

American options: the EPV pricing model

Svetlana Boyarchenko and Sergei Levendorskiĭ

Department of Economics, The University of Texas at Austin, 1 University Station C3100,
Austin, TX 78712-0301, USA
(e-mail: sboyarch@eco.utexas.edu; leven@eco.utexas.edu)

Received: ; revised:

Summary We explicitly solve the pricing problem for perpetual American puts and calls, and provide an efficient semi-explicit pricing procedure for options with finite time horizon. Contrary to the standard approach, which uses the price process as a primitive, we model the price process as the expected present value of a stream, which is a monotone function of a Lévy process. Certain processes exhibiting mean-reverting, stochastic volatility and/or switching features can be modelled in this way. This specification allows us to consider assets that pay no dividends at all when the level of the underlying stochastic factor is too low, assets that pay dividends at a fixed rate when the underlying stochastic process remains in some range, or capped dividends.

Key words Levy processes, option pricing, dividend paying assets.

JEL Classification Numbers: C61, D81, G12.

1 Introduction

The objective of this paper is to provide explicit pricing formulas for perpetual American options and Carr's randomization approach to American options with finite time horizon (and related real options) on assets with the dynamics more general than in the standard models. Explicit analytical results for standard exponential or linear dependence on the underlying stochastic factor X_t have been known both in discrete and continuous time models with infinite time horizon, if X_t is a process with stationary independent increments (random walk and the Brownian motion or, more generally, a Lévy process). However, the geometric Lévy model does not fit empirical data well, and to improve the fit, a process

should exhibit mean-reversion, stochastic volatility, and/or switching features – see Bates (2003), Chernov et al. (2003) and the bibliography therein. Unfortunately, standard mean-reverting, stochastic volatility and switching models lead to analytically efficient pricing formulas for contingent claims of the European type only (see, e.g., Duffie et al. (2000)), and to the lesser extent, for options with a fixed barrier. For American options under mean reverting and stochastic volatility processes, explicit formulas are not known, and standard switching models lead to systems of equations with several unknown functions – values of an option in each state. These models are analytically untractable for general Lévy processes, and even in the Brownian motion case lead to highly non-linear systems. The goal of this paper is to partially fill in the gap between the Lévy model and more general models by constructing analytically tractable one-factor models, which can exhibit any of these features.

As opposed to the classical option pricing theory that uses the price of an asset as the primitive, we model the payoff process as the expected present value (EPV) of a stochastic stream of payoffs, which is a monotone function of a Lévy process. This specification is very flexible, and includes the standard one as a special case, if the stock pays dividends (the case of non-dividend paying stock can be obtained in the limit, as the dividend rate vanishes). From the point of view of economics, the definition of the value of an asset as the EPV of a stream (of revenues/dividends) is a natural one. Our method makes it possible to price options on assets that pay no dividends at all when the value of the firm falls below a certain level, assets that pay capped dividends, and assets with several levels of the dividend rate, depending on the value of the firm or operating profits. Apart from generality and natural economic meaning, our approach has the following advantages against the standard approach which uses the price of an asset as the primitive. Under a natural assumption that the stream of payoffs is a monotone function of the underlying stochastic factor, the simple proofs and interesting general form for the optimal exercise rule emerge. Given a candidate for the optimal exercise threshold, we calculate the option value in terms of the *expected present value operators* (EPV-operators, which calculate the EPV of a stream under the initial process, and its supremum and infimum processes), and the form of the solution suggests the following description of the optimal exercise strategy. If the stream of payoffs is a decreasing function of the underlying stochastic factor (put option case), then it is optimal to exercise the option the first time the EPV of the stream of payoffs calculated for the supremum process instead of the original stochastic process becomes non-negative. Similarly, if the stream of payoffs is an increasing function of the underlying stochastic factor, then it is optimal to exercise a call-like option the first time the EPV of the stream of payoffs calculated for the infimum process instead of the original stochastic process becomes non-negative. This allows us to formulate a general optimal exercise rule: it is optimal to exercise the right for the stream of stochastic payoffs, g_t when the EPV of the stream $\underline{g}_t = \inf_{0 \leq s \leq t} g_s$, starting at the spot level g_0 of the underlying process g_t , becomes non-negative. We call the above statement a *record setting bad news principle*. This principle naturally generalizes and extends Bernanke's (1983) *bad news principle* and record setting news principles spelled out in Boyarchenko (2004). In the latter paper, the principles were stated and proved for the streams of the form $Ae^{X_t} - B$

and $B - Ae^{X_t}$, where X_t is a Lévy process, and A, B are positive constants. Here the result is proved for any monotone function, $g_t = g(X_t)$ of a Lévy process. The method of the paper works for some non-monotone payoff streams as well, and the proof itself becomes fairly simple due to the systematic use of the EPV-operators. In this paper, we consider perpetual American put and call options, and using the variant of analytic method of lines (Carr and Faguet (1994)) or equivalently, Carr's randomization procedure (Carr (1998)), which is, essentially, a sequence of embedded perpetual options, we develop an efficient pricing scheme for American options with finite time horizon. Notice that even in this situation, with arbitrarily many embedded options, the method remains analytically tractable. We generalize the construction in Boyarchenko and Levendorskiĭ (2002b, Chapter 6) and Levendorskiĭ (2004), where less general class of Lévy processes and the standard American put were considered. In the case of the Brownian motion with embedded exponentially distributed jumps, and the stream of payoffs modelled as piece-wise exponential polynomials of the underlying process, the solution of the optimal stopping problems in the sequence reduces to a (longer) sequence of integrals of a simple structure. Using the method of indeterminate coefficients, the calculations reduce to systems of linear (algebraic) equations. In addition, on each time step, it is necessary to find a unique zero of a monotone continuous function. This gives a very fast and accurate numerical procedure.

A simpler version of the same procedure can be applied to price options with finite time horizon and a fixed barrier (barrier options).

The rest of the paper is organized as follows. In Section 2 we introduce the EPV operators, formulate the main results for perpetual calls and puts (the proofs are given in Section 4 and Section 5, with most technical parts delegated to Section 7), and give a short review of the literature on pricing of perpetual American options. In Section 3, we recall the basic definitions of the theory of Lévy processes, calculate action of the EPV operators for the case of diffusions with exponentially distributed jumps, and give explicit realizations of the pricing rules formulated in Section 2. Carr's randomization is considered in Section 6.

2 Perpetual American options

Payoff specification. Assume that the underlying stochastic factor $\{X_t\}$ is a Lévy process under a risk-neutral measure chosen by the market, and denote the corresponding expectation operator by E . The EPV of a stream $g(X_t)$, at the spot level $X_0 = x$, is given by

$$(U_X^q g)(x) := E^x \left[\int_0^\infty e^{-qt} g(X_t) dt \right], \quad (2.1)$$

where $q > 0$ is the riskless rate. For the perpetual American call on an asset, which pays the revenue/dividend stream δ_t , the payoff function is

$$G(x) = (U_X^q \delta)(x) - K = U_X^q(\delta(\cdot) - qK)(x),$$

where K is the strike price, and for the perpetual American put, the payoff function is

$$G(x) = K - (U_X^q \delta)(x) = U_X^q(qK - \delta(\cdot))(x).$$

Notice that we use $G(X_t)$ rather than $\max\{G(X_t), 0\}$, which is admissible because it is not optimal to exercise the option unless $G(X_t)$ is positive (the equivalence of these two specifications was used in Darling et al. (1972)). The standard specification of the payoff for the call is $G(X_t) = e^{X_t} - K$, where X_t is the log-price of the stock, and it can be represented as the EPV of a certain stream as follows. Set $\Psi(z) = \ln E[e^{zX_1}]$ (this is the Lévy exponent of a Lévy process X_t). If $\Psi(1) < q$, which implies that we deal with a dividend-paying asset, then the standard payoff can be represented as the EPV of the stream $g(X_t) = (q - \Psi(1))e^{X_t} - qK$. Similar representation obtains for the put option on a dividend-paying stock.

