REAL OPTIONS AND THE UNIVERSAL BAD NEWS PRINCIPLE

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Abstract. A general framework for pricing of real options in continuous time for wide classes of payoff streams that are monotone functions of a Lévy process is provided. Exercise rules are formulated in terms of statistics of record-setting low payoffs and can be viewed as an extension of Bernanke's bad news principle. To illustrate the framework, we solve analytically the following problems: a capital expansion program when the underlying price exhibits mean reverting features; an entry decision with an option to exit, and a new technology adoption. The effects of industry specific and idiosyncratic risks are separated. The third model is driven by two factors: one describes the dynamics of the frontier technology, the other incorporates non-technological uncertainty. The former factor follows a process with upward jumps. The impact of these factors on new technology adoption is analyzed.

1. Introduction

The goal of this paper is threefold. First, we provide a general framework for pricing of real options in continuous time for fairly general payoff functions when the underlying stochastic factor follows a Lévy process. We demonstrate the importance of the assumption that the value of a firm is the expected present value (EPV) of the stream of revenues; as it turns out, this economically meaningful description provides a natural form for a sufficient condition for optimal exercise rules. Second, we study the possibility of the extension of the standard analytically tractable geometric Gaussian models to more general processes, which may account for fat tails and skewness of probability distributions as well as for the stochastic volatility, mean reversion, or switching features of processes observed in empirical studies of commodity prices (for empirical evidence, see, for example, Deaton and Laroque (1992), Yang and Brorsen (1992), Schwartz (1997), and Dixit and Pindyck (1996)). Finally, we confront the implications of Gaussian models of uncertainty vs. jump models of uncertainty, and demonstrate that the word "uncertainty" does not mean too much in itself: depending on the situation and type of uncertainty, the diffusion uncertainty and jump uncertainty can produce opposite effects.

In the now-classical theory of real options, the price of an underlying asset is modeled as a geometric Brownian motion, and optimal exercise strategies are described by simple explicit formulas. The real options approach recognizes the value of waiting when an irreversible decision has to be made in an uncertain environment and recommends higher (respectively, lower) exercise price for a real call (respectively, put) option than the naive net present value (NPV) rule does. The higher the uncertainty, the more does the exercise price of a real option differ from the one prescribed by the NPV rule. However, practitioners are known (see, for example, Lander and Pinches (1998)) to be uncomfortable with too high trigger prices of investment, which the classical real option theory provides. The results obtained in the paper explain that large differences between the NPV and real options exercise prices can be naturally explained as artifacts of modeling the underlying price as a geometric Brownian motion. The differences decrease if we use more general dependence on the stochastic factor, or if we add jumps and decrease the diffusion uncertainty so that the instantaneous variance, the standard proxy for uncertainty, remains fixed. We argue that this result is not counter-intuitive at all, because modeling the stochastic factor as a Gaussian process is associated with the biggest "loss of information", and therefore with the highest uncertainty, as compared to other Lévy processes with the same instantaneous variance. To see why, it suffices to recall that a normal random variable with 0 mean can be obtained as the limit of $\sum_{j=1}^{n} X_j/\sqrt{n}$, where $X_1, X_2, \dots X_n$ are independent and identically distributed random variables with finite variance, as $n \to +\infty$. The variance is independent of n, and clearly, the larger the n, the more is the information lost, and the higher is the uncertainty. In other words, the Gaussian is the law of maximum entropy - or minimum information - such that its variance is fixed (see Bouchaud and Potters (2000) for details).

There are real options models that allow for jumps of a fixed size, with exponentially distributed time of arrival (see Dixit and Pindyck (1996)), and more general models use geometric Lévy processes. However, in many real life situations, commodity price processes exhibit mean reverting (or stochastic volatility) features, and the dynamics of oil prices in the long run is bimodal of a sort: a long period of moderate fluctuations in the region of high prices may be followed by a period of moderate fluctuations in the region of low prices and vice versa, and the transition periods are typically short. The standard device for situations of this sort are switching models. Unfortunately, the standard mean-reverting models are not analytically tractable, and switching models are analytically tractable in the Gaussian case only. In addition, the switching models lead to systems of unknown functions whereas the method of the paper needs only one unknown function.

This paper extends the classical theory to allow for monotone functions of a general diffusion process with embedded jumps or Lévy process to model prices, and, to some extent, bridges the gap between analytically tractable (geometric) Lévy models and less tractable models, such as mean-reverting processes or switching models with non-Gaussian uncertainty. We demonstrate that it is possible to obtain analytical solutions for the optimal exercise price and the value of a real option when the underlying payoff stream is a monotone function of a Lévy process. In application to timing a capital accumulation program, the use of a more general (than exponential) monotone function makes it unnecessary to impose exogenous restrictions on the capital stock available for investment, or on the returns to capital, which typically arise in the geometric Brownian motion model (see Dixit and Pindyck (1996)). The method of the paper works for some non-monotone payoff streams as well.

If the payoff stream follows a geometric diffusion process with embedded jumps, we present analytical solutions not only for simple real options such as timing an investment of a fixed size or scrapping a production unit, but also for embedded real options, for example, investment with an option of scrapping (or with an option to default if the investment is (partially) financed by debt). We demonstrate that diffusion processes with embedded exponentially distributed jumps are as tractable as Gaussian processes in applications to real options. Modeling the underlying payoff stream as a function of a jump-diffusion process and the method of the paper allow one to separate naturally the contributions of different factors, for instance, industry specific risk (modeled as a diffusion) and idiosyncratic risk (modeled as a jump process) in timing an investment of fixed size with an option of disinvestment, and study dependence of the investment and exit thresholds on the idiosyncratic risk; the model remains a one-factor model, and therefore analytically tractable. We analyze the interaction of two sources of uncertainty in model entry and exit problems, and in the problem of entry with an embedded option to exit. For the latter model, we obtain the following theoretical result. Consider a manager who chooses a time when to invest capital I into a firm that will produce a commodity at a given rate. The manager keeps in mind an option to discontinue the production at some time in the future and scrap the inventory for the value C should the demand for the firm's output deteriorate. If there is an industry producing the same commodity, and, on the industry level, the demand shocks are favorable, then the investment is almost reversible, so that the technology specific capital can be sold to another firm(s) in the industry for C which is close to I. We show that as $C \to I$, the gap between the prices that trigger investment and disinvestment does not vanish.

After that, we consider a capital expansion program, and show that an appropriate choice of the dependence of the output price $P_t = P(X_t)$ on the stochastic factor makes it unnecessary to impose additional fairly stringent conditions on the production function, as in Dixit and Pindyck (1996). To be more precise, it suffices to require that in the region of very high prices, the rate of growth of P becomes smaller than that in the geometric model $P_t = e^{X_t}$. It is worth mentioning that the same choice leads to a lower investment threshold than in the geometric Brownian (more generally, Lévy) model, and for a family of Lévy processes with the same first two instantaneous moments, the Brownian motion gives the highest threshold.

We also solve a problem of new technology adoption, where the manager of a firm chooses not only the optimal capital stock, but also the optimal timing of an upgrade to the frontier technology. The model is driven by two factors: one characterizes the dynamics of the technology frontier, and the other incorporates all other shocks in the economy. Powerfully, the method of the paper preserves the tractability even in this two-factor model. We believe that it is natural to model the dynamics of the frontier technology as a process with upward jumps and not as a pure diffusion process. We analyze how the interaction between the two stochastic factors affects the process of new technology adoption, and show that the differences between the impact of the diffusion component and the impact of the jump component on the adoption threshold are not only quantitative but qualitative as well. This corroborates the conclusion made in Boyarchenko and

Levendorskiĭ (1998) about a model with policy uncertainty and non-Gaussian uncertainty in prices: an interaction of two stochastic factors enhances the impact of jumps on the investment threshold.

Our method uses the definition of the value of an option as the EPV of an instantaneous payoff or a stream of payoffs. If the payoff is instantaneous, we view it as the EPV of a stream of payoffs. Such a representation can be obtained in many situations. Of course, everyone knows how to calculate the EPV of a perpetual stream of payoffs that starts to accrue at a deterministic point in time. We show that the rational price of a payoff stream that starts to accrue at a random time (i.e., after the underlying stochastic variable X_t crosses a certain barrier) can also be obtained in terms of the EPV's of some payoff streams. In some cases, the EPV has to be calculated under the assumption that the underlying stochastic process is replaced by the infimum process $\underline{X}_t = \inf_{0\le s\le t} X_s$. In other instances, it becomes necessary to substitute the supremum process $\bar{X}_t = \sup_{0\le s\le t} X_s$ for the underlying process. Similar results hold for the value of a payoff stream that is lost at a random time. Clearly, one can price (real) perpetual American options using the EPV's of the payoff streams mentioned above or their combinations.

The form of the solution for the option value that we obtain suggests the following description of the optimal exercise strategy. If the payoff stream is a decreasing function of the underlying stochastic factor, then it is optimal to exercise a put-like option the first time the EPV of the stream of payoffs calculated for the supremum process instead of the original stochastic process becomes non-positive. Similarly, if the payoff stream is an increasing function of the underlying stochastic factor, then it is optimal to exercise a call-like option the first time the EPV of the stream of payoffs calculated for the infimum process instead of the original stochastic process becomes non-negative. This allows us to formulate a general optimal exercise rule: it is optimal to exercise the right for (respectively, to give up) the stream of stochastic payoffs, g_t, the first time the EPV of the stream $\underline{g}_t = \inf_{0\le s\le t} g_s$ becomes non-negative (respectively, non-positive). We call the last statement a *universal record-setting bad news principle.* This principle naturally generalizes and extends Bernanke's (1983) bad news principle and record-setting news principles spelled out in Boyarchenko (2004). In the latter paper, the principles were stated and proved for the streams of the form $Ae^{X_t} - B$ and $B - Ae^{X_t}$, where X_t is a Lévy process, and A, B are positive constants. Here the result is obtained for arbitrary monotone (and some non-monotone) functions $g_t = g(X_t)$. When the representation of the instantaneous payoff as the EPV of the stream of payoff is unnatural (an example being the adoption of new technology which is expected to remain fixed for a sizable time period due to fixed costs), the results may differ from the principle stated above, in accordance with general record-setting news principles in Boyarchenko (2004).

The method of the paper differs essentially from the one adopted by the classical real options theory. The classical algorithm can be formalized as follows: use Ito's lemma to write down a second order differential equation for the value of an option, employ economic arguments to add appropriate boundary conditions, such as value matching and smooth pasting, and, using the general solution to the differential equation, reformulate the problem as a system of algebraic equations. In elementary situations, a closed form solution can be derived; in other cases, numerical procedures are available.

Unfortunately, the above approach works only for really simple options. For example, for embedded options, there is no general result about the optimal exercise strategy except the heuristic smooth pasting condition. Moreover, it is not evident that the formal solution satisfying the smooth pasting principle exists at all. In addition, if the underlying stochastic process admits jumps, the intuitive justification for the smooth pasting principle as in Dixit and Pindyck (1996) is no longer valid, and there is no reason to believe that the principle always holds. In fact, it may fail (see Boyarchenko and Levendorskiĭ (2002a, 2002b)).

The method presented in the paper never uses the smooth pasting principle, expands the class of exactly solvable problems in the field of real options, and provides the results in a form that admits meaningful economic interpretation. For example, according to Dixit and Pindyck (1996), in order to find the price that triggers an investment of a marginal unit of capital, one has to multiply the investment threshold, prescribed by the Marshallian law, by a certain correction factor that is greater than one. The correction factor is related to a positive root of the "fundamental quadratic" equation, and has no economic meaning per se. Certainly, this representation is useful since the explicit dependence of the factor on the drift and variance of the underlying Brownian motion allows one to study the dependence of the investment threshold on the parameters of the process; yet, the factor is just a result of some computation, and nothing else. In Boyarchenko (2004), it is shown that the investment threshold can be chosen according to the Marshallian law, provided the EPV of the marginal profit is computed for the infimum process instead of the original price process. Here results of a similar form are obtained for a number of new situations.

The rest of the paper is organized as follows. In Section 2, we consider simple real options; revenue streams are modeled as geometric Lévy processes. Section 3 deals with timing a fixed size investment with an option of scrapping. In Section 4, the underlying price is modeled as an arbitrary monotone function of a Lévy process; timing of exit, timing of entry, and a problem of timing an investment of a marginal unit of capital are solved. In Section 5, a model of new technology adoption is examined. Technical details are presented in the Appendix.

