Financial volatility and independent and identically distributed variables

Annibal Figueiredo[a], Iram Gleria[b], Raul Matsushita[c], Sergio Da Silva[d,*]

[a]*Department of Physics, University of Brasilia, 70910-900 Brasilia DF, Brazil*
[b]*Department of Physics, Federal University of Alagoas, 57072-970 Maceio AL, Brazil*
[c]*Department of Statistics, University of Brasilia, 70910-900 Brasilia DF, Brazil*
[d]*Department of Economics, Federal University of Rio Grande Do Sul, 90040-000 Porto Alegre RS, and National Council for Scientific and Technological Development, Brazil*

Abstract

Given that financial series are poorly described by Gaussian distributions, how can the volatility behavior of such series be explained? Here we put forward a possible explanation to add the existing ones. We focus on a class of reduced variables that are independent and identically distributed. These variables together with an extra exponential law are able to explain the volatility of the intraday Brazilian *real*-US dollar exchange rate for the year 2002.

PACS: 05.40.+j;02.50.-r

Keywords: Financial volatility; Central limit theorem; Independent and identically distributed variables; Exchange rates

1. Introduction

That financial data cannot be satisfactorily addressed by Gaussian distributions is now reasonably well established [1]. Financial systems might be complex. Lévy-stable distributions and their variants have been suggested to model complex systems. Because observed financial quantities are usually the sum of small terms (such as individual asset prices), a Lévy can be used to describe them. This is because of the generalized central limit theorem that states that the non-trivial limit of normalized sums of independent and identically distributed variables is Lévy-stable [2]. And this is at odds with the classic central limit theorem that states that the limit of normalized sums of independent and identically distributed variables with finite variance is Gaussian.

There is too one empirical reason to model financial prices with the Lévy distributions. Data usually exhibit fat tails, sligh skewness, and high kurtosis. Yet although leptokurtosis can be accounted for by stable Lévy distributions, these have never been established in mainstream finance. One reason is related to their property of infinite variance. Since volatility (standard deviation) is a central concept to finance, it is useful for

* Corresponding author.
E-mail address: email@sergiodasilva.com (S. Da Silva).

the variance to be finite. (The debate in the early days of modern finance can be appreciated in Cootner [3]).

To remedy such a deficiency, econophysicists have recently put forward a truncated Lévy flight [4, 5]. This is a stochastic process with finite variance and characterized by scaling relations in a large but finite interval.

Research in econophysics attempts at explaining a number of questions that emerge in such a non-Gaussian agenda. Among these, one of interest is: what is the formation law (if any) for the volatility of a financial series of returns?

There are stylized facts of financial volatility that are known for a long time. An example is volatility clustering. Volatility presents decaying autocorrelation and thus returns activity is clustered in time, a pattern which is easily detected even at the naked eye. Not surprisingly, the stylized facts of financial volatility have received widespread attention [6, 7]. Here we provide another framework to explain it and add to the existing literature on the subject.

If the standard deviation is governed by a formation law, and besides that the asymptotic probability density function is not a Gaussian, it is of interest to learn in addition the conditions under which the classic central limit theorem does not hold [8]. Given that the probability density function is not a Gaussian, how can we learn that the asymptotic regime has been reached? This paper answers questions like that one by focusing on the behavior of the characteristic function.

As for the volatility, we put forward a class of reduced variables that are independent and identically distributed and that seem to fit well a data set from the 15-minute spaced Brazilian *real*-US dollar exchange rate for the year 2002. In particular, such suggested variables together with an extra exponential law are able to explain the volatility behavior of the series fairly well.

The rest of the paper is organized as follows. Benchmark definitions and our reduced variables that are independent and identically distributed are presented in Section 2. Section 3 elaborates further on the latter. Section 4 puts forward one exponential law and one power law to join the analytical framework in order to fit it into experimental data; we also develop a routine to perform that, which is presented in Section 5. Section 6 applies the routine to the *real*-dollar rate. Section 7 studies the asymptotic behavior of the system by focusing on the skewness and kurtosis. Section 8 focuses on the characteristic function. And Section 9 concludes.

2. Benchmark definitions

We first consider a sum of random variables

$$X_n = x_1 + x_2 + \cdots + x_n \tag{1}$$

where a zero mean is assumed for x_i. The probability density of x_i is $f_i(x_i)$, which is assumed to be distinct for every x_i. Second moments of x_i and X_n are, respectively,

$$m_i^2 = \langle x_i^2 \rangle, M_n^2 = \langle X_n^2 \rangle \tag{2}$$

where $M_n^2 = m_1^2 + m_2^2 + \cdots m_n^2$ for independent variables. Here we are particularly interested in "reduced variables", i.e.

$$\bar{x}_i = \frac{x_i}{m_i},\ \bar{X}_n = \frac{X_n}{M_n} \tag{3}$$

where

$$\bar{X}_n = \frac{\sum_{i=1}^{n} m_i \bar{x}_i}{M_n} \tag{4}$$

Characteristic function (CF) of $\bar{x}_i$ is $\bar{\psi}_i(z)$ and that of $\bar{X}_n$ is $\bar{\Psi}_n(z)$. Lévy [9] observes that for finite m_i it holds true that

$$\bar{\psi}_i(z) = e^{-\frac{z^2}{2}(1+w_i(z))}, \bar{\Psi}_n(z) = e^{-\frac{z^2}{2}(1+\Omega_n(z))} \tag{5}$$

where functions $w_i(z)$ and $\Omega_n(z)$ are such that $w_i(0) = 0$, $\Omega_n(0) = 0$. Provided that the x_i are independent, the CF of X_n is $\Psi_n(z) = \psi_1(z)\cdots\psi_n(z)$, where $\psi_i(z)$ is the CF of x_i. For the reduced variables we thus have

$$\bar{\Psi}_n(z) = \bar{\psi}_1(\frac{m_1 z}{M_n})\cdots\bar{\psi}_n(\frac{m_n z}{M_n}) \tag{6}$$

It follows from Eqs. (5) and (6) that

$$\begin{aligned}
&\bar{\Psi}_n(z) = \bar{\psi}_1(\frac{m_1 z}{M_n})\cdots\bar{\psi}_n(\frac{m_n z}{M_n}) \Rightarrow \\
&\bar{\Psi}_n(z) = \exp\left\{-\frac{z^2}{2}\left(\left[\frac{m_1^2}{M_n^2} + \cdots + \frac{m_n^2}{M_n^2}\right] + \left[\frac{m_1^2}{M_n^2} w_1(\frac{m_1}{M_n}) + \cdots + \frac{m_n^2}{M_n^2} w_n(\frac{m_1}{M_n})\right]\right)\right\} \Rightarrow \\
&\Omega_n(z) \equiv \frac{m_1^2}{M_n^2} w_1(\frac{m_1}{M_n}) + \cdots + \frac{m_n^2}{M_n^2} w_n(\frac{m_n}{M_n})
\end{aligned} \tag{7}$$

