preserved)} \tag{3}$$

$$s(y_1) + s(y_2) + s(y_3)y = 0, \text{ (the new probabilities sum to zero)} \tag{4}$$

and

$$s(y_1) \in [0,1], \; s(y_2) \in [-1,0], \; s(y_3) \in [0,1] \text{ (the probability is spread)} \tag{5}$$

Definition 1a is specialized to discrete probability distributions, and it uses the idea of the "3-point mean-preserving spread," developed in Petrakis & Rasmusen (1994) rather than the conventional "4-point mean-preserving spread" of Rothschild & Stiglitz (1970), which have negative probability at two middle points rather than one. The two definitions of spread lead to equivalent definitions of risk (Definition 1b below orders distributions by risk identically whichever defintion of spread is used), but the 3-point definition is simpler and will lead to less clutter in proofs (as well as allowing an easy fix of the error in the main proof in Rothschild & Stiglitz [1970]). Note that Definition 1a does not require that the y_i equal any x_i: the spread can put positive probability on asset values which originally have zero probability (a spread added to $f(x)$ also could result in probabilities that are negative or greater than one, but spreads that do so will not be useful).

Thus, we arrive at Definition 1b, the definition of risk originated in Rothschild & Stiglitz (1970) (though see also Hadar & Russell [1969] and Hanoch & Levy [1969]).

Definition 1b: *Distribution $g(x)$ is* **riskier** *than $f(x)$ iff $g(x)$ can be reached from $f(x)$ by a sequence of mean-preserving spreads.*

This definition of risk has long been conventional, since it is equivalent to saying that the asset becomes less attractive to a risk- averse investor (one with a concave utility function) or that f is like g with noise added, although Definition 1b is only a partial ordering, and many pairs of distributions cannot be ranked by it. In the option context, Bliss (2001) shows the importance of using Definition 1b instead of defining risk as simply higher variance, which is not an equivalent definition. Variance can increase without making an asset less attractive to a risk-averse investor, and option values do not change in a uniform direction with changes in variance.[1]

It is perhaps worth reminding the reader of another statement of risk: in terms of stochastic dominance. Distribution $G(x)$ "first-order stochastically dominates" distribution $F(x)$ if $F(t) \geq G(t)$ for all t, i.e., if F puts more probability on lower values of x than G does. Distribution G "second-order stochastically dominates" distribution F if $\int_0^t F(x)dx \geq \int_0^t G(x)dx$ for all t.[2] A definition of risk equivalent to Definition 1b is that distribution $g(x)$ is **riskier** than $f(x)$ iff $G(x)$ second-order stochastically dominates $F(x)$. We will use densities rather than cumulative distributions in this article, however, because densities are easier to visualize and understand.

[1]An example to show that increased variance can increase utility for a risk-averse person is the following. Let the utility be $U = x$ for $x \leq 10$ $U = 10 + x/2$ for $x \geq 10$, which is weakly concave. Suppose wealth is initially distributed as (.8-7, .2-12), which has mean 8, variance $4(= .8 * 1^2 + .2 * 4^2)$, and utility $7.8(= .8 * 7 + .2 * 11)$. If the distribution is changed to (.2- 0, .8-10), the mean is still 8, the variance increases to $16(= .2*8^2+.8*2^2)$, and utility rises to $8(= .2 * 0 + .8 * 10)$. Kurtosis, which increases when moving weight to the tails of the distribution, is equally unreliable for ranking the riskiness of distributions; it starts as $52(= .8 * 1^4 + .2 * 4^4)$ in this example and rises to $832(= .2 * 8^4 + .8 * 2^4)$.

[2] There is some scope for ambiguity here in whether the inequalities are weak or strong. Four levels that might be defined are "weak stochastic dominance", in which it is possible that the inequality is an equality for all t, so F and G are identical; "semi-weak stochastic dominance," in which the inequality must be strict for at least one value of t; "strict stochastic dominance, " in which the inequality must be strict for all values of t such that $G(t) > 0$ and $G(t) < 1$, so G dominates F everywhere except at the bounds of G's support; and "super- strict stochastic dominance," in which the inequality must be strict for all values of t for which $G(t) \geq 0$ and $F(t) < 1$, so that G dominates F even at the bounds of G's support. For discrete distributions like the ones in the present article, strict and superstrict stochastic dominance are equivalent. Weak and semi-weak stochastic dominance are what are standardly used in economic theorems. See, too, footnote 3 below.

