Idiosyncratic Risk, Systematic Risk and Stochastic Volatility: An Implementation of Merton's Credit Risk Valuation

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Abstract

We extend the credit risk valuation framework introduced by Gatfaoui (2003) to stochastic volatility models. We state a general setting for valuing risky debt in the light of systematic risk and idiosyncratic risk which are known to affect each risky asset in the financial market. The option nature of corporate debt allows then to account for the well-known volatility smile along with two documented determinants, namely stochastic volatility and market risk. Under some regularity conditions, we specify diffusion functionals leading to an asymptotically (relative to time) mean reverting volatility process. The behavior of such a specification is studied along with simulation techniques since debt is valued via a call on the firm assets value. Specifically, our examination resorts to Monte Carlo accelerators to realize related simulations. First, we consider the evolution of stochastic volatility for given parameter values. Then, we assess its impact on both risky debt and the related credit spread.

Keywords : Credit risk, credit spread, idiosyncratic risk, stochastic volatility, systematic risk.
JEL codes : G13, G33

1 Introduction

Originally, Sharpe (1963) stated the dependence of stocks' returns *vis-à-vis* systematic (i.e., market or undiversifiable) risk and idiosyncratic (i.e., specific or diversifiable) risk. Indeed, systematic risk is known to be common to any risky asset in the financial market whereas idiosyncratic risk is peculiar to the asset under consideration. Therefore, credit risky assets (e.g., corporate bonds or debt) should satisfy such a dependence. Many authors investigated this assumption to test wether credit risk is of systematic or idiosyncratic nature. We focus on most recent findings (see Gatfaoui [2003] for a brief survey).

In 1989, Fama & French studied corporate bonds and showed the influence of systematic risk (as represented by business conditions) on default risk premia. Specifically, since systematic risk is correlated with macro-economic atmosphere, its influence on credit risk is studied along with business cycle indicators (see Wilson [1998], Nickell *et al.* [2000], Gatfaoui & Radacal [2001], Bangia *et al.* [2002] among others). For example, consistent with the Fama & French (1989) definition of bonds' risk premia, Spahr *et al.* (2002) study speculative grade debt. Estimating historical default losses along with Altman's actuarial approach, they find that the speculative bond market prices both default risk and systematic risk with efficiency. While studying contemporaneous and first order correlations between frequency and severity of annual defaults, they show that default risk and systematic risk are coincident risks. In the same way, Koopman & Lucas (2003) resort to a multivariate unobserved component approach to describe jointly credit spreads and business failure rates with macro-economic behavior. They find empirical evidence of a correlation between credit risk and macro-economy. Moreover, Elton *et al.* (2001) show the existence of a premium due to systematic risk in corporate spot spreads, namely credit spreads computed as the difference between corporate and Treasury bond yields. To go further, Delianedis & Geske (2001) study the components of corporate credit spreads in the lens of a structural model. First, they find that default risk represents only a small portion of credit spreads. Then, they conclude that default risk and recovery risk fail to explain fully credit risk and credit spreads whereas taxes, jumps (in firm value), liquidity and market risk factors explain mainly such variables. More precisely, Collin-Dufresne *et al.* (2001) find that a common latent factor in corporate bonds, drives mostly credit spreads' changes.

Differently, Campbell & Taksler (2002) study the effect of equity volatility on corporate bond yields. They find that idiosyncratic volatility is as much important as credit ratings in explaining cross-sectional variation in yields. On average, idiosyncratic risk and ratings explain each one one third of such variations. In the same line, Malkiel & Xu (2002) conclude that idiosyncratic volatility explains cross-sectional expected asset returns more than the CAPM beta coefficient or size measures do. The linear as well as the non-linear influences of the beta on returns are mitigated. On the other hand, Goyal & Santa-Clara (2003) study the average stock risk in addition to market risk. To this end, they estimate the average stock risk (i.e., cross-sectional average stock variances) according to the methodology of Campbell *et al.* (2001). First, they find that the average stock risk is mainly driven by idiosyncratic risk (see Campbell & Taksler [2002] for details). Then, even if the market variance has no predictive power for the market return, a significant positive relation prevails between the average stock variance and the market return. Finally, Stein et al. (2003) analyze default risk in the lens of idiosyncratic (i.e., firm specific) risk and systematic (i.e., macro-economic and industry) risk. Given their results, idiosyncratic information is most important for predicting middle market defaults.

The documented research shed light on the typology and components of credit risk. Given the state of the art, credit risk has to be envisioned along with systematic risk and idiosyncratic risk. Such a typology is used by Gatfaoui (2003) to price risky debt in a Merton (1974) framework where diffusion parameters are constant. However, under its constant parameter assumptions, Merton's model leads to implied spreads which are too low in comparison with observed credit spreads. Indeed, Eom *et al.* (2003) show that adding stochastic interest rates correlated with the firm value in Merton's model fails to offset this prediction problem about implied credit spreads. To solve this problem, Hull *et al.* (2003) study the implications of Merton's model about implied at-the-money volatility and volatility skews. Their results lead to several findings which are supported by empirical data. First, implied volatility is sufficient to predict credit spreads. Second, there is a positive relationship between credit spreads and implied volatility, and between volatility skews and both implied credit spreads and implied volatility. Third, implied volatility plays a major role in explaining credit spreads. Finally, as historical volatility leads to implied credit spreads which underestimate their observed counterparts, the implied volatility approach exhibits a superior performance in predicting credit spreads over time. Such findings are consistent with Black & Scholes (1973) option pricing type models. Specifically, such models exhibit a volatility smile effect (i.e., the implied volatility is a U-shaped function of the option's moneyness) which is determined by stochastic volatility, maturity and systematic risk among others (see Äijö [2003], Duque & Lopes [2000] for details, and Psychoyios *et al.* [2003] for a survey about stylized facts of volatility as well as stochastic volatility models).

In the light of such results, we extend the work of Gatfaoui (2003) to stochastic parameters in order to price risky debt in a Merton framework with stochastic volatility. For this purpose, our paper is organized as follows. Section 2 states the basis for the stochastic functionals-based credit pricing model. Then, we underline the link with stochastic volatility models and introduce our pricing methodology. Under the specification of our stochastic functionals introduced in section 3, we formalize our stochastic volatility model. Then, section 4 undertakes a simulation study to assess the impact of stochastic volatility both on risky debt valuation and credit spreads. Finally, section 5 draws some concluding remarks.

2 The general model

We introduce here the dynamic pricing of a firm's risky debt along with the valuation of its total assets (i.e., firm assets value). The mathematical background as well as the pricing methodology are introduced along with the setting proposed by Gatfaoui (2003).