2. Simple options

2.1. **Auxiliary results.** We need several basic facts of the theory of Lévy processes. The moment generating function of a Lévy process can be represented in the form $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$, where $E = E_0$; the function Ψ is called the Lévy exponent. The latter naturally appears when we calculate the action of the infinitesimal generator of X_t, denoted L, on exponential functions: $Le^{zx} = \Psi(z)e^{zx}$. In the paper, we restrict ourselves to the class of jump-diffusion processes introduced in Duffie et al. (2000), with the infinitesimal generator of the form

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x) + \int_{-\infty}^{+\infty} (u(x+y) - u(x))\, F(dy). \tag{2.1}$$

Here the density of jumps, $F(dy)$, or Lévy density, is given by

$$F(dy) = c^+\lambda^+ e^{-\lambda^+ y}\mathbf{1}_{(0,\,+\infty)}(y) - c^-\lambda^- e^{-\lambda^- y}\mathbf{1}_{(-\infty,\,0)}(y), \tag{2.2}$$

$\mathbf{1}_{(a,\,b)}(\cdot)$ denotes the indicator function of the interval $(a,\, b)$, $c^\pm > 0$, and $\lambda^- < 0 < \lambda^+$. The method of the paper can be applied to much more general Lévy processes – see Boyarchenko and Levendorskiĭ (2002a, b) and (2004). As we we will show, the choice (2.2) leads to simple formulas, and the calculations are not much more difficult than in the Gaussian case. At the same time, different terms in (2.1) can represent different stochastic factors. For instance, the Gaussian component, represented by the first two terms, can be used to account for the industry specific uncertainty, and the jump part – for the idiosyncratic one. Should we use a one-factor Gaussian model, and study, for example, how the investment threshold changes due to the change of the variance, we could not separate the impact of the industry specific and idiosyncratic shocks. Also, we can independently change the size and intensity of downward and upward jumps by changing the parameters $c^\pm$ and $\lambda^\pm$. The coefficient c^+ (respectively, c^-) characterizes the intensity of upward jumps (respectively, downward jumps). The parameter λ^+ describes the relative intensity of large jumps: the smaller the λ^+, the larger is the probability of large upward jumps as opposed to small ones. Conversely, the smaller the λ^-, the larger is the probability of large downward jumps. If one of the $c^\pm$ is zero, there are no jumps in the corresponding direction.

Computing the action of the infinitesimal generator (2.1) on e^{zx}, we obtain the exponent $\Psi(z)$ corresponding to the Lévy density (2.2) (for the calculation, see the Appendix):

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}. \tag{2.3}$$

Introduce the EPV-operator (a.k.a. the resolvent operator) of a stochastic process X:

$$U^q_X g(x) = E^x\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right],$$

where $q > 0$ is a constant discount rate. This operator calculates the EPV of a stream $g(X_t)$. Applying U^q_X to $g(x) = e^{zx}$ and using the equality $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$, we obtain that U^q_X acts on exponential functions as the multiplication operator by the number $(q - \Psi(z))^{-1}$:

$$U^q_X e^{zx} = \int_0^{+\infty} e^{-(q-\Psi(z))t+zx}dt = (q - \Psi(z))^{-1}e^{zx}. \tag{2.4}$$

To ensure that the expectation were finite, it is necessary and sufficient that the real part of $q - \Psi(z)$ be positive. Since $(q - L)e^{zx} = (q - \Psi(z))e^{zx}$, we conclude that $q - L$ and U^q_X are mutual inverses on a subspace generated by exponential functions from a wide class.

Hence, on a wide class of functions, the fundamental relation between the infinitesimal generator and the resolvent is valid

$$(q-L)U^q_X = U^q_X(q-L) = I. \tag{2.5}$$

We will also need the EPV-operators of the supremum process $\bar{X}_t = \sup_{0\le s\le t} X_s$ and the infimum process $\underline{X}_t = \inf_{0\le s\le t} X_s$. These EPV-operators act as follows:

$$U^q_{\bar{X}}g(x) := E^x\left[\int_0^\infty e^{-qt}g(\bar{X}_t)dt\right] := E\left[\int_0^\infty e^{-qt}g(\bar{X}_t)dt \mid X_0 = x\right]$$

and

$$U^q_{\underline{X}}g(x) := E^x\left[\int_0^\infty e^{-qt}g(\underline{X}_t)dt\right] := E\left[\int_0^\infty e^{-qt}g(\underline{X}_t)dt \mid X_0 = x\right].$$

It is straightforward to check that $qU^q_{\bar{X}}$ and $qU^q_{\underline{X}}$ also act on an exponential function e^{zx} as multiplication operators by numbers, which we denote $\kappa^+_q(z)$ and $\kappa^-_q(z)$, respectively:

$$qU^q_{\bar{X}}e^{zx} = \kappa^+_q(z)e^{zx}, \quad qU^q_{\underline{X}}e^{zx} = \kappa^-_q(z)e^{zx}. \tag{2.6}$$

These numbers are

$$\kappa^+_q(z) = qE\left[\int_0^\infty e^{-qt}e^{z\bar{X}_t}dt\right], \tag{2.7}$$

$$\kappa^-_q(z) = qE\left[\int_0^\infty e^{-qt}e^{z\underline{X}_t}dt\right]. \tag{2.8}$$

The Wiener-Hopf factorization formula reads

$$\frac{q}{q-\Psi(z)} = \kappa^+_q(z)\kappa^-_q(z) \tag{2.9}$$

(see, e.g., Sato (1999), Section 45). Applying U^q_X, $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$ to $g(x) = e^{zx}$ and using (2.4) and (2.6)–(2.9), we obtain

$$U^q_X g(x) = qU^q_{\bar{X}}U^q_{\underline{X}}g(x) = qU^q_{\underline{X}}U^q_{\bar{X}}g(x). \tag{2.10}$$

By linearity, (2.10) holds for linear combinations of exponents and integrals of exponents, hence for wide classes of functions. Equation (2.10) means that the EPV-operator of a Lévy process admits a factorization into a product of the EPV-operators of the supremum and infimum processes.

For the jump-diffusion process defined by (2.3), $\kappa^+_q(z)$ and $\kappa^-_q(z)$ can be easily calculated (see Boyarchenko and Levendorskiǐ (2002a, b) and Boyarchenko (2004)). Let $\beta^-_{1,2}$ and $\beta^+_{1,2}$ be the negative and positive solutions of the characteristic equation

$$q - \Psi(z) = 0. \tag{2.11}$$

(They are separated by $\lambda^-, 0$, and λ^+: $\beta^-_2 < \lambda^- < \beta^-_1 < 0 < \beta^+_1 < \lambda^+ < \beta^+_2$.) Then

$$\kappa^\pm_q(z) = \frac{\beta^\pm_1}{\beta^\pm_1 - z}\cdot\frac{\beta^\pm_2}{\beta^\pm_2 - z}\cdot\frac{\lambda^\pm - z}{\lambda^\pm}. \tag{2.12}$$

Decomposing $\kappa_q^{\pm}(z)$ into a sum of simple fractions:

$$\kappa_q^{\pm}(z) = \frac{a_1^{\pm}}{\beta_1^{\pm} - z} + \frac{a_2^{\pm}}{\beta_2^{\pm} - z}, \tag{2.13}$$

where $a_{1,2}^{+} > 0$ and $a_{1,2}^{-} < 0$, we derive

$$\left(qU_{\bar{X}}^q g\right)(x) = \sum_{j=1,2} a_j^{+} \int_0^{+\infty} e^{-\beta_j^{+} y} g(x+y)dy, \tag{2.14}$$

$$\left(qU_{\underline{X}}^q g\right)(x) = \sum_{j=1,2} (-a_j^{-}) \int_{-\infty}^{0} e^{-\beta_j^{-} y} g(x+y)dy \tag{2.15}$$

(for the proof and explicit expressions for $a_j^{\pm}$, see the Appendix)

2.2. **Timing exit.** Consider a firm that is hit by adverse demand shocks. The firm's profit Ge^{X_t}, where X_t is a Lévy process, is falling on average, so when the log-price level X_t falls below a certain barrier h, it may be optimal to discontinue operations and sell the firm's inventory for the scrap value C. To ensure that the value of the firm were finite, we require that the EPV of the stream Ge^{X_t} be finite. Using (2.4), we conclude that the equivalent condition is

$$q - \Psi(1) > 0, \tag{2.16}$$

or $\beta_1^{+} > 1$. In the Brownian motion case, (2.11) is the famous "fundamental quadratic" equation, and condition (2.16) is formulated in the following form: the positive root of the fundamental quadratic is greater than one (see Dixit and Pindyck (1996)). Let $\tau = \tau_h^{-} = \inf\{t > 0 |\, X_t \le h\}$ be the exit time. At time τ, the firm looses the right to the stream Ge^{X_τ}, but acquires C, which is the EPV of the stream qC. Thus, at time τ, the value of the option to exit is the EPV $U_X^q g(X_\tau)$ of the stream $g(X_t) = qC - Ge^{X_\tau}$, and at time 0, it is $V^{-}(x;h) = E^x\left[e^{-q\tau_h^{-}}(U_X^q g)(X_{\tau_h^{-}})\right]$. The manager's problem is equivalent to choosing the exit barrier h that maximizes $V^{-}(x;h)$. It was proved in Boyarchenko and Levendorskiǐ (2002a, b; 2004) that if g satisfies certain weak regularity conditions[1], then for any $h \in \mathbb{R}$,

$$V^{-}(x;h) = qU_{\underline{X}}^q \mathbf{1}_{(-\infty,\,h]} U_{\bar{X}}^q g(x). \tag{2.17}$$

The very form of (2.17) indicates the optimal threshold: if g is decreasing and the supermum process is non-trivial, then the optimal exit threshold h_* is a solution to

$$\left(U_{\bar{X}}^q g\right)(x) = 0. \tag{2.18}$$

To argue why, notice that for $x > h_*$, $\left(U_{\bar{X}}^q g\right)(x) = C - q^{-1}\kappa_q^{+}(1)e^x < 0$ (see (2.7)), and $\left(U_{\bar{X}}^q g\right)(x) > 0$ for $x < h_*$. Therefore for $h > h_*$,

$$\begin{aligned} V^{-}(x;h_*) - V^{-}(x;h) &= qU_{\underline{X}}^q \left(\mathbf{1}_{((-\infty,\,h_*]} - \mathbf{1}_{(-\infty,\,h]}\right) U_{\bar{X}}^q g(x) \\ &= qU_{\underline{X}}^q \mathbf{1}_{(h_*,\,h]} \left(-U_{\bar{X}}^q g\right)(x) \ge 0, \ \forall x. \end{aligned}$$

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[1] Piecewise continuous g admitting a bound $|g(x)| \le C(1+e^x)$ are admissible

The last inequality holds as a strict inequality for some $x > h_*$, therefore $h > h_*$ cannot be optimal. For $h < h_*$,

$$V^-(x; h_*) - V^-(x; h) = qU^q_{\underline{X}}\mathbf{1}_{(h,\,h_*]}U^q_{\bar{X}}g(x) \ge 0,\ \forall x.$$

The last inequality is strict for some $x > h$, therefore $h < h_*$ is not optimal as well, and h_* is the optimal exit barrier.

Equation (2.18) is equivalent to $C - q^{-1}G\kappa^+_q(1)e^x = 0$, whence

$$e^{h_*} = \frac{qC}{G\kappa^+_q(1)}. \tag{2.19}$$

Compute the option value of exit:

$$V^-(x) \;=\; V^-(x; h_*) = (qU^q_{\underline{X}}\mathbf{1}_{(-\infty,\,h_*]}U^q_{\bar{X}}g)(x) = qU^q_{\underline{X}}\mathbf{1}_{(-\infty,\,h_*]}(x)\left(C - \frac{G\kappa^+_q(1)e^x}{q}\right)(x).$$

With the help of (2.19), we rewrite the last equation as

$$V^-(x) = CqU^q_{\underline{X}}\mathbf{1}_{(-\infty,\,h_*]}(x)\left(1 - e^{(x-h_*)}\right).$$

Using (2.15), we calculate the action of $qU^q_{\underline{X}}$:

$$\begin{aligned} V^-(x) &= -C\sum_{j=1,2} a^-_j\left[\int_{-\infty}^{h_*-x} e^{-\beta^-_j y}dy - e^{x-h_*}\int_{-\infty}^{h_*-x} e^{(1-\beta^-_j)y}dy\right] \\ &= C\sum_{j=1,2} a^-_j e^{\beta^-_j(x-h_*)}\left(\frac{1}{\beta^-_j} + \frac{1}{1-\beta^-_j}\right). \end{aligned}$$

Therefore

$$V^-(x) = C\sum_{j=1,2}\frac{a^-_j e^{\beta^-_j(x-h_*)}}{\beta^-_j(1-\beta^-_j)} \quad \text{for } x > h_*.$$

Now, the value of the firm whose manager contemplates scrapping the inventory is the EPV of the stream of profits plus the value of the option to exit, i.e.,

$$V(x) = E^x\left[\int_0^\infty e^{-qt}Ge^{X_t}dt\right] + V^-(x),$$

hence for $x > h_*$,

$$V(x) = \frac{Ge^x}{q - \Psi(1)} + C\sum_{j=1,2}\frac{a^-_j e^{\beta^-_j(x-h_*)}}{\beta^-_j(1-\beta^-_j)}.$$

For $x \le h_*$, $V(x) = C$, of course. Using the Wiener-Hopf factorization formula (2.9) and the optimal exit threshold (2.19), we may write

$$\frac{Ge^x}{q - \Psi(1)} = C\kappa^-_q(1)e^{x-h_*},$$

therefore for $x > h_*$,

$$V(x) = C\left[\kappa_q^-(1)e^{x-h_*} + \sum_{j=1,2}\frac{a_j^- e^{\beta_j^-(x-h_*)}}{\beta_j^-(1-\beta_j^-)}\right]. \tag{2.20}$$

It is easy to check that $V(x)$ satisfies the value matching condition $V(h_*-0) = V(h_*+0)$:

$$1 = \kappa_q^-(1) + \sum_{j=1,2}\frac{a_j^-}{\beta_j^-(1-\beta_j^-)} = \sum_{j=1,2}\frac{a_j^-}{\beta_j^- - 1}\left(1 - \frac{1}{\beta_j^-}\right) = \sum_{j=1,2}\frac{a_j^-}{\beta_j^-} = 1; \tag{2.21}$$

and the smooth pasting condition $V'(h_* - 0) = V'(h_* + 0)$:

$$0 = \kappa_q^-(1) - \sum_{j=1,2}\frac{a_j^-}{\beta_j^- - 1}. \tag{2.22}$$

Finally, we notice that $V'(x) > 0$ for $x > h_*$.

