ESTIMATING THE VOLATILITY STRUCTURE OF AN ARBITRAGE-FREE INTEREST RATE MODEL VIA THE FUTURES MARKETS

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ABSTRACT. This paper considers a class of Heath-Jarrow-Morton (1992) term structure models, characterized by time deterministic volatilities for the instantaneous forward rate. The bias that arises from using observed futures yields as a proxy for the unobserved instantaneous forward rate is analyzed. The fact that futures contracts can be viewed as derivative instruments on the forward rate is used to determine the likelihood function for futures prices. The likelihood transformation method of Duan (1994) is then used to obtain the full information maximum likelihood estimator for the observable futures prices. The approach is applied to estimate the volatility structure implied by futures contracts traded on the Chicago Mercantile Exchange.

Key words: Term structure; Heath-Jarrow-Morton; Yield curve; Forward rate volatility function; Estimation bias; FIML; Likelihood transformation; Futures contracts;

JEL classifications: C51, E43, G12, G13

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Interest rate modelling has long been of interest to researchers and practitioners. The arbitrage-free approach to modelling the term structure of interest rates has its origin in Ho and Lee (1986), and is most clearly articulated in Heath, Jarrow and Morton (1992), (hereafter HJM). The HJM model is based on the specification of the term structure of forward rates in terms of the initial forward rate curve and the forward rate volatility function. The condition that rules out arbitrage opportunities determines uniquely the drift of the instantaneous forward rate in terms of the forward rate volatility function and the market price of interest rate risk. The dynamics of the instantaneous spot rate are then developed from the forward rate evolution.

The HJM approach, therefore, has many advantages over the earlier approaches such as Vasicek (1977), Brennan and Schwartz (1979, 1982), Cox et al. (1985). First, the model matches the current term structure by construction. Second, there is no need for any assumptions on investor preferences for pricing purpose. Third, the model offers a parsimonious representation of the market dynamics and requires only specification of the form of the forward rate volatility function. Despite these advantages, there have been very few empirical studies of the HJM model. This is due to the fact that in its most general form, the resulting instantaneous spot rate evolution is not path-independent, ie. it is non-Markovian, and the entire history of the term structure has to be carried forward, thus increasing the computational complexity of most estimation procedures.

In one approach to the empirical study of the HJM model, researchers have relied on implied volatility, most notably Amin and Morton (1994) and Amin and Ng (1997). Under this approach, each day, the volatility parameters are backed out from market prices of derivative instruments, for example, by finding the set of parameters that minimizes the sum of squared errors. The implied volatility approach gives estimates of the model parameters that change day-by-day. This approach is useful from the perspective of market practitioners who need to calibrate the model daily to prevailing market conditions in order to ensure accurate pricing and hedging strategies.

The focus of this paper will rather be on econometric estimation of the (fixed) parameters of a volatility specification across an estimation period, for example to find the "best" from a family of possible volatility specifications. The resulting functional forms could of course then be used by market practitioners in their calibration procedures.

The approach to estimation so far adopted relies on reducing the system dynamics to Markovian form under some particular functional specification of the forward rate volatility. Theoretical work on reduction-to-Markovian form can be found in Björk and Svensson (2001), Bliss and Ritchken (1996), Bhar and Chiarella (1997a), Chiarella and Kwon (2001a, 2001b, 2003), Inui and Kijima (1998), and Ritchken and Sankarasubramanian (1995a). Within these classes of models, empirical work lags behind the cited theoretical developments, so that there is still a dearth of studies on what kinds of forward rate volatility functions are actually suggested by market data.

The HJM class with a time-deterministic instantaneous forward rate volatility function is regarded as a relatively easy one to implement. This is because the instantaneous forward rate process is Markovian, and so may be directly used as the basis of estimation procedures. In this case, there is no need to be concerned with Markovianizing the process for the spot rate of interest[1]. However, if these Markovian forward rate dynamics are used directly in estimation, there still remains a proxy problem.

The proxy problem arises from the un-observability of the instantaneous forward rate, since market traded instruments involve discrete tenor rates, and usually these are essentially futures yields. Using a fixed-maturity futures yield as a proxy for the instantaneous forward rate may result in estimation bias. This paper makes use of the fact that for the class of HJM models where the forward rate volatility function is time deterministic, the evolution of the futures price can be derived from the forward rate evolution. These details are spelt out in Section 1, where the bias due to using fixed-maturity futures yields as a proxy for instantaneous forward rates is quantified. In particular, the bias is decomposed into two components, maturity bias and convexity bias. The maturity bias arises from approximating an instantaneous forward rate by a fixed-maturity forward rate, and is negligible if the fixed-maturity is short. The convexity bias, which is not negligible, arises from using a fixed-maturity futures yield to approximate the fixed-maturity forward rate.

This paper takes advantage of the link between forward and futures rate evolution (due to the time deterministic forward rate volatility function specification) to derive

[1]It should be noted that although there are important classes of HJM with time-deterministic forward volatility whose spot rate dynamics can be transformed into a Markovianized system (of higher dimension), the question as to whether *all* HJM models with time-deterministic volatility have this property remains unresolved as far as we are aware.

the exact likelihood function for the time series of futures prices observed in the market, rather than treating the short maturity futures rate as a proxy for the instantaneous forward rate. A similar approach has been used by Pearson and Sun (1994) in estimating the Cox, Ingersoll, Ross model, and by Ho et al. (2001) in estimating the one factor HJM model with exponential forward rate volatility function. These studies rely on the closed-form solution for bond prices and futures prices to estimate the unobservable instantaneous spot rate and forward rate respectively. The key advance in our approach is that we recognize the observable futures rate as a derivative instrument driven by the same source of uncertainty as that driving the underlying unobservable forward rate. Therefore, despite the fact that we cannot establish a closed-form formula for the futures price (as a function of the forward rate), we are able to derive the exact likelihood function for all model specifications that have deterministic volatility forms, albeit the likelihood will be different in its degree of complexity.

The major contribution of our paper, as a consequence, is a systematic method to estimate an important class of HJM models, where the forward rate volatility function is time deterministic, and the spot rate may or may not be Markovian. An additional important improvement in our estimation approach is that we recognize that futures prices are less than perfectly correlated with each other under a stochastic setting. Therefore, we apply the full information maximum likelihood method to pooled time series and cross-sectional futures price data to estimate our model. By incorporating cross-sectional data, we can exploit the full information content along the yield curve. The data we use is for the Chicago Mercantile Exchange's short term interest rate futures contracts. The market for these instruments is highly liquid and so we would expect it to reflect very closely the "market's view" on forward rate volatility.

The paper is organized as follows. Section 1 reviews the HJM model, discusses the futures rate evolution given the forward rate evolution where the forward rate volatility function is deterministic. This section will also discuss the bias in using futures rates to approximate forward rates. Section 2 then presents the likelihood transformation method, utilizing the results of Duan (1994) to simplify the likelihood calculation. The full information likelihood is derived by transforming market variables to state variables whose transition density function can be found by analytically solving the Kolmogorov partial differential equation, subject to appropriate boundary conditions, as proposed by Lo (1988). Data and models considered are described in Section 3 . We discuss the parameter estimates in Section 4 .Section 5 concludes the paper. Finally, all technical details are relegated to the Appendices.

1. The Forward, Futures and Futures Yield Dynamics within the HJM Framework

In § 1.1 we derive the dynamics for futures prices implied by the instantaneous forward rate dynamics under conditions of no riskless arbitrage. In § 1.2 we quantify the bias that occurs when futures yields are used to proxy instantaneous forward rates. We then suggest an estimation strategy that avoids the proxy problem by finding directly the likelihood function for observed futures prices.

1.1. **The forward and futures link.**

The HJM model starts from the assumption that the instantaneous T-maturity forward rate $f(t,T)$ (for $t \leq T \in \mathbb{R}^+$) evolves according to the stochastic integral equation

$$f(t,T) = f(0,T) + \int_0^t \mu(u,T,\cdot)du + \sum_{i=1}^{I} \int_0^t \sigma_i(u,T)dW_i(u), \tag{1.1}$$

where the $W_i(t)$ are standard Wiener processes under the historical probability measure $\mathcal{Q}$, and $\mu(t,T,\cdot)$ and the $\sigma_i(t,T)$ are respectively the drift and the set of diffusion coefficients for the instantaneous forward rate to maturity T. Here we shall assume that the $\sigma_i(t,T)$ are time deterministic functions.

HJM show that the elimination of arbitrage opportunities amongst traded derivative instruments implies that the drift is uniquely determined by the volatility functions and the market prices of interest rate risk $\phi_i(t)$ according to

$$\mu(t,T,\cdot) = -\sum_i \sigma_i(t,T)\left[\phi_i(t) - \int_t^T \sigma_i(t,s)ds\right]. \tag{1.2}$$

The forward rate evolution can then be described under the equivalent probability measure $\widetilde{\mathcal{Q}}$, where the market price of risk is absorbed into the Wiener process under $\widetilde{\mathcal{Q}}$, by the stochastic integral equation

$$f(t,T) = f(0,T) + \sum_i \left[\int_0^t \sigma_i(u,T)\int_u^T \sigma_i(u,s)ds\,du + \int_0^t \sigma_i(u,T)d\widetilde{W}_i(u)\right], \tag{1.3}$$

or in the more familiar form of a stochastic differential equation as

$$df(t,T) = \sum_i \left[\sigma_i(t,T)\int_t^T \sigma_i(t,s)dsdt + \sigma_i(t,T)d\widetilde{W}(t)\right]. \tag{1.4}$$

The stochastic integral equation for the evolution of the instantaneous spot rate of interest under $\widetilde{\mathcal{Q}}$ can be derived accordingly from (1.3) by setting $T = t$, thus,

$$r(t) = f(0,t) + \sum_i \left[\int_0^t \sigma_i(u,t)\int_u^t \sigma_i(u,s)ds\,du + \int_0^t \sigma_i(u,t)d\widetilde{W}_i(u)\right].$$

The corresponding stochastic differential equation for the instantaneous spot rate of interest under $\widetilde{\mathcal{Q}}$ is

$$\begin{aligned} dr(t) = &\left[f_2(0,t) + \sum_i \left(\frac{\partial}{\partial t}\int_0^t \sigma_i(u,t)\int_u^t \sigma_i(u,s)ds\,du + \int_0^t \frac{\partial\sigma_i(u,t)}{\partial t}d\widetilde{W}_i(u)\right)\right]dt \\ &+ \sum_i \sigma_i(t,t)d\widetilde{W}_i(t). \end{aligned} \tag{1.5}$$

Any derivative instrument can then be priced under the risk neutral measure. A futures contract is a derivative instrument written on a bond, and therefore, its price today is just the expectation of the future payoff under the risk neutral measure.

Let $F(t,T_F,T_B)$ be the price at time t of a futures contract maturing at time $T_F(> t)$. The contract is written on a pure discount instrument which has a face value of \$1 and matures at time $T_B(> T_F)$.

