A Theory for the Term Structure of Interest Rates

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Abstract The Convolution and Master equations governing the time behavior of the term structure of Interest Rates are set up both for continuous variables and for their discretised forms. The notion of Seed is introduced. The discretised theoretical distributions matching the empirical data from the Federal Reserve System (FRS) are deduced from a discretised seed which enjoys remarkable scaling laws. In particular the tails of the distributions are very well reproduced. These results may be used to develop new methods for the computation of the value-at-risk and fixed-income derivative pricing.

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1 Introduction

The problem of modelling the variations of the interest rates is important especially in the context of the evaluations of the value-at-risk and marked-to-market positions in trading floors.

In two recent articles [1], [2], it was shown using empirical data published by the governors of the Federal Reserve System [3] from 1962 until 2002 that the term structure of interest rates decreases essentially as a power for large variations of the interest rates and that this power is of the order three to four. Moreover the distributions seem to obey simple approximate scaling laws as functions of the initial interest rate, of the lag and of the maturity. These findings invalidate many models which predict distributions having either very short tails, generally exponentially decreasing, or very long tails as do Levy type structures [4], [5], [6], [7].

In this paper, a theoretical model is built to serve as a basis for computing the distribution of the variation of interest rates in terms of a few fundamental "microscopic parameters" whose meaning will be highlighted. At the basis of the theory, the "seed" notion is introduced. It is closely related to the variation of the interest rates for very short time intervals.

In order to simplify the presentation, the problem is set in terms of continuous variables. Later, to allow numerical simulations and come closer to the empirical distributions, the variables are discretised.

In the course of this article, in order to make a connection to a real situation, we have chosen to refer systematically to an application of our ideas to the FRS data [3]. Needless to say, we expect our analysis to be extendable, mutatis mutandis, to many other situations.

2 The basic equations with continuous variables

In this section, the basic equations, which govern the continuous time propagation of the term structure of interest rates, are analyzed.

Suppose that, the interest rate for a certain maturity $[m]$ has the value I_0 at time t_0. We want to study the normalized density of probability as seen as at time t

$$p_t^{[m]}(t_f, I_f, t_0, I_0) \tag{1}$$

that, at a later (final) time t_f, thus after a lag

$$L = t_f - t_0 \ , \tag{2}$$

the interest rate has a value I_f. In principle, this density of probability p_t is an unknown function of the five "continuous variables" t, t_f, I_f, t_0, I_0. In order to simplify the notation we will restrict ourselves to a given maturity and suppress the corresponding upper index.

The compounded probability (at time t) $\overline{P}_{t,t_f,I_a \le I_f \le I_b, t_0, I_0}$ that, starting with an initial interest rate I_0 at the initial time t_0, the final interest rate I_f at the final time t_f is in the interval between I_a and I_b is given by the integral on this interval of the probability density

$$\overline{P}_{t,t_f,I_a \le I_f \le I_b, t_0, I_0} = \int_{I_a}^{I_b} p_t(t_f, I_f, t_0, I_0)\, dI_f \tag{3}$$

with, obviously, by normalization

$$\overline{P}_{t,t_f,-\infty \le I_f \le +\infty, t_0, I_0} = 1 \ . \tag{4}$$

We will now make the hypothesis that the interest rate variations satisfy some market laws which are rather stable and are governed by sufficiently smooth equations. Let us try to state and justify our simplifying assumptions.

2.1 Time translation invariance

This is a very delicate hypothesis. It is equivalent in saying that, whatever be the political or economical situation the average behavior will be identical. The effects of the exceptional situations (political or economical crisis) which usually may lead to seemingly incoherent variations of the interest rates are accurately taken into account by the tails of the functions which we will be using. The postulate is that, even if there are extreme situations of various importance, all in all they connect smoothly with the more normal situations which prevail during the peaceful times. It is precisely these extreme situations which are the prime reason for the fat tails of the distributions. And they must be incorporated correctly by the model. The hypothesis is that what happens in the exceptional situations is well taken into account by the smooth tails of the distributions. There is no abrupt transition between really exceptional situations and what we would call the normal situations. Between the extremes, there are situations of intermediate seriousness which lead for the distribution to a smooth passage from a restful period to a chaotic one.

More mathematically, for any time t, the time translation operation is given by

$$t' = t + T \tag{5}$$

where T is the value of the translation time (minutes or days or months or years later). Suppose that at some time t'_0 later than t_0 given by the translation time T

$$t'_0 = t_0 + T \tag{6}$$

the interest rate I'_0 is again exactly equal to I_0. Consider the new final time t'_f

$$t'_f = t_f + T \tag{7}$$

obtained by the same translation time. Technically, the time translation invariance demands that the densities of probability $p_t(t'_f, I_f, t'_0, I_0)$ and $p_t(t_f, I_f, t_0, I_0)$ be equal

$$p_t(t'_f, I_f, t'_0, I_0) = p_t(t_f, I_f, t_0, I_0) \tag{8}$$

whatever be T.

It is not difficult to prove mathematically that this condition essentially implies that p is independent of t and that

$$p_t(t_f, I_f, t_0, I_0) = p(t_f - t_0, I_f, I_0)\ . \tag{9}$$

In other words, the density of probability depends on the lag

$$L = t_f - t_0 \tag{10}$$

In other words, the density of probability p_t is reduced to a function p of three continuous variables only: the lag, the final interest rates I_f and initial interest rate I_0. The variation V of the interest rate during the lag L is defined as

$$V = I_f - I_0\ . \tag{11}$$

For later convenience, a variable change is performed to define the time translation invariant density of probability p and for now on, this form will be used

$$p(L, V, I_0) \equiv p(t_f - t_0, I_f - I_0, I_0) \tag{12}$$

as the basic interest rate distributions.

2.2 The normalization of the probability. Continuous variables

The total probability (4) that after the lag L the rate I_f has any value must be equal to one. This implies the normalization

$$\int_{-\infty}^{+\infty} p(L, I_f - I_0, I_0)\, dI_f \equiv \int_{-\infty}^{+\infty} p(L, V, I_0)\, dV = 1\ . \tag{13}$$

This normalization should hold whatever be the lag and whatever be the initial rate.

2.3 The composition of the probabilities. Continuous variables

The basic equations, which govern the composition of the probability densities p, have to be set up. Intuitively, consider three times t_0, t_i, t_f where the intermediate time t_i lies between the initial time t_0 and the final time t_f.

$$t_0 \leq t_i \leq t_f \ . \tag{14}$$

The initial rate is I_0.

During the time lag $L_1 = (t_i - t_0)$ the interest rate has a density of probability $p(t_i - t_0, I_i - I_0, I_0)$ to reach the intermediate value I_i. Then starting from the intermediate time t_i up to final time t_f (i.e. during the second time lag $L_2 = (t_f - t_i)$) the intermediate observed interest rate I_i at time t_i has a density of probability $p(t_f - t_i, I_f - I_i, I_i)$ to become I_f at the final time t_f. Since the interest rate at the intermediate time t_i can take any value, the density of probability starting from the rate I_0 at initial time t_0 to end up with a rate I_f at the final time t_f is given by the integration on I_i at the intermediate time (convolution of the probabilities)

$$p(t_f - t_0, I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p(t_f - t_i, I_f - I_i, I_i)\, p(t_i - t_0, I_i - I_0, I_0)\, dI_i \ . \tag{15}$$

This is the basic equation which the probability distribution has to fulfill. It shows how the probability distributions of two successive lags $L_1 = (t_i - t_0)$ and $L_2 = (t_f - t_i)$ compose to form the probability distribution for the lag $L = t_f - t_0 = L_1 + L_2$. Eq.(15) should hold whatever be the intermediate time.