Examples of admissible streams of payoffs and dividend streams.

1. The asset pays no dividends when the underlying stochastic factor (say, the value of the firm) is too low: $\delta(x) = 0$, $x \leq x_0$, and dividends increase exponentially after the factor crosses a certain level:

$$\delta(x) = (e^x - e^{x_0})^+ := \max\{0, e^x - e^{x_0}\}. \quad (2.2)$$

2. The asset pays no dividends when the underlying stochastic factor is too low: $\delta(x) = 0$, $x \leq x_0$, and the dividends increase not so fast after the factor crosses a certain level:

$$\delta(x) = (x - x_0)^+. \quad (2.3)$$

3. The asset pays no dividends when the underlying stochastic factor is too low: $\delta(x) = 0$, $x \leq x_0$. When the critical level is crossed, the dividends increase but eventually the growth slows and essentially stops (capped dividends):

$$\delta(x) = (e^{-x_0} - e^{-x})^+. \quad (2.4)$$

4. The asset pays dividends at a fixed rate when the underlying process is within a certain range; when the process arrives in the next range, the dividend rate changes by a jump:

$$\delta(x) = \sum_j \delta_j \mathbf{1}_{[d_j, +\infty)}(x), \quad (2.5)$$

where $\mathbf{1}_{[a, +\infty)}$ is the indicator function of $[a, +\infty)$. The sum can be finite (in this case, the dividends are capped, as in Example 3) or infinite, which allows for unbounded growth of dividends. If the intervals on which δ is constant are large, this model can be viewed as a switching model: when X_t remains inside one of these intervals, the fluctuations of $G(X_t) = U_X^q g(X_t)$ are relatively small, but the differences among levels of $G(X_t)$ for different intervals can be large.

5. If the payoff stream $g(X_t)$ is a convex function of X_t for $X_t < x_0$, and concave for $X_t > x_0$, the process X_t may exhibit a mean-reverting feature. A simple example: $g(X_t) = -1 + e^{X_t}$ for $X_t \leq 0$, and $g(X_t) = 1 - e^{-X_t}$ for $X_t \geq 0$.

6. The dividends are paid in a constant proportion to the firm's value: $\delta(X_t) = \lambda X_t$, but the value itself is an increasing function of a Lévy process, Y_t : $X_t = f(Y_t)$.

7. We can easily generalize examples 1–5 by using $\delta(f(Y_t))$, where f is an increasing function, and Y_t is a Lévy process.

Optimal stopping problem. The rational price of the option with the payoff $G(X_t)$ is given by

$$V(x) = \sup E^x [e^{-q\tau} G(X_\tau)], \quad (2.6)$$

where the supremum is taken over a set \mathcal{M} of all stopping times $\tau = \tau(\omega)$ satisfying $0 \leq \tau(\omega) \leq +\infty$, $\omega \in \Omega$; if $\tau(\omega) = +\infty$, then $G(\tau(\omega)) = 0$ by definition (see, e.g., Shiryaev (1999), XVIII, 2). In the paper, we present the solution to the optimal stopping problem for a wide class of Lévy processes satisfying the (ACP)-property (absolute continuity of potential kernels: see, e.g., Bertoin (1996) or Sato (1999)). The (ACP) property is fairly weak; in particular, if the transition kernel has a density, then this property holds.

The payoff is modelled as $G(X_t) = U_X^q g(X_t)$, where g is an arbitrary monotone function; certain non-monotone g are admissible as well. If g is not continuous, we impose an additional weak condition on the process X_t . The optimal stopping time, τ , turns out to be the hitting time of a semi-infinite interval of the form $(-\infty, h]$ (put options) or $[h, +\infty)$ (call options). We denote these hitting times by τ_h^- and τ_h^+ , respectively. The results are formulated in terms of the EPV operators $U_{\bar{X}}^q$ and $U_{\underline{X}}^q$, of the supremum and infimum processes $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ of X_t . The EPV-operators $U_{\bar{X}}^q$ and $U_{\underline{X}}^q$ are defined by

$$U_{\bar{X}}^q g(x) := E^x \left[\int_0^\infty e^{-qt} g(\bar{X}_t) dt \right] := E \left[\int_0^\infty e^{-qt} g(\bar{X}_t) dt \mid X_0 = x \right]$$

and

$$U_{\underline{X}}^q g(x) := E^x \left[\int_0^\infty e^{-qt} g(\underline{X}_t) dt \right] := E \left[\int_0^\infty e^{-qt} g(\underline{X}_t) dt \mid X_0 = x \right],$$

respectively (explicit analytic formulas for diffusions with exponentially distributed jumps will be given in Section 3; for the general case, see Boyarchenko and Levendorskiĭ (2002a, b)). In the case of the put on a stock which pays no dividends, the result can be deduced as the limit of the result for a dividend-paying option, when the dividend flow vanishes.

Perpetual call options. This is the case of a non-decreasing $G = U_X^q g$. To ensure that G were finite, we impose the condition

$$|g(x)| \leq C(e^{\sigma_+ x} + e^{\sigma_- x}), \quad \forall x, \quad (2.7)$$

where $\sigma_- \leq 0 < \sigma_+$ satisfy the condition

$$E \left[\int_0^{+\infty} e^{-qt + \sigma X_t} dt \right] < +\infty, \quad \sigma = \sigma_\pm. \quad (2.8)$$

In terms of the Lévy exponent of a Lévy process, an equivalent condition is

$$q - \Psi(\sigma_\pm) > 0. \quad (2.9)$$

The optimal stopping problem for the call option is trivial if the supremum process is trivial. Hence, we presume that the supremum process is non-trivial. In Section 5, we prove the optimality in the class \mathcal{M} , under a weak condition that the stream g is a non-decreasing function. (This condition is not necessary; in fact, the proof

in the paper works in some situations when the stream is non-monotone.) Since a non-decreasing g is measurable, and X satisfy the (ACP)-property, the payoff $G(X_t)$ is a continuous function of the factor X_t . Let $w := U_{\underline{X}}^q g$ be the EPV of the stream g under the infimum process. Then w is non-decreasing; if g is increasing, then w is increasing as well. The crucial condition for our method is that w should have a zero, and then this zero is the early exercise boundary. Assume that w changes sign; sufficient conditions are

$$g(-\infty) < 0, \quad g(+\infty) > 0 \quad (2.10)$$

(one or both limits may be infinite). If w is continuous (a sufficient condition is that g is continuous), then the equation

$$w(x) := U_{\underline{X}}^q g(x) = 0 \quad (2.11)$$

has a root, call it h^* . Since g is monotone and satisfies (2.10), w may be locally constant only in a neighborhood of $-\infty$, where it is negative; outside this neighborhood, w is increasing, and a zero of (2.11) is unique. In Section 5, we will show that the optimal stopping time is $\tau_{h^*}^+$, and the rational call price is

$$V^*(x) = qU_{\underline{X}}^q \mathbf{1}_{[h^*, +\infty)} U_{\underline{X}}^q g(x). \quad (2.12)$$

For explicit realizations of formulas (2.11) and (2.12), see Section 3.

For a general monotone g and process X_t satisfying the (ACP)-property, it is possible that w is discontinuous, and a zero does not exist; generically (w.r.t. to the strike price, say) it does exist. So, if g is discontinuous, we assume that X_t enjoys the *call-regularity* property

(CR) $U_{\underline{X}}^q g$ is continuous for any non-decreasing g satisfying (2.7), and $q > 0$.

The same arguments as in Boyarchenko and Levendorskiĭ (2002a, b) show that if the smooth pasting condition for the put option holds, then the (CR) condition holds as well. In particular, it holds if the gaussian component of the process X_t is non-trivial. For further analysis of the smooth pasting condition, see Alili and Kyprianou (2004).