Proposition 1.1. *Under the assumption that the instantaneous forward rate $f(t,T)$ evolves under $\widetilde{\mathcal{Q}}$ according to (1.3), the evolution of $F(t,T_F,T_B)$ is given by the stochastic integral equation*

$$\begin{aligned} F(t,T_F,T_B) = F(0,T_F,T_B)\exp&\left[-\frac{1}{2}\sum_i\int_0^t\left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)^2 du\right. \\ &\left. -\sum_i\int_0^t\left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)d\widetilde{W}_i(u)\right], \end{aligned}$$

or equivalently by the stochastic differential equation

$$\begin{aligned}\frac{dF(t,T_F,T_B)}{F(t,T_F,T_B)} &= -\sum_i \left(\int_{T_F}^{T_B} \sigma_i(t,s)ds \right) d\widetilde{W}_i(t) \\ &\equiv \sum_i \sigma_{Fi}(t,T_F,T_B) d\widetilde{W}_i(t).\end{aligned} \tag{1.6}$$

Proof. Let $P(t,T_B)$ be the price at time t of a pure discount instrument that has a face value of \$1 and matures at time T_B, and let $B(t,T_B)$ be the corresponding log bond price, ie $B(t,T_B) = \ln P(t,T_B)$.

Since futures contracts are marked-to-market, it is shown in Cox et al. (1981) that the futures prices are Martingales under the equivalent measure $\widetilde{\mathcal{Q}}$:

$$\begin{aligned}F(t,T_F,T_B) &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[F(T_F,T_F,T_B)|\mathcal{F}_t\big] \\ &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[P(T_F,T_B)|\mathcal{F}_t\big] \\ &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[\exp(B(T_F,T_B))|\mathcal{F}_t\big] \\ &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\left[\exp\left(-\int_{T_F}^{T_B} f(T_F,s)ds\right)\right].\end{aligned} \tag{1.7}$$

Evaluating the expectation will give us the result. The proof is a straightforward extension of the derivation in Musiela et al. (1992) where only one noise term is considered. Details can be found in Appendix A. □

1.2. **The futures yield and bias decomposition.**

Let $y(t,T_F,T_B)$ be the market quoted "futures yield" corresponding to the futures price $F(t,T_F,T_B)$, ie. the quantity defined according to [2]

$$F(t,T_F,T_B) = 1 - y(t,T_F,T_B)(T_B - T_F). \tag{1.8}$$

Application of Itô's lemma gives the stochastic differential equation for $y(t,T_F,T_B)$ under $\widetilde{\mathcal{Q}}$, viz

$$dy(t,T_F,T_B) = \left(y - \frac{1}{T_B - T_F}\right) \sum_i \sigma_{Fi}(t,T_F,T_B) d\widetilde{W}_i(t). \tag{1.9}$$

[2]This is the "futures yield" quoted as a discount rate, which is appropriate in the U.S. market. In some other markets such as the Australian market, it may be more appropriate to use the "futures yield" quoted as a yield-to-maturity, i.e. according to the formula $F(t,T_F,T_B) = \frac{1}{1+y(t,T_F,T_B)(T_B-T_F)}$. The lines of argument follow similarly.

Under the equivalent measure $\widetilde{Q}$, it follows from (1.2) that the forward rate $f(t, T_F, T_B)$ is distributed normally, whereas it is clear from (1.9) that the futures yield $y(t, T_F, T_B)$ is not distributed normally. The resulting variances of the two processes are different, depending on the maturity of the futures contract (ie. $T_B - T_F$) and the specification of the forward rate volatility function. Since the variance structure is preserved under the transformation from the historical measure to the equivalent measure , using futures yields as a proxy for the instantaneous forward rate (under the historical measure) would impose a wrong variance on the distribution, and therefore, distort the estimation results.

To be more precise, from (1.3), the variance of the instantaneous forward rate is

$$\mathrm{var}\left(f(t, T_B)\right) = \sum_i \int_{t_0}^{t} \sigma_i^2(u, T_B) du, \tag{1.10}$$

whereas the variance of the fixed-maturity futures yield is (see Appendix B)

$$\mathrm{var}\left(y(t, T_F, T_B)\right) = \left(y(0, T_F, T_B) - \frac{1}{T_B - T_F}\right)^2 e^{\bar{\sigma}_f^2 (T_B - T_F)^2} \left(e^{\bar{\sigma}_f^2 (T_B - T_F)^2} - 1\right), \tag{1.11}$$

where

$$\bar{\sigma}_f^2 = \frac{1}{(T_F - T_B)^2} \sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u, T_F, T_B) du.$$

The difference between the two variance measures is the overall bias, which can be decomposed into two components, maturity bias and convexity bias, as illustrated in Figure 1, where we have denoted by $f(t, T_F, T_B)$ the discrete-period forward rate, which is the holding period return between time T_F and $T_B (> T_F)$ of a bond maturing at time T_B, ie. $f(t, T_F, T_B)$ satisfies

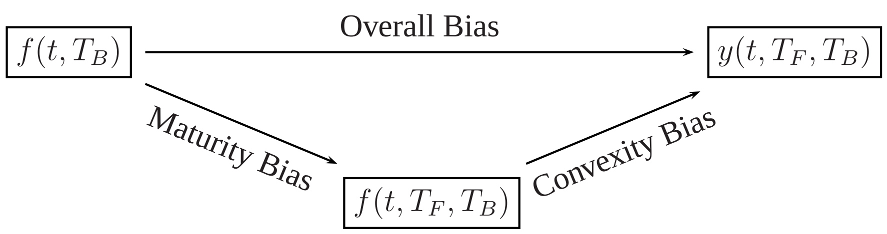
$$P(t, T_F) = P(t, T_B) \exp\left(f(t, T_F, T_B)(T_B - T_F)\right).$$

In Appendix C, we show that the variance of this discrete-period forward rate is

$$\mathrm{var}\left(f(t, T_F, T_B)\right) = \bar{\sigma}_f^2. \tag{1.12}$$

Therefore, from (1.10) and (1.12), the maturity bias component, which arises from approximating the instantaneous forward rate by the discrete-period forward rate, is

FIGURE 1. Bias Decomposition



given by

$$\text{Maturity Bias} = \bar{\sigma}_f^2 - \sum_i \int_{t_0}^{t} \sigma_i^2(u, T_B) du. \tag{1.13}$$

This bias component is negligible when the discrete period is short (ie. $\tau = T_B - T_F \to 0$). This is in agreement with Chapman et al. (1999) who study the bias induced by using fixed tenor short rates as a proxy for the instantaneous spot rate. They also conclude that the bias is not economically significant in the class of linear short rate models, to which the HJM with deterministic volatility belongs.

The convexity bias component, which arises from approximating the fixed-maturity forward rate by a fixed-maturity futures rate, is given by (see (1.11) and (1.12))

$$\text{Convexity Bias} = \left(y(0, T_F, T_B) - \frac{1}{T_B - T_F} \right)^2 e^{\overline{\sigma}_f^2 (T_B - T_F)^2} \left(e^{\overline{\sigma}_f^2 (T_B - T_F)^2} - 1 \right) - \overline{\sigma}_f^2. \tag{1.14}$$

which is non-negligible due to the presence of the initial futures yield value and the convexity of the exponential function. The difference between forward rates and futures rates results from the difference between forward contract prices and futures contract prices. The marking-to-market feature of futures contracts causes their prices to differ from forward contract prices under a stochastic interest rate environment.

We have run a Monte Carlo simulation in order to gauge the level of bias when the futures yield is used as a proxy for the instantaneous forward rate in estimation. The simulation was run for a single factor HJM model with a humped forward volatility curve

$$\sigma(t, T) = [\sigma_0 + \sigma_1 (T - t)] \exp(-\kappa (T - t)),$$

and a constant market price of risk ϕ. The model was simulated (50,000 times) for a time period of one year (252 observations) from an assumed true parameter set[3]. First, we simulated the futures price according to its dynamics (1.6). Then we used this futures price series as the proxy for the instantaneous forward rate, and estimated the model via the likelihood function based on the instantaneous forward rate evolution (1.3) (transformed into the historical measure). The results of the simulation are displayed in Table 1. It can be clearly seen that the proxy method results in quite high mean bias and root mean squared error.

TABLE 1. Estimation Bias from the Proxy Method

This table reports the bias resulting from using "futures yields" calculated from the futures price as a proxy for the instantaneous forward rate in estimation. The simulation is run for 50,000 experiments. "Mean MC" is the mean for all simulated estimates. "MCSD" is the standard deviation of the simulated estimates. "Mean Bias" is the difference between the "Mean MC" and the true parameter value. "RMSE" is the root of the mean squared errors.

Parameters	True value	Mean MC	MCSD	Mean Bias	RMSE
σ_0	0.01	0.0074	0.0036	-0.0026	0.0045
σ_1	0.04	0.0107	0.0121	0.0067	0.0138
κ	0.25	0.5271	0.3873	0.2771	0.4762
ϕ	0.7	1.3128	2.5650	0.6128	2.6372

Thus it is advisable that in empirical work the futures yields should not be used as a proxy for the instantaneous forward rates. One suitable estimation strategy would present itself if we could find a closed-form formula for the futures yields/prices in terms of the latent instantaneous forward rates. Then the method of inverting the density of the observable data to obtain the density of the underlying variable can be applied, as has been done in Pearson and Sun (1994) for their version of the CIR model. However, for HJM model specifications that have (not-very-simple) time-deterministic instantaneous forward rate volatility functions, such closed-form formulae are generally not obtainable. The estimation strategy that we propose here goes beyond the Pearson and Sun (1994) approach by recognizing from (1.7) that the futures contract is a derivative instrument written on the instantaneous forward rate, and therefore, the futures price is driven by the same source of uncertainty as that driving the instantaneous

[3]This assumed true set was chosen to coincide with the estimated values found later in the empirical analysis of Section 4.

forward rate. Knowing the structure of the uncertainty source allows us to derive the likelihood function for the observable futures prices without the need for their closed form formula. In addition, we do not need to Markovianize the interest rate dynamics, which, as far as we are aware, is a requirement in previous studies of the estimation of HJM models. In the next section, we will review the likelihood transformation technique, and derive the likelihood function for quoted futures prices via a state variable for which a closed form for the likelihood function is readily available.

2. Transformation of the Likelihood Function

In this section we derive the likelihood function for the quoted futures prices. In § 2.1 we introduce a state variable whose likelihood function is ready to find. In § 2.2 we write out in detail the transitional likelihood function of this state variable using the pooled time series and cross-sectional data. We then show how to convert this into the full information likelihood function for the observed futures prices.

2.1. **State variables.**

Assume that for each underlying pure-discount interest rate instrument, there are K futures contracts maturing at times T_{Fk} ($k = 1, 2, \ldots, K$). The (observable) quoted futures price in the market is $G(t, T_{Fk}, T_{Bk})$, which is linked with $F(t, T_{Fk}, T_{Bk})$ via an exchange specific function η, so that

$$F(t, T_{Fk}, T_{Bk}) \equiv \eta\big(G(t, T_{Fk}, T_{Bk})\big). \tag{2.1}$$

The link between F and G depends on the quoting convention of each exchange. For example, the Eurodollar futures prices traded on the Chicago Mercantile Exchange are quoted as

$$\begin{aligned} F(t, T_{Fk}, T_{Bk}) &= 1 - \left(1 - \frac{G(t, T_{Fk}, T_{Bk})}{100}\right)(T_{Bk} - T_{Fk}) \\ &\equiv \eta\big(G(t, T_{Fk}, T_{Bk})\big). \end{aligned} \tag{2.2}$$

We are considering the case in which all of the futures contracts are written on the same underlying instrument, and therefore the time to maturity of the underlying contract is $T_{Bk} - T_{Fk} = \tau$ constant for all $k \in [0, K]$.