It is convenient to rewrite the equation by using the new variables I, V and the new integration variable W

$$\begin{array}{rclcrcl} I &=& I_0 &,& I_0 &=& I \\ V &=& I_f - I_0 &,& I_f &=& V + I \\ W &=& I_i - I_0 &,& I_i &=& W + I \\ L_1 &=& t_i - t_0 &,& L_2 &=& t_f - t_i \end{array} \tag{16}$$

as

$$p(L_1 + L_2, V, I) = \int_{-\infty}^{+\infty} p(L_2, V - W, I + W)\, p(L_1, W, I)\, dW \tag{17}$$

This is the basic equation.

2.4 Initial conditions. Continuous variables

The probability distribution has to satisfy an initial condition which can be described in terms of the Dirac δ distribution (see Appendix A). Indeed if the initial rate at time t_0 has the value I_0 (whatever I_0 is), at $t_f = t_0$ (i.e. after a zero lag $L = t_f - t_0 = 0$) we know for sure that the rate is still I_0. The density of probability for the final rate to be different from I_0 is zero. In other words, when the lag is zero, the rate has not moved. It is still I_0 with probability one.

Mathematically, this implies that

$$p(0, V, I_0) = \delta(V) \ . \tag{18}$$

Some properties of the distribution $\delta(V)$ are given in Appendix (A) together with a few useful approximations in terms of more conventional functions which will be used later. In particular, the Dirac distribution is normalized

$$\int_{-\infty}^{+\infty} p(0, V, I_0)\, dV \equiv \int_{-\infty}^{+\infty} \delta(V)\, dV = 1 \ . \tag{19}$$

As it should, in agreement with the composition of probabilities (15) evaluated either for $t_i = t_0$ or for $t_i = t_f$, the Dirac distribution, composed with any distribution, satisfies the identities

$$\begin{aligned} p(t_f - t_0, I_f - I_0, I_0) &= \int_{-\infty}^{+\infty} p(t_f - t_0, I_f - I_i, I_i)\, p(0, I_i - I_0, I_0)\, dI_i \\ &= \int_{-\infty}^{+\infty} p(0, I_f - I_i, I_i)\, p(t_f - t_0, I_i - I_0, I_0)\, dI_i \end{aligned} \tag{20}$$

showing again that an evolution during a zero lag is, in fact, not an evolution.

2.5 The seed. Continuous variables

By the convolution equation (15), if one knows the probability distributions $p(\epsilon, V, I_0)$ for a given lag $L = \epsilon$, whatever be the value of ϵ, the probability distribution for $p(2\epsilon, V, I_0)$ can easily be computed by the convolution of $p(\epsilon)$ with itself. Then $p(3\epsilon, V, I_0)$ is obtained by the convolution of $p(2\epsilon)$ with $p(\epsilon)$. By successive iterations $p(n\epsilon, V, I_0)$ can be computed for any positive integer n. Hence, if one knew the distribution of probability for a very small lag, ϵ much smaller than the empirical lags, the distribution for a lag L much larger than ϵ could be computed approximately by simple successive integration. Essentially a number of integration equal to the integer n closest to $(L/\epsilon - 1)$. This approximation would become better and better when $\epsilon \to 0$.

It is then tempting to let ϵ go to zero and to take the first order approximation, which will be called the seed $S(V, I_0)$, as

$$\begin{aligned} S(V, I_0) &= \lim_{\epsilon\to 0} \frac{p(\epsilon, V, I_0) - p(0, V, I_0)}{\epsilon} \\ &= \partial_L\, p(L, V, I_0)\Big|_{L=0} . \end{aligned} \tag{21}$$

Since the V-integration of the two p terms in the right hand side are equal (before ϵ is put to zero) by (13), the seed S satisfies the normalization condition

$$\int_{-\infty}^{\infty} S(V, I_0)\, dV = 0 \tag{22}$$

Technically, it should be stressed that, as the initial condition (18) is a distribution rather than a function, we should expect that the seed is also a distribution with support restricted to $V = 0$.

If a Taylor expansion could be written, in first approximation for $L = \epsilon$ the probability distribution p could written

$$p(\epsilon, V, I_0) = p(0, V, I_0) + \epsilon S(V, I_0) \tag{23}$$

In the continuous case, this is a distribution. Nevertheless in the discretised case, this form can be used as a very good approximation.

2.6 The Master Equation. Continuous variables

If the seed is known, the density of probability of the interest rate variation can easily be computed by integrating the master equation as an integro-differential equation

$$\partial_{\mathrm{L}}\, p(L, I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p(L, I_f - I_i, I_i)\, S(I_i - I_0, I_0)\, dI_i \tag{24}$$

which follows from the convolution equation (15). With the change of variables (16) the master equation can equivalently be written

$$\partial_{\mathrm{L}}\, p(L, V, I) = \int_{-\infty}^{+\infty} p(L, V - W, I + W)\, S(W, I)\, dW \tag{25}$$

Technically, one obtains Eq.(24) by differentiating the convolution equation (15) with respect to t_i and letting $t_i \to t_0$. The master equation (24) is a consequence of the convolution equation. Conversely, in Appendix (B), it is shown, formally, that the solutions of the master equation satisfy the convolution equation (15).

3 The Gauss distribution as the solution for the simplest seed

The simplest possible (59) (see Appendix (A)) seed is

$$S(V, I_0) = \kappa(I_0)\, \partial_V^2\, \delta(V) \tag{26}$$

This can be justified naïvely as follows. During a very small amount of time ϵ one expects the variation of the interest rate V to be very small, say at most of the order ρ where ρ decreases and goes to zero with ϵ. Introduce the three intervals of length ρ, the left interval $L = [-3\rho/2, -\rho/2]$, the center one $C = [-\rho/2, +\rho/2]$ and the right one $R = [+\rho/2, +3\rho/2]$. Outside these three intervals the probability is essentially zero as is the density of probability. In the left and right intervals the probability is small say of the order κ and in the center it must be $1-2\kappa$ to conserve the normalization of the probability. If the density of probability is then supposed to be constant in the three intervals, the seed becomes κ/ρ in the left and right intervals and $-2\kappa/\rho$ in the center interval. Letting the time interval ϵ go to zero induces ρ to go to zero and the limiting seed becomes, up to κ, the second derivative of the delta function.

Taking this simplest form of the seed, the integro-differential equation (24) becomes a much simpler partial differential equation

$$\partial_L\, p(L, I_f - I_0, I_0) = \kappa(I_0)\, \partial_{I_0}^2\, p(L, I_f - I_0, I_0) \tag{27}$$

which is also written

$$\partial_L\, p(L, V, I_0) = \kappa(I_0) \left(\partial_V^2\, p(L, V, I_0) - 2\, \partial_V\, \partial_{I_0}\, p(L, V, I_0) + \partial_{I_0}^2\, p(L, V, I_0) \right) \tag{28}$$

This equation is still rather complicated.

It is shown in Appendix (C), assuming no I_0 dependence and simplified scaling laws (75) and (76), that the solution of the master equation for the seed (26) is of Gaussian type

$$\widetilde{p}(\widetilde{V}) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{\widetilde{V}^2}{2\sigma^2}} \tag{29}$$

This solution as highlighted in [1] and [2] is completely excluded by the observations.

4 Some considerations about the numerical integration

Technically, given a seed which is a distribution, the convolution equation (17) or the master equation (25) are usually very difficult or impossible to solve analytically. Instead, in this present work, a numerical procedure has been implemented. This numerical integration depends on the discretization of the continuous equation using discretised variables on a grid.