Option Value Does Not Decline with Risk

The fundamental proposition in the theory of risk and options is the well- known Proposition 1: option value is weakly increasing in risk.

Proposition 1 (Merton [1970] Theorem 8, p. 149): *If g is riskier than f, then $V_{call}(f,p) \leq V_{call}(g,p)$ for any p.*

Proof: From (2), the value of the call on the less risky asset, f, is

$$V_{call}(f,p) = \sum_{i=j}^{m} f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \tag{6}$$

and the value of the call on the riskier asset, g, is

$$V_{call}(g,p) = f(x_i)(x_i - p) + s(y_1)Max(y_1 - p, 0) + s(y_2)Max(y_2 - p, 0) + s(y_3)Max(y_3 - p, 0). \tag{7}$$

If

$$0 \leq s(y_1)Max(y_1 - p, 0) + s(y_2)Max(y_2 - p, 0) + s(y_3)Max(y_3 - p, 0). \tag{8}$$

then Proposition 1 is correct.

Note first that from the definition equation (3) the spread is mean-preserving, so $s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0$, and by equation (4) the spread's probabilities add to zero, so $[s(y_1) + s(y_2) + s(y_3)] = 0$. Together, these imply that

$$\begin{aligned} s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p) &= s(y_1)(y_1) + s(y_2)(y_2) + s(y_3)(y_3) \\ &\quad -[s(y_1) + s(y_2) + s(y_3)]p = 0, \end{aligned} \tag{9}$$

a result that will be used below.

(i) Suppose $p \leq y_1$, so inequality (8) becomes

$$0 \leq s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p). \tag{10}$$

Equation (9) tells us that this is true as an equality.

(ii) Suppose $p \geq y_3$, so inequality (8) becomes

$$0 \leq s(y_1)(0) + s(y_2)(0) + s(y_3)(0). \tag{11}$$

This is obviously true as an equality.

(iii) Suppose that $p \in (y_1, y_3)$. Then, since $Max(y_1 - p, 0) = 0$ and $Max(y_3 - p, 0) = y_3 - p$, we can rewrite expression (7) as

$$0 \leq 0 + s(y_2)Max(y_2 - p, 0) + s(y_3)(y_3 - p) \tag{12}$$

(a) If $Max(y_2 - p, 0) = 0$, then inequality (12) is true as a strict inequality, since $s(y_3) > 0$ and $y_3 > p$.

(b) If $Max(y_2 - p, 0) = y_2 - p$, then inequality (12) is true if

$$s(y_2)(y_2 - p) + s(y_3)(y_3 - p) \geq 0 \tag{13}$$

Equation (9) tells us that $s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p) = 0$, so since $s(y_1) > 0$ and, in case (iii), $(y_1 - p) < 0$, it follows that (12) is true as a strict inequality. QED.

Compare Proposition 1 with Proposition 1a, which differs only in the strength of the inequality.

Proposition 1a (false): *If g is riskier than f, then $V_{call}(f, p) < V_{call}(g, p)$ for any strike price p.*

Disproof. Consider a call option with an exercise price of 4.5 and the asset price distribution shown in Figure 1. $V_{call}(f, 4.5) = V_{call}(g, 4.5)$, even though g is riskier than f. The increase in risk has no effect because only changes in the probabilities of terminal values greater than 4.5 would matter to the value of the call, and there are no such changes in the example.