2.1 Basic setting

Consider a probability space (Ω, F, P) with a related natural filtration $F_t = \sigma\left(w_s, 0 \leq s \leq t\right)$ where $w_t^{'} = \left(W_t^X, W_t^I\right)$. $\left(W_t^X\right)$ and $\left(W_t^I\right)$ are two independent P-Brownian motions and represent the public information set at current time t. Let $\mathbb{F} = \left(F_t\right)_{t \in [0,T]}$ be the P-augmentation of F_t with $T < \infty$. All the assumptions prevailing in the Black & Scholes (1973) and Merton (1974) frameworks are supposed to hold except that our diffusion parameters are rather stochastic than constant.[1] Briefly, there is no arbitrage opportunity and the spot risk free rate of interest r is constant.

Consider a firm whose assets value at current time t is V_t, which is an F_t-adapted process. This firm is supposed to issue two kinds of financial assets, namely a risky debt represented by a discount bond maturing at time T with terminal value B (i.e., promised payment to debtholders), and no dividend paying equity. The firm's potential default can only occur at time T. Let X_t and I_t be the systematic and the idiosyncratic risk factors respectively, affecting any financial asset in the market, and therefore the firm value. These two risk factors are F_t-adapted processes whose dynamics are described below:

$$\frac{dX_t}{X_t} = \mu_X\left(t, X_t\right) dt + \sigma_X\left(t, X_t\right) dW_t^X \tag{1}$$

$$\frac{dI_t}{I_t} = \mu_I\left(t, I_t\right) dt + \sigma_I\left(t, I_t\right) dW_t^I \tag{2}$$

where the functionals $\mu_X\left(t, X_t\right)$, $\sigma_X\left(t, X_t\right)$, $\mu_I\left(t, I_t\right)$ and $\sigma_I\left(t, I_t\right)$ are continuous F_t-measurable functions on $[0,T] \times \mathbb{R}$. To ensure strong solutions to the previous SDE, we assume that the functionals are also bounded.[2] For this purpose, we set whatever $t \in [0,T]$ and $X_t, I_t \in \mathbb{R}$:

$$\mu_X^l < \mu_X\left(t, X_t\right) < \mu_X^u \quad \sigma_X^l < \sigma_X\left(t, X_t\right) < \sigma_X^u \tag{3}$$

$$\mu_I^l < \mu_I\left(t, I_t\right) < \mu_I^u \quad \sigma_I^l < \sigma_I\left(t, I_t\right) < \sigma_I^u \tag{4}$$

with μ_X^l, μ_X^u, σ_X^l, σ_X^u, μ_I^l, μ_I^u, σ_I^l, σ_I^u constant values such that $\sigma_X^l > 0$ and $\sigma_I^l > 0$. As introduced in Gatfaoui (2003), the dependence of the firm assets value *vis-à-vis* the two risk factors is as follows:

$$V_t = X_t^{\beta} I_t \tag{5}$$

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[1] Which implies the incompleteness of the financial market.

[2] The reader could refer to Karatzas & Shreve (1991) for explanations.

where β is the beta of the firm assets value (i.e., deterministic constant) as defined by the CAPM. Recall that X encompasses market conditions as well as business cycle, and I encompasses firm specific features such as default risk and liquidity risk.

As underlined in Gatfaoui (2003), observing simultaneously the systematic risk factor X and the idiosyncratic risk factor I is equivalent to observe simultaneously the firm assets value V and its specific risk factor I. Moreover, applying the generalized Ito's lemma leads to the following expression for the firm assets value under the original probability P:

$$\frac{dV_t}{V_t} = \mu_V\left(t, V_t, I_t\right)\, dt + \left[\beta \sigma_X\left(t, X_t\right)\, dW_t^X + \sigma_I\left(t, I_t\right)\, dW_t^I\right] \tag{6}$$

with

$$\mu_V\left(t, V_t, I_t\right) = \beta \mu_X\left(t, \left(\frac{V_t}{I_t}\right)^{\frac{1}{\beta}}\right) + \mu_I\left(t, I_t\right) + \frac{1}{2}\beta\left(\beta - 1\right)\sigma_X^2\left(t, \left(\frac{V_t}{I_t}\right)^{\frac{1}{\beta}}\right) \tag{7}$$

If we set $d\left\langle V, I\right\rangle_t = \rho\left(t, V_t, I_t\right)\, dt$, then the instantaneous correlation between the firm value and its idiosyncratic risk factor is:

$$\rho\left(t, V_t, I_t\right) = \frac{\sigma_I\left(t, I_t\right)}{\sigma_V\left(t, V_t, I_t\right)} \tag{8}$$

where

$$\sigma_V\left(t, V_t, I_t\right) = \sqrt{\beta^2 \sigma_X^2\left(t, \left(\frac{V_t}{I_t}\right)^{\frac{1}{\beta}}\right) + \sigma_I^2\left(t, I_t\right)} \tag{9}$$

is the global volatility of the instantaneous return of the firm assets value. This global stochastic volatility depends on the beta parameter, and the respective volatilities of the two risk factors affecting the firm value.[3] This specification follows the results of Campbell *et al.* (2001) who show that the global volatility of any financial asset has both a systematic (i.e., systematic volatility) component and an idiosyncratic (i.e., a specific volatility) component. Even if idiosyncratic risk plays an increasing role (given market history), the global volatility remains driven by its systematic component (i.e., market volatility).

Moreover, considering the expression (8) of the correlation coefficient and the firm value's dynamic (6), the diffusion of the firm assets value takes the new form in the historical universe:

$$\frac{dV_t}{V_t} = \mu_V\left(t, V_t, I_t\right)\, dt + \sigma_V\left(t, V_t, I_t\right)\left[\begin{array}{c} \varsigma_\beta \sqrt{1 - \rho^2\left(t, V_t, I_t\right)}\, dW_t^X \\ + \rho\left(t, V_t, I_t\right)\, dW_t^I \end{array}\right] \tag{10}$$

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[3] Since the functionals of our diffusions are bounded, the global volatility is therefore bounded as a continuous function of these functionals.

with $\varsigma_\beta = sgn(\beta)$ represents the sign of beta (i.e., $\varsigma_\beta = 1$ if $\beta > 0$ and $\varsigma_\beta = -1$ if $\beta < 0$).

Therefore, describing the firm value's dynamic with relations (1), (2) and (5) is equivalent to characterize the firm assets value with relations (6) or, equivalently (10), and (2). As this specification introduces two risk factors whilst we only observe the firm assets value, we therefore lie in an incomplete market setting. Such a setting appears to be equivalent to a stochastic volatility framework provided that we assume the global volatility $\sigma_V(t, V_t, I_t)$ to be non zero whatever $t, V_t, I_t \in [0,T] \times \mathbb{R}^2$.