We have focussed our attention to the following specific form of the continuous equations (17) for $L = L_1 = L_2$

$$p(2L, V, I) = \int_{-\infty}^{+\infty} p(L, V - W, I + W)\, p(L, W, I)\, dW \tag{30}$$

In the following, the ideas and the delicate points are briefly described. The precise application of the formalism to the FRS data [3] and the results are summarized and highlighted in section (7).

In this section are presented in turn the ideas which are needed to perform the approximate numerical integration of the equation. The continuous equations are discretised by associating a finite lattice (a grid) to the continuous variables. The initial conditions and the seed are then defined on a finite set of points at the nodes of the grid. This allows then to define the χ-squared merit function.

4.1 The ideas behind the grid

The equation (30) depends on three continuous variables. In order to perform numerical integrations, the three variables have to be discretised.

The Lag grid

Suppose that the distribution is known for some value s_L of the lag L. From the first application of (30) one obtains the distribution for $L = 2s_L$. Applying (30) a second time allows the determination of the distribution for $4s_L \equiv 2^2 s_L$. The iteration of the procedure produces the distribution for any L of the form $2^l s_L$ i.e. on a the L-grid of values

$$L = 2^l s_L \quad l = 0, \ldots, N_L \tag{31}$$

Our choice of the Lag grid

We have chosen to obtain a lag $L = 1$ day after $N_L = 10$ iteration. This leads to a value of $s_L = 1/(2^{10})$ days i.e. about 30 seconds. This is a perfect choice if individual contracts are concluded at this rate. If it turns out that the lapse between two successive contracts is smaller, N_L should be increased.

The V grid

For the V variable, the best choice is a regularly spaced grid consisting of points of the form

$$V = vs_V \tag{32}$$

where s_V is the step size in V and v is an integer. A smaller s_L should lead to a choice of a smaller s_V.

Our choice of the V grid

We have chosen to take s_V to be $(\frac{1}{1.4})^{10}$ basis point i.e. around $(\frac{1}{\sqrt{2}})^{10}$ basis point. Indeed a larger s_V turned out not to be small enough while a smaller one was not necessary and increased the computation time and memory requirements with little reward.

The I_0 grid

For the I_0 variable, the best choice is a regularly spaced grid consisting of points of the form

$$I_0 = is_I \tag{33}$$

where s_I is the step size in I_0 and i is an integer.

Our choice of the I_0 grid

Since all the parameters vary with I_0 rather weakly, a choice of s_I = one percent is quite sufficient.

In our actual computations due memory size problems, we have been led to adapt the s_V step size to the level l of the iteration. The step size is progressively increased in such a way as to become one basis point at the tenth iteration. This required delicate numerical adjustments

4.2 The ideas behind the discretised equations

Obviously, on the grid (31), (32), (33) the equation (30) becomes

$$\begin{aligned} p(2^{l+1}s_L, vs_V, is_I) \quad &\approx \quad s_V \left\{ \sum_{w=-\infty}^{\infty} p\left(2^l s_L, (v-w)s_V, is_I + ws_V\right) \times \right. \\ &\left. \times\; p\left(2^l s_L, ws_V, is_I\right) \right\} \end{aligned} \tag{34}$$

To be on the grid, the argument $is_I + ws_V$ which appears in the first p on the right hand side must be an integer multiple of s_I. This would at first sight imply that s_V is an integer multiple of s_I. In fact, our numerical implementation uses, for technical reasons, a more elaborate grid than (32) in the V and a better adapted form of the convolution (34) together with smoothing and interpolation techniques. Moreover the normalization implies

$$\sum_{v=-\infty}^{\infty} p\left(2^l s_L, vs_V, is_I\right) = 1 \tag{35}$$

4.3 The ideas behind the discretised initial condition

The initial condition (18) on the grid is taken as a step-type (61) approximation of the Dirac δ distribution, as explained in Appendix (A). Namely

$$\begin{aligned} p(0,0,is_I) \quad &= \quad \frac{1}{s_V} \\ p(0,vs_V,is_I) \quad &= \quad 0 \qquad \text{for } v \neq 0 \end{aligned} \tag{36}$$

4.4 The ideas behind the discretised seed

Using the discretization initial condition, for $\epsilon = s_L$ chosen sufficiently small, the Taylor expansion (23) of p to the first order in s_L

$$p(s_L, vs_V, is_I) = p(0, vs_V, is_I) + s_L S(vs_V, is_I) \tag{37}$$

The resulting $p(s_L, vs_V, is_I)$ must be a true function (not a distribution anymore). It must be positive and normalized (35)

$$\sum_{v=-\infty}^{\infty} S\left(vs_V, is_I\right) = 0\ . \tag{38}$$

Hence, $S(vs_V, is_I)$ is expected to be a step-type function of vs_V with the following properties

$$
\begin{aligned}
S\left(0, is_I\right) &= -\sum_{v\neq 0} S\left(vs_V, is_I\right) \\
S\left(vs_V, is_I\right) &> 0 \qquad \text{for } v \neq 0 \qquad (39) \\
S\left(0, is_I\right) &< \frac{1}{s_L s_V} \qquad (40)
\end{aligned}
$$

The form of the seed inferred by the FRS data is presented and discussed in section (5).

4.5 The ideas behind the χ-squared function

In order to determine the seed free parameters, a χ-squared minimization method is used.

As usual the generic χ-squared function is defined by

$$
\chi^2 = \sum_k \frac{\left(N_{\text{theory}}(k) - N_{\text{data}}(k)\right)^2}{\sigma(k)^2} \qquad (41)
$$

where the sum is performed on the generic discrete label k indexing all the available data. In (41), $N_{\text{data}}(k)$ and $N_{\text{theory}}(k)$ are respectively the number of events observed and predicted for the label k.

The natural error ($\sigma(k)_{\text{natural}}$) is usually taken as if the distribution was of Poisson type i.e.

$$
\begin{aligned}
\sigma(k)_{\text{natural}} &= \sqrt{N_{\text{data}}(k)} \qquad \text{if } N_{\text{data}}(k) \neq 0 \\
\sigma(k)_{\text{natural}} &= 1 \qquad \text{if } N_{\text{data}}(k) = 0 \qquad (42)
\end{aligned}
$$

The choice of the error function requires more attention in the present case. As has been already stressed, the theory should take into account and reproduce as accurately as possible the tails of the experimental distributions. This is achieved by choosing in the χ-squared a modified form of $\sigma(k)$

$$
\begin{aligned}
\sigma(k)_{\text{modified}} &= \left(N_{\text{data}}(k) N_{\text{theory}}(k)\right)^{1/4} \qquad \text{if } N_{\text{data}}(k) \neq 0 \\
\sigma(k)_{\text{modified}} &= \left(N_{\text{theory}}(k)\right)^{1/4} \qquad \text{if } N_{\text{data}}(k) = 0 \qquad (43)
\end{aligned}
$$

Obviously, if theory and experiment match very closely the σ's in (42) and (43) are essentially identical. This happens in the bulk of the distribution. For the tails of the distributions, when $N_{\rm data}$ is equal to zero or one event, the $\sigma(k)_{\rm modified}$ is smaller than $\sigma(k)_{\rm natural}$. Hence, more emphasis is put on these tail points by the minimization procedure. The tests performed using both forms of $\sigma(k)$ confirm this argument. The use of (43) leads to a very good agreement between theory and data in the tails as can be seen in Figure 9.

5 The Federal Reserve System data. Determination of $N_{\rm data}(v, i_{\rm bin})$

The FRS [3] data gives in successive working days the daily average interest rate $I(\text{day})$ between banks. It is given in percent by a number with exactly two decimal figures, i.e. by a integer number in basis points. It allows the definition of the following meaningful distributions for a lag L of one day. These distributions have to be compared to the corresponding theoretical distributions.