As $n \to \infty$, the CF of the sum variable $\bar{X}_n$ can be written as $\bar{\Psi}(z)$, and thus

$$\bar{\Psi}(z) = e^{-\frac{z^2}{2}(1+\Omega(z))} = \lim_{n\to\infty} \bar{\Psi}_n(z) \tag{8}$$

where it holds true that

$$\Omega(z) = \lim_{n\to\infty} \Omega_n(z) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{{m_i}^2}{{M_n}^2} w_i\left(\frac{m_i}{M_n} z\right) \right) = \sum_{i=1}^{\infty} \lambda_i^2 w_i(\lambda_i z)$$

$$\lambda_i \equiv \lim_{n\to\infty}\left(\frac{m_i}{M_n} \right) \tag{9}$$

If $\lambda_i = 0$, then $\Omega(z) = 0$ and the CF of the reduced sum variable collapses to $e^{-z^2/2}$ as $n \to \infty$, i.e. it collapses to a Gaussian distribution in accordance to the classic central limit theorem (CLT). Results that interest most are those for which the conditions for the CLT does not hold; i.e. $\lambda_i \neq 0$. The CLT falls into the special case to which $\lambda_i = 0$.

3. Reduced variables which are independent and identically distributed

Now we define the class of identically distributed *reduced* variables as follows.

Definition 1. Given random variables x_i, suppose that their distributions $f_i(x_i)$ are such that $f_i(x_i) \neq f_j(x_j), i \neq j$. Yet $\bar{f}_i(\bar{x}_i) = \bar{f}_j(\bar{x}_j)$ for the distributions of the reduced variables. And their CF are such that $\bar{\psi}_i(z) = \bar{\psi}_j(z)$.

Variables x_i are also assumed, without loss of generality, to be ranked, i.e.

$$m_1 \geq m_2 \geq \cdots \geq m_n \tag{10}$$

A permutation operator is defined as

$$P[i_1 \quad \cdots \quad i_n] = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} \tag{11}$$

and a vector $\vec{X}_n$ is defined as

$$\vec{X}_n = (x_1, x_2, \cdots, x_n) \tag{12}$$

Thus it follows that

$$P[i_1 \quad \cdots \quad i_n]\vec{X}_n = (x_{i_1}, \cdots x_{i_n}) \tag{13}$$

We also define

$$S\left(P\begin{bmatrix} i_1 & \cdots & i_n \end{bmatrix}\vec{X}_n\right) = x_{i_1} + \cdots + x_{i_n} \tag{14}$$

So Eq. (1) can be rewritten as

$$X_n = S\left(P\begin{bmatrix} i_1 & \cdots & i_n \end{bmatrix}\vec{X}_n\right) = x_{i_1} + \cdots + x_{i_n} \tag{15}$$

Now we define an event as an n-dimensional real vector generated according to the following rule.

Definition 2. An event $\vec{E}$ is created by first picking a permutation P and then using Eq. (13) to build up a vector in which the kth component is randomly generated by a number from the distribution of x_{i_k} .

Such an event is a realization of vector $\vec{X}_n$ followed by a permutation of any of its components according to Eq. (13). For N events

$$\vec{E}_i = \left(E_{i1}, E_{i2}, \cdots E_{in}\right) \tag{16}$$

and their sum is defined as

$$S_i = S(\vec{E}_i) = E_{i1} + \cdots E_{in} \tag{17}$$

And for N large enough, sequence $S_{1,}, S_2, \cdots, S_N$ has the same distribution of that for the sum variable X_n, as in Eq. (1) .

Now we can perform a series expansion of $w(z)$ and $\Omega_n(z)$ to yield

$$w(z) \equiv \sum_{p=even} K_p z^p + I \sum_{p=odd} K_p z^p \tag{18}$$

$$\Omega_n(z) = \sum_{p=even} L_{np} z^p + I \sum_{p=odd} L_{np} z^p \tag{19}$$

where $I = \sqrt{-1}$. Using Eqs. (5), (7), (18), and (19) produces

$$L_{np} = K_p \sum_{i=1}^{n} \left(\frac{m_i}{M_n}\right)^{p+2} \tag{20}$$

with

$$\sum_{i=1}^{n}\left(\frac{m_i}{M_n}\right)^{p+2} = \frac{m_1^{p+2}+\cdots+m_n^{p+2}}{M_n^{p+2}} = \frac{m_1^{p+2}+\cdots+m_n^{p+2}}{\left(m_1^2+\cdots+m_n^2\right)^{\frac{p+2}{2}}} \tag{21}$$

Note that L_{np} is uniquely determined by K_p and m_i. The K_p is given by Eq. (18) and is entirely determined by the distributions $f_i(x_i)$. And the summation in Eq. (21) is fully determined by the standard deviations.

4. Exponential and power laws

Now we assume that m_i is governed by law $m_i = g(i)$. Then we consider further particular cases in the form of an exponential and a power law.

Case 1. Exponential law: $m_i = Ae^{-Bi}$, $A,B > 0$

where

$$m_i = m_1\left(e^{-2B}\right)^{\frac{i-1}{2}} \equiv m_1 r^{\frac{i-1}{2}} \tag{22}$$

Here it is useful to define

$$\left(\frac{m_{i+1}}{m_i}\right)^{p+2} = r^{\frac{p+2}{2}} \equiv R_p \tag{23}$$

It can be found that

$$M_n = \sqrt{m_1^2+\cdots+m_n^2} = m_1\left(\frac{1-r^n}{1-r}\right)^{1/2} \Rightarrow$$
$$M_n^{p+2} = m_1^{p+2}\left(\frac{1-r^n}{1-r}\right)^{\frac{p+2}{2}} \tag{24}$$

and

$$m_1^{p+2}+\cdots+m_n^{p+2} = m_1^{p+2}\frac{1-R_p^n}{1-R_p} \tag{25}$$

Substituting Eqs. (22), (23), and (24) into Eq. (20) yields

$$L_{np} = K_p \frac{(1-r)^{\frac{p+2}{2}}}{(1-r^n)^{\frac{p+2}{2}}} \frac{\left(1-r^{\frac{p+2}{2}n}\right)}{\left(1-r^{\frac{p+2}{2}}\right)}, 0 < r < 1$$

$$L_{np} = K_p \frac{(r-1)^{\frac{p+2}{2}}}{(r^n-1)^{\frac{p+2}{2}}} \frac{\left(r^{\frac{p+2}{2}n}-1\right)}{\left(r^{\frac{p+2}{2}}-1\right)}, r > 1 \qquad (26)$$

Case 2. Power law: $m_i = \frac{A}{i^B}$, $A,B > 0$

Here function

$$Z(n,r) = 1 + 2^r + 3^r + \cdots + n^r, r \in \Re \qquad (27)$$

is employed to produce

$$m_i^2 = A^2 i^r \rightarrow M_n = A\sqrt{Z(n,r)}, r = -2B \qquad (28)$$

It can also be shown that

$$m_1^{p+2} + \cdots + m_n^{p+2} = A^{p+2} Z\left(n, \frac{p+2}{2} r\right) \qquad (29)$$

Considering Eqs. (27) and (28) together with Eqs. (20) and (21) produces

$$L_{np} = K_p \frac{Z\left(n, \frac{p+2}{2} r\right)}{Z(n,r)^{\frac{p+2}{2}}}, r \le 0 \qquad (30)$$

We will consider an extra power law in Section 6.