2.3. **Timing a fixed size investment.** Consider an investor who chooses a time τ to invest capital I into a technology that produces a commodity at rate G ever after. The output is sold on the spot at the market price e^{X_t}, where X_t follows a Lévy process. We view I as the present value of a stream qI of future expenditures. Let $g(x) = Ge^x - qI$. Fix $h \in \mathbb{R}$ and set $\tau_h^+ = \inf\{t \geq 0 \,|\, X_t \geq h\}$. The investor's problem is equivalent to choosing the investment threshold h so as to maximize the option value of investment:

$$\max_h V^+(x;h) = \max_h E^x\left[e^{-q\tau_h^+}(U_{\underline{X}}^q g)(X_{\tau_h^+})\right].$$

If g satisfies certain weak regularity conditions (as in the previous subsection), then for any h,

$$V^+(x;h) = qU_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}U_{\underline{X}}^q g(x) \tag{2.23}$$

(see Boyarchenko and Levendorskiĭ (2002a, b; 2004) for the proof). The optimal investment threshold h^* is a solution to

$$U_{\underline{X}}^q g(x) = 0. \tag{2.24}$$

Indeed, for $x < h^*$, $U_{\underline{X}}^q g(x) = q^{-1}\kappa_q^-(1)Ge^x - C < 0$ (see (2.8)), and $U_{\underline{X}}^q g(x) > 0$ for $x > h^*$. Therefore, for $h > h^*$,

$$V^+(x;h^*) - V^+(x;h) = qU_{\bar{X}}^q \mathbf{1}_{[h^*,h)}U_{\underline{X}}^q g(x) \geq 0, \quad \forall\, x.$$

The last inequality is strict for some $x < h$. For $h < h^*$,

$$V^+(x;h^*) - V^+(x;h) = qU_{\bar{X}}^q \mathbf{1}_{[h,h^*)}U_{\underline{X}}^q g(x) \geq 0, \quad \forall\, x.$$

For some $x < h^*$, the last inequality holds as a strict one. Hence, h^* is the optimal investment threshold. Equation (2.24) is equivalent to $q^{-1}G\kappa_q^-(1)e^x - I = 0$, therefore the trigger price of investment is

$$e^{h^*} = \frac{qI}{G\kappa_q^-(1)}. \tag{2.25}$$

It remains to compute the option value of investment when the investment threshold is chosen optimally:

$$V^+(x) \;=\; V^+(x;h^*) = qU^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}U^q_{\underline{X}}g(x) = qU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(x)\left(\frac{G\kappa^-_q(1)}{q}e^x - I\right).$$

Using (2.25), we write the option value as

$$V^+(x) = IqU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(x)(e^{x-h^*}-1).$$

Next, we use (2.14) to obtain, for $x < h^*$

$$\begin{aligned} V^+(x) &= I\sum_{j=1,2} a^+_j\left[e^{x-h^*}\int_{h^*-x}^{+\infty} e^{(1-\beta^+_j)y}dy - \int_{h^*-x}^{+\infty} e^{-\beta^+_j y}dy\right] \\ &= I\sum_{j=1,2} a^+_j e^{\beta^+_j(x-h^*)}\left(\frac{1}{\beta^+_j-1} - \frac{1}{\beta^+_j}\right) = I\sum_{j=1,2}\frac{a^+_j e^{\beta^+_j(x-h^*)}}{(\beta^+_j-1)\beta^+_j}. \end{aligned}$$

It is straightforward to check that the value matching and smooth pasting conditions hold in this case.

2.4. **Influence of idiosyncratic uncertainty on exit and entry thresholds.** Consider a family of firms which face the uncertainty represented by processes with the same first two instantaneous moments, $m_1 = \Psi'(0)$ and $m_2 = \Psi''(0)$. Each process has a diffusion component that represents the industry specific uncertainty, and jump component, which models the idiosyncratic risk. If a standard geometric Brownian motion is fitted to each of these price processes, the same Brownian motion is obtained, and entry and exit thresholds will be the same for each firm. However, as we verified in a number of numerical examples, for the firms that face the downward idiosyncratic risk, the entry threshold is lower, and the exit threshold is higher. When the upward jumps prevail, the exit threshold becomes lower. The entry threshold becomes a bit lower as well, which can be explained as follows. We keep the first two moments fixed, therefore if the positive jumps component increases, the drift of the Gaussian component must decrease, and this negative effect dominates the entry decision. The difference between the entry and exit thresholds increases when more positive jumps are added. Notice that on average, the effect of positive jumps on the thresholds is smaller than that of negative ones: bad firm-specific news are more important for investment decisions than good ones. The entry threshold is more sensitive to negative jumps, and the exit one – to positive jumps. Both thresholds can change by more than 10 percent even if a moderate jump component is added (for a significant jump component, they can change by dozens percent); and if one averages over many firms, one observes the thresholds which are lower (entry) and higher (exit) than in the standard Brownian motion model. Notice that practitioners are known to be uncomfortable with too high investment thresholds, which the real option approach recommends, and the use of jump-diffusion processes in investment models can alleviate these concerns.

To illustrate these effects, in Fig. 1, we plot the entry and exit thresholds $H^* = e^{h^*}$ and $H_* = e^{h_*}$ in the jump-diffusion model with fixed m_1, m_2, and either positive jumps only:

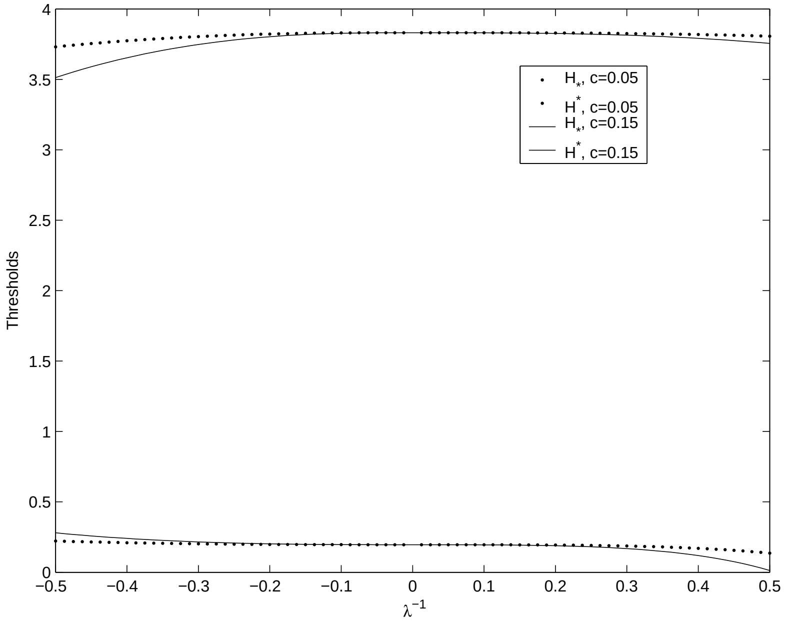

FIGURE 1. Entry and exit thresholds. Parameters: $C = 7, I = 10.5, G = 0.56, q = 0.08, m_1 = -0.6, m_2 = 0.2$.

$\lambda^+ = \lambda$ varies from 2 to $+\infty$, or negative jumps only: $\lambda^- = \lambda$ varies from -2 to $-\infty$. In the limit $\lambda \to \pm\infty$ (the point $1/\lambda = 0$), the Gaussian model is obtained. Parameters $c_\pm = c$ are the same for the cases of upward and downward jumps. When m_1, m_2, c and λ are fixed, the parameters σ and b are uniquely defined.

3. EMBEDDED OPTIONS

Consider an investor who chooses the time τ when to invest capital I into a technology producing output at rate G as in Subsection 2.3. But now the investor has in mind the option of scrapping the inventory for the value C should the things go badly for the firm. It is reasonable to assume that $I > C$, because second hand inventories are less valuable than new inventories. From Subsection 2.2, we know that the optimal exit threshold h_* is given by (2.19), and the value of the firm for $x > h_*$ is given by (2.20). At the time of investment, the investor receives an instantaneous payoff $V(x) - I$, which is the EPV of the stream $g(x) = (q - L)(V(x) - I)$. Here we used the fundamental relation (2.5) between the infinitesimal generator L and the EPV-operator U^q_X.

In Subsection 2.3, the option value of investment, when e^h is chosen as the trigger price, was derived as

$$V^{\mathrm{inv}}(x; h) = qU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}g(x) = qU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(x)U^q_{\underline{X}}(q - L)(V(x) - I).$$

We use the Wiener-Hopf factorization formula (2.9) to proceed:

$$V^{\mathrm{inv}}(x; h) = qU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(x)\left(qU^q_{\bar{X}}\right)^{-1}(V(x) - I). \tag{3.1}$$

Clearly, $V(h) \geq I > C$, otherwise the investment is not optimal, and therefore $h > h_*$. Similarly to (2.24), the investment threshold h is the solution to the equation

$$\left(qU^q_{\bar{X}}\right)^{-1}(V(\cdot) - I)(x) = 0. \tag{3.2}$$

In contrast to simple options in Subsection 2.3, the LHS is not a linear combination of exponents on the whole axis. However, the following argument shows that when calculating the LHS in (3.2) on $[h_*, +\infty)$, we may assume that $V(x)$ is given by the RHS of (2.20) not on $[h_*, +\infty)$ only but on the whole axis, and therefore, the LHS in (3.2) can be calculated as easily as in the case of simple options. Indeed, since

$$(U^q_{\bar{X}}g)(x) = E\left[\int_0^{+\infty} e^{-qt} g(x + \bar{X}_t)dt \big| \, X_0 = x\right],$$

it is obvious that $(U^q_{\bar{X}}g)(x)$ is independent of the values $g(y)$ for $y < x$. Under weak regularity assumptions, it was proved in Boyarchenko and Levendorskiǐ (2002a, 2002b) that $\left(qU^q_{\bar{X}}\right)^{-1} g(x)$ also enjoys the above property. In the case of jump-diffusion processes which we consider here, one can obtain explicit formulas for the actions of $\left(qU^q_{\bar{X}}\right)^{-1}$ and $(qU^q_{\underline{X}})^{-1}$, which are similar to (2.14) and (2.15), and the above property becomes evident for jump-diffusion processes.

Using (2.24), we obtain the following equation for the optimal investment threshold:

$$w(x) = C\left[\frac{\kappa^-_q(1)}{\kappa^+_q(1)} e^{x-h_*} + \sum_{j=1,2} \frac{a^-_j \kappa^+_q(\beta^-_j)^{-1}}{\beta^-_j(1-\beta^-_j)} e^{\beta^-_j(x-h_*)}\right] - I = 0. \tag{3.3}$$

We claim that equation (3.3) has exactly one solution on $[h_*, +\infty)$. Evidently, $w(x)$ is positive for large x. Also, $w(x)$ is convex in x, because its second derivative w.r.t. x is

$$w''(x) = C\left[\kappa^-_q(1)\kappa^+_q(1)^{-1} e^{x-h_*} + \sum_{j=1,2} \frac{a^-_j \kappa^+_q(\beta^-_j)^{-1}\beta^-_j}{1-\beta^-_j} e^{\beta^-_j(x-h_*)}\right] > 0.$$

Hence it suffices to check that $w(h_*) < 0$; for the verification, see the Appendix. From the calculation in the Appendix, in the presence of positive jumps, $w(h_*) < 0$ even if $I = C$, i.e., the investment is completely reversible. If positive jumps are absent, then $w(h_*) = 0$ if $I = C$, however,

$$\begin{aligned} w'(h_*) &= C\left[\kappa^-_q(1)\kappa^+_q(1)^{-1} + \sum_{j=1,2} \frac{a^-_j \kappa^+_q(\beta^-_j)^{-1}}{1-\beta^-_j}\right] \\ &= C\sum_{j=1,2} \frac{a^-_j}{\beta^-_j - 1}\left(\kappa^+_q(1)^{-1} - \kappa^+_q(\beta^-_j)^{-1}\right) < 0, \end{aligned}$$

because $\kappa^+_q(z)$ is increasing in z. Thus, if h^{**} is the solution to (3.3), then the difference $e^{h^{**}} - e^{h_*}$ between the trigger price of investment and the trigger price of scrapping is bounded away from zero even as the investment becomes almost completely reversible: an arbitrary tiny margin between I and C leads to the margin which does not vanish even in the limit.