2.2 Stochastic volatility and Merton's pricing

Indeed, relations (10) and (2) are analogous to the state diffusion and stochastic volatility model proposed by Hofmann *et al.* (1992). In this case, we have more risk factors (i.e., systematic and idiosyncratic risks) than existing or, equivalently, primary assets (i.e., firm value). Consequently, we are unable to give a unique price to any contingent claim on the firm assets value. At best, we can define bounds[4] for such a price or minimize the uncertainty while computing a price. We are going to address these two points therein.

First, to shed light on the stochastic volatility analogy, we need to assume that the firm value's global volatility $\sigma_V(t, V_t, I_t)$ is a $C^{1,2}\left([0,T] \times \mathbb{R}^2\right)$ function (i.e., continuous and, one time derivable relative to time, and two times derivable relative to its two last arguments). Then, consider the global variance of the firm value such that $F(t, V_t, I_t) = \sigma_V^2(t, V_t, I_t)$, and let $F_x(t, V_t, I_t) = \frac{\partial F(t,V_t,I_t)}{\partial x}$ and $F_{xx}(t, V_t, I_t) = \frac{\partial^2 F(t,V_t,I_t)}{\partial x^2}$. Applying multivariate Ito's lemma to the global variance of the firm value's instantaneous return gives:

$$dF(t, V_t, I_t) = Trend\, dt + Vol_1\, dW_t^X + Vol_2\, dW_t^I \tag{11}$$

where

$$\begin{aligned} Trend \;=\;& F_t(t, V_t, I_t) + F_V(t, V_t, I_t) V_t \mu_V(t, V_t, I_t) + F_I(t, V_t, I_t) \mu_I(t, I_t) I_t \\ & + \frac{F_{VV}(t, V_t, I_t)}{2} \sigma_V^2(t, V_t, I_t) V_t^2 + \frac{F_{II}(t, V_t, I_t)}{2} \sigma_I^2(t, I_t) I_t^2 \\ & + F_{VI}(t, V_t, I_t) \sigma_V(t, V_t, I_t) V_t \rho(t, V_t, I_t) \sigma_I(t, I_t) I_t \end{aligned} \tag{12}$$

$$Vol_1 = F_V(t, V_t, I_t) \sigma_V(t, V_t, I_t) V_t \varsigma_\beta \sqrt{1 - \rho^2(t, V_t, I_t)} \tag{13}$$

$$Vol_2 = F_V(t, V_t, I_t) \sigma_V(t, V_t, I_t) V_t \rho(t, V_t, I_t) + F_I(t, V_t, I_t) \sigma_I(t, I_t) I_t \tag{14}$$

Hence, the stochastic volatility framework becomes more evident. Indeed, the dynamics of the firm value and the global variance of the its instantaneous

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[4] See Frey & Sin (1999) among others for a proof.

return depend on two stochastic parts which are correlated.[5] This setting has important implications for Merton type pricing models.

According to Merton (1974), the firm assets value is equal to the sum of its equity value $E(V,\tau)$ and debt value $D(V,\tau)$:

$$V_t = E(V_t,\tau) + D(V_t,\tau) \tag{15}$$

with $\tau = T - t$ time to maturity and given the following conditions:[6]

$$\begin{aligned} E(0,\tau) &= 0 && \text{(16a)} \\ E(V_t,\tau) &\geq 0 && \text{(16b)} \\ E(V_T,0) &= \max(0, V_T - B) = (V_T - B)^+ && \text{(16c)} \end{aligned}$$

The option nature of a firm's balance sheet implies that equity are thought as a European call on the firm value with a strike equal to the promised payment (to debtholders) at the firm debt's maturity. Hence, valuing the risky debt requires to value the European call above mentioned. However, as we lie in an incomplete market setting, there exists an infinity of equivalent martingale measures allowing to price the European call under our assumptions.[7]

Nevertheless, among the set of equivalent martingale measures compatible with V, there exists a unique equivalent martingale measure $\hat{P}$ which minimizes the uncertainty or, equivalently, the relative entropy measure (see Delbaen & Schachermayer [1996], Föllmer & Schweizer [1991], Gouriéroux *et al.* [1998] and Musiela & Rutkowski [1998]). $\hat{P}$ is called the minimal equivalent martingale measure[8] and is uniquely[9] defined by its Girsanov density as follows:

$$\begin{aligned} \hat{L}(t) &= \left.\frac{d\hat{P}}{dP}\right|_{F_t} \\ &= \exp\left\{-\int_0^t \alpha_1(s,V_s,I_s)\,dW_s^X - \int_0^t \alpha_2(s,V_s,I_s)\,dW_s^I\right\} \qquad (17) \\ &\quad \times \exp\left\{-\frac{1}{2}\int_0^t \alpha^2(s,V_s,I_s)\,ds\right\} \end{aligned}$$

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[5] In general, the stochastic variance and the firm value are non perfectly correlated. We would like to underline two points here. First, assume that the volatility of the systematic risk factor X is at best a deterministic function of time, then the global variance of the firm value's instantaneous return is independent of X. In this case, we have a non perfect correlation between the firm value and its variance $Corr\left(d\sigma_V^2(t,V_t,I_t), dV_t\right) = \rho(t,V_t,I_t)$ since $F_V(t,V_t,I_t) = 0$. Second, assume that the volatility of the idiosyncratic risk factor I is at best a deterministic function of time, then the global variance of the firm value's instantaneous return is independent of I. In this case, we have a perfect correlation between the firm value and its variance $Corr\left(d\sigma_V^2(t,V_t,I_t), dV_t\right) = 1$ since $F_I(t,V_t,I_t) = 0$.

[6] We also have $E(V,\tau) = V_t - D(V,\tau)$.

[7] Refer to Mele & Fornari (2000), chapter 3, for details. In the other hand, choosing the risk free asset as a numeraire, the discount price of the firm value becomes a semi-martingale under the historical probability P.

[8] Like Hull & White (1987) framework, the fluctuations of stochastic volatility generate a risk which is not compensated. This feature explains the existence of $\hat{P}$. Refer to the appendix for an explanation about the existence and unicity of the minimal martingale measure.

[9] Refer to Karatzas & Shreve (1991) and Karatzas (1996) for explanations.

where

$$\begin{aligned}
\alpha_1(t,V_t,I_t) &= \frac{\mu_V(t,V_t,I_t)-r}{\sigma_V(t,V_t,I_t)}\varsigma_\beta\sqrt{1-\rho^2(t,V_t,I_t)} && \text{(18a)}\\
\alpha_2(t,V_t,I_t) &= \frac{\mu_V(t,V_t,I_t)-r}{\sigma_V(t,V_t,I_t)}\rho(t,V_t,I_t) && \text{(18b)}\\
\alpha(t,V_t,I_t) &= \frac{\mu_V(t,V_t,I_t)-r}{\sigma_V(t,V_t,I_t)} && \text{(18c)}
\end{aligned}$$

$\alpha(t,V_t,I_t)$ is the global market risk premium due to the global risk (i.e., aggregate risk) borne by the firm value whereas $\alpha_1(t,V_t,I_t)$ and $\alpha_2(t,V_t,I_t)$ are the market risk premia related respectively to the systematic and the idiosyncratic risk factors affecting the firm value.