Perpetual put options. This is the case of a non-increasing $G = U_{\underline{X}}^q g$. To ensure that G were finite, we impose the condition (2.7). We also require that g is a non-increasing function satisfying

$$g(-\infty) > 0, \quad g(+\infty) < 0 \quad (2.13)$$

(one or both limits may be infinite). The optimal stopping problem for a put-like option is trivial if the infimum process is trivial. Hence, we presume that the infimum process is non-trivial. If $w = U_{\underline{X}}^q g$ is continuous (a sufficient condition is that g is continuous), then the equation

$$w(x) = 0, \quad (2.14)$$

has a unique root, call it h_* , and the optimal stopping time is $\tau_{h_*}^-$. In Section 4, we will show that the rational put price is

$$V_*(x) = qU_{\underline{X}}^q \mathbf{1}_{(-\infty, h_*]} U_{\underline{X}}^q g(x). \quad (2.15)$$

If g is discontinuous, we assume that X_t enjoys the *put regularity* property

(PR) $U_{\bar{X}}^q g$ is continuous for any non-increasing g satisfying (2.7), and $q > 0$.

A sufficient condition is: the smooth pasting condition for the call option is valid. In particular, (PR) holds if the gaussian component of the process X_t is non-trivial.

Short review of the literature. Mc Kean (1965) calculated the exercise boundary and price for perpetual call option in the continuous time Gaussian model, Darling et al. (1972) solved the corresponding problem in the discrete time model, for arbitrary random walk, and Merton (1973) solved the problem for the put in the continuous time Gaussian model. Starting from Gerber and Shiu (1994), a series of results for Lévy processes of varying degree of generality were obtained by various authors, using different methods (see the bibliography in Boyarchenko and Levendorskiĭ (2000, 2002a, b) Mordecki (2002a), Alili and Kyprianou (2004)). In Boyarchenko and Levendorskiĭ (2000, 2002a, b), the early exercise boundaries for perpetual American put and call options were calculated for wide classes of Lévy processes used in empirical studies of financial markets, including diffusions with exponentially distributed jumps (used by Duffie et al (2000) in affine term structure models), Hyperbolic processes, Normal Inverse Gaussian processes (see Eberlein et al (1998) and Barndorff-Nielsen (1998), respectively), and the Koponen (1995) model and its extension constructed in Boyarchenko and Levendorskiĭ (2000). Later, the extended Koponen model was used under the name KoBoL family in Boyarchenko and Levendorskiĭ (2002a-c), and under the name CGMY model in Carr et al (2002). The form of the result in Boyarchenko and Levendorskiĭ (2000, 2002a, b) makes sense for any Lévy processes, and later Mordecki (2002a) obtained the result for any Lévy process. However, the proof of one of the key points was omitted; for the missing part, see Mordecki (2004). An independent proof for general Lévy processes was obtained by Alili and Kyprianou (2004).

The standard approach uses the direct link to the first passage problem for a Lévy process or random walk, and it is directly applied for payoffs of the form $(e^{X_t} - K)^+$, $(K - e^{X_t})^+$, and some other payoffs such as $(X_t - K)^+$ and $(K - X_t)^+$ (see e.g., Mordecki (2002a, 2002b)); each type of payoffs should be treated individually. In Boyarchenko and Levendorskiĭ (2000, 2002a, b), the results for standard payoffs were obtained as a byproduct of a general result for more general payoffs. We used the reduction to the free boundary problem and standard analytical argument, which required a certain regularity of a Lévy process. In the present paper, we simplify our method, and use a hybrid of probabilistic and analytical argument to consider more general Lévy processes, and much more general dependence of a payoff on the stochastic factor X_t . For a discrete time analog, see Boyarchenko and Levendorskiĭ (2004).

3 Lévy processes and explicit pricing formulas

General definitions. Recall that a Lévy process is a process with stationary independent increments (for general definitions of the theory of Lévy processes, see, e.g., Bertoin (1996) and Sato (1999)). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density

of jumps, which is called the Lévy density. We denote it by $F(dx)$. If X_t is a Lévy process with finite variation jump component, then the Lévy exponent is given by

$$\Psi(z) = bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{+\infty} (e^{zy} - 1)F(dy), \quad (3.16)$$

where σ^2 and b are the variance and drift of the Gaussian component, and $F(dy)$ satisfies $\int_{\mathbf{R} \setminus \{0\}} \min\{1, |y|\}F(dy) < +\infty$. Equation (3.16) is a special case of the Lévy-Khintchine formula; for the general case, see e.g. Sato (1999). In this paper, we will illustrate our general results for the case of the Brownian motion with embedded jumps used in Duffie et al. (2000). The Lévy density is

$$F(dx) = c^+\lambda^+e^{-\lambda^+x}\mathbf{1}_{(0,+\infty)}(x)dx + c^-(-\lambda^-)e^{-\lambda^-x}\mathbf{1}_{(-\infty,0)}(x)dx, \quad (3.17)$$

where $\lambda^+ > 0 > \lambda^-$, and $c_{\pm} > 0$, and the Lévy exponent is

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+z}{\lambda^+ - z} + \frac{c^-z}{\lambda^- - z}. \quad (3.18)$$

EPV-operators. The Lévy exponent appears when we calculate the action of the infinitesimal generator of X_t , denoted L , on exponential functions: $Le^{zx} = \Psi(z)e^{zx}$. The operator (2.1) calculates the expected present value of a stream. From the fundamental relation between the infinitesimal generator and the resolvent,

$$(q - L)U_X^q = U_X^q(q - L) = I, \quad (3.19)$$

one concludes that U_X^q acts on exponential functions as the multiplication operator by the number $(q - \Psi(z))^{-1}$:

$$U_X^q e^{zx} = (q - \Psi(z))^{-1} e^{zx}. \quad (3.20)$$

One can also derive (3.20) from (2.1) with $g(x) = e^{zx}$, using the definition $E[e^{zX_t}] = e^{t\Psi(z)}$ under the integral sign and evaluating the integral. Similarly, it is straightforward to check that the EPV operators qU_X^q and $qU_{\underline{X}}^q$ also act on an exponential function e^{zx} as multiplication operators by certain numbers, which we denote $\kappa_q^+(z)$ and $\kappa_q^-(z)$, respectively:

$$qU_X^q e^{zx} = \kappa_q^+(z)e^{zx}, \quad qU_{\underline{X}}^q e^{zx} = \kappa_q^-(z)e^{zx}. \quad (3.21)$$

These numbers are

$$\kappa_q^+(z) = qE \left[\int_0^{\infty} e^{-qt} e^{z\bar{X}_t} dt \mid X_0 = x \right], \quad (3.22)$$

$$\kappa_q^-(z) = qE \left[\int_0^{\infty} e^{-qt} e^{zX_t} dt \mid X_0 = x \right]. \quad (3.23)$$

Notice that $\kappa_q^+(z)$ (resp., $\kappa_q^-(z)$) is analytic on the half-plane $\Re z < 0$ (resp., $\Re z > 0$), and continuous up to the boundary. The Wiener-Hopf factorization formula reads (see, e.g., Sato (1999), Section 45)

$$\frac{q}{q - \Psi(z)} = \kappa_q^+(z)\kappa_q^-(z). \quad (3.24)$$

Applying U_X^q , $U_{\underline{X}}^q$ and $U_{\underline{X}}^q$ to an exponential function $g(x) = e^{zx}$ and using (3.20) and (3.21)–(3.24), we obtain

$$U_X^q g(x) = q U_X^q U_{\underline{X}}^q g(x) = q U_{\underline{X}}^q U_X^q g(x). \quad (3.25)$$

By linearity, (3.25) holds for linear combinations of exponents, and integrals of exponents, hence, for wide classes of functions. Equation (3.25) means that the EPV-operator of a Lévy process admits a factorization into a product of the EPV-operators of the supremum and infimum processes.