In order to carry out the estimation, we need the evolution of the futures prices under the physical measure $\mathcal{Q}$, therefore we introduce the market prices, $\phi_i(t)$, for each Wiener process risk into the system dynamics. In addition, in order to capture measurement error in the market (for example, due to bid-ask spread), we introduce into the evolution of $F(t, T_{Fk}, T_{Bk})$ a new Wiener process ε_k which is independent of the processes driving the uncertainty of forward rates. We further assume that the market errors for the return on futures with different maturities are uncorrelated with each other. Thus, in the present context, the stochastic differential equation (1.6) for $F(t, T_{Fk}, T_{Bk})$ becomes [4]

$$\frac{dF(t, T_{Fk}, T_{Bk})}{F(t, T_{Fk}, T_{Bk})} = -\sum_i \left(\int_{T_{Fk}}^{T_{Bk}} \sigma_i(t, s) ds \right) (dW_i(t) + \phi_i(t)dt) + \sigma_\varepsilon d\varepsilon_k. \tag{2.3}$$

It should be noted that we choose a Wiener process representation for the error term clearly with a view towards mathematical tractability. Adopting the standard econometric approach of letting

$$F^{observed}(t, T_{Fk}, T_{Bk}) = F^{model}(t, T_{Fk}, T_{Bk}) + u_k,$$

where $u_k \sim \mathcal{N}(0, \sigma_u)$ would create the difficulty of mixing up the continuous and discrete time settings. Under our setting of continuous measurement error, the error becomes frequency-based (e.g. daily versus weekly data). However, since in most (and in our) empirical analysis, only one frequency data is used, this frequency-based feature is not a significant issue.

If we assume that the market prices of risk are independent of any state variables, then (2.3) implies that the logarithm of the futures price is normally distributed. This is easily seen by introducing the notation

$$\begin{aligned} X(t, T_{Fk}, T_{Bk}) &= \ln(F(t, T_{Fk}, T_{Bk})) \\ &\equiv \zeta(F(t, T_{Fk}, T_{Bk})). \end{aligned} \tag{2.4}$$

Application of Itô lemma gives

$$\begin{aligned} dX(t, T_{Fk}, T_{Bk}) = &-\frac{1}{2} \left[\sum_i \left(\int_{T_{Fk}}^{T_{Bk}} \sigma_i(t, s) ds \right)^2 + \sigma_\varepsilon^2 \right] dt \\ &- \sum_i \int_{T_{Fk}}^{T_{Bk}} \sigma_i(t, s) ds \, (dW_i(t) + \phi_i(t)dt) + \sigma_\varepsilon d\varepsilon_k. \end{aligned} \tag{2.5}$$

[4] In practice, the value of σ_ε should be small (in order and magnitude) in comparison with the forward rate volatility σ, so that any attempt to set up an arbitrage portfolio to trade on this uncertainty source would not result in profits after bid-ask spread and transaction costs are taken into account.

Since X is normally distributed we are able to obtain its likelihood function which implies the normality of X. Details of the actual distribution are given in the next subsection. Then we can apply the likelihood transformation technique twice to derive first the likelihood function for F and then for the market quoted variable G. In Appendix D, we will review the likelihood transformation technique, and utilize Duan's (1994) result to simplify it in the context of our particular application. In the next section, we will write out the exact likelihood function for the quoted futures price, using pooled time series and cross sectional data.

2.2. **The full information maximum likelihood function.**

With a slight abuse of notation, let $X_{j\,k} \equiv X(t_j, T_{Fk}, \tau) \equiv X(t_j, T_{Fk}, T_{Bk})$[5] be the k^{th} unobservable state variable ($k = 1, 2, \ldots, K$) occurring at time $t_j < T_F$ ($j = 0, 1, \ldots, J$).

Due to the Markovian nature of the stochastic process for $X(t, T_{Fk}, T_{Bk})$, the likelihood function for $(X_{0\,k}, X_{1\,k}, \ldots, X_{J\,k})$ [6], for a given parameter vector of interest $\boldsymbol{\theta}$, is

$$p_{\boldsymbol{X}}(X_{0\,k}, X_{1\,k}, \ldots, X_{J\,k}; \boldsymbol{\theta}) = p_{\boldsymbol{X}}(X_{0\,k}, t_0; \boldsymbol{\theta}) \prod_{j=1}^{J} p_{\boldsymbol{X}}(X_{j\,k}, t_j | X_{(j-1)\,k}, t_{j-1}; \boldsymbol{\theta}).$$

With this discrete sample, it is proved in Lo (1988) [7] that the transitional likelihood function has the Gaussian form

$$p_{\boldsymbol{X}}(X_{j\,k}, t_j | X_{(j-1)\,k}, t_{j-1}; \boldsymbol{\theta}) = \left[2\pi\beta_{j(kk)}\right]^{-\frac{1}{2}} \exp\left[-\frac{\left(X_{j\,k} - X_{(j-1)\,k} - \alpha_{j\,k}\right)^2}{2\beta_{j(kk)}}\right],$$

[5] We write $X(t_j, T_{Bk}, \tau) \equiv X(t_j, T_{Fk}, T_{Bk})$ because $T_{Bk} - T_{Fk} = \tau$, a constant for all $k = 1, 2, \ldots, K$

[6] Recall that $X_{j\,k} \equiv X(t_j, T_{Fk}, T_{Bk})$

[7] Lo (1988) proves the case where there is only one noise term. By substitution, it is a straight forward extension to prove the result for the multiple-noise case. In any event, the result is merely a consequence of the fact that the process (2.5) for $X(t, T_{Fk}, T_{Bk})$ is Gaussian due to the assumption of time dependent volatility functions and market prices of interest rate risk.

where the mean $\alpha_{j\,k}$ and the variance $\beta_{j(kk)}$ are given by

$$\begin{aligned}\alpha_{j\,k} = -\frac{1}{2}\left[\sum_i \int_{t_{j-1}}^{t_j} \left(\int_{T_{Fk}}^{T_{Bk}} \sigma_i(u,s)ds\right)^2 du + \int_{t_{j-1}}^{t_j} \sigma_\varepsilon^2 du\right] \\ + \int_{t_{j-1}}^{t_j} \left(\phi(u) \int_{T_{Fk}}^{T_{Bk}} \sigma_i(u,s)ds\right) du,\end{aligned} \tag{2.6}$$

$$\beta_{j(kk)} = \sum_i \int_{t_{j-1}}^{t_j} \left(\int_{T_{Fk}}^{T_{Bk}} \sigma_i(u,s)ds\right)^2 du + \int_{t_{j-1}}^{t_j} \sigma_\varepsilon^2 du. \tag{2.7}$$

If we incorporate cross-sectional data into our study to exploit the full information content of the yield curve, we will have a set of observations with different times to maturity. Denote by $\boldsymbol{x}_j$ the vector of unobservable state variables occurring at time t_j, ie. $\boldsymbol{x}_j = \big(X(t_j,T_1,\tau), X(t_j,T_2,\tau), \ldots, X(t_j,T_K,\tau)\big)$. The transitional likelihood function will have the multi-dimensional Gaussian form

$$\begin{aligned}p_{\boldsymbol{X}}(\boldsymbol{x}_j, t_j|\boldsymbol{x}_{j-1}, t_{j-1};\boldsymbol{\theta}) = (2\pi)^{-\frac{K}{2}}\left|\boldsymbol{\Omega}_j\right|^{-\frac{1}{2}}\times \\ \exp\left(-\frac{1}{2}\left(\boldsymbol{x_j} - \boldsymbol{x}_{j-1} - \boldsymbol{\alpha}_j\right)' \boldsymbol{\Omega}_j^{-1}\left(\boldsymbol{x_j} - \boldsymbol{x}_{j-1} - \boldsymbol{\alpha}_j\right)\right),\end{aligned} \tag{2.8}$$

with mean vector

$$\boldsymbol{\alpha}_j = \left(\alpha_{j\,1}\,,\ \alpha_{j\,2}\,,\ \ldots, \alpha_{j\,k}\,,\ \ldots\ ,\ \alpha_{j\,K}\right)',$$

and covariance matrix

$$\boldsymbol{\Omega}_j = \begin{pmatrix} \beta_{j(11)} & \beta_{j(12)} & \cdots & \beta_{j(1K)} \\ \beta_{j(21)} & \beta_{j(22)} & \cdots & \beta_{j(2K)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{j(K1)} & \beta_{j(K2)} & \cdots & \beta_{j(KK)} \end{pmatrix},$$

where for $k_1 \neq k_2$ [8] the covariances are defined by

$$\beta_{j(k_1 k_2)} = \sum_i \int_{t_{j-1}}^{t_j} \left(\int_{T_{Fk_1}}^{T_{Bk_1}} \sigma_i(u,s)ds\right)\left(\int_{T_{Fk_2}}^{T_{Bk_2}} \sigma_i(u,s)ds\right) du. \tag{2.9}$$

The log likelihood function for the state variable $\boldsymbol{X}$ is

$$\mathcal{L}_{\boldsymbol{X}}(\boldsymbol{\theta}) = \sum_{j=1}^{J} \ln\Big(p_{\boldsymbol{X}}\big(\boldsymbol{x}_j, t_j|\boldsymbol{x}_{j-1}, t_{j-1};\boldsymbol{\theta}\big)\Big). \tag{2.10}$$

[8]Note that the variances $\beta_{j(kk)}$ have already been defined in (2.7).

In the above formula we have ignored the unconditional probability of the first observation at time t_0. As argued in Aït-Sahalia (2002), this unconditional probability is dominated by the sum of all conditional density terms when the sample size becomes large.

However, we do not observe $\boldsymbol{X}$, rather we observe the quoted futures values $\boldsymbol{G}$, which are related to the futures prices $\boldsymbol{F}$ via (2.1). The futures price $\boldsymbol{F}$ is in turn related to $\boldsymbol{X}$ via (2.4). Thus our task now is to achieve the likelihood function for $\boldsymbol{G}$ from the known one for $\boldsymbol{X}$ in (2.10). This involves a repeated application of the likelihood transformation formula (see Appendix D), which in our context is simplified by the fact that the transformations from $\boldsymbol{X}$ to $\boldsymbol{F}$ and from $\boldsymbol{F}$ to $\boldsymbol{G}$ are both on an element-by-element basis.

Recalling that the transformation from $\boldsymbol{X}$ to $\boldsymbol{F}$ (see (2.4)) involves the inverse function $\boldsymbol{\zeta}$ and applying the transformation formula, the likelihood function for $\boldsymbol{F}$ is

$$\mathcal{L}_{\boldsymbol{F}}(\boldsymbol{\theta}) = \sum_{j=1}^{J} \ln \Big(p_{\boldsymbol{X}}\big(\boldsymbol{\zeta}(\boldsymbol{F}_j), t_j | \boldsymbol{\zeta}(\boldsymbol{F}_{j-1}), t_{j-1}; \boldsymbol{\theta}\big)\Big) + \sum_{j=1}^{J} \ln \left| \frac{\partial \boldsymbol{\zeta}_j(\boldsymbol{F}_j; \boldsymbol{\theta})}{\partial \boldsymbol{F}_j} \right|. \quad (2.11)$$

Applying the transformation the second time from $\boldsymbol{F}$ to $\boldsymbol{G}$, the quoted futures price in the market, with the inverse transformation function $\boldsymbol{\eta}$ (see (2.1)) results in the sought log likelihood function for the observed $\boldsymbol{G}$, namely

$$\mathcal{L}_{\boldsymbol{G}}(\boldsymbol{\theta}) = \sum_{j=1}^{J} \ln \Big(p_{\boldsymbol{F}}\big(\boldsymbol{\eta}(\boldsymbol{G}_j), t_j | \boldsymbol{\eta}(\boldsymbol{G}_{j-1}), t_{j-1}; \boldsymbol{\theta}\big)\Big) + \sum_{j=1}^{J} \ln \left| \frac{\partial \boldsymbol{\eta}_j(\boldsymbol{G}_j; \boldsymbol{\theta})}{\partial \boldsymbol{G}_j} \right|. \quad (2.12)$$

Our task is thus to seek the parameter vector $\boldsymbol{\theta}$ that maximizes this likelihood function.