Consequently, the dynamic of $\ln(V)$ under the minimal martingale measure $\hat{P}$ writes after applying generalized Ito's lemma:

$$\begin{aligned}
d\ln(V_t) &= \left[r-\frac{\sigma_V^2(t,V_t,I_t)}{2}\right]dt+\sigma_V(t,V_t,I_t)\varsigma_\beta\sqrt{1-\rho^2(t,V_t,I_t)}d\hat{W}_t^X\\
&\quad+\sigma_V(t,V_t,I_t)\rho(t,V_t,I_t)d\hat{W}_t^I && \text{(19)}
\end{aligned}$$

where

$$\begin{aligned}
d\hat{W}_t^X &= \alpha_1(t,V_t,I_t)\,dt+dW_t^X && \text{(20a)}\\
d\hat{W}_t^I &= \alpha_2(t,V_t,I_t)\,dt+dW_t^I && \text{(20b)}
\end{aligned}$$

are two independent (and F_t-adapted) $\hat{P}$-Brownian motions.[10] Therefore, we know the dynamic followed by the firm value under the minimal martingale measure.

Under our incomplete market assumption, the no arbitrage principle and the minimal martingale measure allow us to give a price to the European call[11] on V (i.e., the firm's equity). Indeed, as the current value of the European call equals the expected discount value of its terminal payoff, the firm's equity are valued as follows:

$$E(V_t,\tau)=E^{\hat{P}}\left[e^{-r\tau}(V_T-B)^+\middle|F_t\right] \qquad (21)$$

The use of Monte Carlo method (see Jäckel [2002]) allows to compute this quantity, and finally, to estimate the debt's value since we have:

$$D(V_t,\tau)=V_t-E(V_t,\tau)=V_t-E^{\hat{P}}\left[e^{-r\tau}(V_T-B)^+\middle|F_t\right] \qquad (22)$$

The stochastic volatility framework may have some nice properties since it adds flexibility to asset pricing, and then can improve Merton's debt valuation. However, the computational cost here may be high since we need to simulate two Brownian motion paths. Nevertheless, such a setting may be extremely simple in some cases and highly useful for debt's pricing. We are going to focus on some kind of useful and optimal simplification for Merton debt pricing.

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[10] See the appendix about the minimal martingale measure for pricing details.

[11] For example, refer to Hofmann *et al.* (1992) for details about option pricing in incomplete markets. See also El Karoui *et al.* (1998) for the properties of the Black & Scholes formula.

3 A stochastic volatility model

In this section, we employ our previous stochastic volatility framework to price risky debt. We start from general case to concentrate on a particular case while specifying our functionals. Our simplified framework allows for a tractable and easy computation of the firm's debt.

3.1 Model specification

To achieve a convenient degree of simplification, we make two major assumptions. First, we assume that the volatility functional of the idiosyncratic risk factor depends only on I. Second, we assume that the drift and volatility functionals of the systematic risk factor are deterministic functions of time. Our assumptions are motivated by the empirical features exhibited by equity volatility. In order to account for realistic features[12] of equity volatility, the stochastic global volatility of the firm value has to be a stationary and mean reverting process (to encompass some shock effects on volatility). We are going to explain therein how our assumptions fit these empirical characteristics.

Let ω and δ depend on prevailing financial and economic conditions, and assume that whatever[13] $t \in]0,T]$ for $\mu_X(t)$ and $\sigma_X(t)$, and whatever $t \in [0,T]$ else:

$$\begin{aligned} \mu_X(t) &= \omega t^{\delta} && \text{(23a)} \\ \sigma_X(t) &= \gamma t^{\alpha} && \text{(23b)} \\ \mu_I(I_t) &= \lambda\left(\frac{\epsilon}{I_t} - 1\right) && \text{(23c)} \\ \sigma_I(I_t) &= \Omega\sqrt{I_t} && \text{(23d)} \end{aligned}$$

where $\gamma > 0$, $\alpha < 0$, $\lambda > 0$, $\epsilon > 0$ and, $\Omega > 0$ are constant parameters such that:

$$\frac{dX_t}{X_t} = \omega t^{\delta}\, dt + \gamma t^{\alpha}\, dW_t^X \tag{24}$$

$$dI_t = \lambda(\epsilon - I_t)\, dt + \Omega I_t \sqrt{I_t}\, dW_t^I \tag{25}$$

We aslo assume that $\mu_X(0) = \mu_0$ and $\sigma_X(0) = \sigma_0$ where μ_0 and σ_0 are bounded constant values. Our setting implies that the variance of the instantaneous return of the firm value takes the new form:

$$\sigma_V^2(t, I_t) = \beta^2\sigma_X^2(t) + \sigma_I^2(I_t) = \beta^2\gamma^2 t^{2\alpha} + \Omega^2 I_t = F(t, I_t) \tag{26}$$

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[12] Refer to Psychoyios *et al.* (2003) and Phoa (2003) among others for brief explanations.

[13] We can also assume that debt is issued at time $t_0 > 0$ and matures at time $T = t_0 + \tau$ where τ is the initial lifetime of debt.

whatever $t \in \left]0, T\right]$ with $F(0, I_0) = \sigma_V^2(0, I_0) = \beta^2\sigma_0^2 + \Omega^2 I_0$ being bounded. Our specification is consistent with Andersen *et al.* (2001) who study model-free measures of volatility and correlation for daily stock prices. Those authors analyze the time-varying features of stock prices' returns (see Bekaert & Wu [2000], Bollerslev & Mikkelsen [1999], Campbell *et al.* [2001] and Christensen & Prabhala [1998] among others). Their work highlight two main findings. First, variances exhibit some systematic common component in their evolution. Second, an asymmetric relationship prevails between returns and volatility.

