Do we understand delta hedging?

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abstract

We show that the delta-hedged portfolio is not actually risk-free even for brownian underlying due to history dependence in the ammount of hold portfolio. We find this ammount explicitly, as a function of underlying price evolution and option price. This shows that even in the B-S world (perfect market and brownian asset price evolution) the B-S equation can only be an approximation.

1 Introduction

As it is well known, Black and Scholes introduced in 1973 their famous equation determining the value of options on assets ("underlying") with log-normal price distribution. They did so by constructing a risk-free portfolio with options and shorted underlying, and using absence of arbitrage they deduced it would evolve at the risk-free rate. The construction of such portfolio rested on our ability of "delta-hedging", that is, on the posibility of balancing options and shares in such a way that the stochastic term vanish.

It is the purpose of this work to analyze the consistence of such procedure. We are supposed to be able of adjusting the ammount of underlying in a continuous way, that is, for each option in the portfolio we hold an ammount of underlying

$$-\Delta = -\frac{\partial V}{\partial S}(S, t)$$

being V the value of the opcion at underlying price S and time t. Is it possible to do so, in a self-financing way, and not spoiling the risk-free feature?

2 The trouble

The question raised at the end of the last section might sound somewhat strange. What migth fail? What we should be sure is that, when we adjust

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the composition of the portfolio in order to keep it delta-hedged, we must buy an ammount of new portfolio for exactly the same value of old portfolio we already hold, without introducing a stochaticity in such ammount. We are in risk of being before an overdetermined problem.

Let us put it concretely. Observe that the portfolio

$$\Pi = V - \Delta S$$

is by itself not hedged nor self-financing [2], both due to the time evolution of Δ. What is supposed to be both risk-free and self-financing is the portfolio

$$\Psi = f(V - \Delta S)$$

where f is the ammount of Π we hold. We are supposed to start with some ammount f_0 of Π at time $t = 0$, and letting f evolve in such a way we keep our portfolio delta-hedged and self-financed. The presence of the nontrivial normalization f is enforced by the fact that the price change of ψ is only driven by time evolution of S, so the time evolution of Δ can only be compensated by a change in the ammount of hold portfolio in the continuous rebalance implied in the continuous delta hedging.

Does the presence of f affect the derivation of the Black-Scholes equation? Just very subtely. If we suppose that with Π delta-hedged we keep Ψ risk-free, and supposing that f is nonzero, then it drops from the equations leading to the B-S equation. So, what we need from f is that it is nonzero, and that it do not ruin the risk-freedom of Ψ. It in turn imply that f should be a function of S and t (a "state function"), because if it depended on the historic evolution of S, at a given price S and time t we could have different ammounts of Ψ depending on the way they get there, and so Ψ would not be risk-free.

Let us establish the conditions on $f(S,t)$. When we let Ψ evolve at fixed f and Δ, we get

$$\delta\Psi = f(V + \delta V - \Delta(S + \delta S))$$

wich leads to the B-S equation if we set $\delta\Psi = r\Psi\delta t$ after dividing out f. This expression is what is ussually called "the self-finnancing condition", meaning that the value of our portfolio is only driven by the evolution of S. Once we let S evolve, we must rebalance our portfolio for keeping it delta-hedged,

[2] By "self-financing" we mean that only the variation in S contribute to price variation of the portfolio, so variations in Δ should be absorbed by a rebalancing. See below.

with the condition that the value of the portfolio we already hold (with "old" f and Δ but "new" S and V) equals the value of the rebalanced one (with only "new" parameters). This "rebalancing condition", reads:

$$f(S-\delta S,t-\delta t)\,(V-\Delta(S-\delta S,t-\delta t)S)=f(S,t)\,(V-\Delta(S,t)S) \quad (1)$$