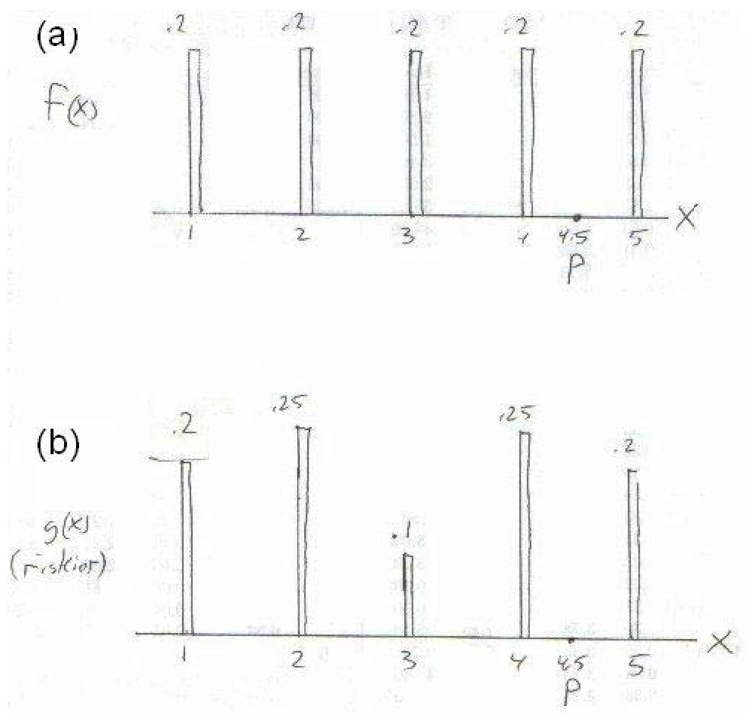

Figure 1: A Counterexample: Risk Does Not Increase Option Value

Propositions 1 and 1a differ only in the weakness of the inequality. That is enough, however, for "Proposition 1a: Option value increases with risk" to be false. Instead, we are left with "Proposition 1: Option value does not fall with risk," which although true, is very weak. That kind of statement can be made of any variable outside the model: "Option value does not fall with wealth," or "Option value does not fall with unemployment," or "Option value does not fall with the temperature in Bloomington."

The statement "Option value does not fall with risk," however, though it does translate the mathematical notation of Proposition 1, is unnecessarily weak. We can instead say that "Option value does not fall with risk, and for at least one value of the strike price it increases." Propostion 1b expresses this in mathematical notation.

Proposition 1b: *If g is riskier than f, then there there exists some exercise price p' such that the associated call option is more valuable under g than under f but no exercise price such that a call option is more valuable under f:*
$\exists p' : V_{call}(f,p') < V_{call}(g,p')$*; but* $\nexists p'' : V_{call}(f,p'') > V_{call}(g,p'')$*.*

Proof:
The proof of Proposition 1 showed that if $p \in (y_1, y_3)$, then the value of the call or put strictly increases. Thus, simply pick p' in (y_1, y_3) for one of the spreads that makes g riskier than f.

That there exists no value p'' for which option value declines is a direct corollary of Proposition 1. QED.

IIb. New Definitions of Risk

Another approach is to find a definition of risk under which something like Proposition 1b is true, and the value of the option does increase with "risk" regardless of the strike price.

Definition 2 (new): *Distribution $g(x)$ is* **pointwise riskier** *than $f(x)$ iff f and g have the same mean and there exist points $\underline{x}$ and $\overline{x}$ in (x_1, x_m) such that*

(a) if $x < \underline{x}$, then $g(x) \geq f(x)$ and if $f(x) > 0$ then $g(x) > f(x)$;

(b) if $x \in [\underline{x}, \overline{x}]$, then $g(x) \leq f(x)$ and if $f(x) > 0$ then $g(x) < f(x)$;

(c) if $x > \overline{x}$, then $g(x) \geq f(x)$ and if $f(x) > 0$ then $g(x) > f(x)$.

Definition 2 says roughly that $g(x)$ is pointwise riskier than $f(x)$ if it takes probability away from *each point* in the middle of the distribution and adds probability to *each point* at the two ends, while preserving the mean. Distribution $g_1(x)$ in Figure 2 is an example. Definition 2 also allows $g(x)$ to add probability to points outside the interval $[x_1, x_m]$– that is, beyond the two extremes of the support of $f(x)$.

This definition can be applied without modification to continuous densities f and g so long as f has convex support– that is, no gaps in its support, at which the definition would require g to have negative density. Note, too, that in the context of a continuous density with $f(x_1) = 0$ and $f(x_m) = 0$ it implies that g has a bigger support on each side, since it then requires that $f(x_m) > 0$ and $f(x_m) > 0$.