Then, we get:

$$\begin{aligned} F_V(t, I_t) &= F_{VV}(t, I_t) = F_{II}(t, I_t) = F_{IV}(t, I_t) = 0 && \text{(27a)} \\ F_I(t, I_t) &= \Omega^2 && \text{(27b)} \\ F_t(t, I_t) &= 2\alpha\beta^2\gamma^2 t^{2\alpha-1} && \text{(27c)} \end{aligned}$$

In this case, the variance satisfies the following SDE:

$$dF(t, I_t) = \left[F_t(t, I_t) + F_I(t, I_t)\,\mu_I(I_t)\,I_t\right] dt + F_I(t, I_t)\,\sigma_I(I_t)\,I_t\,dW_t^I \qquad (28)$$

which writes in the historical universe:

$$dF(t, I_t) = \left[2\alpha\beta^2\gamma^2 t^{2\alpha-1} + \Omega^2\lambda(\epsilon - I_t)\right] dt + \Omega^3 I_t\sqrt{I_t}\,dW_t^I \qquad (29)$$

or, equivalently, under the minimal martingale measure:

$$dF(t, I_t) = \left[2\alpha\beta^2\gamma^2 t^{2\alpha-1} + \Omega^2\lambda(\epsilon - I_t) - \Omega^3 I_t\sqrt{I_t}\alpha_2(t, I_t)\right] dt + \Omega^3 I_t\sqrt{I_t}\,d\hat{W}_t^I \qquad (30)$$

with

$$\begin{aligned} \alpha_2(t, I_t) &= \frac{\mu_V(t, I_t) - r}{\sigma_V(t, I_t)}\rho(t, I_t) && \text{(31a)} \\ \rho(t, I_t) &= \frac{\sigma_I(t, I_t)}{\sigma_V(t, I_t)} && \text{(31b)} \\ \mu_V(t, I_t) &= \beta\mu_X(t) + \mu_I(t, I_t) + \frac{1}{2}\beta(\beta - 1)\sigma_X^2(t) && \text{(31c)} \end{aligned}$$

In the original universe, such a diffusion process behaves almost like a mean reverting square-root process except that the random shocks affecting its trend are higher in magnitude.[14] The next section undertakes some simulations about

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[14] When choosing $\mu_I(I_t) = \frac{\lambda\epsilon}{\Omega} - \left(\frac{\Omega}{2} - \lambda\right)\ln(I_t)$ and $\sigma_I(I_t) = \sqrt{\Omega\ln(I_t)}$ provided that $I_t > 1$, we would have a stochastic variance such that $F(t, I_t) = \beta^2\gamma^2 t^{2\alpha} + \Omega\ln(I_t)$. Hence, for t tending towards infinity, we have $F(t, I_t) = \Omega\ln(I_t)$ in which case the variance follows asymptotically a mean reverting square root process such that $dF(t, I_t) = \lambda\left[\epsilon - F(t, I_t)\right] dt + \Omega\sqrt{F(t, I_t)}\,dW_t^I$. However, we prefer to avoid logarithmic specifications which require values for random variables to be above unity.

I_t and $F(t, I_t)$ for given values of parameters and with varying β and λ parameters. The related values of the correlation coefficient $\rho(t, I_t)$ are also displayed in the appendix.

Moreover, when time t tends towards infinity, the stochastic variance of the firm value tends towards $F(t, I_t) = \Omega^2 I_t$ such that the volatility reads $\sqrt{F(t, I_t)} = \Omega\sqrt{I_t}$. In the same way, the diffusion above mentioned takes asymptotically the new form presented underneath:

$$dF(t, I_t) = \left[\Omega^2 \lambda (\epsilon - I_t)\right] dt + \Omega^3 I_t \sqrt{I_t}\, dW_t^I \tag{32}$$

which rewrites:

$$dF(t, I_t) = \lambda \left[\Omega^2 \epsilon - F(t, I_t)\right] dt + F(t, I_t) \sqrt{F(t, I_t)}\, dW_t^I \tag{33}$$

Such a specification has an important implication. If the variance tends asymptotically to be zero, then its diffusion becomes $dF(t, I_t) = \lambda\Omega^2\epsilon\, dt > 0$. Therefore, when t tends towards infinity and the variance is zero, the zero threshold becomes a reflecting barrier for the variance of the firm value's instantaneous return.[15]

Our specification implies that the variance depends only on the idiosyncratic risk factor in terms of randomness. Indeed, the stochastic volatility here comes clearly from the non observability of the idiosyncratic risk factor. Namely, stochastic volatility is due to the intrinsic risk of the firm value because such a risk is non tradable. This feature leads to some nice properties for the pricing of the firm's debt.

3.2 Implication for debt pricing

The stochastic volatility framework we introduced previously has some nice implications. Indeed, the randomness of the firm value's variance depends only on the idiosyncratic risk factor (see relation (29)). Moreover, the correlation coefficient expresses:

$$\rho(t, I_t) = \frac{\Omega\sqrt{I_t}}{\sqrt{\beta^2\gamma^2 t^{2\alpha} + \Omega^2 I_t}} \tag{34}$$

and the firm value's dynamic under the minimal martingale measure $\hat{P}$ then reads:

$$\begin{aligned} d\ln(V_t) &= \left[r - \frac{\sigma_V^2(t, I_t)}{2}\right] dt + \sigma_V(t, I_t)\, \varsigma_\beta \sqrt{1 - \rho^2(t, I_t)} d\hat{W}_t^X \qquad (35) \\ &\quad + \sigma_V(t, I_t)\, \rho(t, I_t)\, d\hat{W}_t^I \end{aligned}$$

---

[15] This observation is important insofar as empirical features of equity volatility exhibit some stationarity and mean reverting characteristics. In the asymptotic case, $\Omega^2\epsilon$ can be thought as the long-run mean of the firm value's stochastic variance, and λ may be seen as the velocity to revert to the long-run mean.

After integrating on the time subset $[t, T]$, we get:

$$\begin{aligned} \ln\left(\frac{V_T}{V_t}\right) &= \left(r - \frac{\bar{\sigma}_V^2}{2}\right)\tau + \int_t^T \sigma_V\left(s, I_s\right)\varsigma_\beta\sqrt{1-\rho^2\left(s, I_s\right)}d\hat{W}_s^X \\ &\quad + \int_t^T \sigma_V\left(s, I_s\right)\rho\left(s, I_s\right)d\hat{W}_s^I \end{aligned} \tag{36}$$

with

$$\bar{\sigma}_V^2 = \frac{1}{\tau}\int_t^T \sigma_V^2\left(s, I_s\right)ds \tag{37}$$

where $\bar{\sigma}_V^2$ is the firm value's average variance over the time to maturity of the debt (i.e., the remaining life of the European call or firm's equity).

Let introduce $G_t = F_t \cup \{I_s, t \leq s \leq T\}$ and compute the two first moments of the probability distribution of $\ln\left(\frac{V_T}{V_t}\right)$ conditional on G_t. We obtain the next expressions:

$$\begin{array}{r} E^{\hat{P}}\left[\ln\left(\frac{V_T}{V_t}\right)\middle| G_t\right] = \left(r - \frac{\bar{\sigma}_V^2}{2}\right)\tau \\ Var^{\hat{P}}\left[\ln\left(\frac{V_T}{V_t}\right)\middle| G_t\right] = \bar{\sigma}_V^2\,\tau \end{array} \tag{38}$$

Hence, conditional on G_t, the firm value's natural logarithm $\ln\left(\frac{V_T}{V_t}\right)$ follows a normal law (i.e., the firm value follows a lognormal law) with a volatility parameter equal to $\sqrt{\bar{\sigma}_V^2}$.