5. Measure of the standard deviations m_i

For a series of events $E_1, E_2, \cdots E_N$, a list of $N' = nN$ numbers can be obtained, i.e.

$$\vec{u} = \lfloor u_1, \cdots, u_n, u_{n+1}, \cdots, u_{2n}, \cdots, u_{(N-1)n+1}, \cdots, u_{nN}, \rfloor \qquad (31)$$

where $u_1 = E_{11}, u_2 = E_{12}, \cdots, u_n = E_{1n}, u_{n+1} = E_{21}$. Eq. (31) can be thought of as an n-periodic stochastic process. If the period is known with certainty, Eq. (31) can be used to make a list as follows.

$$\vec{U} = [U_1, \cdots, U_N] \tag{32}$$

where $U_1 = u_1 + \cdots + u_n, U_2 = u_{n+1} + \cdots + u_{2n}, \cdots$, and so on.

It can be shown [8] that the skewness and kurtosis of a statistical list $\vec{X}$ are, respectively,

$$\begin{aligned} Skew(\vec{U}) &= \frac{m_1^3 + \cdots + m_n^3}{\left(m_1^2 + \cdots + m_n^2\right)^{3/2}} Skew(\vec{u}), \\ Kurt(\vec{U}) &= \frac{m_1^4 + \cdots + m_n^4}{\left(m_1^2 + \cdots + m_n^2\right)^2} Kurt(\vec{u}) \end{aligned} \tag{33}$$

At this point a practical difficulty emerges. How to get the standard deviations m_i? We are aware by the very nature of the process that they are not ordered. Besides, it is not always possible to evaluate a standard variation of a stochastic variable from a single measure. But such problems can be overcome by a rationale as follows.

Take the jth period of our list, i.e. $\vec{u} : u_{(j-1)n+1}, \cdots, u_{jn}$. Calculate the standard deviations $SD(\vec{X})$ from pairs of a bidimensional list

$$SD\left(\left[u_{(j-1)n+i}, u_{(j-1)n+i+1}\right]\right), i = 1, \cdots, n-1 \tag{34}$$

and summarize the output as σ_{ij}. Then rank the output as

$$\sigma_{j1} \geq \sigma_{j2} \geq \cdots \geq \sigma_{jn-1}, j = 1,2, \cdots, N \tag{35}$$

Finally, evaluate m_i from

$$m_i = \frac{\sigma_{1i} + \sigma_{2i} + \cdots + \sigma_{Ni}}{N} \quad i = 1,2, \cdots, n-1 \tag{36}$$

Thus such an approach can be summarized as follows.

$$
\overbrace{\begin{matrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{N1} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{1n-1} & \sigma_{2n-1} & \cdots & \sigma_{Nn-1} \end{matrix}}^{\text{N periods of size n-1}} \quad \begin{matrix} m_1 = \sum_{j=1}^{N} \frac{\sigma_{j1}}{N} \\ \vdots \\ m_{n-1} = \sum_{j=1}^{N} \frac{\sigma_{jn-1}}{N} \end{matrix} \tag{37}
$$

The routine above is nonstandard and produces only an approximate value for m_i. Yet when applied to experimental data it seems to be fairly effective, as next section will show.

6. Application to the Brazilian *real*-US dollar exchange rate

Here the above routine is applied to the 15-minute spaced Brazilian *real*-US dollar exchange rate for the year 2002 (Fig. 1). By employing Eq. (28) we get list $\vec{u}$ from such a set of data. This list is made up of $N' = 6140$ figures. Then we use Eq. (31) to get list $\vec{U}$ from a "daily" set of data (Fig. 2) which is built up as follows. A "day" is considered to possess 20 data points of the original 15-minute series. So the period of the process is $n = 20$. Since $N' = nN$, the list $\vec{U}$ is made up of 307 figures.

The skewness and kurtosis of the two lists are

$$
\begin{aligned}
&Skew(\vec{u}) = 3.0653, Kurt(\vec{u}) = 114.4593 \\
&Skew(\vec{U}) = 1.5288, Kurt(\vec{U}) = 19.3846
\end{aligned} \tag{38}
$$

As can be seen, the hypothesis of an independent and identically distributed (IID) process is promptly discarded. Indeed for an IID process we have

$$
\begin{aligned}
&m_1 = m_2 = \cdots = m_n \Rightarrow \\
&Skew(\vec{U}) = \frac{1}{n^{1/2}} Skew(\vec{u}), \\
&Kurt(\vec{U}) = \frac{1}{n} Kurt(\vec{u}),
\end{aligned} \tag{39}
$$

Using $n = 20$ and the results for $\vec{u}$ produces

$$
\begin{aligned}
&Skew(\vec{U}) = \frac{1}{20^{1/2}} 3.0653 = 0.6854, \\
&Kurt(\vec{U}) = \frac{1}{20} 114.4593 = 5.2965
\end{aligned} \tag{40}
$$

which obviously does not hold.

We then evaluate whether the process can be explained in terms of our suggested independent and identically distributed reduced (IIDR) variables. To apply the technique summarized in Eq. (37) we consider N = 309 periods of size n – 1 = 19. The standard deviations m_i are shown in Fig. 3. Applying Eq. (33) and using the values of m_i produces

$$\begin{aligned} Skew(\vec{U}) &= 1.3497, \\ Kurt(\vec{U}) &= 25.6804 \end{aligned} \tag{41}$$

which is in good agreement with Eq. (38).