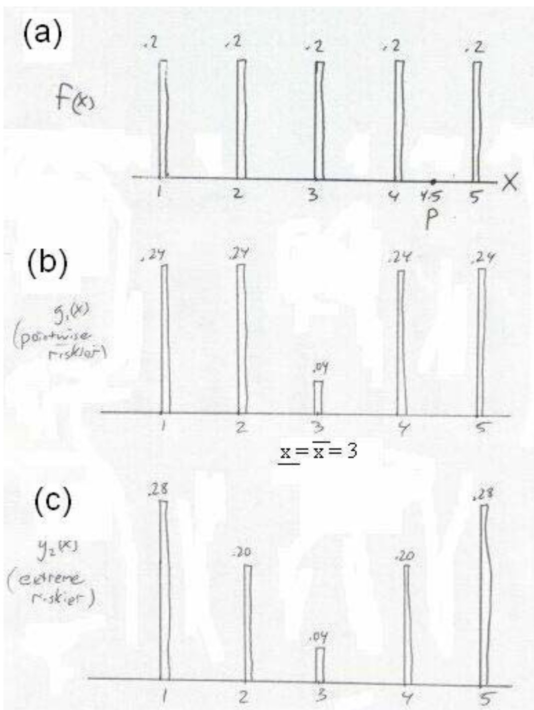

Figure 2: Pointwise Riskiness

Pointwise riskiness will be not a necessary condition but a sufficient one for option value to increase with risk for all strike prices, as we will see in Proposition 4. Distribution $g_2(x)$ in Figure 2 is an example in which g is not pointwise riskier than f, but $V_{call}(f, p) < V_{call}(g, p)$ nonetheless for all p.

Note also that if p is fixed, $g(x)$ does not even have to be a mean-preserving spread to increase the value of the call. But we are asking what changes to the asset distribution will increase the value of *any* call written on the asset.[3]

Our other new definition of risk is one which is *necessary* for extra risk to increase option value: extremum risk.

Definition 3 (new): *Distribution* $g(x)$ *is* **extremum riskier** *than* $f(x)$ *iff (a) either* $f(x_1) < g(x_1)$*, or* $g(x) > 0$ *for some* $x < x_1$*;*
and
(b)either $f(x_m) < g(x_m)$*, or* $g(x) > 0$ *for some* $x > x_m$*.*

The distribution in part (b) of Figure 3 is extremum-riskier than the distribution in part (a) of Figure 3.

[3] Since pointwise riskiness and second order stochastic dominance both both can alternatively be defined in terms of functions that cross a limited number of times, the reader may wonder if pointwise riskiness is the same as the strict second-order stochastic dominance of footnote 2. Distribution G strictly second-order stochastically dominates F if $\int_0^t F(x)dx \geq \int_0^t G(x)dx$ for all t and the inequality is strict for all values of t such that $G(t) > 0$ and $G(t) < 1$. But for G to be reached from F by a mean-preserving spread, it must be that at the upper point T of the spread, $\int_0^T F(x)dx = \int_0^T G(x)dx$ (see "Riskiness," http://cepa.newschool.edu/het/essays/uncert/increase.htm, viewed August 20, 2004.) If G is pointwise riskier than F it is still riskier, so $\int_0^T F(x)dx = \int_0^T G(x)dx$ and G does not strictly second- order dominate F; the two concepts are not the same.

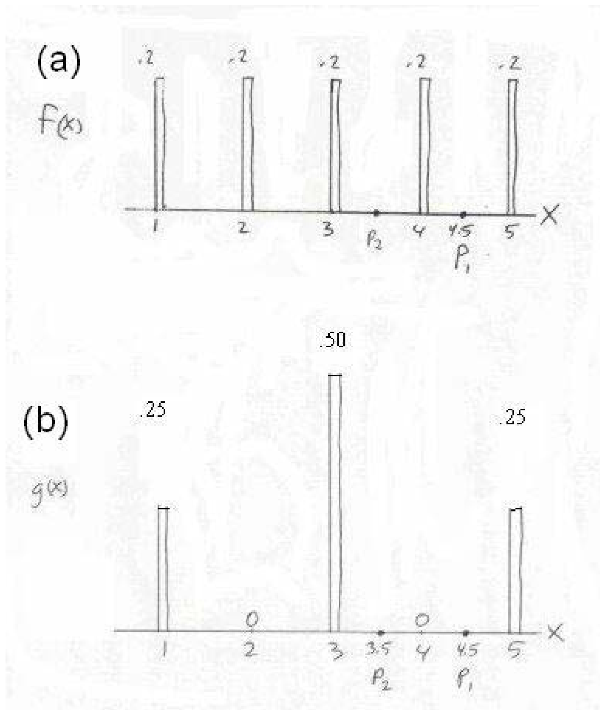

Figure 3: Extremum Risk versus Risk

Proposition 2: *Consider two distributions f and g. A necessary condition for it to be true that $V_{call}(g,p) > V_{call}(f,p)$ for any strike price p is that g be extremum-riskier than f.*

Proof:
(i) If $f(x)$ and $g(x)$ are identical for all $x > p$ or for all $x < p$, then $V_{call}(g,p) = V_{call}(f,p)$.