On the other hand, recall the value (21) of the firm's equity or, equivalently, of the European call on the firm value under the minimal martingale measure. Applying the iterated expectations theorem allows to write that $E\left(V_t, \tau\right) = E^{\hat{P}}\left[E^{\hat{P}}\left[e^{-r\tau}\left(V_T - B\right)^+\middle| G_t\right]\middle| F_t\right]$. However, given the law of $\ln\left(\frac{V_T}{V_t}\right)$ conditional on G_t, the equity value then reads:

$$E\left(V_t, \tau\right) = E^{\hat{P}}\left[C_{BS}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right)\middle| F_t\right] \tag{39}$$

where $C_{BS}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right)$ is the Black & Scholes (1973) price[16] employed with an average time-dependent volatility. Consequently, the equity value is the average Black & Scholes European call price over each possible volatility path.

Our deterministic systematic risk volatility assumption leads to an optimal Monte Carlo European call pricing. Indeed, it requires only to generate one Brownian motion, namely the randomness affecting the stochastic volatility (i.e., the idiosyncratic risk Brownian motion). From this setting follows a simple computation of the price of the firm's debt since:

$$D\left(V_t, \tau\right) = V_t - E^{\hat{P}}\left[C_{BS}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right)\middle| F_t\right] \tag{40}$$

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[16] Refer to appendix for details.

Given the information set available at current time, we are able to give a price to the firm's debt in such a way that uncertainty is minimized. Moreover, as our functional diffusion parameters are bounded on $]0,T]$ and their starting values at time $t=0$ are bounded, the stochastic volatility is also bounded whatever $t \in [0,T]$:

$$\sigma_V^l < \sigma_V\left(t, I_t\right) < \sigma_V^u \tag{41}$$

with $\sigma_V^l = \beta^2 \sigma_X^{l2} + \sigma_I^{l2} > 0$ and $\sigma_V^u = \beta^2 \sigma_X^{u2} + \sigma_I^{u2}$. Because of this, the Black & Scholes call price is bounded[17] by:

$$C_{BS}\left(\tau, r, V_t, B, \sigma_V^l\right) < C_{BS}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right) < C_{BS}\left(\tau, r, V_t, B, \sigma_V^u\right) \tag{42}$$

which implies that both equity and debt values of the firm are bounded.

Hence, we are able to price corporate debt under the minimal martingale measure. We are also able to give bounds to such a debt by establishing a value bracket which depends on the magnitude of the variations of the volatility of the firm value's rate of return. Our assumptions imply that the volatility's trend is driven by systematic risk whereas idiosyncratic risk affects this trend through shocks. Consequently, the magnitude of the variations of the firm value's global (i.e., aggregate) volatility is driven by the impact of systematic and idiosyncratic risk factors. We are going to observe such impacts using simulation techniques in the rest of the paper.

4 Simulation study

We undertake a simulation study using a Monte Carlo methodology. For this purpose, Monte Carlo accelerators are used to examine the behaviors of the debt's pricing as well as its related credit spread in a stochastic volatility setting.

4.1 Volatility and debt

We simulate successively the behaviors of the stochastic volatility and its impact on the evolution of the firm's debt and equity. Then, we plot the paths obtained for each of these random variables or display their average values in tables. Assuming that the initial debt's time to maturity[18] is $\tau = T - t = 10$ years, we state the following parameter values:

$$\begin{array}{lll} \alpha = -\frac{1}{4} & \epsilon = 0.5 & \gamma = 1 \\ I_t = 3.5 & \Omega = \sqrt{\frac{0.2}{\epsilon}} & F\left(t, I_t\right) = \beta^2 t^{-\frac{1}{2}} + 0.4\, I_t \end{array} \tag{43}$$

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[17] See Frey & Sin (1999) for a rigorous proof.

[18] We assume that debt is issued at time $t > 0$ and matures at time $T = t + \tau$.

Daily values $\sqrt{F(t, I_t)}$ are computed for different values of beta[19] and lambda parameters (i.e., $\beta = 0, 0.5, 1, 1.5$ and $\lambda = 0.2, 1, 5$) for time t running from $T-\tau$ to T. The related plots are displayed below for the different values taken by lambda parameter.

[Insert figures 1, 2 and 3 here]

The higher the lambda parameter is, the more stable are the evolutions and convergence to their long-run means of our stochastic variance $F(t, I_t)$, idiosyncratic factor I_t and correlation coefficient $\rho(t, I_t)$. On the other hand, the greater the beta parameter is, the lower the firm value's variance is for any fixed λ. In contrast, the higher the beta is, the higher the correlation coefficient becomes.[20] These results are summarized in the table below which displays the average values of our simulated variables on $[t, T]$.

Table 1: Average values for daily simulated variables

Variable	$\lambda \setminus \beta$	0	0.5	1	1.5
σ_V (%)	0.2	71.69	80.53	101.19	127.24
ρ	0.2	1.00	0.85	0.65	0.51
σ_V (%)	1	64.10	73.36	94.87	121.76
ρ	1	1.00	0.84	0.63	0.49
σ_V (%)	5	47.79	58.57	82.68	111.76
ρ	5	1.00	0.81	0.58	0.43

As I is independent of β, we give successively its average simulated values, namely 2.10, 1.50 and 0.60 for λ being equal to 0.2, 1 and 5 respectively.

We further set $I_t = 0.1$ and assume $\omega = 0$ which implies that $\mu_X(t) = 0$ whatever t. The diffusion of the idiosyncratic factor under the minimal martingale measure then writes:

$$dI_t = \left[\lambda(\epsilon - I_t) - \frac{\Omega^2 I_t^2}{\sigma_V^2(t, I_t)}(\mu_V(t, I_t) - r)\right] dt + \Omega I_t \sqrt{I_t}\, d\hat{W}_t^I \tag{44}$$

with $\sigma_V^2(t, I_t) = F(t, I_t)$ and $\mu_V(t, I_t) = \lambda\left(\frac{0.5}{I_t} - 1\right) + \frac{1}{2}\beta(\beta - 1)t^{-\frac{1}{2}}$. And, the related average stochastic variance $\bar{\sigma}_V^2$ conditional on G_t reads:

$$\bar{\sigma}_V^2 = \frac{1}{\tau}\int_t^T \sigma_V^2(s, I_s)\, ds = \frac{2\beta^2}{\tau}\left(\sqrt{T} - \sqrt{t}\right) + \frac{\Omega^2}{\tau}\int_t^T I_s ds \tag{45}$$

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[19] Recall that when $\beta = 0$, we have $V_t = I_t$.