Now we turn to the question of what the formation law for the standard deviations in Fig. 3 is. Here we sketch an approximate answer. We start with tentative exponential law $m_i = Ae^{-Bi}$. Fig. 4 displays the data in Fig. 3 together with $m_i = Ae^{-Bi}$. By using such an exponential law it can be shown with some algebra that

$$\begin{aligned} Skew(\vec{U}) &= Skew(\vec{u})\frac{(1-r)^{\frac{3}{2}}}{\left(1-r^{3/2}\right)}\frac{\left(1-r^{\frac{3}{2}n}\right)}{\left(1-r^{n}\right)^{3/2}}, \\ Kurt(\vec{U}) &= Kurt(\vec{u})\frac{(1-r)^{2}}{\left(1-r^{2}\right)}\frac{\left(1-r^{2n}\right)}{\left(1-r^{n}\right)^{2}}, \\ r &= e^{-2B} < 1 \end{aligned} \tag{42}$$

Thus employing Eq. (42) together with B (given in Table 1) yields

$$\begin{aligned} Skew(\vec{U}) &= 1.3086, \\ Kurt(\vec{U}) &= 23.9674 \end{aligned} \tag{43}$$

which is in good agreement with the experimental data.

Secondly we try out the fitting of power law $m_i = Ai^{-B}$. Fig. 5 shows the data using $m_i = Ai^{-B}$. Here it can be shown that

$$\begin{aligned} Skew(\vec{U}) &= Skew(\vec{u})\frac{Z(n,3r/2)}{Z(n,r)^{3/2}}, \\ Kurt(\vec{U}) &= Kurt(\vec{u})\frac{Z(n,2r)}{Z(n,r)^{2}}, \\ r &= -2B < 0 \end{aligned} \tag{44}$$

In our example $A \approx 0.0149, B \approx 0.7711$ (Table 1), from which we obtain

$$
\begin{aligned}
&Skew(\vec{U}) = 1.4296, \\
&Kurt(\vec{U}) = 31.0246
\end{aligned}
\tag{45}
$$

which departures from the experimental data.

Just in case, we consider another power law, namely $m_i = (A + iB)^{-1/C}$. Fig. 6 presents results for this case. The skewness and kurtosis are obtained directly from Eq. (33); as for m_i, we take the values in Fig. 6, with $m_i \approx (4.2267 + 0.4973)^{-1/0.3607}$. We obtain

$$
\begin{aligned}
&Skew(\vec{U}) = 1.3493, \\
&Kurt(\vec{U}) = 25.5359
\end{aligned}
\tag{46}
$$

Table 2 gives a summary of results. Thus the best fitting is obtained with the assumption that the process is an IIDR together with an exponential law describing the behavior of the second moment.

7. Asymptotic behavior

If the standard deviation is governed by a formation law then the asymptotic probability density function (PDF) may not be a Gaussian. Indeed from Eq. (9) one can realize that, if $\lambda_i \neq 0$, the conditions for the CLT to hold are not fulfilled. We focus on the asymptotic behavior of the L_{np}. We simply make $n \to \infty$ in both the exponential law (Eq. (26)) and the power law in Eq. (30). This produces

$$
\begin{aligned}
&L_p \equiv \lim_{n\to\infty} L_{np} = \frac{K_p (1-r)^{\frac{p+2}{2}}}{\left(1 - r^{\frac{p+2}{2}}\right)}, 0 < r < 1 \\
&L_p \equiv \lim_{n\to\infty} L_{np} = \frac{K_p (r-1)^{\frac{p+2}{2}}}{\left(r^{\frac{p+2}{2}} - 1\right)}, r > 1
\end{aligned}
\tag{47}
$$

from Eq. (26), and

$$
\begin{aligned}
&L_p \equiv \lim_{n\to\infty} L_{np} = \frac{K_p \zeta\left(\left|\frac{p+2}{2} r\right|\right)}{\zeta(|r|)^{\frac{p+2}{2}}}, r < -1; \\
&L_p = 0, r \geq -1
\end{aligned}
\tag{48}
$$

from Eq. (30), where $\zeta(r)=\sum_{p=0}^{\infty}\frac{1}{p^{|r|}}$ is the Riemman zeta function. Note that we have $L_p = 0 \Rightarrow \Omega(z) = 0$ for $r \geq -1$, and the asymptotic PDF is Gaussian.

For the other power laws we consider the behavior of the skewness and kurtosis [8]. We make $n >> 1$ in Eq. (33); here n = 1000 is considered to be large enough for practical purposes. To compare our results we present in Table 3 the output for $n \to \infty$. As can be seen, no process is able to approach zero.

8. Characteristic function

Note that the CF of the reduced variable can be written as

$$\varphi(z) = e^{-\frac{z^2}{2}(1+W(z))}, W(z) = W_R(z) + IW_I(z) \tag{49}$$

The function associated with $\vec{u}$ is $W_u(z)$, and that related to $\vec{U}$ is $W_U(z)$. Thus the functions related to an IID and an IIDR process are, respectively,

$$W_{IID}(z) = W_u(z/\sqrt{n}) \tag{50}$$

and

$$W_{IIDR}(z) = \sum_{i=1}^{n} \frac{m_i^2}{M_n^2} W_u(\frac{m_i}{M_n} z), \quad M_n^2 = m_1^2 + \cdots + m_n^2 \tag{51}$$

where m_i is obtained from Eq. (37), as displayed in Fig. 3. Fig. 7 shows both the real and imaginary part of Eqs. (50) and (51), where n = 19 has been used.

9. Concluding remarks

If financial data cannot be satisfactorily addressed by Gaussian distributions, a question of interest is: what is the formation law (if any) for the volatility of the series? This paper tackles such a problem and sketches some tentative answers. We then add to the existing literature on the subject. To do that, we put forward a class of reduced variables that are independent and identically distributed and that seem to fit a financial data set well. The set is sampled from the intraday Brazilian *real*-US dollar exchange rate for the year 2002. We find that our suggested variable together with an exponential law are able to explain the volatility behavior of the series reasonably well.

In particular, we find that the best fitting is obtained with the assumption that the process is independent and identically distributed for our reduced variable together with an exponential law, in which case the value for the kurtosis is closer to the experimental data.

As for the asymptotic behavior of volatility, we find that our reduced variable cannot reach zero as the sample size approaches infinity; and this holds for all of our suggested exponential and power laws.

Note that our results are based on the assumption that the stochastic process exhibits a characteristic period and in some cases that might not be true. Furthermore, even in case of periodicity one might erroneously identify the period. And when assuming that the standard deviation is governed by a formation law, the role of the autocorrelation function in the study of the properties of the system is neglected.

Acknowledgements

We are grateful to Felipe Beys (Agora Senior Consultants) for providing the data. We also thank Martha Scherer and Aline Gandon for research assistance, and an anonymous referee for comments. Iram Gleria acknowledges support from FAPEAL (Fund of Assistance to Research of the Brazilian State of Alagoas).

References

[1] A. Lo, A. C. MacKinlay, Stock market prices do not follow random walks: evidence from a simple specification test, Rev. Fin. Stud. 1 (1988) 41-66.