3. Models and Data

In § 3.1 we lay out the volatility function we seek to estimate. In § 3.2 we discuss the data and how it is structured for estimation purposes.

3.1. **Models.**

In this paper, we are interested in the short term interest rate futures market. Since futures contracts are usually actively traded for maturities less than 5 years, Amin and Morton (1994) argue that there is usually insufficient variation in the term structure across different maturities to separate the effect of different uncertainty sources. In addition, they cite Dybvig (1990), who shows that almost all of the variation in forward rates with maturities less than five years can be explained by a dominant single factor. Therefore, we will estimate a single-factor HJM model, ie. there is only a single source of uncertainty[9]. In addition, since the focus of this paper is on the volatility function, we treat the market price of risk as a constant[10].

The class of HJM model with which we are working is determined by the specification of the volatility function. We choose a fairly general "time-invariant" humped-volatility curve, ie. the volatility $\sigma(t,T)$ depends on $T-t$ only, not on the calendar date t, thus we set

$$\sigma(t,T) = [\sigma_0 + \sigma_1(T-t)] \exp(-\kappa(T-t)). \tag{3.1}$$

The model (3.1) nests many of the time-deterministic volatility forms considered in the literature so far:

- The exponential model (Hull and White (1990) Extended Vasicek Model): $\sigma(t,T) = \sigma_0 \exp(-\kappa(T-t))$
- The linear absolute model: $\sigma(t,T) = \sigma_0 + \sigma_1(T-t)$
- The absolute (or constant) model (Ho and Lee (1986) model): $\sigma(t,T) = \sigma_0$

Despite the fact that the implied volatility functions obtained from caps and swaptions data often exhibit a humped volatility structure (Amin and Morton (1994), p. 160, and Hull and White (1996), p. 33), as far as we are aware, there has so far only been the attempt of Ritchken and Chuang (1999) to estimate the humped-volatility model of the form (3.1) in the HJM framework[11]. The estimation approach used by Ritchken and Chuang (1999) requires the Markovianization of the interest rate dynamics, ie. they need to use the spot rate of interest evolution, which is not Markovian, and therefore they Markovianize the system containing the evolution of the spot rate of interest[12]. Even though we estimate the same model, we do not need to rely on

[9] Even though a single source of uncertainty is appropriate for the volatility structure of futures contracts, in other markets such as the bond market, previous studies (eg. Duan and Simonato (1999), Dai and Singleton (2000)) have found that allowance should be made for at least two factors driving the term structure.

[10] As we discuss later in the data section, the estimation is carried out for each year. The assumption of a constant market price of risk over a one year horizon seems plausible.

[11] The Moraleda and Vorst (1997) model is also a humped-volatility model. However, they follow Amin and Morton (1994) by finding the implied volatility parameters day-by-day rather than estimating the fixed model parameters over an estimation period.

[12] Markovianization is a property of certain classes of HJM models. Even within the time deterministic volatility class, it is not clear whether all models can be Markovianized.

any Markovianization property of the system dynamics since our approach proceeds directly from the instantaneous forward rate dynamics.

The analytical expression for the log likelihood function of the quoted futures price under this volatility specification involves performing the integrations in (2.6), (2.7), and (2.9), details of which can be found in Appendix E.

3.2. **Data.**

We apply the method outlined above to short term interest rate futures contracts traded on the Chicago Mercantile Exchange (CME). The CME contracts are written on Eurodollar Time Deposits with a three-month maturity. The last trading day for each contract is the second London bank business day before the third Wednesday of the contract month, which rests in the March, June, September, December cycle. The data are taken from Datastream^{TM}.

The CME Eurodollar futures contracts are chosen for their extreme liquidity. Table 2 reports the average daily trading volume of the contracts used in our study.

TABLE 2. CME Eurodollar Futures Contracts

This table reports the set of contracts used in our estimation each year. Each contract in one set is three quarters apart in maturity from the next one in the sequence. Due to liquidity reasons, the number of contracts included in each set is different. The average daily trading volume is measured as the number of contracts traded.

Year	Begin Contract	End Contract	Total Number of Contracts	Observation per Series	Average Daily Trading Volume
1988	03/1989	12/1989	2	211	7,100
1989	03/1990	09/1991	3	211	7,119
1990	03/1991	09/1992	3	213	8,216
1991	03/1992	06/1994	4	212	8,238
1992	03/1993	06/1995	4	213	14,913
1993	03/1994	12/1997	6	210	11,840
1994	03/1995	12/1998	6	210	19,434
1995	03/1996	12/1999	6	210	15,397
1996	03/1997	12/2000	6	214	15,883
1997	03/1998	12/2001	6	210	16,990
1998	03/1999	12/2002	6	213	18,709
1999	03/2000	12/2003	6	209	16,497
2000	03/2001	12/2004	6	211	17,926
2001	03/2002	12/2005	6	210	30,762

The CME data cover the 14-year period from January 1, 1988 to December 31, 2001. The period is chosen so that the first 5-year period coincides with the data used in Amin and Morton (1994). We first estimate the model using both time series and cross sectional data during the whole period of 1988-2001. We then estimate our model for each year period separately, since the volatility parameters must reflect the current market condition, as also argued in Bühler et al. (1999). The choice of one year period is certainly arbitrary, however we choose the period of one year in a belief that this might be what practitioners would do, ie. do the estimation every year to choose the best family of models for that period, and do the calibration every shorter time period (say, a week) to obtain up-to-date parameters values.

Since a futures contract has a relatively short life, we roll over futures contracts along the 14-year sample period. For each trading year, the futures series considered starts from the March contract maturing the following year, until the last actively traded contracts. To ensure a sufficient variation in futures prices, and so avoid possible singularity of the covariance matrix, the set of contracts used are spaced three quarters apart. For example, to estimate volatility parameters for 2001, we use March 2002, December 2002, September 2003, June 2004, March 2005 and December 2005 contracts (see Figure 2)[13]. Since the amount of trading activity in each year differs, the number of contracts included in our analysis varies with time, as shown in Table 2. From 1993 to 2001, 6 contracts are included in our analysis. On average, there are 211 observations for each series.

4. Empirical Results

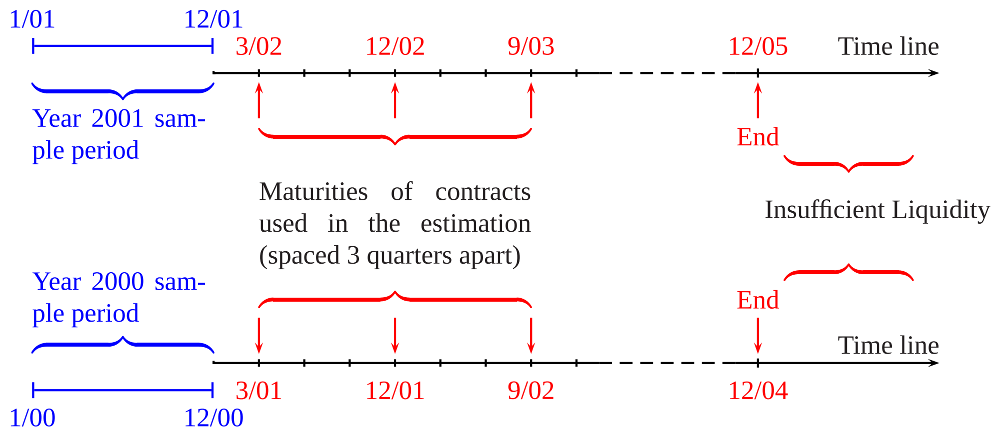
§ 4.1 lays out the estimation results and discusses their implications. We consider both the full sample period and various subperiods related to different phases of interest rate policy. In § 4.2 we give some analysis of the stability of our parameter estimates. § 4.3 considers the issue of model goodness of fit by carrying out various tests on the residuals.

All of our empirical work was carried out using Ox^{TM}, a matrix-oriented programming language[14].

[13]For all of the years, we have repeated the estimation using different combination of futures price series (such as different starting contracts and different spacing between contracts), and the estimation results are not significantly different

[14]See Doornik (1996).

FIGURE 2. Research design - Futures contracts used in sample period January 2001 - December 2001.



4.1. **The estimation.**

We first ran the estimation for the whole 14-year sample period. The estimation would indicate an "average" of volatilities over the period. We then ran the estimation for each year, which would reflect the variation in market conditions more closely. The results of the estimations can be found in Table 3.

The estimated average volatility curve over the entire sample period is

$$\sigma(t,T) = \underset{(0.0005)}{[0.0096} + \underset{(0.0006)}{0.0041}(T-t)]\exp(\underset{(0.0200)}{-0.2380}(T-t)).$$

The model estimates the instantaneous volatility of the short rate to be 1% per annum. A humped forward volatility curve implies that the instantaneous volatility of the spot rate is lower than short-term forward volatilities. However, forward volatility gradually decreases as time to maturity increases, and finally reaches a lower level than the spot rate volatility. Figure 3 shows that the hump occurs at 1.5-2 years to maturity.

The estimated standard deviation of the error term is 0.0009 (standard error of 1.7×10^{-5}), which is very small in magnitude and therefore it would indeed not be possible to exploit the noise term source to set up strategies leading to arbitrage profits.

The estimation using yearly data reveals that the volatility curve is changing every year. For the years 1988 to 1992, the estimation using the humped shaped volatility

TABLE 3. Estimation result for the entire period 1988-2001

This table reports the best model for each year and the corresponding parameter values. The unrestricted model is the humped model, where the forward volatility function is $\sigma(t,T) = (\sigma_0 + \sigma_1(T-t))\exp(-\kappa(T-t))$, the restricted model is the exponential model (where $\sigma_1 = 0$), the linear absolute model (where $\kappa = 0$), or the constant model (where $\sigma_1 = 0$ and $\kappa = 0$). Robust asymptotic standard errors of the estimate are given in parentheses. The p-values for the likelihood ratio test (between the humped model and the model reported here) are given in square brackets and under the corresponding likelihood value.