If f and g are identical for all $x > p$, then clearly the call value of equation (2) must be equal for f and g, since then f and g are identical for $x \geq x_j$, and only such values of x enter into equation (2), reproduced below.

$$V_{call}(f,p) = \sum_{i=j}^{m} f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \tag{14}$$

If, on the other hand, f and g are identical for all $x < p$, then since their unconditional means, the expected values taken over all possible values of x, are equal, and their means conditional on $x < p$ are equal, their means conditional on $x > p$ must also be equal. This, too implies that the values of calls on f and g are equal.

Thus, for $V_{call}(g, p) > V_{call}(f, p)$ to be true, it is necessary that $g(x) > f(x)$ for some but not all $x > p$.

Since this is true for every p, it must be true for $p = x_m - \epsilon$, for any small value ϵ. Thus, for some $x \geq x_m$, $g(x) > f(x)$. But this can be true only if either $g(x_m) > f(x_m)$, or if $g(x) > 0$ for some $x > x_m$. That is condition (b) in Definition 3.

Similarly, if $f(x)$ and $g(x)$ are identical for all $x < p$, then $V_{call}(f, p) = V_{call}(f, p)$. Thus, if $V_{call}(g, p) > V_{call}(f, p)$ is to be true, it must also be true that for any p, for some $x < p$ it must be true that $g(x) > f(x)$. Since this is true for every p, it must be true for $p = x_1 + \epsilon$, for any small value ϵ. Thus, we need that for some $x \leq x_1$ it is true that $g(x) > f(x)$. But this can be true only if either $g(x_1) > f(x_1)$ or if $g(x) > 0$ for some $x < x_1$. That is condition (a) in Definition 3.

Thus, for the value of the call to be greater under $g(x)$ for all p, it is necessary that g satisfy the conditions for being extremum riskier than f. QED.

Why is this just a necessary condition, and not sufficient? Figure 3 shows why. In Figure 3, $g(x)$ has more probability at the extremes than $f(x)$ does– the probability of each extreme is .25 instead of .20— but it is not riskier in the conventional sense, because it cannot be reached from $f(x)$ by a sequence of mean-preserving spreads. If the strike price is 4.5, then the call's value is higher under distribution $g(x)$, because the outcome $x = 5$ occurs with probability .25 instead of $f(x)$'s .20. $V_{call}(f, 4.5) = .20(5-4.5) = .10 < V_{call}(g, 4.5) = .25(5 - 4.5) = .125$. If the strike price is 3.5, however, the call's value is higher under distribution $f(x)$, because under $g(x)$ the outcomes $x = 4$ and $x = 5$ together occur with probability .25 instead of

.40 and $V_{call}(f, 3.5) = .20(4 - 3.5) + .20(5 - 3.5) = .40 > V_{call}(g, 3.5) = .00(4 - 3.5) + .25(5 - 3.5) = .375$.

The solution is simple— if g is riskier than f as well as extremum-riskier, that is sufficient for all call options on g to be more valuable. (Note that extremum riskiness already implies that g is not less risky than f, since more weight is in the far tail of the distribution in g, but it might be that f and g are not ordered by risk.)

Proposition 3: *Consider two distributions f and g. A sufficient condition for it to be true that $V_{call}(g, p) > V_{call}(f, p)$ for any strike price p is that*
(a) g is extremum-riskier than f; and
(b) g is riskier than f.

Proof:
From Proposition 1 we know that if condition (b) is true, then $V_{call}(g, p) \geq V_{call}(f, p)$, that is, Proposition 3's inequality is true at least weakly. Thus, all that we need show is that condition (a) makes the inequality strict.

Suppose Proposition 3 is false, and there does exist some value p such that $V_{call}(g, p) = V_{call}(f, p)$. Recall that equation 1) says that $g(x)$ has the same unconditional mean as $f(x)$:

$$Ex = \sum_{i=1}^{m} f(x_i)x_i = \sum_{i=1}^{m} g(x_i)x_i + \sum_{i=m+1}^{n} g(x_i)x_i,$$

where $x_{m+1} < x_{m+2} < ... < x_n$ are points in the support of g but not f. If $V_{call}(g, p) = V_{call}(f, p)$, then it is also true (using (2) for the value of a call) that

$$\sum_{i=j}^{m} f(x_i)(x_i - p) = \sum_{i=j}^{m} g(x_i)(x_i - p) + \sum_{i=m+k}^{n} g(x_i)(x_i - p), \; where \; j : x_{j-1} < p < x_j and k : x_{m+k-1} \tag{15}$$

so f and g have the same means conditional on x being greater than p.