[20] Our comments also concern the graphs displayed in the appendix. Notice that the idiosyncratic risk factor I_t depends only on λ and not on β.

From formula (40), the computation of the firm's debt requires the estimation of the call's price. To achieve this estimation, we use monte carlo simulation methods with monte carlo accelerators based on antithetic variables.[21] Namely, let $nsim$ be the number of simulations and $C_{BS,k}(\cdot)$ be the call's price associated to the k-th simulation. Then, the estimated value for equity conditional on the information set available at time t is the arithmetic mean of the simulated variables, namely:

$$
\begin{aligned}
E(V_t,\tau) &= E^{\hat{P}}\left[C_{BS}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right) \middle| F_t \right] \qquad (46) \\
&= \frac{1}{nsim} \sum_{k=1}^{nsim} \frac{C_{BS,k}\left(\tau, r, V_t, B, \sqrt{\bar{\sigma}_V^2}\right) + C_{BS,k}\left(\tau, r, V_t, B, -\sqrt{\bar{\sigma}_V^2}\right)}{2}
\end{aligned}
$$

Therefore, we realize our monthly simulations with the following parameter values where V and B are expressed in billions of dollars:

$$r = 8\% \qquad B = 13 \qquad V_t = 52 \tag{47}$$

Notice that our examination uses varying moneyness (i.e., a varying ratio $\frac{V_t}{B}$ whatever t), volatility and time to maturity insofar as we focus here on the combined effect of these determinants. After $nsim = 1000$ simulations, we get the following average values of the firm's debt for various β and λ parameters' levels:

Table 2: Average monthly simulated values of the firm's debt

$\lambda \setminus \beta$	-1.5	-1	-0.5	0	0.5	1	1.5
0.2	2.25	4.93	8.02	8.94	7.65	4.24	1.79
1	3.48	4.54	7.47	8.94	6.86	3.75	1.82
5	3.70	4.34	7.08	8.94	6.72	3.65	1.88

Whatever λ values, the firm's average debt is a concave function of β with a maximal value reached for $\beta^* = 0$. When $|\beta| < 1.5$, the debt is generally a decreasing function of λ. In contrast, when $|\beta| = 1.5$, the reverse behavior takes place. Moreover, the average debt's value remains constant whatever λ when $\beta = 0$ since our debt's evolution does not depend on λ under the minimal martingale measure. Indeed, this comes from the fact that our debt's value is computed from equity, and equity value does no depend on λ under the minimal martingale measure.[22]

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[21] Refer to Jäckel (2002), Ripley (1987) and Rubinstein (1981) for details.

[22] Refer to the appendix for details. In fact, equity are considered as a function of both the firm assets value and the idiosyncratic risk factor as indicated by the European call's expression. However, neither the firm assets value nor the idiosyncratic factor depends on λ parameter when β is zero.

Table 3: Average monthly simulated values of the firm's equity

$\lambda \setminus \beta$	-1.5	-1	-0.5	0	0.5	1	1.5
0.2	50.07	61.55	70.32	71.52	56.83	59.55	129.46
1	51.19	60.47	67.87	71.52	52.76	58.64	135.57
5	54.15	62.77	70.97	71.52	57.51	65.90	154.30

Generally speaking, equity are a non monotonous function of β parameter. This function increases for growing $\beta \leq 0$, decreases in $\beta = 0.5$, and goes on increasing for growing $\beta \in]0.5, 1.5]$. We also notice a varying behavior for equity relative to λ parameter. Specifically, equity are a convex function of λ for $|\beta| < 1.5$ with a minimum reached in $\lambda^* = 1$. Finally, they become an increasing function of λ for $|\beta| = 1.5$.

We also display underneath the conditional expected value of the average monthly stochastic volatility $\bar{\sigma}^e_V = E^{\hat{P}}\left[\sqrt{\frac{1}{\tau}\int_t^T \sigma^2_V(s, I_s)\, ds}\,\middle|\, G_t\right] = E^{\hat{P}}\left[\sqrt{\bar{\sigma}^2_V}\,\middle|\, G_t\right]$ over the remaining time to maturity of the debt and under the minimal martingale measure. The stochastic integral composing the firm value's variance is computed using the finite difference method and the average stochastic volatility is computed always using Monte Carlo simulation methodology.

Table 4: Average monthly simulated values of path dependent stochastic volatility (percent)

$\lambda \setminus \beta$	-1.5	-1	-0.5	0	0.5	1	1.5
0.2	85.32	68.82	45.45	27.22	50.20	71.45	97.46
1	84.53	73.00	51.80	27.22	58.28	75.32	100.10
5	82.96	74.39	54.19	27.22	56.56	74.97	100.26

Whatever λ values, the average stochastic volatility is a convex function of β with a minimal value reached for $\beta^* = 0$. When $-1.5 < \beta < 0$ and $\beta = 1.5$, the average stochastic volatility is an increasing function of λ. In contrast, when $\beta = -1.5$, the average stochastic volatility decreases as a function of λ. On the other hand, when $\beta = 0.5$ or $\beta = 1$, the average stochastic volatility is a concave function of λ. Moreover, the average stochastic volatility remains also constant whatever λ when $\beta = 0$. Such a behavior of stochastic volatility has some impact on the term structure of corporate credit spreads. We address this influence in the rest of the paper.

4.2 Credit spread

In this subsection, we complete the study started above while assessing the impact of a stochastic volatility on the evolution of credit spreads. In particular, we focus on the term structure of credit spreads.

Let $y(\tau)$ be the yield-to-maturity of the firm's risky debt or, equivalently, the default risky debt of the firm. Such a yield is linked with the current value of the firm's debt in the following way:

$$D(V_t, \tau) = e^{-y(\tau)\tau} B \tag{48}$$

which implies that

$$y(\tau) = \frac{-1}{\tau} \ln\left(\frac{D(V_t, \tau)}{B}\right) \tag{49}$$

Hence, the related credit spreads (i.e., yield spreads against government bonds) take the form:

$$S(\tau) = y(\tau) - r = \frac{-1}{\tau} \ln\left(\frac{D(V_t, \tau)}{B}\right) - r \tag{50}$$

Then, our previous simulations of debt's values allow us to compute the monthly related values of credit spreads with varying moneyness and time to maturity. The results we get are first displayed in the table underneath. And second, part of these results are summarized in the graph below which plots the related credit spreads for lambda being equal to 5 and with a varying beta parameter.