It remains to show that h^{**}, the solution to (3.3), is the optimal investment barrier. Using (3.1), we write

$$V^{\mathrm{inv}}(x; h) = q(U^q_{\bar{X}} \mathbf{1}_{[h,+\infty)} w)(x).$$

Since $w(h_*) < 0$, w is convex, and $w(x) > 0$ for sufficiently large x, we conclude that $w(x) > 0$ for $x > h^{**}$, and $w(x) < 0$ for $x \in [h_*, h^{**})$. Let $h \in [h_*, h^{**})$, then

$$V^{\mathrm{inv}}(x; h^{**}) - V^{\mathrm{inv}}(x; h) = q(U^q_{\bar{X}}(-\mathbf{1}_{[h,h^{**})} w))(x) \geq 0 \ \forall \ x.$$

The last inequality holds as a strict inequality for some $x < h^{**}$. If $h > h^{**}$, then

$$V^{\mathrm{inv}}(x; h^{**}) - V^{\mathrm{inv}}(x; h) = q(U^q_{\bar{X}} \mathbf{1}_{[h^{**},h)} w)(x) \geq 0 \ \forall \ x,$$

and strict inequality holds for some $x < h$. Therefore h^{**} is the optimal investment threshold. Notice that $h^* > h^{**}$, which is a natural result, because the investor must be more cautious if the investment is absolutely irreversible. For the proof, it suffices to show that $w(h^*) > 0$ (for the verification, see the Appendix).

In Fig. 2, we plot the graphs of the entry threshold with an option to exit, $H^{**} = e^{h^{**}}$, the exit threshold, $H_* = e^{h_*}$, and the entry threshold without option to exit, $H^* = e^{h^*}$, in the model with jumps in one direction (either positive or negative, cf. Fig. 1), for $C/I = 0.4, 0.7, 0.999$. It is clearly seen that H^{**} grows and approaches H^* as more positive jumps are added. The last effect is clearly seen from the formula for $w(h^*)$ in the Appendix, and the observation that $h^* - h_*$ increases as more positive jumps are added. The economic intuition is as follows. If the idiosyncratic uncertainty involves sizable probability of large upward jumps, then at time of investment, the exit boundary is far away, and the value of the embedded option to exit is small; hence, the investment threshold does not differ much from the one in the model of irreversible investment. Finally, even when the scrap value is almost equal to the investment cost, the gap between H^{**} and H_* is quite sizable.

4. Beyond the geometric Lévy processes

As it was already mentioned in the Introduction, the workhorse for modeling the price of the underlying asset is the geometric Brownian motion. More general models use geometric Lévy processes. However, in many real life situations, commodity price processes exhibit mean reverting features, and the dynamics of oil prices in the long run is bimodal of a sort: a long period of moderate fluctuations in the region of high prices may be followed by a period of moderate fluctuations in the region of low prices, and the transition periods are typically short. However, the standard mean-reverting models are not analytically tractable. In this Section, we consider timing an entry, timing an exit, and problem of optimal capital expansion program to demonstrate that monotone functions of the underlying stochastic factor can be almost as tractable as exponents (geometric Lévy processes) or linear functions (Lévy processes). At the same time, such functions may account for more realistic features of price behavior, for example, mean reverting features and bimodal behavior described above. Simple examples are $P_t = \bar{P}(1 + \arctan(X_t)/\pi)$ and $P_t = \bar{P}e^{\epsilon x}(1 + \arctan(X_t)/\pi)$, where $\epsilon > 0$ is small. In the first case, the price may fluctuate for long time either in a neighborhood of $\bar{P}/2$ or $3\bar{P}/2$, and the fluctuations

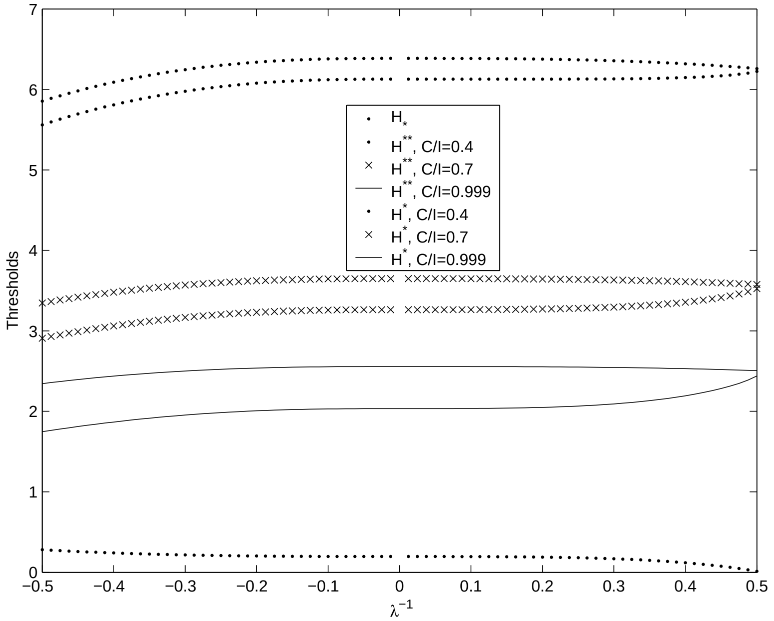

FIGURE 2. Investment with the option to exit: entry threshold with an option to exit, H^{**}, and entry and exit thresholds H^* and H_*. Parameters: $C = 7, G = 0.56, q = 0.08, m_1 = -0.6, m_2 = 0.2, c = 0.20$.

around $\bar{P}$ are more prominent. In addition, when the price is low, the sizes of upward jumps are larger than those of the downward ones, on average, and when the price is high, the opposite relation between downward and upward jumps is observed. Thus, the price behavior exhibits mean-reverting features. The introduction of the factor $e^{\epsilon X_t}$ allows for unbounded but controllable growth of the price; in the intermediate region in the neighborhood of $\bar{P}$, the dynamics is, approximately, as in the Lévy model or geometric Lévy model. In the model of gradual capital expansion, modeling the price as an appropriate monotone function of the stochastic factor eliminates the necessity of exogenous constraint on the overall capital stock in the economy and on the returns to capital in the production function.

4.1. **Timing exit and entry.** Assume that the price $P(x)$ is a non-decreasing function satisfying the estimate

$$P(x) \le c_1 e^{\gamma x}, \tag{4.1}$$

where $c_1 > 0$, $\gamma \ge 0$, and

$$q - \Psi(\gamma) > 0. \tag{4.2}$$

Condition (4.2) generalizes (2.16), and ensures that the value of the firm is finite. Formulas (2.17) (with $g(x) = qC - GP(x)$) and (2.23) (with $g(x) = GP(x) - qC$) for the option value of exit and entry hold, and the optimal disinvestment and investment boundaries are determined from the same equations (2.18) and (2.24), respectively. As in the geometric

Lévy case, the LHS's in these equations are monotone functions but for a general P, it is necessary to require explicitly that $U^q_{\underline{X}}g(x)$ and $U^q_{\bar{X}}g(x)$, respectively, change sign. Under these conditions, the proofs in Subsection 2.2 and Subsection 2.3 can be repeated word by word. Finally, notice that these proofs show that h_* and h^* are optimal exercise boundaries under the assumption that the optimal inaction region is a semi-infinite interval. It can be proved (see Boyarchenko and Levendorskiĭ (2004)) that if g is monotone (this condition is not necessary) then this is really the case.

4.2. **Timing an investment of a marginal unit of capital.** Consider a firm whose production function depends only on capital: $G = G(K)$. We assume that G is differentiable, increasing, concave, and satisfies the Inada conditions. The firm's output is sold on the spot at the market price P_t. A similar situation was considered in Dixit and Pindyck (1996) for the geometric Brownian motion model and extended by Boyarchenko (2004) for geometric Lévy processes. In the present paper, we assume that $P_t = P(X_t)$ is an increasing function of the stochastic factor X_t that follows a Lévy process. The revenue flow is $R(K_t, X_t) = P(X_t)G(K_t)$. In particular, such a payoff may account for the case when the firm chooses both capital and costlessly adjustable labor as in Abel and Eberly (1999) for the geometric Brownian motion model. Should the firm decide to invest a unit of capital, it suffers the installation cost C. The firm's objective is to choose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t\geq 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K,x) = \sup_{\mathcal{K}} E^x\left[\int_0^{+\infty} e^{-qt}(P(X_t)G(K_t) - qCK_t)dt\right]. \tag{4.3}$$

For the time being, to ensure that firm's value (4.3) were bounded, we impose a resource constraint: there exists $\bar{K} < \infty$ such that $K_t \leq \bar{K}$, $\forall t$. Also, assume that

$$E\left[\int_0^{+\infty} e^{-qt}P(X_t)dt\right] < \infty. \tag{4.4}$$

In the case of jump diffusion processes, (4.4) is equivalent to (4.1) with $\gamma < \beta_1^+$. Later, we will show that if γ is sufficiently small, then the resource constraint is redundant: the expected rate of growth of the optimal capital is not very large, and the value of the firm is finite even if the firm has unlimited access to capital. Notice that if P is bounded ($\gamma = 0$), then there exists $\bar{K}$ such that the firm would never want to choose $K_t > \bar{K}$.

It is well-known (see, for example, Dixit and Pindyck (1996)) that in order to determine the optimal capital expansion program, it is only necessary to decide when to invest at any given stock of capital. Equivalently, one needs to find the investment threshold $h(K)$, which is the boundary between two regions in the state variable space (K, x): the action and the inaction ones. To derive the equation for the investment boundary, suppose first that every new investment can be made in chunks of capital, ΔK, only[2]. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the revenue gain due to this investment can be represented in the form of the EPV of the stream $g(X_t) =$

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[2] The authors are indebted for this simplifying trick to Mike Harrison; the initial proof (for geometric Lévy case) in Boyarchenko (2004) was more involved.

$(G(K+\Delta K)-G(K))P(X_t)-qC\Delta K$. From Subsection 2.3, we know that it is optimal to invest capital $C\Delta K$ the first time the price of the firm's output crosses the investment barrier $h(K;\Delta K)$ that satisfies (2.24). For g defined above, (2.24) can be written as

$$U^q_{\underline{X}}\left[\left(G(K+\Delta K)-G(K)\right)P(\cdot)-qC\Delta K\right](x)=0,$$

or

$$\left(G(K+\Delta K)-G(K)\right)U^q_{\underline{X}}P(x)=C\Delta K. \tag{4.5}$$

Dividing (4.5) by ΔK and passing to the limit as $\Delta K\to 0$, we obtain the following equation for the optimal investment threshold $h^*=h^*(K)$:

$$G'(K)U^q_{\underline{X}}P(h^*)=C, \tag{4.6}$$

or

$$G'(K)E\left[\int_0^{+\infty}e^{-qt}P(h^*+\underline{X}_t)dt\big|X_0=0\right]=C. \tag{4.7}$$

The last equation says that it is optimal to invest into a marginal unit of capital the first time the EPV of the marginal profit, calculated under the assumption that the underlying stochastic process $\{X_t\}$ is replaced by the infimum process $\{\underline{X}_t\}$, becomes non-negative[3].