Diffusions with exponentially distributed jumps. Let X_t be a process with the characteristic exponent (3.18). Then $q - \Psi(z)$ is a rational function, which has 4 real zeroes; two of them are positive, and two negative. We will call them β_j^- and β_j^+ , $j = 1, 2$, respectively. It is easy to show that λ^- separates the negative roots, and λ^+ – the positive ones: $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$. Since $q - \Psi(z)$ is rational, the factors $\kappa^\pm(z)$ can easily be obtained by representing the LHS in (3.24) as the fraction of two polynomials, factorizing these polynomials out, and collecting the factors with positive (respectively, negative) zeroes. For details of these calculations and calculations below, see, e.g., Levendorskii (2004). We obtain

$$\kappa_q^+(z) = \frac{\beta_1^+ \beta_2^+ (\lambda^+ - z)}{(\beta_1^+ - z)(\beta_2^+ - z)\lambda^+} = \sum_{j=1,2} \frac{a_j^+}{\beta_j^+ - z}, \quad (3.26)$$

where

$$a_1^+ = \frac{\beta_1^+ \beta_2^+ (\lambda^+ - \beta_1^+)}{(\beta_2^+ - \beta_1^+)\lambda^+}, \quad a_2^+ = \frac{\beta_1^+ \beta_2^+ (\lambda^+ - \beta_2^+)}{(\beta_1^+ - \beta_2^+)\lambda^+}, \quad (3.27)$$

are positive, and

$$\kappa_q^-(z) = \frac{\beta_1^- \beta_2^- (\lambda^- - z)}{(\beta_1^- - z)(\beta_2^- - z)\lambda^-} = \sum_{j=1,2} \frac{a_j^-}{\beta_j^- - z}, \quad (3.28)$$

where

$$a_1^- = \frac{\beta_1^- \beta_2^- (\lambda^- - \beta_1^-)}{(\beta_2^- - \beta_1^-)\lambda^-}, \quad a_2^- = \frac{\beta_1^- \beta_2^- (\lambda^- - \beta_2^-)}{(\beta_1^- - \beta_2^-)\lambda^-} \quad (3.29)$$

are negative. The operators qU_X^q and $qU_{\underline{X}}^q$ act as follows:

$$qU_X^q u(x) = \sum_{j=1,2} a_j^+ \int_0^\infty e^{-\beta_j^+ y} u(x+y) dy, \quad (3.30)$$

$$qU_{\underline{X}}^q u(x) = \sum_{j=1,2} (-a_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y) dy. \quad (3.31)$$

To see this, it suffices to insert $u(x) = e^{zx}$, and use the definition of the numbers $\kappa_q^+(z)$, $\kappa_q^-(z)$. If u is a piece-wise exponential polynomial, then the calculation of the integrals in (3.30)–(3.31) is straightforward. Using the method of indeterminate coefficients, these calculations can be reduced to systems of linear algebraic equations. For details, see Levendorskii (2004).

Explicit formulas for the perpetual call: the case of exponential jump-diffusions.
 As an example, consider the call option on a stock which pays dividends at rate $\delta(X_t)$, where δ is a non-decreasing function satisfying $\delta(+\infty) > qK > \delta(-\infty)$. We use (3.31), and conclude that h^* is the solution to the equation

$$q^{-1} \sum_{j=1,2} (-a_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} \delta(x+y) dy - K = 0, \quad (3.32)$$

which can easily be found by standard numerical methods. In particular, in Examples 1–5 of Section 2 (and in many others), the integral in (3.32) can be calculated explicitly, and we have to solve an algebraic equation. Consider, for instance, Example 1 with $x_0 = 0$. For $x \leq 0$, the LHS in (3.32) is $-K$, therefore a unique zero is on the positive half-axis. For $x > 0$, we change the variables

$$q^{-1} \sum_{j=1,2} (-a_j^-) e^{\beta_j^- x} \int_{-\infty}^x e^{-\beta_j^- y} \delta(y) dy - K = 0, \quad (3.33)$$

and calculate

$$\int_{-\infty}^x e^{-\beta_j^- y} \delta(y) dy = \int_0^x e^{-\beta_j^- y} (e^y - 1) dy = \frac{e^{(1-\beta_j^-)x} - 1}{1 - \beta_j^-} - \frac{e^{-\beta_j^- x} - 1}{-\beta_j^-}.$$

We see that the first term on the RHS of (3.33) is

$$\begin{aligned} \sum_{j=1,2} \frac{(-a_j^-)}{q} \left\{ \frac{e^x - e^{-\beta_j^- x}}{1 - \beta_j^-} - \frac{1 - e^{-\beta_j^- x}}{-\beta_j^-} \right\} &= q^{-1} \sum_{j=1,2} \left\{ \frac{-a_j^- e^x}{1 - \beta_j^-} - \frac{a_j^-}{\beta_j^-} + \frac{a_j^- e^{-\beta_j^- x}}{(1 - \beta_j^-) \beta_j^-} \right\} \\ &= q^{-1} \left\{ \kappa_q^-(1) e^x - 1 + \sum_{j=1,2} \frac{a_j^- e^{-\beta_j^- x}}{(1 - \beta_j^-) \beta_j^-} \right\}. \end{aligned}$$

Thus, the equation for h^* is

$$\kappa_q^-(1) e^x + \sum_{j=1,2} \frac{a_j^-}{(1 - \beta_j^-) \beta_j^-} e^{-\beta_j^- x} = qK + 1, \quad x > 0,$$

and it can be solved quite easily (numerically). The equation being solved, we represent $w = U_{\underline{X}}^q \delta - K$ in (2.11) in the form

$$w(x) = q^{-1} \sum_{j=1,2} (-a_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} (\delta(x+y) - \delta(h^* + y)) dy,$$

and calculate the rational call price using (3.30):

$$\begin{aligned} V^*(x) &= \sum_{k=1,2} a_k^+ \int_0^{+\infty} e^{-\beta_k^+ y} \mathbf{1}_{[h^*, +\infty)}(x+y) w(x+y) dy \\ &= \sum_{k=1,2} a_k^+ \int_{h^*-x}^{+\infty} e^{-\beta_k^+ y} w(x+y) dy \end{aligned}$$

$$\begin{aligned}
&= \sum_{j,k=1,2} \frac{a_k^+(-a_j^-)}{q} \int_{h^*-x}^{+\infty} \int_{-\infty}^0 e^{-\beta_k^+ y - \beta_j^- z} [\delta(x+y+z) - \delta(h^*+z)] dz dy \\
&= q^{-1} \sum_{j,k=1,2} a_k^+(-a_j^-) e^{\beta_k^+(x-h^*)} \\
&\quad \cdot \int_0^{+\infty} \int_{-\infty}^0 e^{-\beta_j^- z - \beta_k^+ y} [\delta(h^*+y+z) - \delta(h^*+z)] dz dy \\
&= q^{-1} \sum_{k=1,2} a_k^+ \mathcal{D}_k^+ e^{\beta_k^+(x-h^*)},
\end{aligned}$$

where the constants

$$\mathcal{D}_k^+ = \sum_{j=1,2} (-a_j^-) \int_0^{+\infty} \int_0^{+\infty} e^{\beta_j^- z - \beta_k^+ y} [\delta(h^* - z) - \delta(h^* + y - z)] dz dy$$

can be easily calculated in Examples 1-5 of Section 2, and in many other examples.

The calculations for the put options are similar.

4 Proof of the main results for the perpetual put

We use the following general lemma and theorem (Lemma 5.1, Theorem 2.12 and Remark 2.1 in Boyarchenko and Levendorskii (2002b); Lemma 7.1 and Theorem 2.1 in Boyarchenko and Levendorskii (2002b)). We formulate them for a special case of a process on the line, and an optimal stopping region of the form $(-\infty, h]$.