The empirical $N(v, I)$ and $\overline{N}(v, i)$ density distributions

For v and I both expressed in basis points, the empirical distribution $N(v, I)$ is the number of occurrences, between 1962 and 2003, when the interest rate of the FRS on some day was I and the next day $I + v$. These numbers are statistically very small. It is useful to consider the compounded empirical distribution $\overline{N}(v, i)$ where v is still expressed in basis points but the interest i is expressed in percent by grouping the days when the interest rate is almost i

More precisely, for the data, the $N(v, I)$ (v and I in basis points) density distribution is defined as follows

$$N(v, I) = \text{number of days when } I(\text{day}) = I \text{ and } I(\text{day} + 1) = I + v \ . \quad (44)$$

Note that, in order to obtain $I(\text{day}+1)$, non working days are simply discarded. The average discretised $\overline{N}(v, i)$ (v in basis points and i in percent) density distribution is precisely defined by

$$\overline{N}(v, i) = \sum_{I=100i-49}^{100i+50} N(v, I) \ . \quad (45)$$

It is the number of days when the interest rate was between $i - 1/2$ percent (exactly $100i - 49$ basis points) and $i + 1/2$ percent (exactly $100i + 50$ basis points) and the interest rate has moved by an amount v basis points by the next day.

i	1	2	3	4	5	6	7	8	9
$\overline{w}(i)$	239	262	664	1269	1674	2147	1110	1147	582
i	10	11	12	13	14	15	16	17	18
$\overline{w}(i)$	469	176	202	105	160	131	65	38	0

Table 1: The empirical $\overline{w}(i)$

i_{bin}	1	2	3	4	5	6	7	8
$i \in i_{\text{bin}}$	1-3	4	5	6	7	8	9-10	11-17
$\overline{w}(i_{\text{bin}})$	1165	1269	1674	2147	1110	1147	1051	877

Table 2: The empirical $\overline{w}(i_{\text{bin}})$

The empirical $w(I)$ and $\overline{w}(i)$ interest distributions

The empirical interest distribution $w(I)$ is defined as

$$w(I) = \sum_{v_{min}}^{v_{max}} N(v, I) \tag{46}$$

where I is in basis point. It is the number of days between 1962 and 2003 when the interest rate was I in basis point. The average discrete version $\overline{w}(i)$ of (46) is defined by

$$\overline{w}(i) = \sum_{I=100i-49}^{100i+50} w(I) \tag{47}$$

where i is an integer giving the interest rate in percent. it is simply the number of days when the interest rate was between $(i - 1/2)$ and $(i + 1/2)$ percents. The total number of events $\overline{w} = \sum_i \overline{w}(i)$ is 10,440.

The $\overline{w}(i)$ are given in Table (1). In preceding papers it was realized that bins with less than about one thousand events present too much statistical fluctuations. Hence it is useful to group the data in bins. In Table (2), the $\overline{w}(i_{\text{bin}})$ are given together with a definition of the bins.

The empirical $\overline{N}_{\text{data}}(v, i_{\text{bin}})$ density distributions

The final empirical density distribution $\overline{N}(v, i_{\text{bin}})$ which enters the χ-squared merit function is taken as

$$\overline{N}_{\text{data}}(v, i_{\text{bin}}) = \sum_{i \in i_{\text{bin}}} \overline{w}(i)\, \overline{N}(v, i) \tag{48}$$

6 The definition of N_{theory} and the χ-squared merit function

The empirical distribution $\overline{N}_{\text{data}}(v, i)$ has to be compared to the effective distribution $\overline{N}_{\text{theory}}(v, i)$ computed from the theoretical distribution evaluated at lag $L = 1$ day, which means for $l = N_L$. This distribution $\overline{p}(N_L, v, i)$ is computed from the seed by N_L successive convolutions.

We have

$$\begin{aligned} \overline{N}_{\text{theory}}(v, i) &= \overline{w}(i)\, \overline{p}(N_L, v, i) \\ \overline{N}_{\text{theory}}(v, i_{\text{bin}}) &= \sum_{i \in i_{\text{bin}}} \overline{N}_{\text{theory}}(v, i) \end{aligned} \tag{49}$$

The χ-squared merit function

The following χ-squared function is minimized

$$\chi^2 = \sum_{i_{\text{bin}}=1}^{8} \sum_{v=-300}^{300} \frac{(\overline{N}_{\text{theory}}(v, i_{\text{bin}}) - \overline{N}_{\text{data}}(v, i_{\text{bin}}))^2}{\sigma(v,_{\text{bin}})^2} \tag{50}$$

In this formula, the numerator is the square of the difference between the empirical distribution (45) and the theoretical distribution (49). It is a measure of the discrepancy between theory and experiment. The denominator is the square of the modified error $\sigma(v,_{\text{bin}})$ According to the previous discussion around (43), this $\sigma(v,_{\text{bin}})$ is taken as

$$\begin{aligned} \sigma(v, i_{\text{bin}})_{\text{modified}} &= (N_{\text{data}}(v, i_{\text{bin}}) N_{\text{theory}}(v, i_{\text{bin}}))^{1/4} \quad &\text{if } N_{\text{data}}(v, i_{\text{bin}}) \neq 0 \\ \sigma(v, i_{\text{bin}})_{\text{modified}} &= (N_{\text{theory}}(v, i_{\text{bin}}))^{1/4} \quad &\text{if } N_{\text{data}}(v, i_{\text{bin}}) = 0 \end{aligned} \tag{51}$$

Finally, let us note that the limits on the summation over v, $[-300, 300]$, have been chosen at the point where the distributions in v have become empirically completely negligible. For example, increasing the limits from 300 to 400 basis points does not change the values of the parameters provided by the minimization method in any appreciable way.

7 Seed Parametrization

In preceding articles [1], [2], distributions fitting very closely the FRS data have been obtained using Padé Approximants [0, 4] i.e. a polynomial of zero degree in v in the numerator divided by a polynomial of fourth degree in the denominator. Moreover, it is shown that the Padé coefficients follow rather simple scaling laws. Extrapolating these scaling laws for values of the lag small compared to one day leads to guess forms for the seed.

7.1 The Padé guess for the seed

The naïve form suggested by the preceding results was a Padé form

$$s_L\, S(vs_V, is_I) = \frac{a(i)}{1 + b(i)v^2 + c(i)v^4} \qquad \text{for } v \neq 0 \tag{52}$$

The $v = 0$ bin being determined by the normalization condition (40). As a function of the interest rate the suggested form of the parameters is

$$\begin{aligned} a(i) &= a_1 e^{a_2(i-i_0)} \\ b(i) &= b_1 e^{b_2(i-i_0)} \\ c(i) &= c_1 e^{c_2(i-i_0)} \end{aligned} \tag{53}$$

where $i_0 = 6$ percents is an arbitrarily but conveniently chosen parameter. Minimizing the χ-squared function, the parameter b_1 becomes so small that $b(i)$ can be safely discarded. Moreover, the theoretical tails of the distributions are somewhat small compared to the data. This conclusion comforts and even strengthen our earlier findings. Though, the v^{-d} with d equal four is not incompatible with the data, Hill estimators were pointing towards a somewhat smaller value of d. Precise values can be found in [1].