If X is a jump-diffusion process defined by (2.3), then we proceed using (2.15):

$$G'(K)\sum_{j=1,2}(-a_j^-)\int_{-\infty}^0 e^{-\beta_j^- y}P(h^*+y)dy=qC. \tag{4.8}$$

Let $h=h(K;\Delta K)$ be a solution to (4.5). Then at the price level $P(x)$, the option value associated with the chunk of capital ΔK is

$$qU^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}\Big[\left(G(K+\Delta K)-G(K)\right)U^q_{\underline{X}}P(\cdot)-C\Delta K\Big](x).$$

As $\Delta K\to 0$, we have $h(K;\Delta K)\to h^*(K)$. Therefore, dividing the above option value by ΔK and passing to the limit as $\Delta K\to 0$, we obtain the formula for the marginal option value of capital:

$$V^{\mathrm{opt}}_K(K,x)=qU^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}(x)\Big(G'(K)U^q_{\underline{X}}P(x)-C\Big).$$

Substituting C from (4.6) into the above equation, we arrive at

$$V^{\mathrm{opt}}_K(K,x)\;=\;G'(K)qU^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}(x)U^q_{\underline{X}}\left(P(x)-P(h^*)\right)=G'(K)q(U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}w)(x),$$

where

$$w(x)=E\left[\int_0^{+\infty}e^{-qt}\left(P(x+\underline{X}_t)-P(h^*+\underline{X}_t)\right)dt\big|\,X_0=0\right].$$

For the jump-diffusion process,

$$w(x)=q^{-1}\sum_{j=1,2}(-a_j^-)\int_{-\infty}^0 e^{-\beta_j^- y}\left(P(x+y)-P(h^*+y)\right)dy,$$

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[3] For the rigorous justification of the limiting argument see Boyarchenko (2004).

and

$$\begin{aligned} V^{\mathrm{opt}}_K(K,x) &= G'(K)\sum_{j=1,2} a^+_j \int_{h^*-x}^{+\infty} e^{-\beta^+_j y} w(x+y)dy \\ &= G'(K)\sum_{j=1,2} a^+_j e^{\beta^+_j(x-h^*)} \int_0^{+\infty} e^{-\beta^+_j y} w(h^*+y)dy. \end{aligned} \tag{4.9}$$

Integrating (4.9) w.r.t. K, we find the option value

$$V^{\mathrm{opt}}(K,x) = \int_{\bar{K}}^{K} V^{\mathrm{opt}}_K(K',x)dK'. \tag{4.10}$$

If we want to remove the resource constraint $K \le \bar{K}$, we need to prove that the limit of the integral (4.10) exists as $\bar{K} \to +\infty$, and then the value of the firm is given by (4.10) with $\bar{K} = +\infty$. In the Appendix, we show that if (4.1) holds, then a sufficient condition for the convergence is

$$\int_1^{+\infty} G'(K')^{\beta^+_1/\gamma} dK' < +\infty. \tag{4.11}$$

In the geometric Lévy case, when $P_t = e^{X_t}$ and $\gamma = 1$, this condition is necessary. In particular, if the production function is a Cobb-Douglas one, i.e., $G(K) = dK^\theta$ ($d > 0$, $\theta \in (0,1)$), then for the convergence of the integral in the case of the jump-diffusion process, we must have $\theta < 1 - 1/\beta^+_1$. In other words, θ must be sufficiently less than one, which means that the returns to capital must decrease sufficiently fast. As Dixit and Pindyck (1996) show in the geometric Brownian motion case, for typical parameters of a process, this condition requires for θ to be too small. If the jump component is not very strong, then the same conclusion holds.

Now, suppose that the price process is fitted well by a geometric jump-diffusion process, and $\theta \ge 1 - 1/\beta^+_1$. To ensure that the value of the firm be finite, we may assume that above a certain high level P_c of the price, the rate of growth of $P(X_t)$ slows down, and (4.1) holds with sufficiently small $\gamma > 0$ so that $\theta < 1 - \gamma/\beta^+_1$. Then the integral (4.11) converges, and the value of the firm is finite, even if the resource constraint is dropped. Finally, assume that P is uniformly bounded from above: $P(x) \le c_2$, then the LHS in (4.7) admits an upper bound via $G'(K)c_2q^{-1}$. Since G satisfies the Inada conditions, $G'(K) \to 0$ as $K \to +\infty$, therefore for sufficiently large K, the LHS in (4.7) will be smaller than the RHS for any h^*, hence it is not optimal to increase the capital stock above a certain level, and the resource constraint becomes redundant.

4.3. **An example.**

Consider the following dependence of the price process on the stochastic factor. As the price remains below a certain critical value P_c, the dynamics of the price is given by the geometric Lévy process:

$$P(X_t) = P_c e^{X_t}, \quad X_t \le 0. \tag{4.12}$$

However, in the region above the critical level P_c, the rate of growth of P_t slows down:

$$P(X_t) = P_c[\gamma^{-1}(e^{\gamma X_t} - 1) + 1], \quad X_t > 0, \tag{4.13}$$

where $\gamma \in (0,1)$. In the limit $\gamma \to 1$, we recover the standard geometric Lévy case.

Consider equation (4.7) for the investment threshold. From our general result, we know that (4.7) has a unique solution, $h^* = h^*(K)$. If $h^* \le 0$, then the LHS is independent of the values of $P(x)$ for positive x, hence h^* is determined from the same equation as in the geometric Lévy case:

$$G'(K)E\left[\int_0^{+\infty} e^{-qt}P_c e^{h^*+\underline{X}_t}dt\right] = C, \tag{4.14}$$

which is

$$G'(K)q^{-1}\kappa_q^-(1)P_c e^{h^*} = C. \tag{4.15}$$

From (4.15), it is evident that $h^* \le 0$ iff $G'(K)q^{-1}\kappa_q^-(1)P_c \ge C$.

Let $G'(K)q^{-1}\kappa_q^-(1)P_c < C$, then (4.15) has no non-positive solutions. Therefore, the investment threshold h^* is positive, and we have to use both (4.12) and (4.13). For $h^* > 0$, it suffices to calculate $(qU^q_{\underline{X}}P)(x)$ for $x > 0$:

$$qU^q_{\underline{X}}P(x) = P_c\left[\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma x} - \gamma^{-1}(1-\gamma) + \sum_{j=1,2} d_{\gamma,j}e^{\beta_j^- x}\right], \tag{4.16}$$

where $d_{\gamma,j}$ are positive constants (see the Appendix). The investment threshold is the solution to equation (4.6) therefore, as $K \to \infty$, $U^q_{\underline{X}}P(h^*) = C/G'(K) \to \infty$, hence $e^{h^*(K)} \to \infty$, and $qU^q_{\underline{X}}P(h^*) \sim P_c\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*}$. Now we can write an approximate equation

$$G'(K)q^{-1}P_c\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*} = C$$

instead of (4.6) and obtain

$$H^* = P(h^*) \sim P_c e^{\gamma h^*} \sim \frac{qC\gamma}{\kappa_q^-(\gamma)G'(K)}.$$

The smaller the $\gamma > 0$, the lower is the trigger price of investment, $P(h^*)$. In the upper panel of Fig. 3, we plot the graph of $P(x)$ for $\gamma = 0.999$ (which is close to the geometric Lévy case $\gamma = 1$), $\gamma = 0.5$ and $\gamma = 0.1$. In the lower panel, we plot the investment threshold $H^* = H^*(K)$, for the Cobb-Douglas production function $G(K) = dK^\theta$. The stochastic parameter is a diffusion with exponentially distributed jumps, the same as in Fig. 1. Finally, notice that as $K \to +\infty$, $H^*(K)$ grows as $K^{(1-\theta)/\gamma}$, hence the value of the firm is finite iff $\theta < 1 - \gamma/\beta_1^+$.

5. New technology adoption

In this Section, we assume that the manager of a firm chooses not only the optimal capital stock, but also the optimal timing of an upgrade to the frontier technology. This model is more complicated than the ones of the previous Sections because it is driven by two factors: one characterizes the dynamics of the technology frontier, and the other incorporates all other shocks in the economy. Powerfully, the method of the paper preserves the tractability even in this two-factor model. Timing new technology adoption

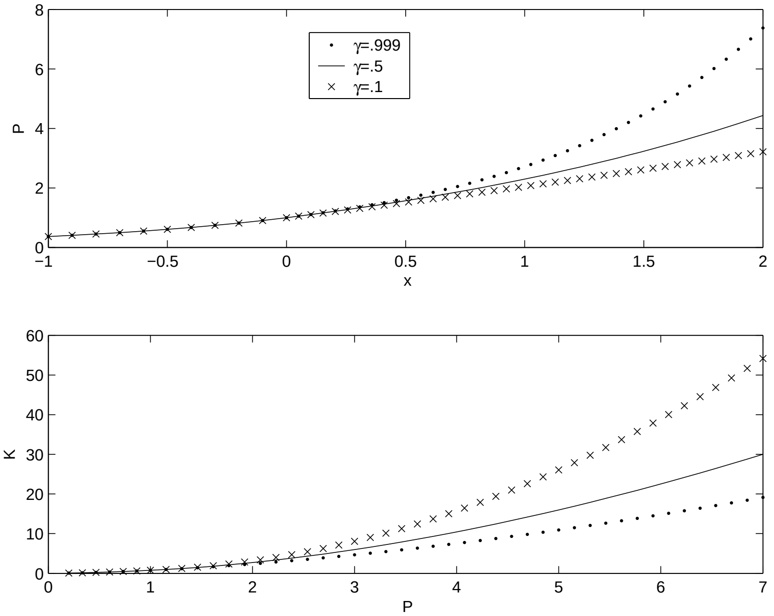

FIGURE 3. Upper panel: price as a function of the underlying stochastic factor. Lower panel: investment threshold $H^* = H^*(K)$ for the Cobb-Douglas production function. Parameters: $d = 1, \theta = 0.4, q = 0.08, \sigma^2 = 0.2, b = -0.6, c^- = 0.10, \lambda^- = -2, c^+ = 0$.

is one of the applications where it is essential to model a stochastic technology factor as a process with jumps, because the new technology is not introduced continuously. We believe that the most important component in the evolution of the technology frontier is a compound Poisson process with upward jumps, with possible inclusion of a small diffusion component. One may think about the diffusion component in the technological process as moderate innovations in technology, which may be caused by (or lead to) small fluctuations in non-technological uncertainty; in this case, the interaction between the technological factor and (small) innovations to non-technological factor is modeled as in the standard Gaussian model. However, major technological breakthroughs should be modeled as a jump process, and then it is natural to presume that if there is a correlation between technological and non-technological factors, it should be described by a bivariate jump process.

5.1. **Model specification.** We follow fairly closely the setup of Abel and Eberly (2002). There are no costs of adjustment of the stock of capital, and the stock is chosen optimally, therefore we may concentrate solely on the timing of adoption of the frontier technology. Let A_t be the technology in place, and $\hat{A}_t$ be the frontier technology at date t. Let the variable S_t incorporate all the sources of non-technological uncertainty (for more detailed exposition, see Abel and Eberly (2002)). Suppose that the updating happens at stopping times $\tau_1 < \tau_2 < \cdots$, so that between the updates the level of technology remains constant: for $t \in [\tau_{i-1}, \tau)$, $A_t = A_{\tau_{i-1}}$. Abel and Eberly (2002) show that the firm's cash flow can be described as $C_t = A_t S_t$. Updating to the frontier technology is costly, and the cost of updating is proportional to the updated cash stream: $\theta A_{\tau_i} S_{\tau_i}$, $\theta \in (0, 1)$. Let $V(A_{\tau_{i-1}}, \hat{A}_t, S_t)$ be the value of the firm net of the value of its capital stock for $t \in [\tau_{i-1}, \tau)$. Following Abel and Eberly (2002), we assume that the value admits a representation

$$V(A_{\tau_{i-1}}, \hat{A}_t, S_t) = A_{\tau_{i-1}} S_t V^1(\hat{A}_t / A_{\tau_{i-1}}), \tag{5.1}$$

and that updating occurs when the ratio $\hat{A}_t / A_{\tau_{i-1}}$ reaches a certain threshold, call it A^*.

In Abel and Eberly (2002), the technological factor $\hat{A}_t$ and non-technological factor S_t are modeled as geometric Brownian motions: $\hat{A}_t / A_{\tau_{i-1}} = e^{a_t}$, $S_t = e^{X_t}$, where (a_t, X_t) is a two-dimensional Gaussian process with non-trivial correlation between components. We assume that $\hat{A}_t / A_{\tau_{i-1}} = e^{X_t^1}$, $S_t = e^{X_t^2}$, where $X_t = (X_t^1, X_t^2)$ is a two-dimensional Lévy process driven by compound Poisson processes and two independent standard Brownian motions W_t^1 and W_t^2. To be more specific, we model X as a solution to the stochastic differential equation

$$d \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} + \sum_k \begin{bmatrix} 1 \\ \gamma_k \end{bmatrix} dJ_{c_k, \lambda_k; t}, \tag{5.2}$$

where $c_k > 0$, $\lambda_k > 0$, $\gamma_k \in \mathbb{R}$, and $J_{c,\lambda;t}$ denotes the compound Poisson process with the Lévy density $ce^{-\lambda x}\mathbf{1}_{(0,+\infty)}(x)$. We may identify $\sum_k J_{c_k,\lambda_k;t}$ as the jump component of the innovation process (creation of essentially new technologies), and then γ_k describe the impact of unexpected innovations on the dynamics of non-technological factor. If $\gamma_k < 0$ (respectively, $\gamma_k > 0$), then a positive jump in the frontier technology is accompanied by a negative (respectively, positive) jump in the non-technological stochastic factor. The diffusion part of the process describes small fluctuations in the non-technological factor, and related fluctuations in minor technological improvements. If $\sigma_{12} = 0$, then σ_{21} describes the impact of the process of small technological innovations on small fluctuations in non-technological uncertainty, and if $\sigma_{21} = 0$, then σ_{12} describes the impact of the latter on the former. The Lévy exponent of X_t, $\Psi(z) = \Psi(z_1, z_2)$, is defined by

$$E\left[e^{\langle z, X_t \rangle}\right] = E\left[e^{z_1 X_t^1 + z_2 X_t^2}\right] = e^{t\Psi(z)}.$$

For the process given by (5.2),

$$\Psi(z) = \frac{1}{2}||\Sigma' z||^2 + \langle b, z \rangle + \int_{\mathbb{R}^2 \setminus \{0\}} \left(e^{\langle z, y \rangle} - 1\right) F(dy), \tag{5.3}$$

where $\Sigma = [\sigma_{j,k}]$; $b = (b_1, b_2)$ and $\Sigma\Sigma'$ are the drift and variance-covariance matrix of the Gaussian component of the process, and

$$F(dy) = \sum_k c_k \lambda_k e^{-\lambda_k y_1} \mathbf{1}_{[0,+\infty)}(y_1) \delta_0(y_2 - \gamma_k y_1) dy_1 \tag{5.4}$$

is the Lévy density. Here δ_0 is the one-dimensional Dirac delta-function.