Lemma 4.1 *Let h_* and a function V_* satisfy the following conditions:*

$$(q-L)V_*(x) = 0, \quad x > h_*; \quad (4.34)$$

$$V_*(x) = G(x), \quad x \leq h_*; \quad (4.35)$$

$$V_*(x) \geq \max\{G(x), 0\}, \quad x \in \mathbf{R}; \quad (4.36)$$

$$(q-L)V_*(x) \geq 0, \quad x < h_*; \quad (4.37)$$

$$W_* := (q-L)V_* \quad \text{is universally measurable}; \quad (4.38)$$

$$U_X^q W_* = V_*. \quad (4.39)$$

Then $\tau_{h_*}^-$, is the optimal stopping time for (2.6) in the class \mathcal{M} , and V_* is the rational put price.

Theorem 4.2 *Let X_t satisfy the (ACP)-property, and let G be a continuous function satisfying (2.7). Then for any h_* , $V_0^-(h_*; x) := E^x \left[e^{-q\tau_h^-} G(X_{\tau_h^-}) \right]$ is a unique solution of the problem (4.34)-(4.35) in the class of function satisfying (2.7) and continuous on $(h_*, +\infty)$.*

It remains to show that if h_* is the solution to (2.14), then $V_*(x) := V_0^-(h_*; x)$ satisfies (4.36)-(4.39). A relatively short and simple verification below is based on the following theorem.

Theorem 4.3 *Let X_t satisfy the (ACP)-property, and let g be measurable, locally bounded, non-negative, and satisfy (2.7). Then*

$$E^x \left[e^{-q\tau_h^-} U_X^q g(X_{\tau_h^-}) \right] = qU_{\underline{X}}^q \mathbf{1}_{(-\infty, h]} U_{\underline{X}}^q g(x), \quad \forall x > h. \quad (4.40)$$

Proof This theorem was proved in Boyarchenko and Levendorskii (2002a, 2002b) for wide classes of Lévy processes. In Section 7, we give the proof for a much wider class of Lévy processes satisfying the (ACP)-property.

Set $V^-(h_*; x) := qU_{\underline{X}}^q \mathbf{1}_{(-\infty, h_*]} U_{\underline{X}}^q g(x)$.

Corollary 4.4 *For any x , $V_*(x) = V^-(h_*; x)$.*

Proof By Theorem 4.3, the equality holds for $x > h$, and since V_* satisfies (4.35), it suffices to show that $V^-(h_*; x) = G(x)$ for $x \leq h_*$ as well. We use (3.25) to represent $V^-(h_*; x)$ in the form

$$\begin{aligned} V^-(h_*; x) &= qU_{\underline{X}}^q U_{\underline{X}}^q g(x) - q(U_{\underline{X}}^q \mathbf{1}_{(h_*, +\infty)} U_{\underline{X}}^q g)(x) \\ &= U_{\underline{X}}^q g(x) - v(h_*; x) = G(x) - v(h_*; x), \end{aligned} \quad (4.41)$$

where $v(h_*; x) = q(U_{\underline{X}}^q \mathbf{1}_{(h_*, +\infty)} U_{\underline{X}}^q g)(x)$. For $x \leq h_*$,

$$v(h_*; x) = qE \left[\int_0^{+\infty} e^{-qt} (\mathbf{1}_{(h_*, +\infty)} U_{\underline{X}}^q g)(\underline{X}_t) dt \mid X_0 = x \right] = 0,$$

therefore $V^-(h_*; x) = G(x)$, $x \leq h_*$.

Verification of (4.36). We assumed that g is continuous or X satisfies the (CR)-condition, hence $w := U_{\underline{X}}^q g$ is continuous, and h_* , the solution to the equation (2.14), exists, and it is unique. It follows that $\mathbf{1}_{(-\infty, h]} w$ is continuous if and only if $h = h_*$, therefore this choice of the exercise boundary makes $V^-(h; x)$ maximally regular. This observation extends the smooth pasting principle as the informal principle for the choice of the optimal exercise boundary. Due to the choice of h_* , $\mathbf{1}_{(-\infty, h_*]} w$ is continuous and non-negative. Therefore, $V_*(\cdot) = V^-(h_*; \cdot)$ is continuous, and

$$\begin{aligned} V^-(h_*; x) &= qU_{\underline{X}}^q \mathbf{1}_{(-\infty, h_*]} w(x) \\ &= qE \left[\int_0^{+\infty} e^{-qt} (\mathbf{1}_{(-\infty, h_*]} w)(x + \underline{X}_t) dt \right] \geq 0, \quad \forall x. \end{aligned} \quad (4.42)$$

Since w is non-increasing, $\mathbf{1}_{(h_*, +\infty)} w$ and $v(h_*; \cdot)$ in (4.41) are non-positive, and we conclude from (4.41) that $V^-(h_*; x) \geq G(x)$ for all x . Thus, V_* satisfies (4.36).

Verification of (4.37)-(4.39). Since (4.34) holds, we need to check (4.37) on $(-\infty, h_*)$. Below, we will show that

$$W_* = (q - L)V_* \quad \text{is non-increasing on } (-\infty, h_*), \quad (4.43)$$

and

$$W_*(h_* - 0) \geq 0. \quad (4.44)$$

Conditions (4.37) and (4.38) follow immediately from (4.43) and (4.44), and it remains to check (4.39). Since $\mathbf{1}_{(-\infty, h_*]} U_X^q g$ is continuous, $V_* = V^-(h_*; \cdot)$ given by the LHS in (4.40) is continuous, and since W_* is universally measurable and X satisfies the (ACP)-property, $U_X^q W_*$ is continuous. Therefore it suffices to prove (4.39) in the sense of generalized functions:

$$\int_{-\infty}^{+\infty} U_X^q W_*(x) u(x) dx = \int_{-\infty}^{+\infty} V_*(x) u(x) dx, \quad \forall u \in C_0^\infty(\mathbf{R}). \quad (4.45)$$

By the standard duality argument,

$$\int_{-\infty}^{+\infty} U_X^q (q - L) V_*(x) u(x) dx = \int_{-\infty}^{+\infty} V_*(x) (q - \tilde{L}) U_{\tilde{X}}^q u(x) dx,$$

where \tilde{X} is the dual process and \tilde{L} its generator. Since $u \in C_0^\infty(\mathbf{R})$, $(q - \tilde{L}) U_{\tilde{X}}^q u = u$, hence (4.45) holds, and the proof of (4.39) is finished.

Proof of (4.43) and (4.44). Represent W_* in the form

$$W_* = (q - L)G - (q - L)w^1,$$

where $w^1 = q U_X^q \mathbf{1}_{(h_*, +\infty)} w$. Since $\text{supp } w^1 \subset [h_*, +\infty)$, and the Gaussian part of the infinitesimal generator is a local (differential) operator, we have for $x < h_*$

$$\begin{aligned} W_*(x) &= (q - L)G(x) \\ &\quad + \int_{-\infty}^{+\infty} (w^1(x + y) - w^1(x) - \mathbf{1}_{[-1, 1]}(y)(w^1)'(x)) F(dy) \\ &= g(x) + \int_{h_* - x}^{+\infty} w(x + y) F(dy), \end{aligned}$$

where $F(dy)$ is the Lévy density. Since $g = (q - L)G$ and $w = U_X^q g$ are non-increasing, both terms on the RHS are non-increasing, and (4.43) is proved.

Finally, assume that (4.44) fails. On the strength of (4.43), W_* must be negative on some interval (h, h_*) , where $h < h_*$. By applying U_X^q to $W_* = (q - L)V_*$ and using (4.40) and (3.25), we obtain

$$U_X^q W_* = U_X^q (q - L) q U_X^q \mathbf{1}_{(-\infty, h_*]} w = \mathbf{1}_{(-\infty, h_*]} w.$$

For $x \in (h, h_*)$, we have $U_X^q W_*(x) \leq 0$, but $\mathbf{1}_{(-\infty, h_*]} w(x) > 0$, a contradiction. Thus, (4.44) holds, and the proof of the optimality of the stopping time $\tau_{h_*}^-$ has been completed.