7.2 Our educated guess

Following the above discussion, the following simpler form has been used

$$s_L\, S(vs_V, is_I) = \frac{\alpha(i)}{1 + \gamma(i)\, v^{d(i)}} \qquad \text{for } v \neq 0 \tag{54}$$

with

$$\alpha(i) = \alpha_1 e^{\alpha_2(i-i_0)}$$

α_1	α_2	γ_1	γ_2	d_1	d_2	d_3
(2.65 $\pm$0.04) 10^{-5}	(9.17 $\pm$0.56) 10^{-2}	(5.40 $\pm$0.11) 10^{-6}	$-$(1.76 $\pm$0.70) 10^{-2}	2.993 $\pm$0.004	$-$(9.29 $\pm$0.43) 10^{-3}	$-$(6.16 $\pm$0.71) 10^{-4}

Table 3: The seed parameters. The standard errors correspond to the increase of χ-squared by one unit

$$
\begin{aligned}
\gamma(i) &= \gamma_1 e^{\gamma_2(i-i_0)} \\
d(i) &= d_1 e^{d_2(i-i_0)+d_3(i-i_0)^2}
\end{aligned}
\tag{55}
$$

Again i is in percent and i_0 is arbitrarily chosen to be six percents

The values of the seven parameters (α_1, α_2, γ_1, γ_2, d_1, d_2, d_3) obtained by minimization of the χ-squared merit function are given in table (3).

The value of the minimal χ-squared is about 680 which taking into account the number of degrees of freedom leads to an excellent normalized value about 0.7.

7.3 Comparison with the data

With the set of parameters obtained in table (3), we have drawn the theoretical and data curve for every bin. The connection between the bins and the initial interest rates (expressed in percent) they contain is found in table (2). These curves appear in the eight figures (1-8). In figure 9, the tail for the bin 7 as an example is zoomed to show the perfect agreement for large v's obtained by the minimization. In figure 10 and 11, the distributions for a lag of two days are given as an illustration of an eleventh iteration ($l = N_L + 1 = 11$ in (31)). In figure 12, the crucial exponent d is drawn as a function of the initial interest rate.

8 Conclusions

In this paper, a microscopic theory of the term structure of interest rates has been developed. Convolution techniques, implying about ten successive convolutions, combined with time translation invariance lead to a time scaled theory where the term structure for practical lags (one day or more) can be deduced from a seed function living at a microscopical lag scale (a few seconds).

The previously discovered scaling laws suggest forms for the seed related to the Padé form i.e. ratios of polynomials.

Our results show that the scaling law assumptions are even simpler at the microscopical lag scale. Indeed, it is shown that the FRS data are amazingly well reproduced (for a lag of one day but the results easily extend to arbitrary higher lags) by assuming that the seed has the critical form of a self-organized [8] econophysical system . In other words, a very simple power law behavior emerges with essentially only one v^{-d} term. The exponent d is of the order of three in close agreement with the tail behaviors obtained previously using the Hill [9] estimator.

These results open the door to two major issues, one rather theoretical and the other more practical and pragmatic.

Since the seed has such a simple scaling form, it suggests the existence of an underlying statistical model. The discovery of the basic ingredients and laws leading in a natural and systematic way to the scaling would be a major achievement which, if attained, may also lead to new theoretical insights in related but different contexts.

Finally, one may wonder how this theory and its predictions can be efficiently used in the context of risk management (e.g. Value-at-Risk computation) and fixed-income derivative pricing. For the time being, studies are under way to measure the add-value of using this model instead of the traditional ones. First results will be provided in a near future.

A A naïve introduction to Dirac type distributions

A.1 Formal definition

A distribution $G(V)$ is a continuous linear functional on some space of functions of a real variable V. The Dirac $\delta(V)$ distribution, which has compact support, is defined on a continuous C^∞ function $f(V)$ at $V=0$ by

$$(\delta(V), f(V)) = f(0) \tag{56}$$

The jth derivative $G^{[j]}$ of a distribution of compact support $G(V)$ is defined as

$$\left(G^{[j]}(V), f(V)\right) = (-1)^j \left(G(V), f^{[j]}(V)\right) \tag{57}$$

where

$$f^{[j]}(V) = \frac{d^j f(V)}{dV^j} \tag{58}$$

is the jth derivative of the function $f(V)$.

In particular, the second derivative of the Dirac distribution

$$\left(\delta^{[2]}(V), f(V)\right) = f^{[2]}(0) \tag{59}$$

is the second derivative of $f(V)$ evaluated at $V=0$.

Without being mathematically rigorous, these definitions can be seen in a more intuitive way. As Laurent Schwartz, the inventor of the general distribution concept, used to caution : The formulae given below, with physicist notations, have to be used with great care and have to be justified in a detailed and precise way. But he also recognized that they were very convenient to guess properties of distributions.

A.2 Discussion and approximations for the Dirac distributions

The distribution $\delta(V)$ is a generalized function which can be thought as being zero for $V<0$ and $V>0$ and infinite at $V=0$ in such a way that, in formal agreement with (56)

$$\int_{-\infty}^{+\infty} \delta(V)\, f(V)\, dV = f(0)\ . \tag{60}$$

There are many functions which can approximate the delta function. Let us give two.

1. The step-type function

$$\delta(V) = \lim_{s\to 0} D(V,s) \begin{cases} D(V,s) = 0 & \text{for} \quad V < -\frac{s}{2} \\ D(V,s) = \frac{1}{s} & \text{for} \quad -\frac{s}{2} \le V \le \frac{s}{2} \\ D(V,s) = 0 & \text{for} \quad \frac{s}{2} < V \end{cases} \tag{61}$$

has compact support. It is not difficult do the Riemann integration and prove that for some $\hat{V}$

$$-\frac{s}{2} \le \hat{V} \le \frac{s}{2} \tag{62}$$

one has

$$\begin{aligned} \int_{-\infty}^{+\infty} D(V,s)\, f(V)\, dV &= f(\hat{V}) \left(\int_{-\frac{s}{2}}^{+\frac{s}{2}} D(V,s)\, dV \right) \\ &= f(\hat{V}) \; . \end{aligned} \tag{63}$$

The limit for $s \to 0$ reproduces clearly (60) as $f(\hat{V}) \to f(0)$.

2. The Dirac distribution can also be approximated by the continuous function

$$\begin{aligned} \delta(V) &= \lim_{\Delta\to 0} E(V,\Delta) \\ E(V,\Delta) &= \frac{\Delta}{\pi\,(\Delta^2 + V^2)} \; . \end{aligned} \tag{64}$$

This form is very interesting, as it shows that the $\delta(V)$ function is the boundary value of a continuous function of two variables. For positive Δ, this function is purely positive for all V.

A step-type approximation for the second derivative of the delta function is given as follows. Consider the intervals $A = [-3s/2, -s/2], B = [-s/2, +s/2], A = [+s/2, +3s/2]$, the step function which is zero outside the three intervals, has value $1/s$ in the intervals A and C and $-2/s$ in the interval B and let $s \to 0$.