Year	Model	$\hat{\sigma}_0$	$\hat{\sigma}_1$	$\hat{\kappa}$	$\hat{\sigma}_\varepsilon$	ϕ	Log Lik.
All	Humped	0.0096	0.0041	0.2380	0.0009	0.6706	36096
		(0.0005)	(0.0006)	(0.0200)	(0.0000)	(0.2720)	
1988	Exponential	0.0144	-	0.1425	0.0008	0.4441	710.34
		(0.0017)	-	(0.0622)	(0.0001)	(1.1062)	[0.0351]
1989	Exponential	0.0201	-	0.2442	0.0012	1.4254	974.83
		(0.0023)	-	(0.0508)	(0.0001)	(1.1107)	[1.0000]
1990	Exponential	0.0128	-	0.1179	0.0012	0.3226	1027.4
		(0.0016)	-	(0.0507)	(0.0002)	(1.1256)	[0.3173]
1991	Exponential	0.0113	-	0.1589	0.0008*	2.6739	1754.0
		(0.0010)	-	(0.0273)	(0.0000)	(1.1552)	[1.0000]
1992	Exponential	0.0160	-	0.1111	0.0010	0.9848	1529.0
		(0.0015)	-	(0.0334)	(0.0001)	(1.1112)	[0.0000]
1993	Humped	0.0077	0.0050	0.2884	0.0007	2.2447	2850.5
		(0.0012)	(0.0012)	(0.0421)	(0.0000)	(1.1157)	
1994	Humped	0.0119	0.0028	0.1533	0.0006	-1.8507	3065.6
		(0.0012)	(0.0009)	(0.0335)	(0.0000)	(1.0818)	
1995	Humped	0.0139	0.0060	0.3528	0.0008	2.5302	2674.0
		(0.0021)	(0.0017)	(0.0431)	(0.0000)	(1.0747)	
1996	Humped	0.0136	0.0046	0.2600	0.0006	-0.5006	2984.9
		(0.0020)	(0.0014)	(0.0415)	(0.0000)	(1.0679)	
1997	Humped	0.0063	0.0033	0.2168	0.0004	0.9650	3525.0
		(0.0006)	(0.0006)	(0.0217)	(0.0000)	(1.1078)	
1998	Humped	0.0041	0.0063	0.3022	0.0007	1.4143	2914.3
		(0.0024)	(0.0019)	(0.0363)	(0.0001)	(1.1100)	
1999	Humped	0.0043	0.0064	0.2654	0.0007	-1.5252	2944.1
		(0.0009)	(0.0008)	(0.0199)	(0.0000)	(1.1078)	
2000	Humped	0.0065	0.0033	0.2122	0.0007	0.9207	2951.3
		(0.0008)	(0.0007)	(0.0369)	(0.0000)	(1.1004)	
2001	Humped	0.0106	0.0023	0.1288	0.0010	1.5715	2475.5
		(0.0014)	(0.0015)	(0.0581)	(000000)	(1.1431)	

curve did not result in significant parameter estimates, suggesting that the forward volatility is over-parameterized. Therefore we re-estimated the volatility curve using the exponential, linear absolute and absolute specification. We choose the exponential specification for the forward volatility curve for its ability to deliver stable estimates, even though the likelihood ratio tests do not support the restriction in the years 1988 and 1992. This 1988-1992 estimation period happens to coincide with the estimation period used in Amin and Morton (1994). In that study, Amin and Morton also reported that the estimates had high standard error, and they concluded in favour of the absolute model (i.e. constant volatility) for its ability to deliver stable parameter values.

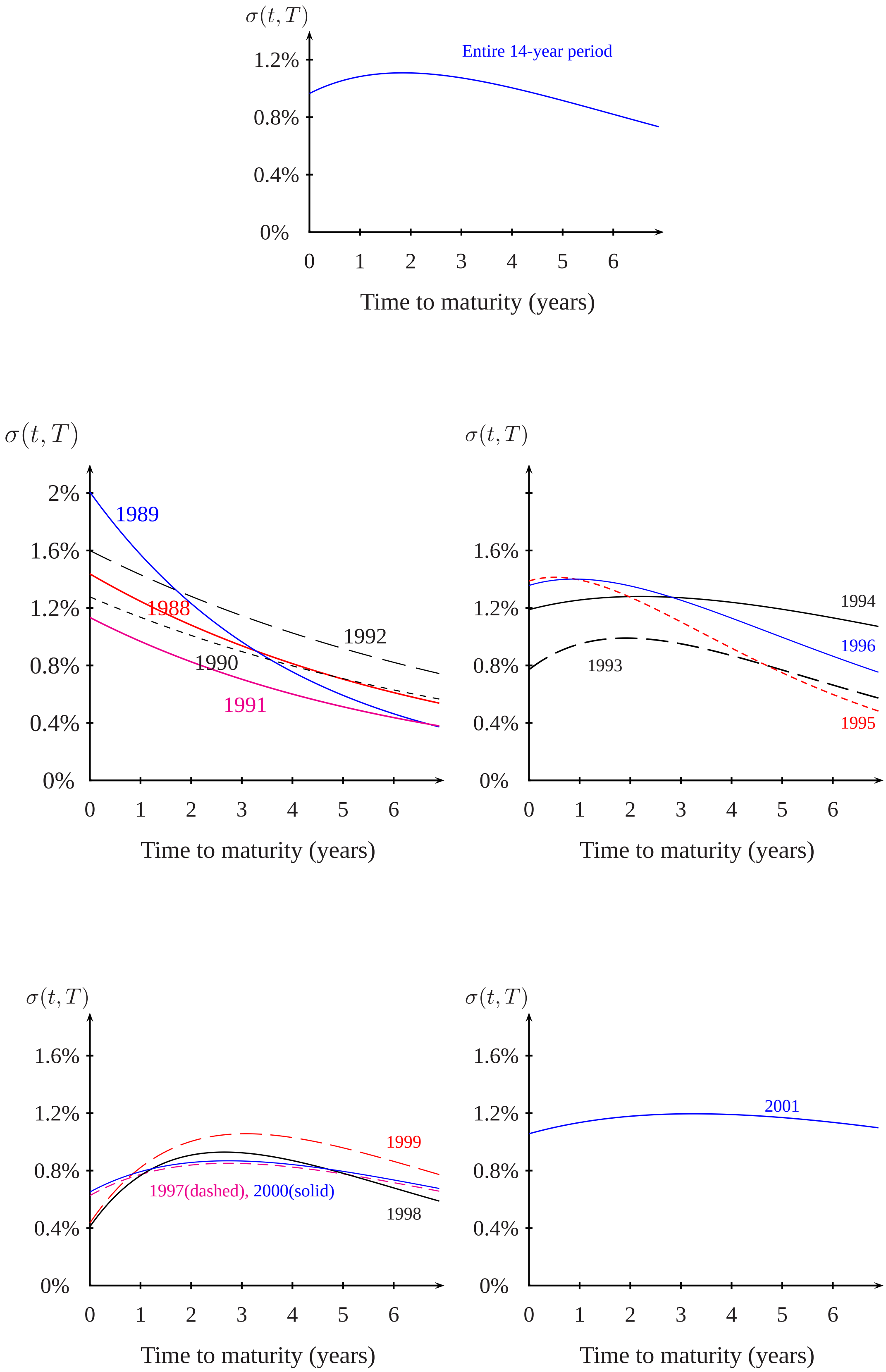
The changes in the forward rate volatility curve are best illustrated in Figure 3. During the 14-year sample period, the interest rate market appears to experience four distinct phases. The first one spanned the five years of 1988-1992, the second one lasted from 1993-1996, the third one came into effect from 1997-2000, and the fourth one just started from 2001.

The first sub-period of 1988-1992 is characterized by the rate-reduction policy of the Federal Reserve, starting at around 9.5% and coming down to 3% by the end of 1992. In this period, the volatility of the interest rate market was high, with the volatility for the instantaneous spot rate averaging at around 1.5%. The market apparently understood the trend of the reducing interest rate, therefore the volatility of longer-term forward rates were smaller than the short term counterpart, resulting in an exponential shape for the volatility curve.

The year of 1992 also marked the near-end of the economic recession period. The economy started to pick up in 1993 and recovery gained momentum from 1994. The volatility of the spot rate of interest reduced compared to the previous period, averaging at about 1.15%. The market seemed to have confidence in the long run recovery, but was still somewhat uncertain about the short term, this would explain why the long run volatility displayed in the years 1995 and 1996 is much smaller than the short term volatility.

The next period, 1997-2000 was characterized by price stability. Volatility in the interest rate market was therefore at its lowest level with the volatility for the instantaneous spot rate averaging at 0.53%. However, from the second half of the year 2000, the economy started to soften, with signs of possible recession appearing in 2001. As

FIGURE 3. Instantaneous forward volatility - Second sample period

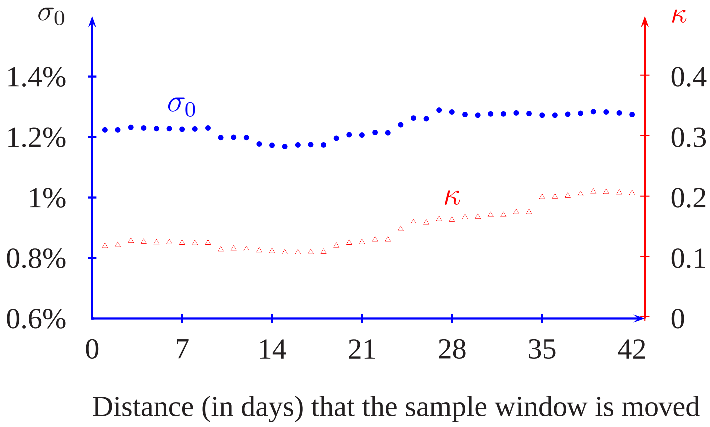


a result, in 2001, volatility of the interest rate market picked up again, and the instantaneous spot rate volatility rose to the 1% level.

4.2. **Stability of the estimates.**

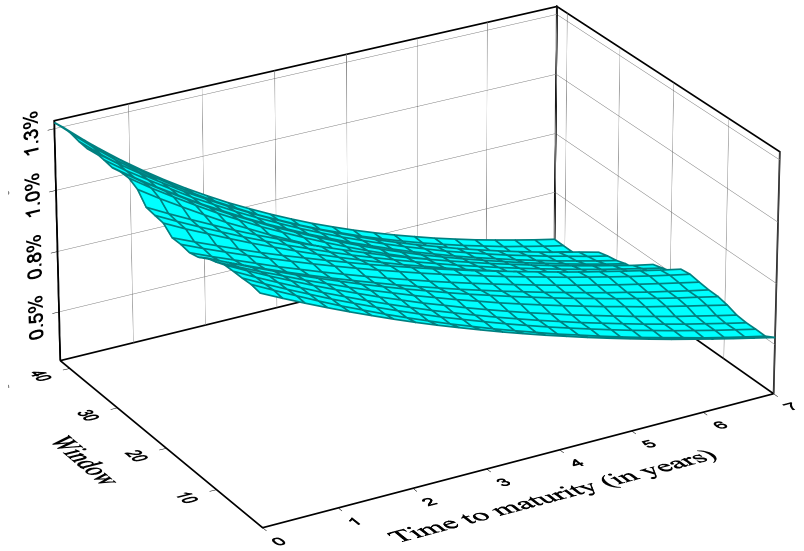
To check the stability of our estimates, we used a “moving window” approach. We used trading data from January to October each year to estimate our model. Then we moved our window sample by 1 day, keeping sample size constant (ie. the drop-one/add-one method) to compute sequential estimates until the end of December each year. It should be noted that even though we only moved our sample by one day, the sample length was more than 200 days, therefore, the futures prices we dropped and the ones we added are usually very different in magnitude. Figure 4 plots the series of instantaneous spot rate volatility σ_0 and the decay factor κ obtained in 1990. There are some fluctuations in the series, but overall the estimates do not seem to be unstable. The results for other years are reported in table 4, which shows that the sequential estimates have small ranges.