5 Proof of the main results for the perpetual call

We assume that the supremum process is non-trivial, and g is piece-wise continuous, and satisfies (2.7). Since X satisfies the (ACP)-property, $G = U_X^q g$ is continuous, and it satisfies (2.7).

Fix $h \in \mathbf{R}$, and set $\tau = \tau_h^+ = \inf\{t \mid X_t > h\}$. If we change the direction on the real line, a neighborhood of $-\infty$ becomes a neighborhood of $+\infty$, the supremum process becomes the infimum process, and vice versa. Hence, by symmetry, we obtain an analog of Theorem 5.1.

Theorem 5.1 *Let X_t satisfy the (ACP)-property, and let g be measurable, locally bounded, non-negative, and satisfy (2.7). Then*

$$E^x \left[e^{-q\tau_h^+} U_{X^q}^q(X_{\tau_h^+}) \right] = qU_{X^q}^q \mathbf{1}_{[h, +\infty)} U_{\underline{X}^q}^q g(x), \quad \forall x < h. \quad (5.46)$$

The proof of the optimality of the early exercise boundary (2.11) is the mirror reflection of the proof for the put option.

6 Carr's randomization

In the finite time horizon case, exact formulas for the early exercise boundary are not available even in the Brownian motion case. There are several approximate methods – see, e.g., the discussion and references in Levendorskii (2004). An approximate method for a finite horizon problem is based on the time discretization. One of the first methods of this kind is the analytical method of lines suggested by Carr and Faguet (1994); another interpretation of the same pricing procedure was given by Carr (1998). Carr and Faguet (1994) and Carr (1998) considered the Brownian motion case. Boyarchenko and Levendorskii (2002b, Chapter 6) generalized this method for wide classes of Lévy processes, and in Levendorskii (2004), an efficient pricing procedure for the put under exponential jump-diffusion processes was suggested. A different generalization of Carr's randomization method for spectrally negative Lévy processes was used in Avram et al. (2002).

In this section, we generalize results in Boyarchenko and Levendorskii (2002b) and Levendorskii (2004). We consider both calls and puts with payoffs of the form $G = U_{X^r}^r g$, where $r > 0$ is the riskless rate (in this section, it is convenient to use r , and reserve the label q for other purposes).

Call option. We assume that a stream g is non-decreasing, and satisfies (2.10) and (2.7), and either g is continuous or the process X_t satisfies the (CR)-condition discussed at the end of Section 2.

Let T be the maturity date. We divide the period $[0, T]$ into n sub-periods of length $\Delta = T/n$, set $t_j = j\Delta$, and find approximations h_j and v_j to the optimal exercise boundary and option value at time t_j using the backward induction, as follows. At time T , $v_n(x) = G(x)^+$. For $j = n-1, n-2, \dots, 0$, the function v_j solves the time-discretized version of the generalized Black-Scholes equation on $[t_j, t_{j+1}]$, and the early exercise boundary is determined so as to make $v_j(x)$ maximal. For the call option, the problem is

$$\Delta^{-1}(v_{j+1}(x) - v_j(x)) + (L - r)v_j(x) = 0, \quad x < h_j, \quad (6.47)$$

$$v_j(x) = G(x), \quad x \geq h_j. \quad (6.48)$$

In (6.48), we may write $G(x)$ instead of $G(x)^+$, since it is non-optimal to exercise the option at negative levels of the payoff. The sequence of problems (6.47)–(6.48) is the analytical method of lines, and Carr's randomization is a less formal argument, which yields the same sequence. For details, see Chapter 6 in Boyarchenko and Levendorskii (2002b).

The backward induction procedure for the call option is:

I. Define $v_n(x) = G(x)^+$, $q = \Delta^{-1} + r$, and calculate $U_X^r g$ and

$$\tilde{g} = (q - L)G = (q - L)U_X^r g = (\Delta^{-1} + r - L)U_X^r g = g + \Delta^{-1}U_X^r g.$$

II. For $j = n - 1, n - 2, \dots, 0$, calculate

$$z_j = v_{j+1} - \Delta \cdot \tilde{g}; \quad (6.49)$$

$$w_j = U_X^q z_j; \quad (6.50)$$

$$h_j, \quad \text{the zero of } w_j; \quad (6.51)$$

$$v_j = \Delta^{-1}qU_X^q \mathbf{1}_{(-\infty, h_j)} U_X^q z_j + U_X^r g. \quad (6.52)$$

Notice that G and \tilde{g} satisfy the same conditions (2.10), (2.7) and monotonicity/continuity conditions as g , and that in the process of the proof, it will be shown that a zero exists, and it is unique. We rewrite (6.47)–(6.48) in the form

$$(q - L)v_j(x) = \Delta^{-1}v_{j+1}(x), \quad x < h_j, \quad (6.53)$$

$$v_j(x) = U_X^q \tilde{g}(x), \quad x \geq h_j. \quad (6.54)$$

Introduce $V_j = v_j - \Delta^{-1}U_X^q v_{j+1}$. Since $(q - L)U_X^q = I$, V_j solves (6.53)–(6.54) with 0 and $\tilde{g} - \Delta^{-1}v_{j+1}$ in the RHS, respectively. Since X_t satisfies the (ACP)-condition, the continuous solution to (6.53)–(6.54), which is bounded on $(-\infty, h_j]$, is

$$V_j = E^x \left[e^{-q\tau_{h_j}^+} U_X^q (\tilde{g} - \Delta^{-1}v_{j+1})(X(\tau_{h_j}^+)) \right]$$

(This follows from the mirror reflection of Theorem 4.2). Theorem 5.1 gives

$$\begin{aligned} v_j &= qU_X^q \mathbf{1}_{[h_j, +\infty)} U_X^q (\tilde{g} - \Delta^{-1}v_{j+1}) + \Delta^{-1}U_X^q v_{j+1} \\ &= qU_X^q \mathbf{1}_{(-\infty, h_j)} U_X^q (\Delta^{-1}v_{j+1} - \tilde{g}) + qU_X^q U_X^q (\tilde{g} - \Delta^{-1}v_{j+1}) + \Delta^{-1}U_X^q v_{j+1} \\ &= qU_X^q \mathbf{1}_{(-\infty, h_j)} U_X^q (\Delta^{-1}v_{j+1} - \tilde{g}) + U_X^r g, \end{aligned}$$

since $qU_X^q U_X^q = U_X^q$, and $U_X^q \tilde{g} = U_X^q (q - L)U_X^r g = U_X^r g$. Define $z_j = v_{j+1} - \Delta \cdot \tilde{g}$, and write v_j in the form (6.52). For $j = n - 1$, $v_{j+1} = (U_X^r g)^+$. Hence, by using the equalities $\tilde{g} = (I + \Delta^{-1}U_X^r)g$ and $\Delta(q - r) = 1$, we obtain

$$z_{n-1} = (U_X^r g)^+ - \Delta \cdot \tilde{g} = (U_X^r g)^+ - U_X^r g - \Delta \cdot g = (U_X^r g)^- - \Delta \cdot g,$$

where $(U_X^r g)^-(x) = \min\{0, U_X^r g(x)\}$. Due to the conditions imposed on g , z_{n-1} satisfies the following conditions

$$z_j \quad \text{is non-increasing, and continuous if } g \text{ is continuous;} \quad (6.55)$$

$$z_j(-\infty) > 0, \quad z_j(+\infty) < 0; \quad (6.56)$$

$$|z_j(x)| \leq C(e^{\sigma+x} + e^{\sigma-x}). \quad (6.57)$$

Set $w_j = U_X^q z_j$. Clearly, if z_j satisfies (6.55)–(6.57), then w_j does, and moreover, due to our assumption on g and X , w_j is continuous. Hence, the equation

$$w_j(x) = 0 \quad (6.58)$$

has a zero, call it h_j . On the strength of (6.56), w_j may be locally constant only in a neighborhood of $+\infty$, where it is negative. Hence, the zero is unique. So far, we have established these properties of z_j and w_j for $j = n - 1$. This is the basis for our backward induction.