B Formal discussion of the Master Equation. Continuous variables

In the following it is shown that the formal solution of the master equation is fully compatible with the convolution equation (15) whatever be the intermediate time

Suppose that, the probability distribution has a Taylor power expansion, around $L = 0$, of the form

$$p(L, V, I_0) = \sum_{n=0}^{\infty} \frac{L^n}{n!}\, p^{(n)}(V, I_0)\ . \tag{65}$$

where $p^{(n)}(V, I_0)$ is the n'th order derivative of p with respect to L evaluated at $L = 0$

$$p^{(n)}(V, I_0) = \partial_L^n\, p(L, V, I_0)\Big|_{L=0}\ . \tag{66}$$

The zeroth order $p^{(0)}(V, I_0)$ is known by Eq.(18) and the firth order $p^{(1)}(V, I_0)$ by Eq.(21)

$$\begin{aligned} p^{(0)}(V, I_0) &= p(0, V, I_0) = \delta(V) \\ p^{(1)}(V, I_0) &= S(V, I_0) \end{aligned} \tag{67}$$

Introducing this power expansion in the master equation (24) and equating, in the left and right hand sides, the coefficients of the same powers in L, one gets the recurrence equations

$$p^{(n)}(I_f - I_0, I_0) = \int_{-\infty}^{+\infty} p^{(n-1)}(I_f - I_i, I_i)\, S(I_i - I_0, I_0)\, dI_i\ , \quad n = 1, \ldots, \infty\ . \tag{68}$$

It is easily seen that the equation for $n = 1$ is an identity. For $n = 2$, one finds

$$p^{(2)}(I_f - I_0, I_0) = \int_{-\infty}^{+\infty} S(I_f - I_1, I_1)\, S(I_1 - I_0, I_0)\, dI_1 \tag{69}$$

where the dummy integration variable has been called I_1. For arbitrary $n \geq 1$, one obtains

$$\begin{aligned} p^{(n+1)}(I_f - I_0, I_0) &= \int_{-\infty}^{+\infty} dI_n \int_{-\infty}^{+\infty} dI_{n-1} \ldots \int_{-\infty}^{+\infty} dI_1 \\ &\qquad S(I_f - I_n, I_n)\, S(I_n - I_{n-1}) \ldots S(I_1 - I_0, I_0) \\ &= \int_{-\infty}^{+\infty} \left(\prod_{p=1}^{n} dI_p\right) \left(\prod_{q=0}^{n} S(I_{q+1} - I_q, I_q)\right)\ , \quad I_{n+1} = I_f \end{aligned} \tag{70}$$

It is not difficult to check that the formal power expansion (65), (67), (69), (70) satisfies the convolution equation (15) without any further condition. Hence, it can reasonably be supposed that one can focus on the master equation (24) together with its natural boundary conditions.

C Master Equation with a simplified Scaling Law and no I_0 dependence. The Gauss distribution solution

As suggested in section (3), it is convenient to study analytically the solution of the master equation with the seed (26)

$$S(V, I_0) = \kappa(I_0)\,\partial_V^2\,\delta(V)\ . \tag{71}$$

In order to simplify the problem let us limit ourselves to suppose that the distribution p does not depend on I_0

$$p(L, V, I_0) = \widehat{p}(L, V) \tag{72}$$

as well as the seed (26) and hence

$$\kappa(I_0) = \widehat{\kappa}\ . \tag{73}$$

Eq.(27) becomes

$$\partial_{\mathrm{L}}\,\hat{p}(L, V) = \widehat{\kappa}\,\partial_V^2\,\widehat{p}(L, V)\ . \tag{74}$$

To simplify the problem even further, "scaling laws" which are approximately but not exactly verified by the data [2], are assumed. It has to be emphasized that the data do not support exactly these scaling laws. Hence the results we will obtain in this section cannot be hoped to be correct. On the other hand, it is worth to quote them since they have as a consequence the very common but wrong belief that, in a natural way, distributions should fall off at large V as exponentials.

The ultra simplification of the empirical results found in [1], [2] is equivalent to the statement that the distributions depend on the reduced variable $\widetilde{V}$ (76) and scales (75) in the natural way as $1/\sqrt{L}$

$$p(L, V, I_0) = \frac{\widetilde{p}(\widetilde{V})}{\sqrt{L}} \tag{75}$$

$$\widetilde{V} = \frac{V}{\sqrt{L}} \tag{76}$$

Replacing p by its guess (75) in (27), one finds the final equation

$$2\kappa\,\partial^2_{\widetilde{V}}\,\widetilde{p}(\widetilde{V}) + \partial_{\widetilde{V}}\left(\widetilde{V}\,\widetilde{p}(\widetilde{V})\right) = 0 \tag{77}$$

The solutions of this equation can easily be found. Indeed, it can be written

$$\partial_{\widetilde{V}}\left(2\kappa\partial_{\widetilde{V}}\,\widetilde{p}(\widetilde{V})+\widetilde{V}\,\widetilde{p}(\widetilde{V})\right)=0 \tag{78}$$

whose general solution is

$$2\kappa\partial_{\widetilde{V}}\,\widetilde{p}(\widetilde{V})+\widetilde{V}\,\widetilde{p}(\widetilde{V})=R_1 \tag{79}$$

where R_1 is an arbitrary constant. To solve this first order differential equations, the homogeneous equation is solved

$$2\kappa\partial_{\widetilde{V}}\,\widetilde{p}^h(\widetilde{V})+\widetilde{V}\,\widetilde{p}^h(\widetilde{V})=0 \tag{80}$$

giving the homogeneous solution $\widetilde{p}^h(\widetilde{V})$

$$\widetilde{p}^h(\widetilde{V})=H\,e^{-\frac{\widetilde{V}^2}{4\kappa}} \tag{81}$$

Assuming (the method of variation of constants) that H depends on $\widetilde{V}$, introduce (81) in (79) to obtain the equation

$$\partial_{\widetilde{V}}\,H(\widetilde{V})=\frac{R_1}{2\kappa}e^{\frac{\widetilde{V}^2}{4\kappa}} \tag{82}$$

which can be solved by a simple integration

$$H(\widetilde{V})=\frac{R_1}{2\kappa}\int_0^{\widetilde{V}}dx\,e^{\frac{x^2}{4\kappa}}+R_2 \tag{83}$$

where R_2 is the second constant of integration. One has now to plot the final solution obtained by replacing the solution for H (83) in the homogeneous solution (81)

$$\begin{aligned}\widetilde{p}(\widetilde{V}) &= H(\widetilde{V})\,e^{-\frac{\widetilde{V}^2}{4\kappa}}\\ &= \left(\frac{R_1}{2\kappa}\int_0^{\widetilde{V}}dx\,e^{\frac{x^2}{4\kappa}}+R_2\right)e^{-\frac{\widetilde{V}^2}{4\kappa}}\end{aligned} \tag{84}$$

In fact, we are looking for a distribution function $p(L,V,I_0)$ which is, in first order symmetric in V, hence for a scaled function $\widetilde{p}(\widetilde{V})$ which is also symmetrical in $\widetilde{V}$. Since by (81) the homogeneous solution $\widetilde{p}^h(\widetilde{V})$ is symmetrical, one looks for a symmetrical $H(\widetilde{V})$ solution. This clearly implies that R_1 should be zero.

We thus conclude that the solution of the master equation with strict scaling law leads to the familiar Gauss type behavior

$$\widetilde{p}(\widetilde{V}) = R_2\, e^{-\frac{\widetilde{V}^2}{4\kappa}} \tag{85}$$

which is not at all sustained by the data as the main result of our preceding investigation has shown that the asymptotically the term structure decreases as a power of $1/V^d$ with d around three to four.

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Figure Caption

Figure 1.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 1, 2, 3$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 2.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 4$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 3.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 5$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 4.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 6$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 5.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 7$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 6.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 8$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 7.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 9, 10$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 8.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 11 - 17$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 9.

The comparison between the theoretical curve (line) compared with the empirical curve (crosses) for the tail (i.e. for v greater than 5 basis points) for the seventh bin which contains the initial interest rates $i = 9, 10$. The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves of the other bins present fits of equivalent quality.

Figure 10.

The comparison between the theoretical curve for a Lag of 2 days (line) compared with the empirical curve (crosses) for the first bin which contains the initial interest rates $i = 1, 2, 3$. The related total number of events is found in (1), (2). The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events.

Figure 11.