[2] R. Weron, Lévy-stable distributions revisited: tail index > 2 does not exclude the Lévy-stable regime, International Journal of Modern Physics C 12 (2001), 209-223.

[3] P. Cootner, The Random Character of Stock Market Prices, The MIT Press, Cambridge, 1964.

[4] R.N. Mantegna, H.E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: the truncated Lévy flight, Phys. Rev. Lett. 73 (1994) 2946-2949.

[5] R.N. Mantegna, H.E. Stanley, Scaling behavior in the dynamics of an economic index, Nature 376 (1995) 46-49.

[6] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C. K. Peng, H. E. Stanley, The statistical properties of the volatility of price fluctuations, Phys. Rev. E 60 (1999) 1390-1400.

[7] Z. Ding, C. W. J. Granger, R. F. Engle, A long memory property of stock market returns and a new model, J. Empirical Finance 1 (1993) 83-105.

[8] A. Figueiredo, I. Gleria, R. Matsushita, S. Da Silva, Lévy flights, autocorrelation, and slow convergence, Physica A 337 (2004) 369-383.

[9] D. Dugué, Oeuvres de Paul Lévy, Vol. III: Eléments Aléatoires, Gauthiers-Villars, Paris, 1976.

Table 1. Estimated models

	Estimated value ± standard error
IIDR Exponential law $m_i = Ae^{-Bi}$	A = 0.0156 ± 0.000599 B = 0.2070 ± 0.0101
IIDR Power law $m_i = Ai^{-B}$	A = 0.0149 ± 0.000899 B = 0.7711 ± 0.0521
IIDR Power law $m_i = (A + iB)^{-1/C}$	A = 4.2267 ± 1.3063 B = 0.4973 ± 0.3191 C = 0.3607 ± 0.0807

Table 2. Skewness and kurtosis for experimental data under alternative assumptions

	Experimental data	IID (39)	IIDR (41)	Exponential law (43)	Power law (45)	Power law (46)
Skewness	1.5287	0.6854	1.3497	1.3086	1.4296	1.3493
Kurtosis	19.3846	5.7230	25.6804	23.9674	31.0246	25.5359

Note: "IID (39)" is meant results obtained for the hypothesis of an IID process according to Eq. (39), and so on. As can be seen, the exponential law provides a value for the kurtosis that is closer to the experimental data.

Table 3. Comparison of results

	Skewness	Kurtosis
IID	$n \to \infty$ 0.0000	$n \to \infty$ 0.0000
IIDR Exponential law $m_i = Ae^{-Bi}$	$n \to \infty$ 1.3008	$n \to \infty$ 23.1422
IIDR Power law $m_i = Ai^{-B}$	$n \to \infty$ 1.1274	$n \to \infty$ 22.2813
IIDR Power law $m_i = (A + iB)^{-1/C}$	$n = 1000$ 1.3378	$n = 1000$ 25.2400

It is apparent that no law can lead a process to the Gaussian zero.

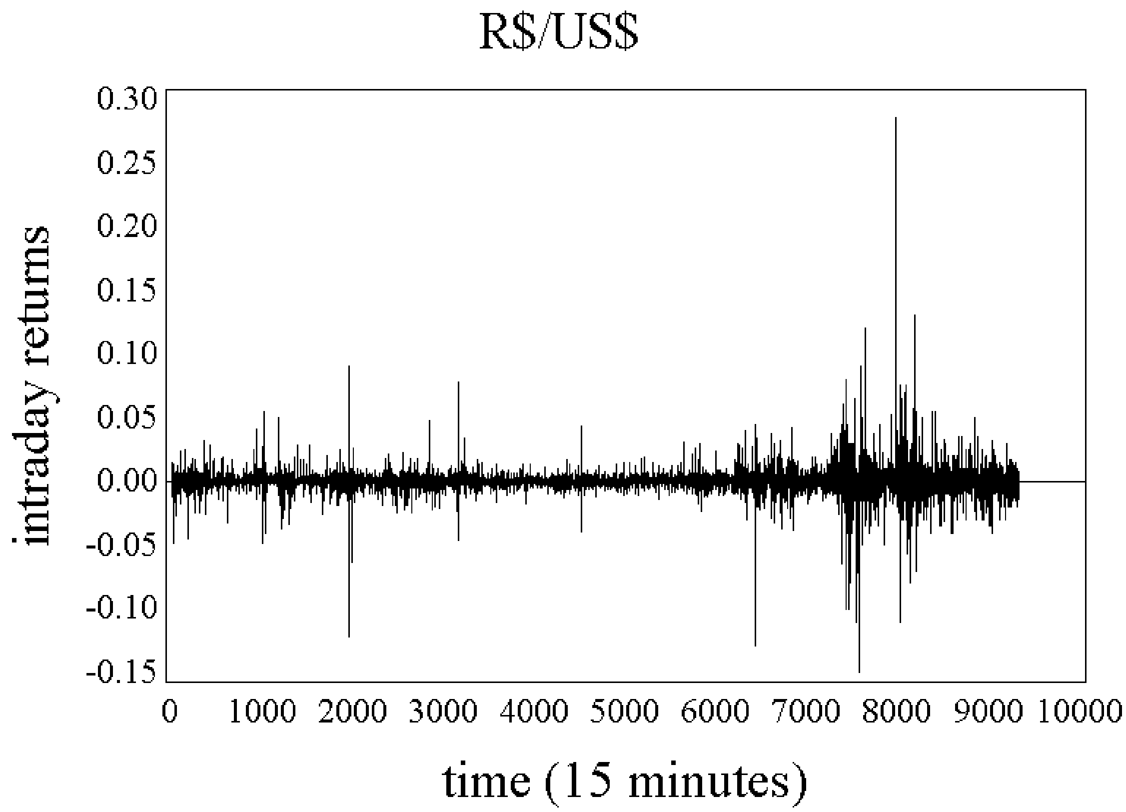


Fig. 1. Brazilian *real*-US dollar 15-minute returns for the year 2002.