Lemma 6.1 *Assume that at step j , z_j satisfies (6.55)–(6.57). Then h_j is the optimal exercise boundary at time t_j .*

Proof The proof above shows that w_j is a non-increasing continuous function, which changes sign only once; hence, h_j is well-defined. Denote by $v_j(x, h_j)$ the solution to (6.53)–(6.54). We compare $v_j(x, h_j)$ and $v_j(x, h'_j)$. They are given by (6.52) with different boundaries h_j and h'_j but the same z_j . Suppose that $h'_j < h_j$. Then

$$v_j(x, h_j) - v_j(x, h'_j) = \Delta^{-1} q U_X^q \mathbf{1}_{[h'_j, h_j]} U_X^q z_j(x). \quad (6.59)$$

Since $w_j = U_X^q z_j$ is positive on $(-\infty, h_j)$, the RHS in (6.59) is non-negative for all x , and positive for some $x < h_j$. Hence, the choice $h'_j < h_j$ is non-optimal. Similarly, if $h'_j > h_j$, then

$$v_j(x, h'_j) - v_j(x, h_j) = \Delta^{-1} q U_X^q \mathbf{1}_{[h_j, h'_j]} U_X^q z_j(x). \quad (6.60)$$

Since $w_j = U_X^q z_j$ is negative on $(h_j, +\infty)$, the RHS in (6.59) is non-negative for all x , and positive for some $x < h'_j$. Hence, the choice $h'_j < h_j$ is non-optimal.

We conclude that h_{n-1} is the optimal exercise boundary on step $n-1$. Now it remains to prove that if (6.55)–(6.57) hold for $j = n-1, n-2, \dots, l+1$, and h_j is chosen as the solution to (6.58), $j = n-1, n-2, \dots, l+1$, then z_l satisfies (6.55)–(6.57) as well. Using (6.52) and the equality

$$U_X^r g - \Delta \cdot \tilde{g} = U_X^r g - \Delta \cdot (g + \Delta^{-1} U_X^r g) = -\Delta \cdot g,$$

we obtain

$$\begin{aligned} z_l &= v_{l+1} - \Delta \cdot \tilde{g} \\ &= \Delta^{-1} q U_X^q \mathbf{1}_{(-\infty, h_{l+1})} U_X^q z_{l+1} + U_X^r g - \Delta \cdot \tilde{g} \\ &= \Delta^{-1} q U_X^q \mathbf{1}_{(-\infty, h_{l+1})} U_X^q z_{l+1} - \Delta \cdot g. \end{aligned}$$

Due to the choice of h_{l+1} , the function $\mathbf{1}_{(-\infty, h_{l+1})} U_X^q z_{l+1}$ satisfies conditions (6.55)–(6.57), with the only exception: the limit at $+\infty$ is 0. Since $\Delta > 0$, and $-g$ satisfies (6.55)–(6.57), we conclude that z_l satisfies (6.55)–(6.57).

Put option. We assume that a stream g is non-increasing, and satisfies (2.13) and (2.7), and either g is continuous or the process X_t satisfies the (PR)-condition discussed at the end of Section 2. By changing the direction on the real line, we transform the pricing procedure for the call into the pricing procedure for the put; the infimum process becomes the supremum process, and vice versa:

I. Define $v_n(x) = G(x)^+$, $q = \Delta^{-1} + r$, and calculate $U_X^r g$ and

$$\tilde{g} = (q - L)G = (q - L)U_X^r g = (\Delta^{-1} + r - L)U_X^r g = g + \Delta^{-1} U_X^r g.$$

II. For $j = n-1, n-2, \dots, 0$, calculate

$$\begin{aligned} z_j &= v_{j+1} - \Delta \cdot \tilde{g}; \\ w_j &= U_X^q z_j; \\ h_j, &\quad \text{the zero of } w_j; \\ v_j &= \Delta^{-1} q U_X^q \mathbf{1}_{(-\infty, h_j)} U_X^q z_j + U_X^r g. \end{aligned}$$

7 Proof of Theorem 4.3

Fix $h \in \mathbf{R}$, and set $\tau = \tau_h^- = \inf\{t \mid X_t < h\}$. For z in the upper right-plane $\Re z \geq 0$, define functions $u(z; x) = u(h, z; x) = \mathbf{1}_{(-\infty, h]}(x)e^{zx}$, and

$$\begin{aligned} f(z; x) &= f(q, h, z; x) = E^x[\exp(zX_\tau - q\tau)], \\ f^1(z; x) &= f^1(q, h, z; x) = \kappa_q^-(z)^{-1}qU_{\underline{X}}^q u(z; x). \end{aligned}$$

Lemma 7.1 For $x > h$, $f(z; x) = f^1(z; x)$.

Proof For a fixed x , both functions are analytic in the half-plane $\Re z > 0$, and continuous up to the boundary. Hence, it suffices to prove the equality for $z \in [0, \epsilon]$, where ϵ is some positive number. If $z = 0$, then $\kappa_q^-(z)^{-1} = 1$, and the equality

$$E^x[\exp(zX_\tau - q\tau)] = qE^x \left[\int_0^{+\infty} e^{-qt} \mathbf{1}_{(-\infty, h]}(\underline{X}_t) dt \right]$$

holds. Thus, the lemma is proved in the case $z = 0$. The proof for small positive z consists of the following steps: $f(z; \cdot)$ is RCLL on $(h, +\infty)$ (right continuous with left limits); $f^1(z; \cdot)$ is RCLL on $(h, +\infty)$; the Laplace transforms of these two functions are equal.

Function $f(0; \cdot)$ is q -excessive (Proposition 41.5 (ii) and (viii) in Sato (1999)). Since X satisfies the (ACP)-property, a q -excessive function is lower semi-continuous (Theorem 41.5 (4) in Sato (1999)), but $f(0; \cdot)$ is evidently non-increasing; hence, $f(0; \cdot)$ and $f^1(0; \cdot)$ are RCLL on $(h, +\infty)$. Consider sufficiently small $z > 0$ so that $q - \Psi(z) > 0$. Introduce $\Psi_z(w) := \Psi(w + z) - \Psi(z)$. This is the Lévy exponent of X_t under a generalization of the Esscher transform of the measure \mathbf{Q} (the Esscher transform uses a particular value z , which makes e^{-qt+X_t} a local martingale); denote this transform by \mathbf{Q}_z . Let $E^{\mathbf{Q}}$ and $E^{\mathbf{Q}_z}$ be the expectation operators under \mathbf{Q} and \mathbf{Q}_z , respectively. We have

$$f(z; x) := E^{\mathbf{Q}; x}[\exp(-q\tau_h^- + zX_{\tau_h^-})] = e^{zx} E^{\mathbf{Q}_z; 0}[\exp(-(q - \Psi(z))\tau_h^-)].$$

Since X satisfies the (ACP)-property under \mathbf{Q} , it satisfies the (ACP) property under \mathbf{Q}_z . Hence, the last factor on the RHS is RCLL on $(h, +\infty)$, and $f(z; \cdot)$, its product with a continuous function, is RCLL on $(h, +\infty)$ as well.

To prove that $f^1(z; \cdot)$ is RCLL on $(h, +\infty)$, we change the variables $x \mapsto x + h$, so that h becomes 0, and represent $f^1(z; \cdot)$ in the form

$$f^1(z; x) = \kappa_q^-(z)^{-1} \left(qU_{\underline{X}}^q \mathbf{1}_{(-\infty, 0]}(x) + qU_{\underline{X}}^q \mathbf{1}_{(-\infty, 0]}(e^{z\cdot} - 1)(x) \right), \quad (7.61)$$

The function $\mathbf{1}_{(-\infty, 0]}(x)(e^{zx} - 1)$ is continuous, therefore the second term in the brackets on the RHS in (7.61) is continuous. The first term in the brackets equals $f(0; x)$, hence it is RCLL on $(h, +\infty)$. We conclude that the sum is RCLL on $(h, +\infty)$.