This condition leads to two differential equations, using Ito's lemma. That coming from the stochastic term:

$$fS\frac{\partial^2 V}{\partial S^2}=(V-\Delta S)\frac{\partial f}{\partial S} \quad (2)$$

and the deterministic term:

$$fS\left(\frac{\partial^2 V}{\partial t\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^3 V}{\partial S^3}\right)-\left(V-S\frac{\partial V}{\partial S}\right)\left(\frac{\partial f}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}\right)-\sigma^2 S^3\frac{\partial f}{\partial S}\frac{\partial^2 V}{\partial S^2}=0 \quad (3)$$

These sistem of equations form, together with the B-S equations, an uncompatible system of differential equations on f and V. Thus, delta-hedging does not lead to a risk-free self financing portfolio.

3 What is going on?

Let us analyze the reason why there is no $f(S,t)$. Let us remark that we didn't show that we cannot delta-hedge, only that f is not a function on S and t alone. What really happen is that f depends on the whole history of S evolution. In order to see that, let us calculate f explicitly.

A way in wich we can write the self-financing condition is the following:

$$(1+\epsilon)(V-\Delta(S-\delta S,t-\delta t)S)=V-\Delta S \quad (4)$$

where ϵ is the infinitesimal change on the amount of hold portfolio. We can write

$$\epsilon=\chi_d dt+\chi_s dW \quad (5)$$

where χ_d and χ_s are the deterministic and stochastic components respectively. After a series of rebalancing operations, we can express f as

$$f=(1+\epsilon_1)(1+\epsilon_2)...(1+\epsilon_N) \quad (6)$$

in the limit where N goes to infinity. We thus find

$$f = e^{\Sigma_i \ln(1+\epsilon_i)} \tag{7}$$

and since we are in the limit of vanishing ϵ, we can replace the logaritms by its first order Ito expansion:

$$f = e^{\int_c [(\chi_d - \frac{\chi_s^2}{2})dt + \chi_s dW]} \tag{8}$$

Using $dS = \mu S dt + \sigma S dW$ we get

$$f = e^{\int_c [(\chi_d - \frac{\chi_s^2}{2} - \frac{\mu}{\sigma}\chi_s)dt + \frac{\chi_s}{\sigma S} dS]} \tag{9}$$

The condition for f to be history-independent is that the argument of the integral in expression 9 is an exact differential, that is

$$\frac{\partial}{\partial S}\left(\chi_d - \frac{\chi_s^2}{2} - \frac{\mu}{\sigma}\chi_s\right) = \frac{\partial}{\partial t}\left(\frac{\chi_s}{\sigma S}\right) \tag{10}$$

Let us solve for χ_d and χ_s from expression 4. We get, from the stochastic term

$$\chi_s = -\frac{\sigma S^2}{\left(V - S\frac{\partial V}{\partial S}\right)}\frac{\partial^2 V}{\partial S^2} \tag{11}$$

and from the deterministic one

$$\chi_d = \left(\frac{\sigma S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}}\right)^2 - \frac{S}{\left(V - S\frac{\partial V}{\partial S}\right)}\left(\mu S\frac{\partial^2 V}{\partial S^2} + \frac{\partial^2 V}{\partial t \partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^3 V}{\partial S^3}\right) \tag{12}$$

Then condition 10 reads

$$-\frac{\partial}{\partial t}\left(\frac{S}{\left(V - S\frac{\partial V}{\partial S}\right)}\frac{\partial^2 V}{\partial S^2}\right) =$$

$$\frac{\partial}{\partial S}\left(\frac{1}{2}\left(\frac{\sigma S^2}{\left(V - S\frac{\partial V}{\partial S}\right)}\frac{\partial^2 V}{\partial S^2}\right)^2 - \frac{S}{\left(V - S\frac{\partial V}{\partial S}\right)}\left(\frac{\partial^2 V}{\partial t \partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^3 V}{\partial S^3}\right)\right) \tag{13}$$

wich is not satisfied identically. Thus, f is history-dependent and explicitly given by 9, 11 and 12.