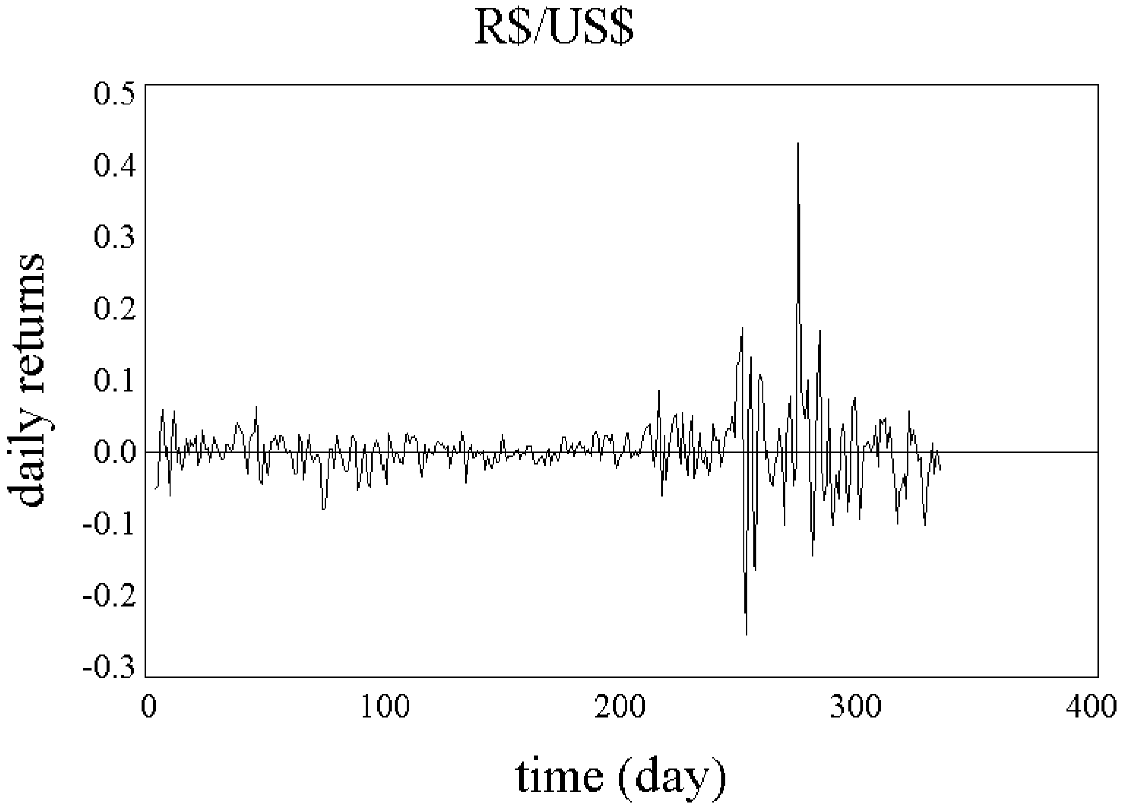


Fig. 2. Brazilian *real*-US dollar "daily" returns for the year 2002.

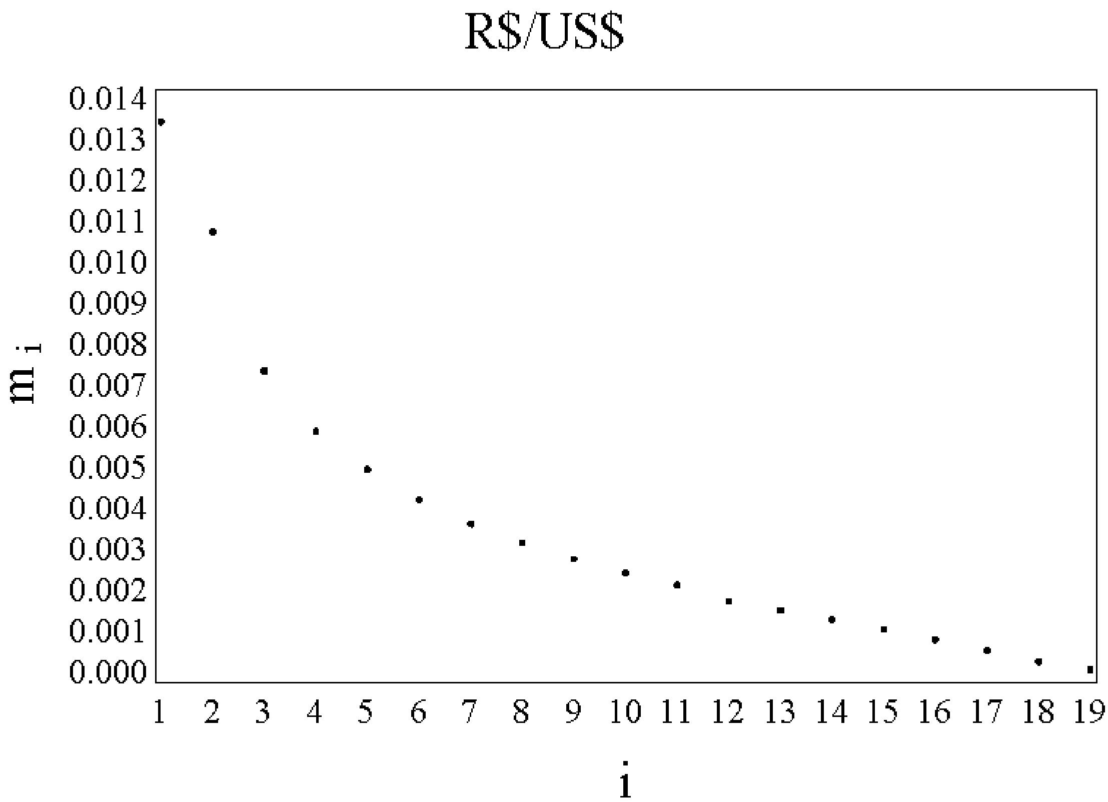


Fig. 3. Standard deviations m_i against i.