FIGURE 4. Moving Window Approach: Parameter Estimates for 1990

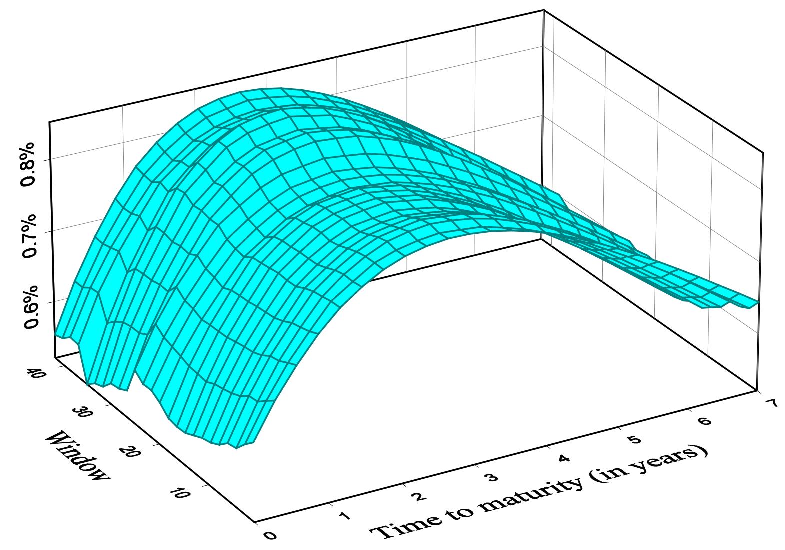


The very small changes of our parameter estimates implies that the resulting forward volatility curve experiences only slight and smooth movement over time. Figure 5 graphs two representative volatility curves, one having a simple exponential shape and the other having the humped shape. As time evolves, the instantaneous forward volatility of long-maturity forward rates is more volatile than that of short-maturity forward rates. Overall, the smooth volatility surfaces indicate adequate stability in our estimation results.

FIGURE 5. Variation in the estimated forward volatility using the moving window approach



1990 (exponential shaped)



2000 (humped shaped)

TABLE 4. Sequential estimates

This table reports the sequential estimates from the moving window approach. Each year the first estimation window started with trading data covering January to October. Then the sample was moved by 1 day (add-one/drop-one method), and the model was re-estimated. The process was repeated until all trading days in November and December were included in the samples, ie. on average there are 43 moving windows. The average, minimum and maximum values of the parameter estimates for each moving window series are reported. The values of σ_0 and σ_1 are reported as percentages.

Year	$\hat{\sigma}_0$ (%)			$\hat{\sigma}_1$ (%)			$\hat{\kappa}$		
	Avr	Min	Max	Avr	Min	Max	Avr	Min	Max
1988	1.450	1.419	1.489	-	-	-	0.171	0.141	0.187
1989	2.037	2.011	2.051	-	-	-	0.249	0.241	0.266
1990	1.284	1.217	1.338	-	-	-	0.145	0.107	0.206
1991	1.107	1.071	1.142	-	-	-	0.170	0.159	0.191
1992	1.542	1.446	1.608	-	-	-	0.101	0.083	0.112
1993	0.615	0.399	0.838	0.624	0.405	0.812	0.321	0.263	0.380
1994	1.134	1.058	1.193	0.355	0.269	0.447	0.181	0.151	0.207
1995	1.266	1.069	1.410	0.597	0.544	0.672	0.345	0.333	0.360
1996	1.176	0.892	1.360	0.679	0.460	0.914	0.313	0.260	0.364
1997	0.604	0.560	0.628	0.338	0.319	0.365	0.225	0.216	0.240
1998	0.431	0.409	0.460	0.596	0.552	0.635	0.302	0.284	0.315
1999	0.406	0.351	0.445	0.667	0.616	0.724	0.273	0.257	0.293
2000	0.589	0.536	0.647	0.390	0.334	0.467	0.241	0.214	0.269
2001	0.751	0.505	0.106	0.751	0.234	1.153	0.271	0.129	0.345

4.3. **Model fit.**

The model's goodness of fit was assessed by tests on residuals. Since the residuals of our estimates had different variances at each point of time by model construction, we carried out goodness of fit tests by checking the estimated standardized residuals.

To test whether the standardized residuals came from a multivariate normal distribution, we employed the Omnibus test. The test is derived by Doornik and Hansen (1994) based on Shenton and Bowman (1977), who give the sample kurtosis a gamma distribution, and D'Agostino (1970), who approximates the distribution of sample skewness by the Johnson S_u system. In addition, the test has been corrected for small sample bias and adapted to the multivariate case. Under this test, we can reject the null hypothesis of normal distribution for all sample periods, at the 99% confidence level.

In addition, we calculated the serial correlation for the estimated standardized residuals and carried out multivariate Portmanteau tests. As can be seen from the results reported in Table 5, we do not have zero autocorrelation in most of the years. We report the result for lag length of 30, since the test requires a large lag length. The results for other lag lengths (20, 50) are not qualitatively different.

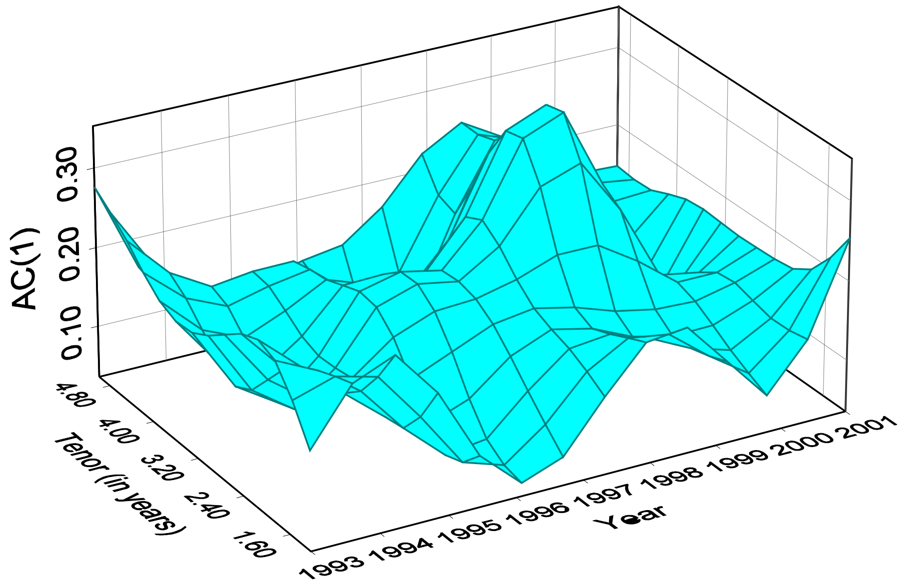
TABLE 5. The Multivariate Portmanteau Tests

This table reports the p-value associated with the multivariate Portmanteau statistics (30 lags) for the autocorrelation test in the standardized residuals from the fitted model for each year. From 1988 to 1992, the best model is the exponential model, where $\sigma(t,T) = \sigma_0 \exp(-\kappa(T-t))$, whereas from 1993 to 2001, the best model is the humped model, where $\sigma(t,T) = (\sigma_0 + \sigma_1(T-t))\exp(-\kappa(T-t))$. The test results for other lag lengths are not qualitatively different.

Year	P-Value	Year	P-Value
1988	0.0032	1995	0.3333
1989	0.8419	1996	0.0000
1990	0.3418	1997	0.0001
1991	0.0231	1998	0.0000
1992	0.0030	1999	0.0001
1993	0.0062	2000	0.1496
1994	0.0249	2001	0.0000

The correlation in the standardized errors is not negligible. The absolute value of the first order correlation coefficient averages at 0.16. The correlation reduces as the lag order increases, but is still at 0.05 at lag order 30. Figure 6, which plots the absolute value of the first order residual serial correlation, shows that for most of the years, the residual serial correlation is higher in the mid-range maturities, whereas for the short rates and long rates, the serial correlation is weaker.

FIGURE 6. First order serial correlation in estimated standardized residuals
(The graph plots the absolute value of the correlation coefficients)



The existence of serial correlation in the estimated standardized residuals up to very long lags suggests that the model is misspecified. There are at least three possible ways to account for this autocorrelation feature. First we could remain in the time deterministic volatility framework and allow for higher order polynomials in front of the exponential term in equation (3.1). This would allow for more than one hump in the volatility function. Second, we could consider other HJM specifications where the instantaneous forward rate depends on the whole history of the term structure. This could be done, for example, by including either the instantaneous forward rate itself or the instantaneous spot rate into the specification of the instantaneous forward rate volatility. Third, we could consider forward rate models with jumps. The omission of a jump component when it exists will also result in autocorrelation in the standardized residuals. We leave these issues for future research[15].

[15]An initial empirical study of the HJM model with jump components is undertaken in Chiarella and Tô (2004)

5. Conclusion

In this paper we focus on a method of estimation of the forward rate volatility function for an important family of Heath-Jarrow-Morton term structure models, namely where the instantaneous forward rate volatility function is time deterministic. For such a family of models the evolution of the instantaneous forward rate is Markovian, even though the evolution of the instantaneous spot rate may not be.

Among different methods of estimation, the Maximum Likelihood Estimator has favourable asymptotic properties. However, it cannot be applied directly in the HJM framework due to the unobservability of instantaneous forward rates. We show that the attempt to use futures yields as a proxy for the instantaneous forward rates leads to non-negligible estimation bias, which can be decomposed into a (negligible) maturity bias component and a (non-negligible) convexity bias component.

The major contribution of this paper rests on the realization that a futures contract can be viewed as a derivative instrument written on the instantaneous forward rate, and therefore is driven by the same source of uncertainty as that driving the forward rate evolution. Using a likelihood transformation technique, and utilizing the result of Duan (1994) to simplify the likelihood function, we are able to derive the exact likelihood function for all model specifications that have deterministic volatility forms, albeit the likelihood function will be different in its degree of complexity.

The advantage of our approach over many earlier empirical studies is that we do not need to concern ourselves with the problem of Markovianizing the process for the instantaneous spot rate. This was necessary in such earlier studies since they relied on use of a bond price formula, which in turn is a function of the instantaneous spot rate.

To demonstrate our method, we focus on the humped-forward rate volatility specification suggested by the hump that is often revealed when an implied volatility function is backed out from caps and swaptions data. We use 14 years (from 1988-2001) of data for CME Eurodollar futures contracts, which is a highly liquid market, to estimate our model. We not only use time series, but also pool in cross-sectional data, ie. futures contracts that have different tenors at each point of time, in order to exploit the full information content of the yield curve.

For our sample period, we find that the exponential-volatility model provides a better fit for the first five years (1988-1992), whereas the humped-volatility model is preferable for the later period (1993-2001). The volatility curve appears to experience four different phases, the first spanning five years from 1988-1992, the next two periods each lasting for four years, and the last period starting in 2001. Our estimates remain stable with respect to sample windows. However, goodness of fit tests on the standardized residuals indicate that the chosen volatility functional form does not fully capture all the features of the data. It is our conjecture that such features could be captured by allowing forward rate volatility functions containing jump components or non-deterministic forms for forward rate volatilities, for example depending on a set of fixed maturity forward rates. We intend to explore these issues in subsequent research.