W.l.o.g., set $\tau_{i-1} = 0$ and denote $\tau = \tau_i = \inf\{t > 0 |\, X^1_t > h\}$, where $h = \log A^*$. Then the value of the firm satisfies

$$V(A_0, \hat{A}_t, S_t) = E_t\left[\int_t^\tau e^{-q(s-t)} A_0 S_s ds\right] + E_t\left[e^{-q(\tau-t)}\left(V(\hat{A}_\tau, \hat{A}_\tau, S_\tau) - \theta \hat{A}_\tau S_\tau\right)\right].$$

Substitute (5.1) into the last equation and divide it by $A_0 S_t$. Let

$$v(X_t) = V^1\left(e^{X^1_t}\right) = V^1(\hat{A}_t/A_0).$$

Notice that at the time of updating, $A_\tau = \hat{A}_\tau$, hence $V^1(\hat{A}_\tau/A_\tau) = V^1(1) = v(0)$. Now for $t \in [0, \tau)$, we have

$$v(X_t) = E_t\left[\int_t^\tau e^{-q(s-t)+X^2_s - X^2_t} ds\right] + E_t\left[e^{-q(\tau-t)+X^2_\tau - X^2_t} e^{X^1_\tau}(v(0) - \theta)\right]. \tag{5.5}$$

5.2. **One source of uncertainty.** First, we consider the case when only innovations to technology occur, i.e., the factor X^2_t is constant. The underlying stochastic process is a one-dimensional Lévy process. Examining only technological innovations is not only instructive by itself, but as we will show it in the next Subsection, the general case reduces to this special case. Of course, the Lévy exponent of a one-dimensional process that appears after the reduction will depend on the Lévy exponent of the initial two-dimensional process. In Subsection 5.3, we will discuss the impact of interaction between the two processes on the new technology adoption threshold.

Let h be the threshold for updating. The objective of the firm is to choose h so as to maximize the value

$$v(x; h) = E_t\left[\int_t^\tau e^{-q(s-t)} ds |\, X_t = x\right] + E_t\left[e^{-q(\tau-t)} e^{X_\tau}(v(0; h) - \theta) |\, X_t = x\right]. \tag{5.6}$$

To ensure that the value of the firm is finite, assume that X satisfies (2.16). In the Appendix, we show that it is possible to rewrite (5.6) in the form

$$v(x; h) = q^{-1} + q\left(U^q_{\bar{X}} \mathbf{1}_{[h, +\infty)}\left[\kappa^+_q(1)^{-1}(v(0; h) - \theta)e^{\cdot} - q^{-1}\right]\right)(x), \tag{5.7}$$

where $e^{\cdot}$ denotes the exponential function $x \mapsto e^x$. Introduce

$$v_{\text{opt}}(x; h) = v(x; h) - q^{-1}.$$

Recall that given the new technology is adopted at the threshold h, the value of the firm is

$$V(A_0, \hat{A}_t, S_t; h) = A_0 S_t v(X_t; h) = \frac{A_0 S_t}{q} + A_0 S_t v_{\text{opt}}(X_t; h).$$

The first term, $A_0 S_t/q$, is the EPV of the stream of profits, which the firm will generate provided the current technology stays in place forever, and the second term is the option value of upgrading to the frontier technology. In order to find the option value, we rewrite (5.7) in terms of $v_{\text{opt}}(x; h)$:

$$v_{\text{opt}}(x; h) = q\left(U^q_{\bar{X}} \mathbf{1}_{[h, +\infty)}\left[\kappa^+_q(1)^{-1}(v_{\text{opt}}(0; h) + q^{-1} - \theta)e^{\cdot} - q^{-1}\right]\right)(x). \tag{5.8}$$

Suppose for a moment that we know the option value $V_0 := v_{\mathrm{opt}}(0;h)$ at the moment of updating. Assuming that $V_0 + q^{-1} - \theta > 0$ (a sufficient condition is $q\theta < 1$, that is, the cost of updating is not too high), and arguing as in the proof of (2.24), we conclude that the optimal updating threshold h satisfies

$$\kappa_q^+(1)^{-1}(v_{\mathrm{opt}}(0;h) + q^{-1} - \theta)e^h - q^{-1} = 0. \tag{5.9}$$

Using (5.9), we can simplify (5.8) for $x < h$:

$$v_{\mathrm{opt}}(x;h) = e^{-h}\left(U_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}\left(e^{\cdot} - e^h\right)\right)(x) = \left(U_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(x). \tag{5.10}$$

Equation (5.9) has two unknowns: h and $v_{\mathrm{opt}}(0;h)$, however we can add the second equation by letting $x = 0$ in (5.10):

$$v_{\mathrm{opt}}(0;h) = \left(U_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(0). \tag{5.11}$$

By substituting (5.11) into (5.9), and multiplying by $q\kappa_q^+(1)$, we obtain the equation for h:

$$e^h\left(qU_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(0) + (1-q\theta)e^h - \kappa_q^+(1) = 0. \tag{5.12}$$

We claim that if $q\theta < 1$, then this equation has a unique solution on $(0,+\infty)$. Indeed, as $h \to +\infty$, the LHS tends to $+\infty$, and at $h = 0$, the LHS is negative:

$$(qU_{\bar{X}}^q(e^{\cdot} - 1))(0) + (1-q\theta) - \kappa_q^+(1) = \kappa_q^+(1) - 1 + (1-q\theta) - \kappa_q^+(1) = -q\theta < 0.$$

Hence, a solution exists, and to see that it is unique, it suffices to check that the LHS is convex. We will verify this, and obtain explicit formulas for h and $v_{\mathrm{opt}}(0;h)$ after we specify a process for the frontier technology.

Suppose that X is a diffusion process with exponentially distributed upward jumps. The Lévy density is

$$F(dy) = c\lambda e^{-\lambda y}\mathbf{1}_{(0,+\infty)}(y)dy, \tag{5.13}$$

where $c > 0$ and $\lambda > 1$ (the last inequality is necessary for (2.16) to hold). Then the Lévy exponent is $\Psi(z) = \sigma^2 z^2/2 + bz + cz/(\lambda - z)$, and (2.16) is satisfied provided $q > \sigma^2/2 + b + c/(\lambda - 1)$. The characteristic equation has three roots: $\beta^- < 0 < 1 < \beta_1^+ < \lambda < \beta_2^+$. The factor $\kappa_q^-(z)$ is defined by $\kappa_q^-(z) = \beta^-/(\beta^- - 1)$, and $\kappa_q^+(z)$ is given by (2.12) or (2.13). The option value $v_{\mathrm{opt}}(x;h)$ satisfying (5.10) can be computed in exactly the same manner as the value $V^+(x)$ in Subsection 2.3:

$$v_{\mathrm{opt}}(x;h) = q^{-1}\sum_{j=1,2}\frac{a_j^+ e^{\beta_j^+(x-h)}}{\beta_j^+(\beta_j^+ - 1)}, \text{ for } x < h,$$

and (5.12) assumes the form

$$\sum_{j=1,2}\frac{a_j^+ e^{(1-\beta_j^+)h}}{\beta_j^+(\beta_j^+ - 1)} + (1-q\theta)e^h - \kappa_q^+(1) = 0. \tag{5.14}$$

Denote by $f(h)$ the LHS in (5.14). We have shown for the general case above that $f(h)$ changes sign on $(0,+\infty)$, and the root of (5.14) exists. To show the uniqueness of the root, we prove that f is convex:

$$f''(h)=\sum_{j=1,2}\frac{a_j^+(\beta_j^+-1)e^{(1-\beta_j^+)h}}{\beta_j^+}+(1-q\theta)e^h>0.$$

5.3. **Two sources of uncertainty.** For simplicity, assume that there is only one term in the jump component. Set $c=c_k, \lambda=\lambda_k, \gamma=\gamma_k$, assume that $\gamma<\lambda-1$, and denote by a_{jk} the entries of the variance-covariance matrix $\Sigma\Sigma'$. In the Appendix, we show that the new technology adoption threshold in the two-factor model (5.2) is the same as in the one-factor model with the characteristic exponent

$$\Psi^1(z_1)=\frac{a_{11}}{2}z_1^2+b^1z_1+\frac{c^1z_1}{\lambda^1-z_1}, \tag{5.15}$$

where $b^1=a_{12}+b_1$, $c^1=c\lambda/(\lambda-\gamma)$, and $\lambda^1=\lambda-\gamma$. To ensure that the value of the firm were finite, we need to impose two conditions ((A.9) and (A.10)), which in the case of one jump component assume the form

$$q^1:=q-\frac{a_{22}}{2}-b_2-\frac{c\gamma}{\lambda-\gamma}>0, \tag{5.16}$$

and

$$q-\frac{a_{11}}{2}-a_{12}-\frac{a_{22}}{2}-b_1-b_2-\frac{c(1+\gamma)}{\lambda-\gamma-1}>0. \tag{5.17}$$

Notice that both (5.16) and (5.17) imply that γ cannot be too close to λ, equivalently, if positive technological jumps are accompanied by vigorous positive jumps in the non-technological factor, then the value of the firm becomes infinite: the prospects are too good to be true. Probably, the advocates of the New Economy had in mind similar models for shocks in technology and non-technological uncertainty. We also need to require $1-q^1\theta>0$; if this condition is violated, then new technology adoption is never optimal.

If the Gaussian component in the dynamics of the technology frontier is non-trivial, then the characteristic equation has three roots $\beta^-<0<1<\beta_1^+<\lambda<\beta_2^+$, and the equation for the technology adoption frontier is (cf. (5.14))

$$\sum_{j=1,2}\frac{a_j^+e^{(1-\beta_j^+)h}}{\beta_j^+(\beta_j^+-1)}+(1-q^1\theta)e^h-\kappa_{q^1}^+(1)=0, \tag{5.18}$$

where a_j^+ and $\kappa_{q^1}^+(1)$ are defined by the same formulas as in Section 2 with q^1 in place of q. The existence and uniqueness of the solution h to (5.18) is proved in Subsection 5.2.

5.4. **Dependence of the new technology adoption threshold, A^*, on diffusion and jump uncertainty.** We start with the study of the dependence of A^* on the jump component when the technological process has no Gaussian component: $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. For the calculation of A^* in this case, see the Appendix. First, we fix the Gaussian component of the non-technological factor, σ_{22}, and change c, λ and γ (Fig. 4). Then we fix λ, and change c, σ_{22}, and γ (Fig. 5). The increase in c means that the total uncertainty of the technological factor increases, the increase in λ^{-1} means that the average jump size becomes larger (hence, the technological uncertainty increases), and the increase in σ_{22} means the increase in non-technological uncertainty. Finally, the increase in γ means that the correlation between the two factors goes up. In these figures, it is clearly seen (and the same effect is observed for other parameters' values) that the new technology adoption threshold is

(a) an increasing function of (c, λ^{-1}), that is, of the uncertainty in the technological factor;
(b) a decreasing function of σ_{22}, that is, of the uncertainty in the non-technological factor;
(c) a decreasing function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors.