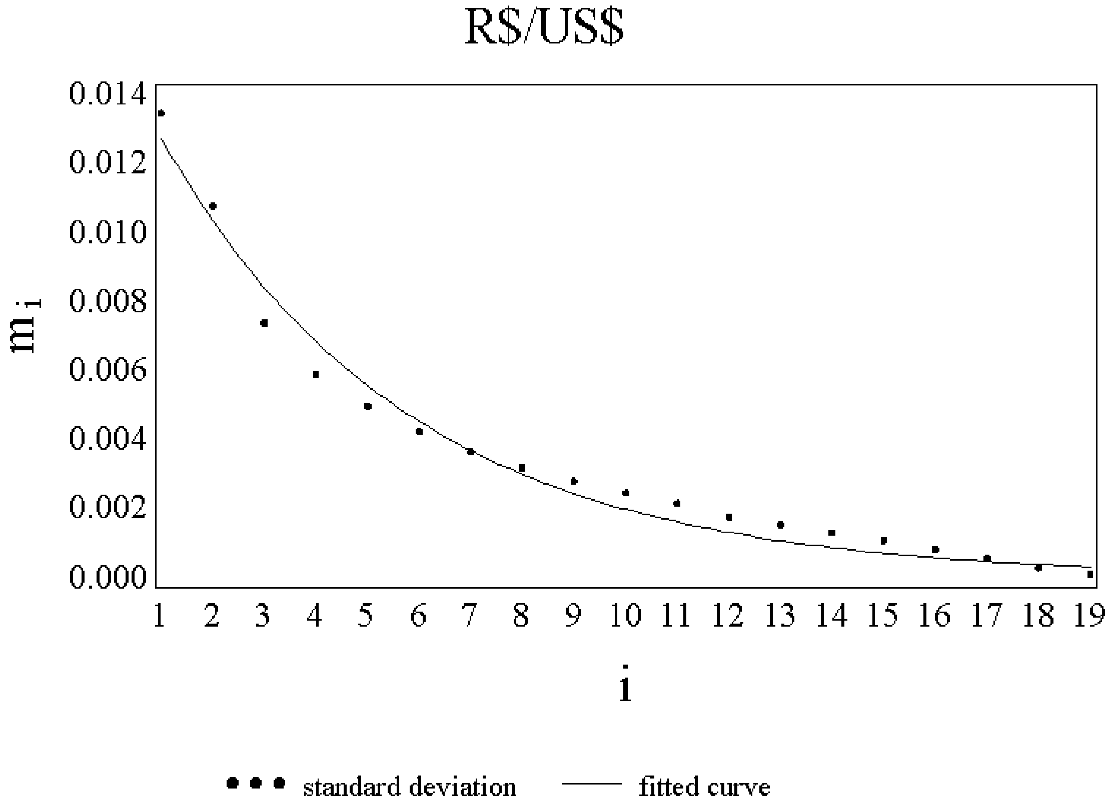


Fig. 4. Fitting exponential law $m_i = Ae^{-Bi}$.

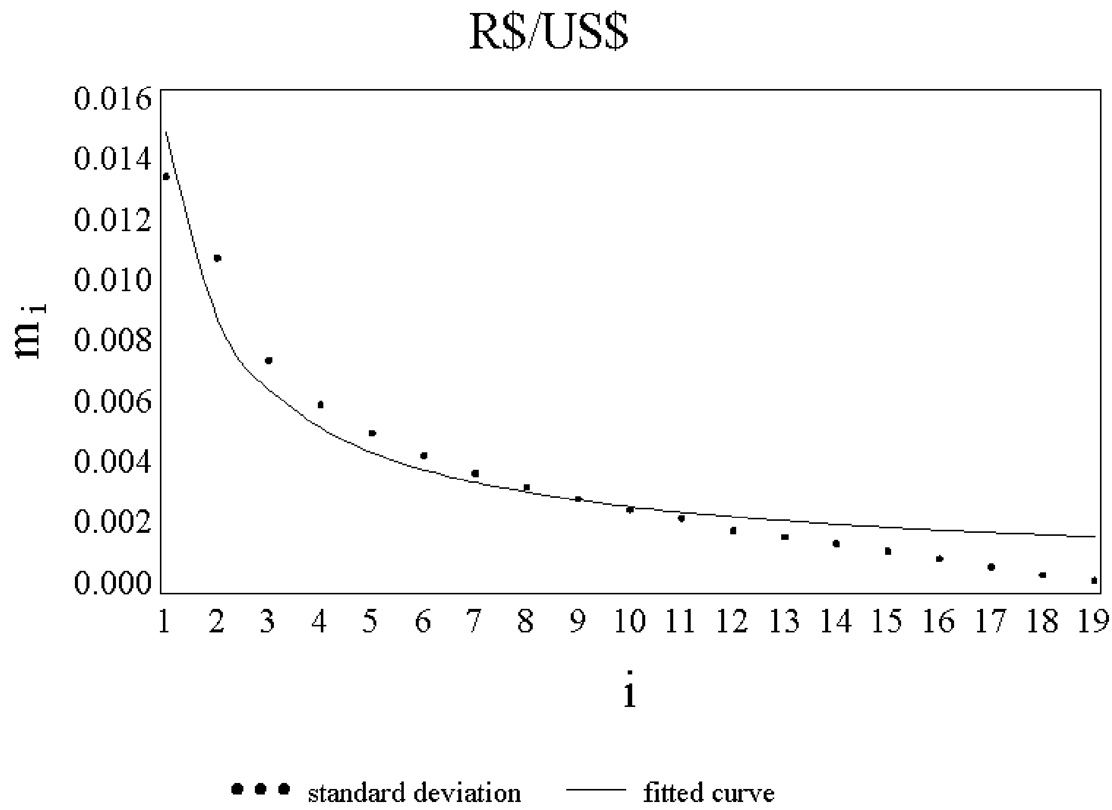


Fig. 5. Fitting power law $m_i = Ai^{-B}$.