Putting these two equations together implies that

$$\sum_{i=1}^{j-1} f(x_i)(x_i - p) = \sum_{i=1}^{j-1} g(x_i)(x_i - p) + \sum_{i=m+1}^{m+k-1} g(x_i)(x_i - p), \; where\ j : x_{j-1} < p < x_j and k : x_{m+k-1} \tag{16}$$

so f and g have the same means conditional on x being less than p.

Since g is extremum-riskier, though, we know from Definition 3 that either $f(x_1) < g(x_1)$, or $g(x) > 0$ for some $x < x_1$– that g puts more weight on the far left tail of the distribution. Thus, equation (16) is false– g's mean conditional on $x < p$ is lower than f's. In turn, this implies that equation (16) ia false, and no p exists for which $V_{call}(g,p) = V_{call}(f,p)$. Every call on g is strictly more valuable than it would be on f. Q.E.D.

You might ask why I did not write Proposition 3 to say that conditions (a) and (b) are jointly necessary and sufficient, rather than just sufficient. If options on g are to be always more valuable than options on f, isn't it necessary that g be both riskier and extremum- riskier than f? No.

Proposition 3a (false): *Consider two distributions f and g. A necessary and sufficient condition for it to be true that $V_{call}(f,p) < V_{call}(g,p)$ for any strike price p is that*
(a) g is extremum-riskier than f; and
(b) g is riskier than f.

Disproof: Conditions (a) and (b) are jointly sufficient, as Proposition 3 says. Condition (a) by itself is necessary, as Proposition 2 says. Thus, what we need to show to disprove Proposition 3a is that there exist distributions f and g such that Condition (b) is violated but nonetheless $V_{call}(f,p) < V_{call}(g,p)$ for any p. That is, we must show that g's options are always more valuable, but g is not riskier than f.

Consider the example in Figure 4. Distribution g is extremum-riskier than distribution f, but it is not riskier, because it has more probability at the mean, $x = 5$ (in fact, $g(5) = 0$). The distributions f and g cannot be ordered by risk.

The value of a call option on an asset with density f and strike price $p \in (2, 8)$ is, from equation (2),

$$\begin{aligned} V_{call}(f,p) &= Max\{0, .25(2-p)\} + Max\{0, .25(4-p)\} + Max\{0, .25(6-p)\} \\ &\quad + Max\{0, .25(8-p)\} \end{aligned} \tag{17}$$

and the value of a call option on an asset with density g and strike price p is

$$V_{call}(g,p) \quad = \quad Max\{0, .25(1-p)\} + Max\{0, .40(5-p)\} + Max\{0, .30(9-p)\} \tag{18}$$

The possible values of p go from $p = 2$ to $p = 8$, where the endpoints are not possible (as the option would then be always or never exercised). We will split this up into four intervals and examine each in turn.

Lemma 1: $V_{call}(f,p) < V_{call}(g,p)$ for $p \in (2, 4]$.

Proof: Then $V_{call}(f,p) = .25(4-p) + .25(6-p) + .25(8-p) = .25(18-p) = 4.5 - .75p$. On the other hand, $V_{call}(g,p) = .40(5-p) + .30(9-p) = 4.7 - .70p$, which is greater, so g has the more valuable options.

Lemma 2: $V_{call}(f,p) < V_{call}(g,p)$ for $p \in (4, 5]$.

Proof: Then $V_{call}(f,p) = .25(6-p).25(8-p) = 3.5 - .50p$. On the other hand, $V_{call}(g,p) = .40(5-p) + .30(9-p) = 4.7 - .70p$. It is true that $3.5 - .50p < 4.7 - .70p$ if $.20p < 1.2$, which is true if $p < 6$, and in particular if $p \in [4, 5]$, so g has the more valuable options.

Lemma 3: $V_{call}(f,p) < V_{call}(g,p)$ for $p \in [5, 6]$.