Appendix A. The evolution of futures price under the HJM model

Let $P(t, T_B)$ be the price at time t of a pure discount instrument that has a face value of \$1 and matures at time T_B, and let $B(t, T_B)$ be the corresponding log bond price, ie $B(t, T_B) = \ln P(t, T_B)$.

Denote by $F(t, T_F, T_B)$ the price at time t of a futures contract written on the pure discount instrument. The futures contract matures at time T_F.

Using the fact that futures contracts are marked-to-market, Cox et al. (1981) show that the futures prices are a Martingale under the equivalent measure $\widetilde{\mathcal{Q}}$:

$$\begin{aligned} F(t, T_F, T_B) &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[F(T_F, T_F, T_B)|\mathcal{F}_t\big] \\ &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[P(T_F, T_B)|\mathcal{F}_t\big] \\ &= \mathbb{E}_t^{\widetilde{\mathcal{Q}}}\big[\exp(B(T_F, T_B))|\mathcal{F}_t\big]. \end{aligned}$$

We know that under $\widetilde{Q}$

$$\begin{aligned} B(T_F, T_B) &= -\int_{T_F}^{T_B} f(T_F, s)ds \\ &= -\int_{T_F}^{T_B} f(0, s)ds - \sum_i \int_{T_F}^{T_B}\int_0^{T_F} \sigma_i(u, s)\int_u^s \sigma_i(u, v)dv\,du\,ds \\ &\quad - \sum_i \int_{T_F}^{T_B}\left(\int_0^{T_F} \sigma_i(u, s)d\widetilde{W}_i(u)\right)ds. \end{aligned}$$

Then by an application of the stochastic Fubini theorem

$$B(T_F,T_B) = -\int_{T_F}^{T_B} f(0,s)ds - \sum_i \int_0^{T_F}\int_{T_F}^{T_B} \sigma_i(u,s)\int_u^s \sigma_i(u,v)dv\,ds\,du$$
$$-\sum_i \int_0^{T_F}\left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)d\widetilde{W}_i(u).$$

Therefore[16]

$$F(t,T_F,T_B) = \exp\left[-\int_{T_F}^{T_B} f(0,s)ds - \sum_i \int_0^{T_F}\int_{T_F}^{T_B} \sigma_i(u,s)\int_u^s \sigma_i(u,v)dv\,ds\,du \right.$$
$$\left. -\sum_i \int_0^t\int_{T_F}^{T_B}\sigma_i(u,s)ds\,d\widetilde{W}_i(u) + \frac{1}{2}\sum_i \int_t^{T_F}\left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)^2 du\right].$$

Using the expansion obtained as a result of substituting $t = 0$, the above formula can be reduced to

$$F(t,T_F,T_B) = F(0,T_F,T_B)\exp\left[-\frac{1}{2}\sum_i \int_0^t \left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)^2 du \right.$$
$$\left. -\sum_i \int_0^t\int_{T_F}^{T_B}\sigma_i(u,s)ds\,d\widetilde{W}_i(u)\right].$$

Taking stochastic differentials of this last expression gives the stochastic differential equation for $F(t,T_F,T_B)$ as (1.6) in the text:

$$\frac{dF(t,T_F,T_B)}{F(t,T_F,T_B)} = -\sum_i \int_{T_F}^{T_B}\sigma_i(t,s)ds\,d\widetilde{W}_i(t).$$

APPENDIX B. VARIANCE OF FUTURES YIELD

To ease the notation set $\sigma_{Fi}(t) \equiv \sigma_{Fi}(t,T_F,T_B)$.

The futures yield follows a stochastic differential equation

$$dy(t,T_F,T_B) = \sum_i \left(y - \frac{1}{T_B - T_F}\right)\sigma_{Fi}(t)d\widetilde{W}_i(t).$$

[16]We remind the reader that at time t the integral $\int_0^t \left(\int_{T_F}^{T_B}\sigma_i(u,s)ds\right)d\widetilde{W}_i(u)$ is a realized quantity

Let $z(t, T_F, T_B) = \frac{1}{T_B - T_F} + y(t, T_F, T_B)$, then $\operatorname{var}(y(t, T_F, T_B)) = \operatorname{var}(z(t, T_F, T_B))$, and

$$dz(t, T_F, T_B) = \sum_i z\sigma_{Fi}(t) d\widetilde{W}_i(t).$$

With a view to calculating $\mathbb{E}_0[z(t, T_F, T_B)]$ and $\operatorname{var}_0[z(t, T_F, T_B)]$ we set

$$m(t) = \ln z(t, T_F, T_B) \tag{B.1}$$

and

$$n(t) = \ln \left(z(t, T_F, T_B)\right)^2 = 2m(t). \tag{B.2}$$

Application of Itô's lemma to (B.1) followed by an integration yields

$$m(t) = m(t_0) + \sum_i \int_0^t \sigma_{Fi}(u) d\widetilde{W}_i(u), \tag{B.3}$$

and it follows from (B.2) that

$$n(t) = 2m(t_0) + \sum_i \int_0^t \sigma_{Fi}(u) d\widetilde{W}_i(u). \tag{B.4}$$

Since $\sigma_{Fi}(t)$ are deterministic functions of time, (B.3) and (B.4) imply that both $m(t)$ and $n(t)$ are normally distributed and we readily calculate that

$$m(t) \sim \mathcal{N}\left(m(0)\ ,\ \sum_i \int_{t_0}^t \sigma_{Fi}^2(u) du\right), \tag{B.5}$$

and

$$n(t) \sim \mathcal{N}\left(2m(0)\ ,\ 4\sum_i \int_{t_0}^t \sigma_{Fi}^2(u) du\right). \tag{B.6}$$

We recall that if a random variable $v(t)$ is distributed $\mathcal{N}\left(\mu(t), \sigma^2(t)\right)$ then

$$\mathbb{E}\left[e^{v(t)}\right] = e^{\mu(t) + \frac{1}{2}\sigma^2(t)}.$$

Using this result we calculate from (B.5) and (B.6) that

$$\mathbb{E}_0\left[z(t, T_F, T_B)\right] = \mathbb{E}_0\left[e^{m(t)}\right] = \exp\left(m(0) + \frac{1}{2}\sum_i \int_{t_0}^t \sigma_{Fi}^2(u) du\right), \tag{B.7}$$

and

$$\mathbb{E}_0\left[z(t,T_F,T_B)^2\right] = \mathbb{E}_0\left[\mathrm{e}^{n(t)}\right] = \exp\left(2m(0) + 2\sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du\right). \tag{B.8}$$

Using (B.7) and (B.8) and the relationship

$$\begin{aligned}\mathrm{var}\left[y(t,T_F,T_B)\right] &= \mathrm{var}\left[z(t,T_F,T_B)\right] \\ &= \mathbb{E}_0\left[z(t,T_F,T_B)^2\right] - \Big(\mathbb{E}_0\left[z(t,T_F,T_B)\right]\Big)^2,\end{aligned}$$

and some minor manipulations, we obtain

$$\begin{aligned}\mathrm{var}\big(y(t,T_F,T_B)\big) = {}& \left(y(0,T_F,T_B) - \frac{1}{T_B - T_F}\right)^2 \times \exp\left(\sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du\right) \\ & \times \left[\exp\left(\sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du\right) - 1\right]. \end{aligned} \tag{B.9}$$

If we define

$$\bar{\sigma}_f^2 \equiv \frac{1}{(T_B - T_F)^2} \sum_i \int_{t_0}^{t} \sigma_{Fi}^2(u)du,$$

then (1.11) is obtained.

Appendix C. Fixed-maturity forward rate evolution

Consider an investor who holds a bond maturing at T_B and seek the return he or she would earn between T_F and $T_B(> T_F)$ by contracting now at time t. The required rate of return is the discrete period forward rate $f(t,T_F,T_B)$ defined by

$$P(t,T_F) = P(t,T_B)\exp\big(f(t,T_F,T_B)(T_B - T_F)\big),$$

ie.

$$\begin{aligned}f(t,T_F,T_B) &= \frac{1}{T_B - T_F}\ln\left[\frac{P(t,T_F)}{P(t,T_B)}\right] \\ &= \frac{1}{T_B - T_F}\int_{T_F}^{T_B} f(t,s)ds.\end{aligned}$$

Recall that the evolution of the instantaneous forward rate is

$$f(t,T_B) = f(0,T_B) + \sum_i \left[\int_0^t \sigma_i(u,T_B)\int_u^{T_B} \sigma_i(u,v)dv\,du + \int_0^t \sigma_i(u,T_B)d\widetilde{W}_i(u)\right].$$

Therefore, the discrete period forward rate $f(t, T_F, T_B)$ evolves according to

$$\begin{aligned}
f(t, T_F, T_B) = &\frac{1}{T_B - T_F} \int_{T_F}^{T_B} f(0, s) ds \\
&+ \frac{1}{T_B - T_F} \sum_i \int_{T_F}^{T_B} \int_0^t \sigma_i(u, s) \int_u^s \sigma_i(u, v) dv\, du\, ds \\
&+ \frac{1}{T_B - T_F} \int_{T_F}^{T_B} \int_0^t \sigma_i(u, s) d\widetilde{W}_i(u)\, ds \\
= &\frac{1}{T_B - T_F} \int_{T_F}^{T_B} f(0, s) ds \\
&+ \frac{1}{T_B - T_F} \sum_i \int_0^t \int_{T_F}^{T_B} \sigma_i(u, s) \int_u^s \sigma_i(u, v) dv\, ds\, du \\
&+ \frac{1}{T_B - T_F} \int_0^t \int_{T_F}^{T_B} \sigma_i(u, s) ds\, d\widetilde{W}_i(u).
\end{aligned}$$

The variance of the discrete period forward rate is thus readily calculated as

$$\begin{aligned}
\operatorname{var}\big(f(t, T_F, T_B)\big) &= \frac{1}{(T_B - T_F)^2} \sum_i \int_{t_0}^t \left(\int_{T_F}^{T_B} \sigma_i(u, s) ds \right)^2 du \\
&= \overline{\sigma}_f^2.
\end{aligned}$$

Appendix D. The likelihood transformation formula

With a slight abuse of notation, let $X_{j\,k} \equiv X(t_j, T_{Fk}, \tau) \equiv X(t_j, T_{Fk}, T_{Bk})$[17] be the k^{th} unobservable state variable ($k = 1, 2, \ldots, K$) occurring at time $t_j < T_F$ ($j = 0, 1, \ldots, J$).

Denote by $\boldsymbol{x}_j$ the vector of unobservable state variables occurring at time t_j, ie. $\boldsymbol{x}_j = \big(X(t_j, T_1, \tau), X(t_j, T_2, \tau), \ldots, X(t_j, T_K, \tau)\big)$. Denote by $\boldsymbol{x}$ the unobservable

[17] We write $X(t_j, T_{Bk}, \tau) \equiv X(t_j, T_{Fk}, T_{Bk})$ because $T_{Bk} - T_{Fk} = \tau$ constant for all $k = 1, 2, \ldots, K$

state vector of size $K(J+1)\times 1$ at time t_J, ie.

$$\begin{aligned}\boldsymbol{x} &= \text{vec}\begin{pmatrix} \boldsymbol{x}_0 & \boldsymbol{x}_1 & \dots & \boldsymbol{x}_J \end{pmatrix} \\ &= \text{vec}\begin{pmatrix} X(t_0,T_{F1},\tau) & X(t_1,T_{F1},\tau) & \dots & X(t_J,T_{F1},\tau) \\ X(t_0,T_{F2},\tau) & X(t_1,T_{F2},\tau) & \dots & X(t_J,T_{F2},\tau) \\ \vdots & \vdots & \ddots & \vdots \\ X(t_0,T_{FK},\tau) & X(t_1,T_{FK},\tau) & \dots & X(t_J,T_{FK},\tau) \end{pmatrix},\end{aligned}$$

where vec is the standard matrix operator that, when applied to a matrix, transforms the matrix into a vector by stacking the columns of the matrix on top of each other.