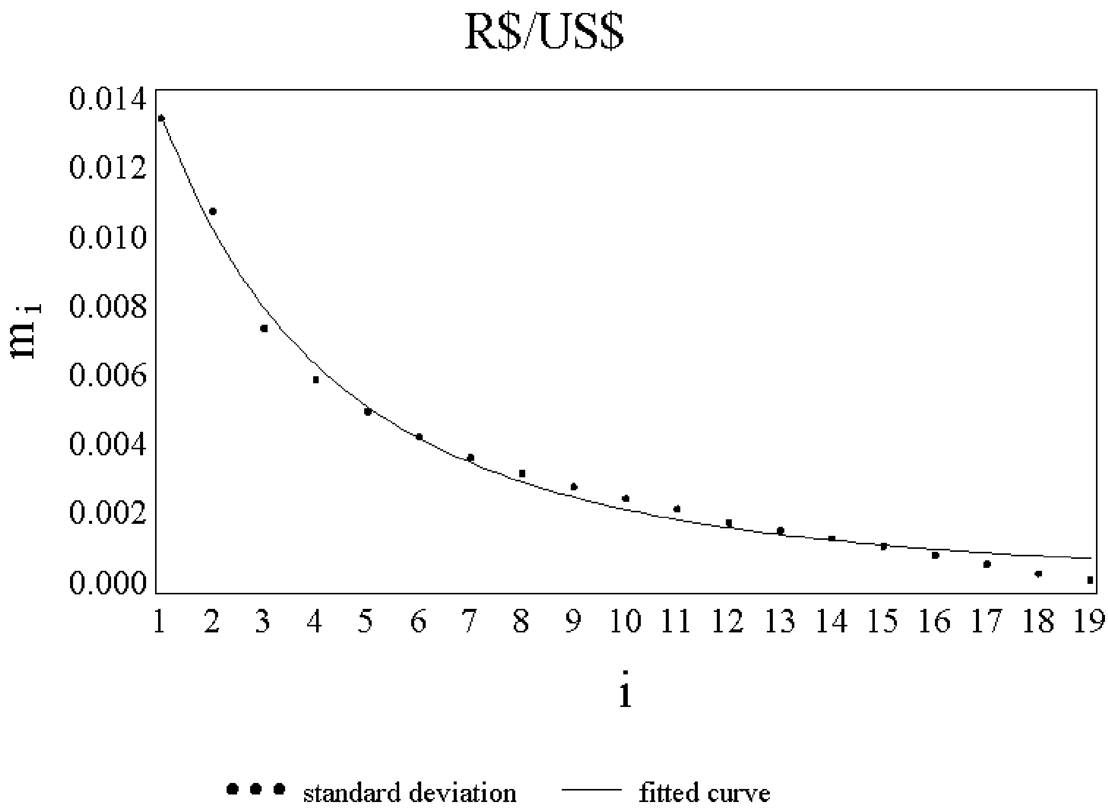
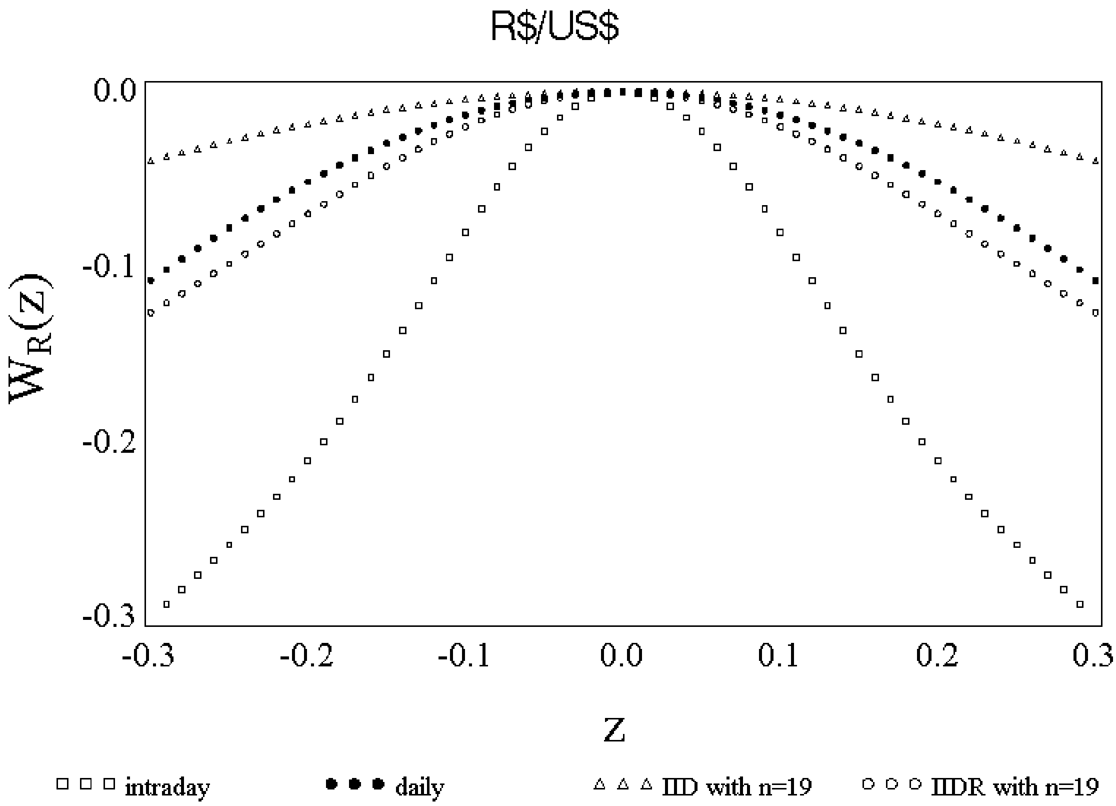
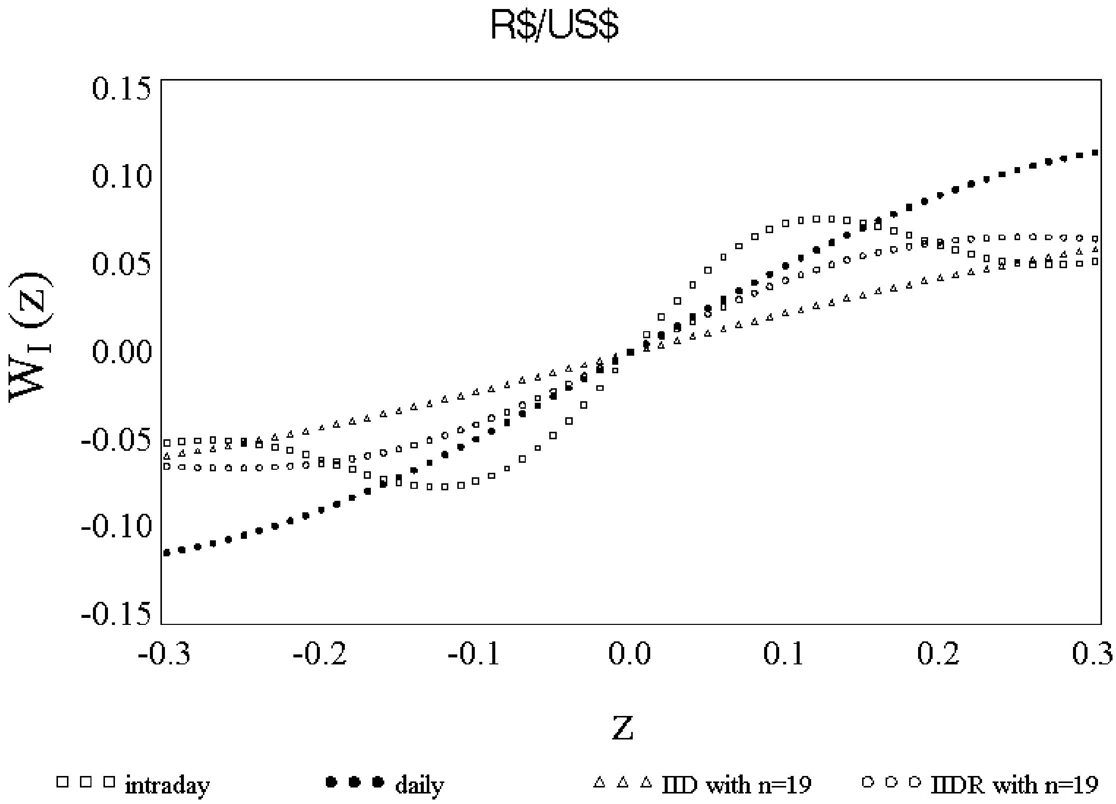


Fig. 6. Fitting power law $m_i = (A + iB)^{-1/C}$.



R$/US$
0.0
-0.1
-0.2
-0.3
$W_R(z)$
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
Z
intraday
daily
IID with n=19
IIDR with n=19

(a)



(b)

Fig. 7. Behavior of $W(z)$, which is related to an IID and an IIDR process. Real and imaginary parts are in Figs. 7a and 7b respectively.