Thus, the uncertainty in the technological factor and uncertainty in the non-technological one affect the threshold in opposite directions. The dependence on the technological uncertainty can be naturally explained in the framework of the record-setting news principles in Boyarchenko (2004) as follows. In a situation similar to the call option with an instantaneous (random) payoff, the record-setting good news principle applies, and the higher the uncertainty of good news, the higher is the threshold. Clearly, this is the situation with new technology adoption: once the new technology is in place, it remains fixed for a sizable time period. The feature (b) is not as transparent as (a). According to the record-setting news principles in Boyarchenko (2004), if the option gives the right to a *stream* of payoffs (a cash flow here), then the record-setting bad news principle applies, and the higher the uncertainty of bad news (the lower the trajectories of the infimum process), the higher is the threshold. It may seem that the increase in σ_{22} means the increase in the overall uncertainty in S_t, the non-technological factor, hence in the uncertainty of bad news, and so the threshold should increase. Notice, however, that the threshold is derived for the technological factor, but not for S_t, and the standard intuition may be non-applicable. If σ_{22} increases, then $b_2 + \sigma_{22}^2/2$, the rate of growth of S_t increases; therefore, the higher the expected rate of growth of S_t (hence, of the revenue), the sooner should the firm take the advantage of adoption of the frontier technology. The reader may wonder if the difference between the ways the new technology factor and non-technological one influence the threshold is an artifact of the different ways these factors are modeled: pure jump process and diffusion process with embedded jumps, respectively. In Fig. 6, we demonstrate how the adoption threshold changes if we add the diffusion component to technological process so that the Gaussian uncertainty in the non-technological factor drives the Gaussian uncertainty in the technological factor (similar effects are observed when the latter driver the former). We also show the threshold when there is no Gaussian uncertainty in the technological factor. The conclusions made earlier remain valid. The new technology adoption threshold is

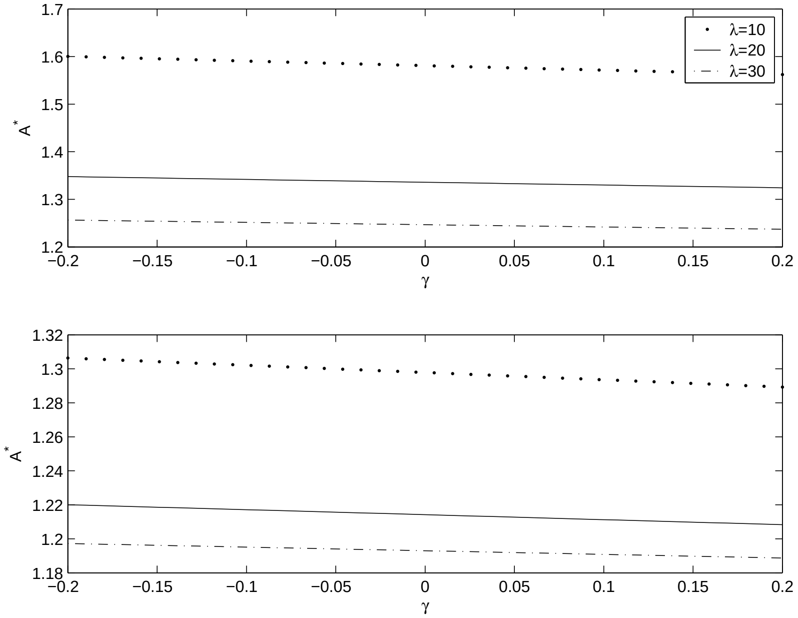

FIGURE 4. Adoption of new technology threshold A^* as a function of the correlation parameter γ, intensity of jumps c^+, and steepness parameter λ. The technology factor is compound Poisson. Parameters: $\theta = 5, q = 0.08, \sigma_{22}^2 = 0.10, b_1 = -0.01, b_2 = 0.00$. Upper panel: $c^+ = 0.25$; lower panel: $c^+ = 0.10$.

(a) an increasing function of the uncertainty in the technological factor;
(b) a decreasing function of the uncertainty in the non-technological factor;
(c) a decreasing function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors;
(d) an increasing function of the covariance coefficients, σ_{12} and σ_{21}, between the Gaussian components in the technological and non-technological factors.

Notice the important difference between the impact of the "correlation" between the Gaussian and non-Gaussian sources of uncertainty on the threshold: A^* is a **decreasing** function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors, and an **increasing** function of the correlation coefficients σ_{12} and σ_{21} between the Gaussian components of technological and non-technological innovations. Hence, the interaction between Gaussian sources of uncertainty, and the one between non-Gaussian sources of uncertainty are not just qualitatively different: they are of **opposite signs**.

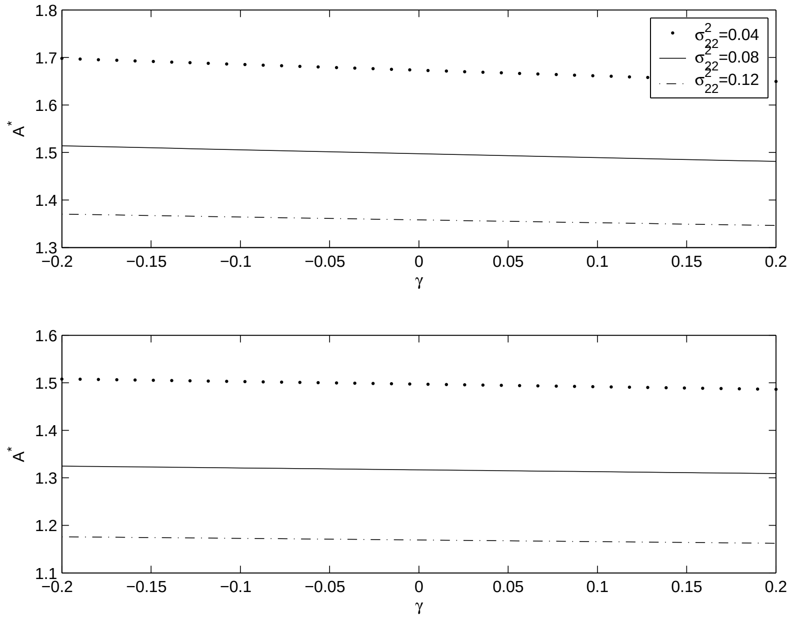

FIGURE 5. Adoption of new technology threshold A^* as a function of the correlation parameter γ, intensity of jumps c^+, and Gaussian uncertainty, σ_{22}. The technology factor is compound Poisson. Parameters: $\theta = 5, q = 0.08, \lambda = 15, b_1 = -0.01, b_2 = 0.00$. Upper panel: $c^+ = 0.25$; lower panel: $c^+ = 0.10$.

APPENDIX A

Proof of (2.3) Computing the action of the infinitesimal generator (2.1) on e^{zx}, we obtain the exponent $\Psi(z)$ corresponding to the Lévy density (2.2):

$$\begin{aligned} Le^{zx} &= \left[\frac{\sigma^2}{2}z^2 + bz + c^+\lambda^+\int_0^{+\infty}\left(e^{(z-\lambda^+)y} - e^{-\lambda^+y}\right)dy\right. \\ &\quad \left. -c^-\lambda^-\int_{-\infty}^{0}\left(e^{(z-\lambda^-)y} - e^{-\lambda^-y}\right)dy\right]e^{zx} \\ &= \left(\frac{\sigma^2}{2}z^2 + bz + \frac{c^+z}{\lambda^+-z} + \frac{c^-z}{\lambda^--z}\right)e^{zx} = \Psi(z)e^{zx}. \end{aligned}$$

Formulas for $a_j^{\pm}$

$$a_1^+ = \frac{\beta_1^+\beta_2^+}{\beta_2^+-\beta_1^+}\cdot\frac{\lambda^+-\beta_1^+}{\lambda^+}, \qquad a_2^+ = \frac{\beta_1^+\beta_2^+}{\beta_1^+-\beta_2^+}\cdot\frac{\lambda^+-\beta_2^+}{\lambda^+}, \tag{A.1}$$

$$a_1^- = \frac{\beta_1^-\beta_2^-}{\beta_2^--\beta_1^-}\cdot\frac{\lambda^--\beta_1^-}{\lambda^-}, \qquad a_2^- = \frac{\beta_1^-\beta_2^-}{\beta_1^--\beta_2^-}\cdot\frac{\lambda^--\beta_2^-}{\lambda^-}. \tag{A.2}$$

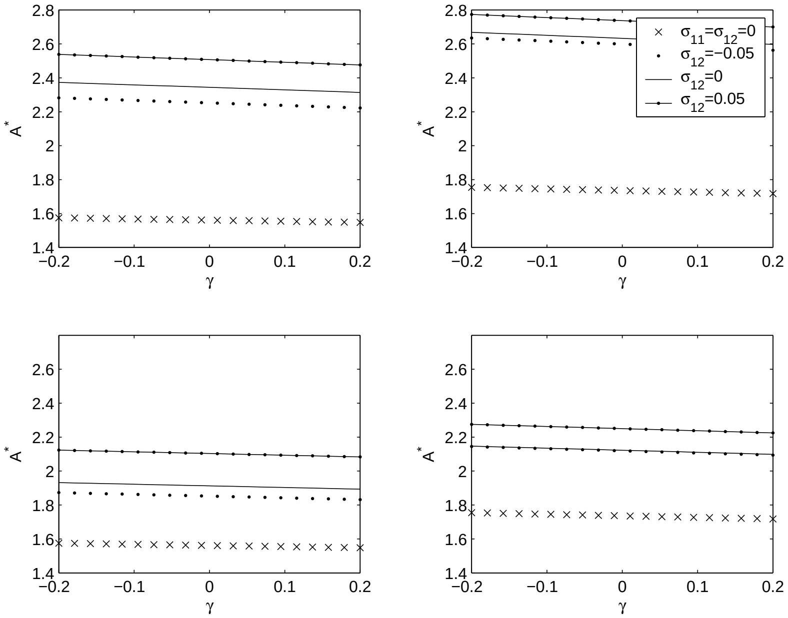

FIGURE 6. Adoption of new technology threshold A^* as a function of the correlation parameter γ, and the correlation coefficient σ_{12}; $\sigma_{21} = 0.0$ (demand/prices influence small improvements in the technology but not vice versa). Crosses: no Gaussian uncertainty in the technological factor: $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. Other parameters: $\theta = 5, q = 0.08, \lambda = 15, b_1 = -0.01, b_2 = 0.00$. Panels: (a) $c^+ = c^- = 0.2$; (b) $c^+ = 0.2, c^- = 0.1$; (c) $c^+ = 0.1, c^- = 0.2$; (d) $c^+ = c^- = 0.1$.

If $g(x) = e^{zx}$, then

$$\int_0^{+\infty} e^{-\beta_j^+} e^{z(x+y)} dy = e^{zx} \int_0^{+\infty} e^{(z-\beta_j^+)y} dy = \frac{e^{zx}}{\beta_j^+ - z},$$

therefore

$$qU^q_{\bar{X}} e^{zx} = \kappa_q^+(z) e^{zx} = \sum_{j=1,2} \frac{a_j^+}{\beta_j^+ - z} e^{zx}.$$

Thus, (2.14) is proved for exponential functions. By expanding an arbitrary (sufficiently regular) function g as a Fourier integral, we obtain (2.14). (2.15) is proved similarly.

Verification of $w(h_*) < 0$. First, consider the case when there are no positive jumps. Then $\kappa_q^+(z) = \beta^+/(\beta^+ - z)$, and for $C < I$,

$$\begin{aligned} w(h_*) &= C\left(\frac{\kappa_q^-(1)(\beta^+-1)}{\beta^+} + \sum_{j=1,2}\frac{a_j^-(\beta^+-\beta_j^-)}{\beta^+\beta_j^-(1-\beta_j^-)}\right) - I \\ &= \frac{C}{\beta^+}\sum_{j=1,2}\frac{a_j^-\beta^+(\beta_j^- - 1)}{\beta_j^-(\beta_j^- - 1)} - I = C\kappa_q^+(0) - I = C - I < 0. \end{aligned}$$

If positive jumps exist, then we will show that

$$\phi(x) = \kappa_q^-(1)\kappa_q^+(1)^{-1}e^{x-h_*} + \sum_{j=1,2}\frac{a_j^-\kappa_q^+(\beta_j^-)^{-1}}{\beta_j^-(1-\beta_j^-)}e^{\beta_j^-(x-h_*)} - 1$$

is negative at h_*. This will immediately imply that $w(h_*) = C\phi(h_*) - (I - C) < 0$ for $I \geq C$. First, we use (2.12) for $\kappa_q^+(z)$ and compute

$$\kappa_q^+(z)^{-1} - 1 = \frac{z(\lambda^+ z + \beta_1^+\beta_2^+ - \lambda^+(\beta_1^+ + \beta_2^+)}{\beta_1^+\beta_2^+(\lambda^+ - z)}.$$

Next, we use the value matching condition (2.21) to write

$$1 = \kappa_q^-(1) + \sum_{j=1,2}\frac{a_j^-}{\beta_j^-(1-\beta_j^-)}.$$

Finally, compute

$$\begin{aligned} \phi(h_*) &= \kappa_q^-(1)\left(\kappa_q^+(1)^{-1} - 1\right) + \sum_{j=1,2}\frac{a_j^-\left(\kappa_q^+(\beta_j^-)^{-1} - 1\right)}{\beta_j^-(1-\beta_j^-)} \\ &= \frac{1}{\beta_1^+\beta_2^+}\sum_{j=1,2}a_j^-\left[\frac{\lambda^+ + \beta_1^+\beta_2^+ - \lambda^+(\beta_1^+ + \beta_2^+}{\lambda^+ - 1} - \frac{\lambda^+\beta_j^- + \beta_1^+\beta_2^+ - \lambda^+(\beta_1^+ + \beta_2^+)}{\lambda^+ - \beta_j^-}\right] \\ &= \frac{(\lambda^+ - \beta_1^+)(\lambda^+ - \beta_2^+)}{\beta_1^+\beta_2^+(\lambda^+ - 1)}\sum_{j=1,2}\frac{a_j^-}{\beta_j^- - \lambda^+} = \frac{(\lambda^+ - \beta_1^+)(\lambda^+ - \beta_2^+)}{\beta_1^+\beta_2^+(\lambda^+ - 1)}\kappa_q^-(\lambda^+) < 0. \end{aligned}$$

Notice that exactly the same result holds if there are no negative jumps, the only difference being that in this case, $\kappa_q^-(z) = \beta^-/(\beta^- - z)$.