Denote the density function of $\boldsymbol{X}$ by

$$p_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta}) = p_{\boldsymbol{X}}(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_J;\boldsymbol{\theta}),$$

where $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ is the parameter vector of interest.

Suppose that a transformation $\boldsymbol{\Upsilon}$ exists, which applied to $\boldsymbol{X}$, produces a vector $\boldsymbol{Z}$ that is observable in the market, so that

$$\boldsymbol{Z} = \boldsymbol{\Upsilon}(\boldsymbol{X};\boldsymbol{\theta}) : \mathbb{R}^{K(J+1)\times 1} \rightarrow \mathbb{R}^{K(J+1)\times 1},$$

where

$$\begin{aligned}\boldsymbol{z} &= \text{vec}\begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \dots & \boldsymbol{z}_J \end{pmatrix} \\ &= \text{vec}\begin{pmatrix} Z_{01} & Z_{11} & \dots & Z_{J1} \\ Z_{02} & Z_{12} & \dots & Z_{J2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{0K} & Z_{1K} & \dots & Z_{JK} \end{pmatrix}.\end{aligned}$$

Assume that this transformation is one-to-one for every $\boldsymbol{\theta}\in\boldsymbol{\Theta}$.

Since $\boldsymbol{\Upsilon}$ is one-to-one, there exists an inverse $\boldsymbol{\Upsilon}^{-1} = \boldsymbol{\Psi}(\boldsymbol{Z};\boldsymbol{\theta})$. Applying the standard change of variable technique, the density functions in terms of $\boldsymbol{Z}$ and $\boldsymbol{X}$ are related by

$$p_{\boldsymbol{Z}}(\boldsymbol{z},\boldsymbol{\theta}) = p_{\boldsymbol{X}}\left(\boldsymbol{\Psi}(\boldsymbol{z};\boldsymbol{\theta})\right) \times \left|\mathrm{J}\left(\boldsymbol{\Psi}(\boldsymbol{z};\boldsymbol{\theta})\right)\right|,$$

where J is the Jacobian of the transformation from $\boldsymbol{X}$ to $\boldsymbol{Z}$, ie.

$$\mathrm{J}\left(\boldsymbol{\Psi}(\boldsymbol{z};\boldsymbol{\theta})\right) = \left|\frac{\partial\boldsymbol{\Psi}(\boldsymbol{z};\boldsymbol{\theta})}{\partial\boldsymbol{z}'}\right|.$$

Duan (1994) proves that if the transformation is on an element-by-element basis, ie. $Z_{j\,k} = \Upsilon_{j\,k}(X_{j\,k})$ (and $X_{j\,k} = \Psi_{j\,k}(Z_{j\,k})$) for all $j \in [0, J]$ and $k \in [1, K]$, then the first-derivative matrix is diagonal, therefore

$$\mathrm{J}\left(\boldsymbol{\Psi}(\boldsymbol{z};\boldsymbol{\theta})\right) = \prod_{j=0}^{J}\prod_{k=1}^{K}\frac{d\Psi_{j\,k}(Z_{j\,k};\boldsymbol{\theta})}{dZ_{j\,k}}.$$

Furthermore, if $\boldsymbol{X}$ is “joint-Markovian”, ie.

$$p_{\boldsymbol{X}}(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_J; \boldsymbol{\theta}) = p_{\boldsymbol{X}}(\boldsymbol{x}_0, t_0; \boldsymbol{\theta}) \prod_{j=1}^{J} p_{\boldsymbol{X}}(\boldsymbol{x}_j, t_j | \boldsymbol{x}_{j-1}, t_{j-1}; \boldsymbol{\theta})$$

then upon substitution, the likelihood for Z can be compactly written as

$$p_{\boldsymbol{Z}}(\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_J; \boldsymbol{\theta}) = p_{\boldsymbol{Z}}(\boldsymbol{z}_0, t_0; \boldsymbol{\theta}) \prod_{j=1}^{J} p_{\boldsymbol{Z}}(\boldsymbol{z}_j, t_j | \boldsymbol{z}_{j-1}, t_{j-1}; \boldsymbol{\theta}),$$

where

$$p_{\boldsymbol{Z}}(\boldsymbol{z}_0, t_0; \boldsymbol{\theta}) = p_{\boldsymbol{X}}\big(\boldsymbol{\Psi}_0(\boldsymbol{y_0}), t_0; \boldsymbol{\theta}\big) \times \Big|\mathrm{J}\big(\boldsymbol{\Psi}_0(\boldsymbol{z}_0; \boldsymbol{\theta})\big)\Big|,$$

$$p_{\boldsymbol{Z}}\big(\boldsymbol{z}_j, t_j | \boldsymbol{z}_{j-1}, t_{j-1}; \boldsymbol{\theta}\big) = p_{\boldsymbol{X}}\big(\boldsymbol{\Psi}_j(\boldsymbol{z}_j), t_j | \boldsymbol{\Psi}_{j-1}(\boldsymbol{z}_{j-1}), t_{j-1}; \boldsymbol{\theta}\big) \times \Big|\mathrm{J}\big(\boldsymbol{\Psi}_j(\boldsymbol{z}_j; \boldsymbol{\theta})\big)\Big|,$$

and

$$\mathrm{J}\big(\boldsymbol{\Psi}(\boldsymbol{z}_j;\boldsymbol{\theta})\big) = \prod_{k=1}^{K}\frac{d\Psi_{j\,k}(Z_{j\,k};\boldsymbol{\theta})}{dZ_{j\,k}} \quad (j = 0, 1, \ldots, J).$$

APPENDIX E. FULL INFORMATION LOG LIKELIHOOD FUNCTION FOR QUOTED FUTURES PRICES

The main task in deriving the log likelihood function is to calculate the Jacobian of the transformation and write out the drift vector and covariance matrix for each transition log likelihood function. These quantities then can be substituted directly into the formulae in the text (equations (2.8), (2.10), (2.11) and (2.12)) to write out the likelihood function for observable futures prices.

From (2.4)

$$X_{jk} = \ln(F_{jk}) \equiv \zeta(F_{jk}),$$

we have

$$\frac{\partial \zeta(F_{jk};\boldsymbol{\theta})}{\partial F_{jk}} = \frac{1}{F_{jk}}.$$

From (2.2)

$$F_{jk} = 1 - \left(1 - \frac{G_{jk}}{100}\right)\tau \equiv \eta\big(G_{jk}\big),$$

where $\tau = 90/360$ for CME Eurodollar futures, we find that

$$\frac{\partial \eta(G_{jk};\theta)}{\partial G_{jk}} = \frac{\tau}{100}$$

The variance is calculated as

$$\begin{aligned}\beta_{j(kk)} &= \int_{t_{j-1}}^{t_j} \left(\int_{T_{F_k}}^{T_{B_k}} \sigma(u,s)ds\right)^2 du + \int_{t_{j-1}}^{t_j} \sigma_\varepsilon^2 du \\ &= N^2 I_{02} + 2NRI_{12} + R^2 I_{22} + \sigma_\varepsilon^2(t_j - t_{j-1}),\end{aligned}$$

where

$$\begin{aligned} N &= -\left(\frac{\sigma_0}{\kappa} + \frac{\sigma_1}{\kappa^2}\right)\left(\mathrm{e}^{-\kappa T_{Bk}} - \mathrm{e}^{-\kappa T_{Fk}}\right) - \frac{\sigma_1}{\kappa}\left(T_{Bk}\,\mathrm{e}^{-\kappa T_{Bk}} - T_{Fk}\,\mathrm{e}^{-\kappa T_{Fk}}\right), \\ R &= \frac{\sigma_1}{\kappa}\left(\mathrm{e}^{-\kappa T_{Bk}} - \mathrm{e}^{-\kappa T_{Fk}}\right), \\ I_{ab} &= \int_{t_{j-1}}^{t_j} \tau^a\,\mathrm{e}^{\kappa b \tau}\,d\tau \\ &= \left(-\,\mathrm{e}^{\kappa b\tau}\left[\frac{1}{(-\kappa b)}\tau^a + \frac{a}{(-\kappa b)^2}\tau^{a-1} + \frac{a(a-1)}{(-\kappa b)^3}\tau^{a-2} + \dots \right.\right. \\ &\qquad \left.\left. \dots + \frac{a(a-1)\dots 2}{(-\kappa b)^a}\tau + \frac{a(a-1)\dots 1}{(-\kappa b)^{a+1}}\tau^0\right]\right)\Bigg|_{t_{j-1}}^{t_j}. \end{aligned}$$

The covariances (where $k_1 \neq k_2$) are given as

$$\begin{aligned}\beta_{j(k_1k_2)} &= \int_{t_{j-1}}^{t_j} \left(\int_{T_{Fk_1}}^{T_{Bk_1}} \sigma(u,s)ds\right)\left(\int_{T_{Fk_2}}^{T_{Bk_2}} \sigma(u,s)ds\right) du \\ &= N_1N_2I_{02} + (N_1R_2 + R_1N_2)I_{12} + R_1R_2I_{22},\end{aligned}$$

where

$$N_1 = -\left(\frac{\sigma_0}{\kappa} + \frac{\sigma_1}{\kappa^2}\right)\left(\mathrm{e}^{-\kappa T_{Bk_1}} - \mathrm{e}^{-\kappa T_{Fk_1}}\right) - \frac{\sigma_1}{\kappa}\left(T_{Bk_1}\,\mathrm{e}^{-\kappa T_{Bk_1}} - T_{Fk_1}\,\mathrm{e}^{-\kappa T_{Fk_1}}\right),$$

$$N_2 = -\left(\frac{\sigma_0}{\kappa} + \frac{\sigma_1}{\kappa^2}\right)\left(\mathrm{e}^{-\kappa T_{Bk_2}} - \mathrm{e}^{-\kappa T_{Fk_2}}\right) - \frac{\sigma_1}{\kappa}\left(T_{Bk_2}\,\mathrm{e}^{-\kappa T_{Bk_2}} - T_{Fk_2}\,\mathrm{e}^{-\kappa T_{Fk_2}}\right),$$

$$R_1 = \frac{\sigma_1}{\kappa}\left(\mathrm{e}^{-\kappa T_{Bk_1}} - \mathrm{e}^{-\kappa T_{Fk_1}}\right),$$

$$R_2 = \frac{\sigma_1}{\kappa}\left(\mathrm{e}^{-\kappa T_{Bk_2}} - \mathrm{e}^{-\kappa T_{Fk_2}}\right),$$

and I_{ab} are defined as in the variance formulae.

The drift term is

$$\alpha_{jk} = -\frac{1}{2}\beta_{j(kk)} + \phi(t)\left(NI_{01} + RI_{11}\right).$$

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