The comparison between the theoretical curve for a Lag of 2 days (line) compared with the empirical curve (crosses) for the tail (i.e. for v greater than 5 basis points) for the first bin which contains the initial interest rates $i = 7$. The horizontal axis is in basis points. The vertical axis represents the number of events. The size of the arms of the crosses are estimated errors equal approximatively to the square root of the number of events. The analogous curves of the other bins present fits of equivalent quality.

Figure 12.

The parameter d as a function of the initial interest rate for i between 1 and 17.

Figure (1)

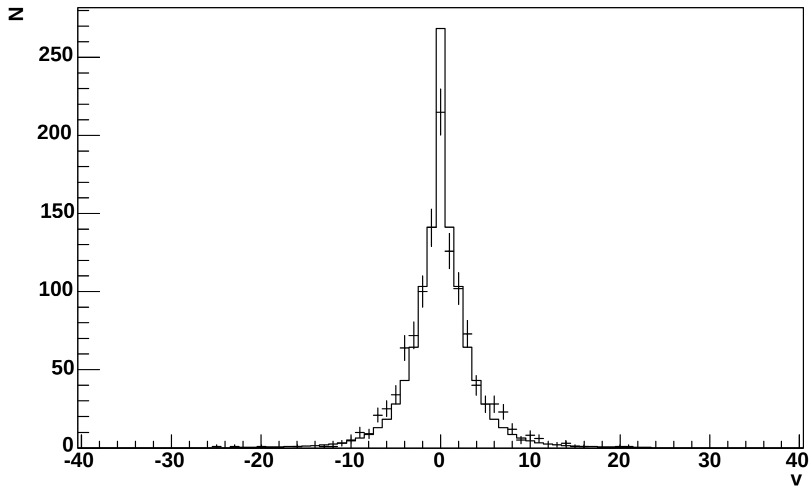


Figure (2)

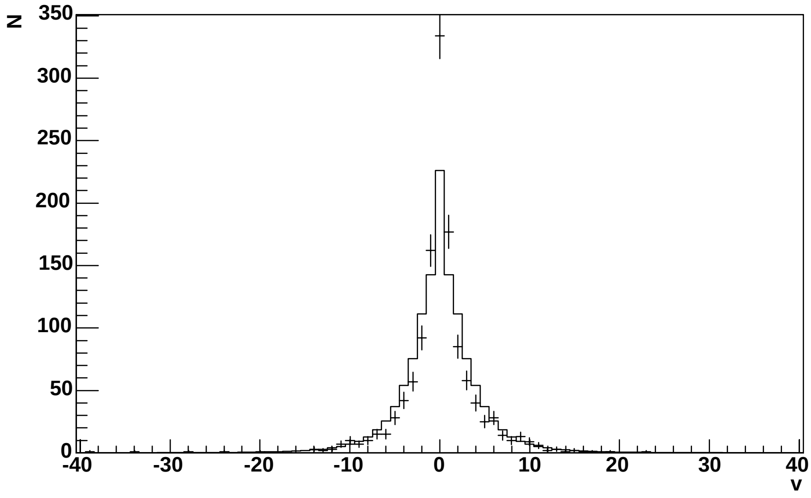
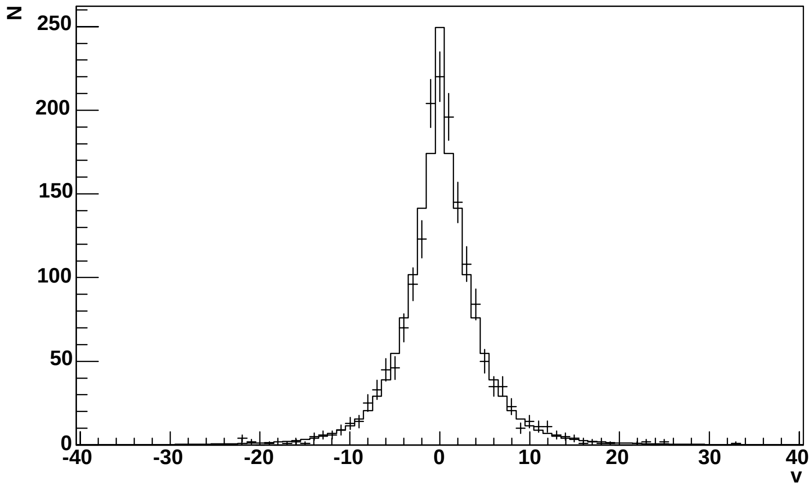


Figure (3)



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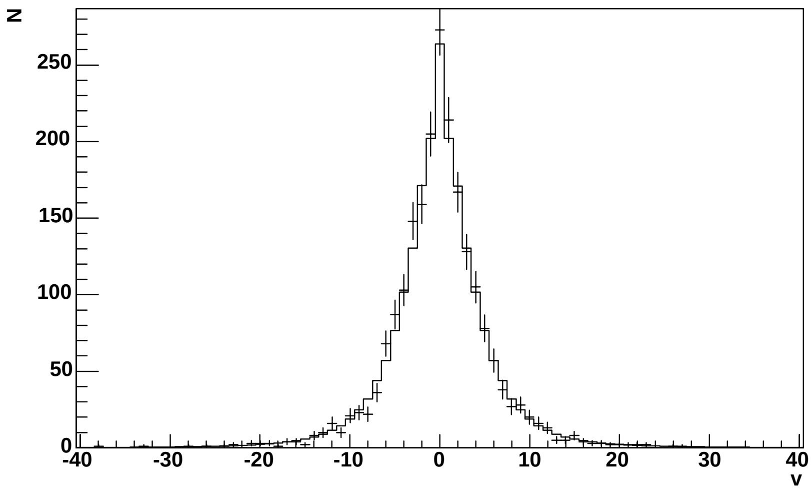

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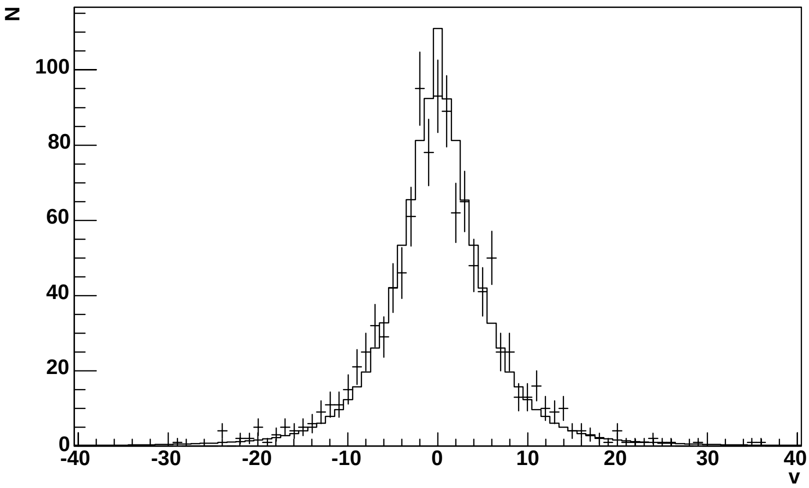

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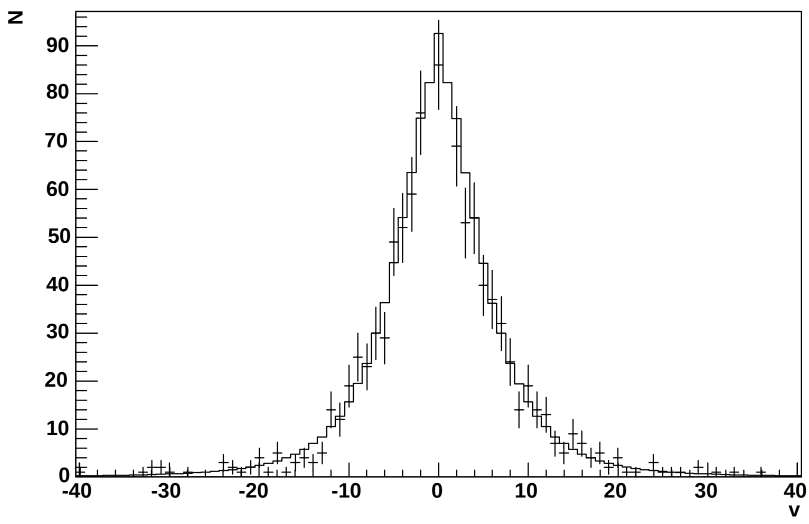