Now we consider the Laplace transforms. The fluctuation identity (3.13) in Hilberink–Rogers (2002) states that for any $\mu > z$,

$$\int_0^\infty e^{-\mu x} f(z; x) dx = \frac{1}{\mu - z} \left[1 - \frac{\kappa_q^-(\mu)}{\kappa_q^-(z)} \right],$$

and therefore we need to prove the same equality for f^1 . We have

$$f^1(z; x) = \kappa_q^-(z)^{-1} qU_{\underline{X}}^q(1 - \mathbf{1}_{(0, +\infty)}(x)) e^{zx} = e^{zx} - \kappa_q^-(z)^{-1} qU_{\underline{X}}^q v(z; x),$$

where $v(z; x) = \mathbf{1}_{(0, +\infty)}(x)e^{zx}$. For w in the upper half-plane $\Re w > \Re z$,

$$\hat{v}(z; w) = \int_0^{+\infty} e^{-wx} v(z; x) dx = \int_0^{+\infty} e^{-x(w-z)} dx = (w - z)^{-1}$$

is well-defined; therefore, using the inverse Laplace transform, we obtain

$$f^1(z; x) = e^{zx} - (2\pi i)^{-1} \int_{-i\infty + \sigma}^{+i\infty + \sigma} e^{wx} \kappa_q^-(z)^{-1} \kappa_q^-(w) (w - z)^{-1} dw, \quad (7.62)$$

for any $\sigma > \Re z$. Take $\mu > z$ (and $\sigma \in (z, \mu)$), and calculate the Laplace transform

$$\int_0^{+\infty} e^{-\mu x} f^1(z; x) dx = \frac{1}{\mu - z} - \frac{\kappa_q^-(\mu)}{\kappa_q^-(z)(\mu - z)} = \frac{1}{\mu - z} \left[1 - \frac{\kappa_q^-(\mu)}{\kappa_q^-(z)} \right].$$

Thus, the Laplace transforms of $f(z; \cdot)$ and $f^1(z; \cdot)$ are equal.

Proof of Theorem 4.3. If the infimum process is trivial, both sides of (4.40) are 0, so we assume that it is non-trivial. Consider first $g(x) = e^{zx}$, where $z \in i\mathbf{R}$. Using (3.20) and (3.21), we can rewrite (4.40) as $(q - \Psi(z))^{-1} f(z; x) = q^{-1} \kappa_q^+(z) f^1(z; x)$, where f and f^1 are the functions in Lemma 7.1. This equality holds on the strength of (3.25) and Lemma 7.1, hence (4.40) is proved for oscillating exponents.

Next, consider $g \in C_0^\infty(\mathbf{R})$. We represent g as the Fourier integral, use Lemma 7.1 under the integral sign, and obtain (4.40). Finally, an arbitrary non-negative locally bounded measurable function can be approximated (point-wise) by a non-decreasing sequence of compactly supported locally bounded measurable functions. Using the Dominant Convergence Theorem, we conclude that it suffices to consider a compactly supported g . Such a g can be approximated in L^∞ -topology by indicator functions of open bounded sets, hence, it suffices to consider these indicator functions. An indicator function of an open set can be approximated by a non-decreasing sequence of functions of the class $C_0^\infty(\mathbf{R})$, therefore it remains to prove Theorem 4.3 for these functions, which we have done already.

References

1. Alili, L., Kyprianou, A.E.: Some remarks on first passage of Lévy processes, the American put and pasting principles. Working paper (2004)
2. Avram, F., Chan, T., and Usabel, M.: On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's randomization for American puts. *Stochastic Processes and their Applications* **100**, 75-107 (2002)
3. Barndorff-Nielsen, O.E.: Processes of Normal Inverse Gaussian Type. *Finance and Stochastics* **2**, 41-68 (1998)
4. Bates, D.S.: Empirical option pricing: a retrospection. *Journal of Econometrics* **116**, 387-404 (2003)
5. Bernanke, B.S.: Irreversibility, uncertainty, and cyclical investment. *Quarterly Journal of Economics* **98**, 85-106 (1983)

6. Bertoin, J.: Lévy processes (Cambridge tracts in mathematics, vol. 121). Cambridge: Cambridge University Press 1996.
7. Boyarchenko, S.I.: Irreversible decisions and record setting news principles. *American Economic Review* (2004), *Forthcoming*
8. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Option Pricing for Truncated Lévy Processes. *International Journal of Theoretical and Applied Finance* **3**, 549–552 (2000)
9. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Perpetual American Options under Lévy Processes. *SIAM Journal on Control and Optimization* **40**, 1663–1696 (2002a)
10. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Non-Gaussian Merton-Black-Scholes theory. Singapore: World Scientific 2002b
11. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability* **12**, 1261–1298 (2002c)
12. Boyarchenko, S.I., Levendorskiĭ, S.Z.: Practical guide to real options in discrete time. Submitted to *International Economic Review* (2004)
<http://ssrn.com/abstract=510324>
13. Carr, P.: Randomization and the American put. *Review of Financial Studies* **11**, 597–626 (1998).
14. Carr, P., and Faguet, D.: Fast accurate valuation of American options. Working paper, Cornell University, Ithaca, 1994
15. Carr, P., Geman, H., Madan, D.B., and Yor, M.: The fine structure of asset returns: an empirical investigation. *Journal of Business* **75**, 305–332 (2002)
16. Chernov, M., Gallant, A.R., Ghysels, E., Tauchen, G.: Alternative models for stock price dynamics. *Journal of Econometrics* **16**, 225–257 (2003)
17. Darling, D. A., Liggett, T., and Taylor, H. M.: Optimal stopping for partial sums. *Annals of Mathematical Statistics* **43**, 1363–1368 (1972)
18. Duffie, D., Pan, J., and Singleton, K.: Transform analysis and option pricing for affine jump-diffusions. *Econometrica* **68**, 1343–1376. (2000)
19. Eberlein, E., Keller, U., and Prause, K.: New insights into smile, mispricing and value at risk: The hyperbolic model. *Journ. of Business* **71**, 371–406 (1998)
20. Gerber, H.U., Shiu, E.S.W.: Martingale approach to pricing perpetual American options on two stocks. *Mathematical Finance* **6**, 303–322 (1996)
21. Hilberink, B., and Rogers, L.C.G.: Optimal capital structure and endogenous default. *Finance and Stochastics* **6**, 237–263 (2002)
22. Koponen, I.: Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physics Review E* **52**, 1197–1199 (1995)
23. Mc Kean, H. P., Jr.: Appendix: A free boundary problem for the heat equation arising from a problem in Mathematical Economics. *Industrial Management Review* **6**, 32–39 (1965)
24. Levendorskiĭ, S.: Pricing of American put under Lévy processes. *International Journal of Theoretical and Applied Finance* (2004)
25. Merton, R. C.: The theory of rational option pricing. *Bell Journal of Economics* **4**, 141–183 (1973)
26. Mordecki, E.: Optimal stopping and perpetual options under Lévy processes. *Finance and Stochastics* **6**, 473–493 (2002a)
27. Mordecki, E.: Perpetual Options for Lévy Processes in the Bachelier Model. *Proceedings of the Steklov Mathematical Institute* **237**, 256–264 (2002b)
28. Mordecki, E.: A note on “Optimal stopping and perpetual options under Lévy processes”. Working paper, May 11 (2004)
<http://www.cmat.edu.uy/~mordecki/articles/note>
29. Sato, K.: Lévy processes and infinitely divisible distributions. Cambridge: Cambridge University Press 1999