Verification of $w(h^*) > 0$:

$$\begin{aligned} w(h^*) &= C\left[\kappa_q^-(1)\kappa_q^+(1)^{-1}e^{h^*-h_*} + \sum_{j=1,2}\frac{a_j^-\kappa_q^+(\beta_j^-)^{-1}}{\beta_j^-(1-\beta_j^-)}e^{\beta_j^-(h^*-h_*)}\right] - I \\ &= C\sum_{j=1,2}\frac{a_j^-\kappa_q^+(\beta_j^-)^{-1}}{\beta_j^-(1-\beta_j^-)}e^{\beta_j^-(h^*-h_*)} > 0 \end{aligned}$$

(here we used (2.25) and (2.19) to obtain $e^{\beta_j^-(h^*-h_*)} = I\kappa_q^+(1)/(C\kappa_q^-(1))$).

Proof of (4.11). Let $P(x)$ satisfy (4.1), then

$$w(x) \le c_1 E\left[\int_0^{+\infty} e^{-qt+\gamma(x+\underline{X}_t)}dt\right] \le c_1 q^{-1}\kappa_q^-(\gamma)e^{\gamma x} \le c_1 q^{-1} e^{\gamma x}.$$

Therefore

$$\int_0^{+\infty} e^{-\beta_j^+ y} w(h^* + y)dy \le c_1 q^{-1} e^{\gamma h^*} \int_0^{+\infty} e^{-\beta_j^+ y + \gamma y}dy = \frac{c_1 e^{\gamma h^*}}{q(\beta_j^+ - \gamma)},$$

and

$$V_K^{\mathrm{opt}}(K,x) \le \frac{c_1 G'(K) e^{\gamma h^*}}{q} \sum_{j=1,2} \frac{a_j^+}{\beta_j^+ - \gamma} e^{\beta_j^+(x-h^*)}.$$

Since $\gamma \in (0,1]$ and $1 < \beta_1^+ < \beta_2^+$, we obtain

$$V_K^{\mathrm{opt}}(K,x) \le D(x)G'(K)e^{(\gamma-\beta_1^+)h^*(K)}, \tag{A.3}$$

where $D(x)$ depends on $x \le h^*(K)$ but not on K. Next, we notice that if $P(x) \le Q(x)$ for any x, then $h_P^*(K) = h^*(K;P) \ge h^*(K;Q) = h_Q^*(K)$. This result follows immediately if one compares (4.8) for Q

$$G'(K)\sum_{j=1,2}(-a_j^-)\int_{-\infty}^0 e^{-\beta_j^- y}Q(h_Q^* + y)dy = qC$$

with

$$G'(K)\sum_{j=1,2}(-a_j^-)\int_{-\infty}^0 e^{-\beta_j^- y}P(h_Q^* + y)dy \le qC.$$

For $Q(x) = c_1 e^{\gamma x}$, we have from (4.8) $G'(K)\kappa_q^-(\gamma)e^{\gamma h_Q^*(K)} = qC$, therefore the RHS in (A.3) admits a bound via $D_1(x)G'(K)^{\beta_1^+/\gamma}$, and we conclude that (4.11) is a sufficient condition for the convergence of the integral (4.10) with $\bar{K} = +\infty$. In the geometric Lévy case, we obtain that $V_K^{\mathrm{opt}}(K,x) = D_1(x)G'(K)^{\beta_1^+/\gamma}$, where $\gamma = 1$, therefore (4.11) is necessary as well.

Proof of (4.16). W.l.o.g., $P_c = 1$. We have

$$\begin{aligned} qU_{\underline{X}}^q P(x) &= \sum_{j=1,2}(-a_j)\int_{-\infty}^0 e^{-\beta_j^- y}P(x+y)dy \\ &= \sum_{j=1,2}(-a_j)e^{\beta_j^- x}\int_{-\infty}^x e^{-\beta_j^- y}P(y)dy = \sum_{j=1,2}(-a_j)e^{\beta_j^- x}f_j(x), \end{aligned}$$

where

$$\begin{aligned} f_j(x) &= \int_{-\infty}^{0} e^{-\beta_j^- y} e^y dy + \int_0^x e^{-\beta_j^- y}(\gamma^{-1}e^{\gamma y} + (1-\gamma^{-1}))dy \\ &= \frac{1}{1-\beta_j^-} + \frac{1}{\gamma(\gamma-\beta_j^-)}(e^{(\gamma-\beta_j^-)x} - 1) - \frac{1-\gamma}{\gamma(-\beta_j^-)}(e^{-\beta_j^- x} - 1) \\ &= \frac{1}{1-\beta_j^-} - \frac{1}{\gamma(\gamma-\beta_j^-)} + \frac{1-\gamma}{\gamma(-\beta_j^-)} + \frac{e^{(\gamma-\beta_j^-)x}}{\gamma(\gamma-\beta_j^-)} - \frac{(1-\gamma)e^{(-\beta_j^-)x}}{\gamma(-\beta_j^-)}. \end{aligned}$$

Using

$$\sum_{j=1,2} \frac{-a_j}{\gamma(\gamma-\beta_j^-)} = \gamma^{-1}\kappa_q^-(\gamma), \quad \sum_{j=12,} \frac{-a_j}{-\beta_j^-} = \kappa_q^-(0) = 1,$$

we obtain (4.16) with

$$d_{\gamma,j} = \frac{(-a_j)(1-\gamma)}{(-\beta_j^-)(1-\beta_j^-)(\gamma-\beta_j^-)}.$$

Proof of (5.7) If $X_t = x$, then using (2.23),

$$\begin{aligned} E_t\left[\int_t^\tau e^{-q(s-t)}ds\right] &= E_t\left[\int_t^{+\infty} e^{-q(s-t)}ds\right] - E_t\left[\int_\tau^{+\infty} e^{-q(s-t)}ds\right] \\ &= q^{-1} - q(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}1)(x) = q^{-1} - q(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}q^{-1})(x). \end{aligned}$$

Use the fundamental relationship between the infinitesimal generator and the EPV-operator (2.5) to write the payoff $e^{X_\tau}(v(0;h)-\theta)$ as the EPV of a stream $g(x) = (q-L)e^x(v(0;h)-\theta)$, substitute $U^q_X g(X_\tau)$ into (5.6), and apply (2.23) in order to write the second term in (5.6) as

$$\begin{aligned} q\left(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}\right)g(x) &= q\left(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}(q-L)(v(0;h)-\theta)e^{\cdot}\right)(x) \\ &= q\left(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(qU^q_{\bar{X}})^{-1}(v(0;h)-\theta)e^{\cdot}\right)(x) \\ &= q\left(U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}(v(0;h)-\theta)\kappa_q^+(1)^{-1}e^{\cdot}\right)(x). \end{aligned}$$

(Here we used the Wiener-Hopf factorization formula (2.9) and (2.6).) Now it becomes possible to rewrite (5.6) in the form (5.7).

Proof of (5.15). For the sake of brevity, assume that $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$; the proof in the general case is similar. For any $s \geq t$ and $z = (z_1, z_2)$,

$$\begin{aligned} E_t\left[e^{z_1X_s^1+z_2X_s^2}\right] &= e^{z_1X_t^1+z_2X_t^2+(s-t)\Psi(z_1,z_2)} \\ &= e^{z_2X_t^2+(s-t)\Psi(0,z_2)}e^{z_1X_t^1+(s-t)[\Psi(z_1,z_2)-\Psi(0,z_2)]}. \end{aligned} \tag{A.4}$$

It is easy to see that for a fixed z_2, $\Psi^1_{z_2}(z_1) = \Psi(z_1,z_2) - \Psi(0,z_2)$ is the Lévy exponent of a one-dimensional Lévy process. In particular, if $||\Sigma' z||^2 = \sigma_{22}^2 z_2^2$, then from (5.3), we obtain

$$\begin{aligned}\Psi(z) &= \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k \int_{y_1>0} \left(e^{z_1y_1+z_2y_2} - 1\right) e^{-\lambda_k y_1}\delta_0(y_2 - \gamma_k y_1)dy_1 \\ &= \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k \int_0^{+\infty} \left(e^{(z_1+z_2\gamma_k)y_1} - 1\right) e^{-\lambda_k y_1}dy_1 \\ &= \frac{\sigma_{22}^2}{2}z_2^2 + b_2z_2 + \sum_k c_k\lambda_k \left[\frac{1}{\lambda_k - z_2\gamma_k - z_1} - \frac{1}{\lambda_k}\right].\end{aligned}$$

Hence,

$$\Psi(z) = \frac{\sigma_{22}^2}{2}z_2^2 + b_1z_1 + b_2z_2 + \sum_k c_k\lambda_k \left[\frac{1}{\lambda_k - z_1 - z_2\gamma_k} - \frac{1}{\lambda_k}\right], \tag{A.5}$$

and

$$\Psi^1_{z_2}(z_1) = b_1z_1 + \sum_k \frac{c_k\lambda_k z_1}{(\lambda_k - \gamma_k z_2)(\lambda_k - \gamma_k z_2 - z_1)}, \tag{A.6}$$

which is the characteristic exponent of a pure jump process with exponentially distributed upward jumps, and the Lévy density which depends on z_2. Let $Q^1_{z_2}$ be the probability measure which corresponds to $\Psi^1_{z_2}$, that is,

$$E^{Q^1_{z_2}}\left[e^{z_1X^1_t}\right] = e^{t\Psi^1_{z_2}(z_1)}.$$

Then we can write (A.4) as

$$E_t\left[e^{z_1X^1_s + z_2X^2_s}\right] = e^{z_2X^2_t + (s-t)\Psi(0,z_2)}E_t^{Q^1_{z_2}}\left[e^{z_1X^1_s}\right].$$

Decomposing a sufficiently regular function $f(X^1_s)$ as a Fourier integral, we have

$$E_t\left[e^{z_2X^2_s}f(X^1_s)\right] = e^{z_2X^2_t + (s-t)\Psi(0,z_2)}E_t^{Q^1_{z_2}}\left[f(X^1_s)\right]. \tag{A.7}$$

Set $\Psi^1 = \Psi^1_1, Q^1 = Q^1_1$, and $q^1 = q - \Psi(0,1)$, and apply (A.7) with $z_2 = 1$ to (5.5):

$$v(X^1_t) = E_t^{Q^1}\left[\int_t^\tau e^{-q^1(s-t)}ds\right] + E_t^{Q^1}\left[e^{-q^1(\tau-t)}e^{X^1_\tau}(v(0) - \theta)\right]. \tag{A.8}$$

If

$$q - \Psi(0,1) > 0, \tag{A.9}$$

then (A.8) is of the same form as (5.6), which we have studied already. The condition for the value of the firm to be finite is

$$q^1 - \Psi^1(1) = q^1 - (\Psi(1,1) - \Psi(0,1)) > 0,$$

which is equivalent to

$$q - \Psi(1,1) > 0. \tag{A.10}$$

Thus, we require both (A.9) and (A.10). If there is only one term in (5.4), then from (A.6), we derive (5.15).

The case of a pure jump technological process

Assume first that there is no Gaussian component in the technological factor ($\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$), and $b^1 = b_1 < 0$, that is, upward jumps in the frontier technology are followed by periods of decline in the effectiveness of innovations. Then the characteristic equation

$$q^1 - \Psi^1(z_1) = 0$$

has two roots $\beta^- < 0 < 1 < \beta^+$:

$$\beta^{\pm} = \frac{q^1 + c^1 + \lambda^1 b_1 \mp \sqrt{(q^1 + c^1 + \lambda^1 b_1)^2 - 4q^1 b_1 \lambda^1}}{2b_1}, \tag{A.11}$$

(recall that we assume $b_1 < 0$), and $\kappa^+_{q^1}(z_1) = \beta^+(\lambda^1 - z_1)/(\lambda^1(\beta^+ - z_1)) = \beta^+/\lambda^1 + a^+/(\beta^+ - z_1)$, where $a^+ = \beta^+ - (\beta^+)^2/\lambda^1$. The equation for the technology adoption frontier is obtained in the same manner as (5.14), but there is no summation because there is a unique positive root:

$$\frac{a^+ e^{(1-\beta^+)h}}{\beta^+(\beta^+ - 1)} + (1 - q^1\theta)e^h - \kappa^+_{q^1}(1) = 0. \tag{A.12}$$

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