Proof: Then $V_{call}(f,p) = .25(8-p) = 2 - .25p$. On the other hand, $V_{call}(g,p) = .30(9-p) = 2.7 - .30p$. It is true that $2 - .25p < 2.7 - .30p$ if $.05p < .7$, which is true if $p < 14$, and in particular if $p \in [5, 6]$, so g has the more valuable options.

Lemma 4: $V_{call}(f,p) < V_{call}(g,p)$ for $p \in [6,8)$.

Proof: Then $V_{call}(f,p) = 0$. On the other hand, $V_{call}(g,p) = .30(9-p) = 2.7 - .30p$. It is true that $0 < 2.7 - .30p$ if if $p < 9$, and in particular is true if $p \in [6,8]$, so g has the more valuable options.

Thus, for any $p \in (2,8)$, the set for which an option on f is risky, g has more valuable options, which was to be proved.

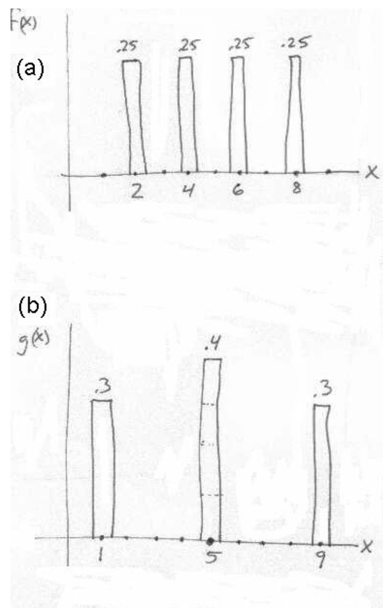

Figure 4: Why Riskiness and Endpoint Riskiness Are Not Necessary for Options To Increase in Value

For many applications, it is convenient to specify a simple sufficient condition for one option to be riskier than another. Indeed, my first motivation for this paper was to identify such a sufficient condition in the context of information acquisition during an auction (see Rasmusen [2004]). Pointwise riskiness is a sufficient condition that is both simple and plausible.

Proposition 4: *If g is pointwise riskier than f, then for any p, $V_{call}(f,p) < V_{call}(g,p)$.*

Proof: If g is pointwise riskier than f, then it is also riskier and extremum riskier. It is riskier because we can move from f to g by a series of mean-preserving spreads that take probability away from the middle interval $[\underline{x}, \overline{x}]$ and move it to the extremes. It is extremum riskier because $x_1 < \underline{x}$ and $x_m > \overline{x}$, so g puts more probability on x_1 and x_m than f does. It follows from Proposition 3, proved above, that calls on g will be more valuable than calls on f. Q.E.D.

We have, of course, already found one sufficient condition for options on g to be more valuable than options on f. Proposition 3 said that riskiness plus extremum riskiness provides a sufficient condition. Proposition 3, in fact, is a tighter sufficient condition. If g is pointwise riskier than f it is always both riskier and extremum riskier– but g can be riskier and extremum riskier than f without being pointwise riskier. Nonetheless, pointwise riskiness is a useful concept, because it is simpler and more intuitive than standard plus extremum riskiness.

Propositions similar to Propositions 2, 3, and 4 are easy to derive for put options as well as for call options. The propositions do not extend to exotic options that convey purchase or sale rights over ranges of prices that do not slice the real line in two (e.g., the right to buy if the price is either in the interval [3, 5.6] or in [7, 26]). Neither the intuition nor the rigorous propositions extend to that kind of option, since an exotic option such as in my parenthetic example can increase in value when probability shifts from the extremes to the middle, a reduction in risk.

IV. Concluding Remarks

If distribution g is riskier than distribution f, then any call option on an asset whose value has distribution g will be at least as valuable as the equivalent option on an asset with distribution f. But the option on g might not be *more* valuable, because the values might be equal. This paper has developed a necessary condition for all call options on an asset whose value has distribution g to be strictly more valuable than the equivalent option on an asset with distribution f, and two sufficient conditions for it, differing in strength and convenience. The necessary condition is that g be "extremum riskier": it must put more probability on the extreme values of the asset. One sufficient condition is that g be not only extremum riskier, but also riskier under the conventional definition of risk– that g can be reached from f by a series of mean-preserving spreads. A second sufficient condition, more restrictive but simpler, is that g be "pointwise riskier": asset values in the middle of g have higher probability than under f, and asset values outside the middle have lower probability.

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