Figure (7)

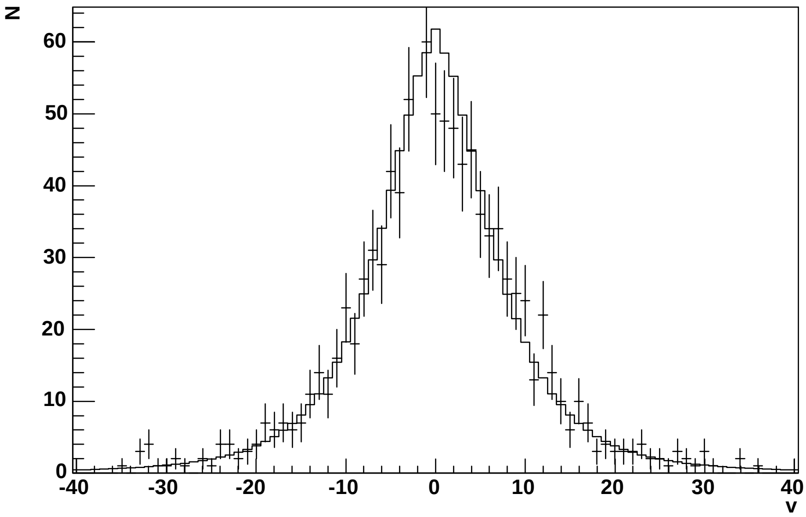


Figure (8)

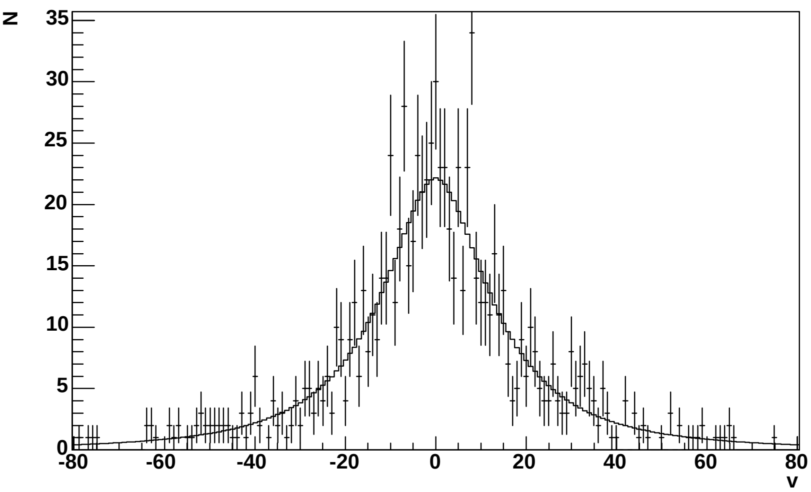


Figure (9)

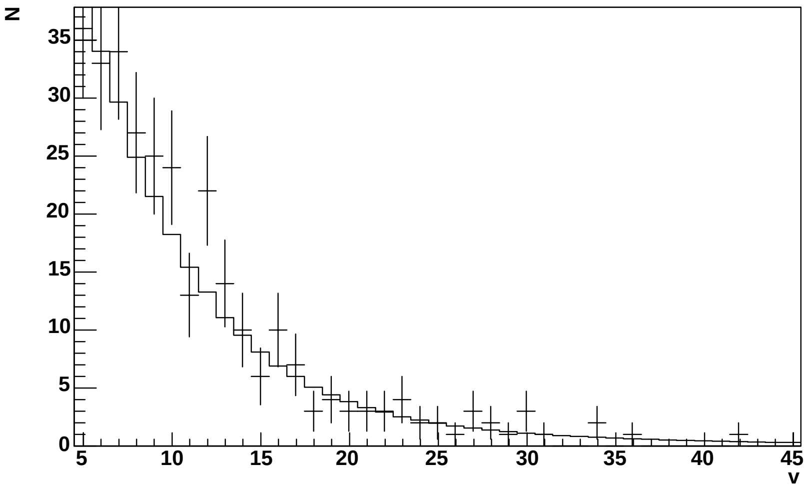
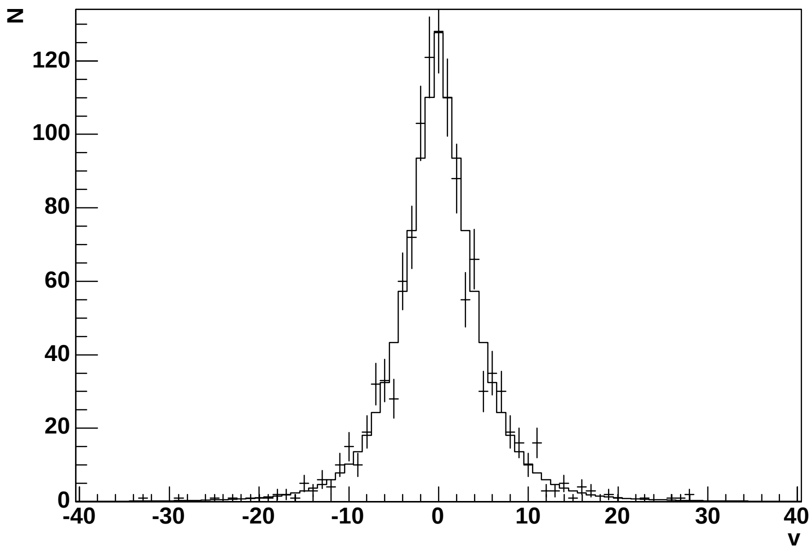


Figure (10)



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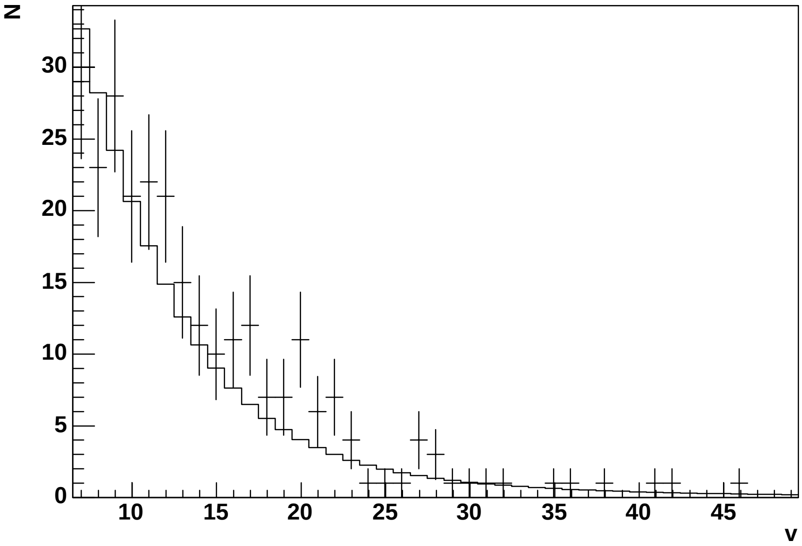

Figure (12)