Table 5: Average monthly simulated values of credit spreads in basis points

$\lambda \setminus \beta$	-1.5	-1	-0.5	0	0.5	1	1.5
0.2	4265.33	1921.20	345.89	2.86	507.29	2486.03	5285.82
1	2945.94	2194.89	566.89	2.86	862.23	2908.93	5156.47
5	2748.30	2348.05	743.01	2.86	938.75	2982.52	5030.70

Whatever the value of λ, credit spreads are a convex function of β with a minimum reached for $\beta^* = 0$. Moreover, credit spreads' behaviors relative to λ are mitigated. For $|\beta| = 1.5$, credit spreads are decreasing functions of λ whereas the reverse takes place for $|\beta| = 0.5$ or 1. Finally, the credit spread becomes a constant function of λ when $\beta = 0$ due to the independence of our dynamics relative to λ under the minimal martingale measure.

[Insert figure 4 here]

Two features have to be underlined here. First, the higher the absolute value of beta is, the wider the related credit spread becomes for a given level of lambda. Second, the credit spread's level related to a given negative value of beta is slightly under the credit spread's level related to the corresponding positive value of beta.

Since we lie in a bounded volatility framework with bounded diffusion parameters, we could establish bounds for credit spreads' evolutions (i.e., term structure of credit spreads). Hence, we could have an alternative approach to the study of Chen & Huang (2002). Those authors give analytical bounds to the term structure of credit spreads in order to solve the problem of negative implied default probabilities. Such a problem arises when calibrating credit models to empirical data, and comes from the no arbitrage principle's violation.

In the same way, we establish bounds for our implied credit spreads. Indeed, formulae (40) and (42) allow us to write:

$$D^l(\tau) < D(V_t, \tau) < D^u(\tau) \tag{51}$$

with

$$\begin{aligned} D^l(\tau) &= V_t - E^{\hat{P}}\left[C_{BS}(\tau, r, V_t, B, \sigma_V^u) \middle| F_t \right] && \text{(52a)} \\ D^u(\tau) &= V_t - E^{\hat{P}}\left[C_{BS}\left(\tau, r, V_t, B, \sigma_V^l\right) \middle| F_t \right] && \text{(52b)} \end{aligned}$$

Thus, the risky yield-to-maturity becomes bounded as follows:

$$\frac{1}{\tau} \ln\left(\frac{B}{D^u(\tau)}\right) < y(\tau) < \frac{1}{\tau} \ln\left(\frac{B}{D^l(\tau)}\right) \tag{53}$$

which implies the boundedness of the related credit spread, namely:

$$S^l(\tau) < S(\tau) < S^u(\tau) \tag{54}$$

where

$$\begin{aligned} S^l(\tau) &= \frac{1}{\tau} \ln\left(\frac{B}{D^u(\tau)}\right) - r && \text{(55a)} \\ S^u(\tau) &= \frac{1}{\tau} \ln\left(\frac{B}{D^l(\tau)}\right) - r && \text{(55b)} \end{aligned}$$

Therefore, we are able to give an interval of possible variation for the corporate credit spread at each time from the debt's issue date to its maturity. The evolutions of the bounds of our corporate credit spread over time can be viewed as extreme scenarii for corporate credit spreads' evolution.

5 Conclusion

We have focused on the credit risk valuation model of Gatfaoui (2003). This modeling proposes to value corporate debt in a Merton framework while accounting for both systematic and idiosyncratic risk. Specifically, the option nature of debt allows the author to price corporate debt through a call on the firm assets value while encompassing the constant parameters-based dynamics of the systematic and idiosyncratic risk factors. Our work addressed then the extension of such a framework along with two key points.

First, we considered the general case of stochastic parameters-based dynamics of the two risk factors above mentioned. Under some regularity conditions, we showed that this kind of formulation is equivalent to a stochastic volatility option pricing (i.e., stochastic credit pricing) model. Indeed, it is equivalent to specify the diffusion of the firm assets value and the diffusion of the idiosyncratic risk factor affecting risky debt. In such a case, we consider two risk sources affecting the firm assets value whereas we can only observe the firm value. Consequently, we lie in an incomplete market framework where the incompleteness is due to the non observable idiosyncratic part of the firm assets value. This incompleteness is materialized by a stochastic volatility for the firm value. Hence, given the no arbitrage principle and the option theory in incomplete markets which introduces the notion of minimal martingale measure, we are able to give a price to the firm's equity and therefore to its debt. Such a valuation becomes possible in the historical universe as well as the minimal martingale measure's universe given our bounded volatility assumption. The equivalent minimal martingale measure reveals to be useful insofar as this notion allows us to reduce the global risk to its minimal component, namely the intrinsic (i.e., idiosyncratic) risk.

Second, we proposed an illustration of such a framework while stating some given specifications for the stochastic parameters of the diffusions under consideration. This setting allows us to realize the pricing of corporate debt in a bounded volatility case. Under our functional assumptions, we obtain an asymptotically mean reverting stochastic volatility process relative to time. Moreover, an interesting implication of our model consists of a stochastic correlation coefficient prevailing between the firm value and the idiosyncratic risk factor. Such a dependence as well as the stochastic volatility process are illustrated by a simulation study. Further illustrations are also given in order to show the implications of our framework on equity/debt pricing and on credit spreads computation. Specifically, accelerator-based Monte Carlo simulations are undertaken to study the various behaviors of equity, debt and credit spreads as functions of our model's parameters. In the same way, we also simulated the path-dependent average stochastic volatility related to our pricing framework. The advantage of such a setting is the flexibility given by the employed parameters since we are able to account for many risk scenarii and various market-linked firms. Moreover, the boundedness of the firm value's stochastic volatility implies the boundedness of related equity, debt and credit spreads. Such bounds can be viewed as extreme scenarii levels (i.e., worst/minimal potential losses due to increased/reduced global risk where the global risk level of the firm depends on the systematic and idiosyncratic risk factors). Further investigation can however be undertaken. In particular, our stochastic setting can allow for a more accurate computation of historical conditional default probabilities (i.e., conditional on the current available information set). Since default probabilities allow to assess the creditworthiness of counterparties, the possible boundedness of such probabilities given the likely scenarii has some non negligible importance and significance.

Our paper presents then a series of non negligible advantages. As a first step, our work is based on the fundamental notion of volatility. Volatility is important for asset valuation, risk management and portfolio diversification (see Eberlein *et al.* [2002/03] for example). Indeed, stochastic volatility models are useful tools to account for fundamental time-varying volatility (i.e., latent volatility component) of financial assets (see Hwang & Satchell [2000] for explanations). Moreover, volatility is commonly thought as a liquidity indicator. Hence, incorporating a stochastic volatility in credit risk modeling allows to account implicitly for the liquidity effects describing credit risky assets (refer to Collin-Dufresne *et al.* [2001] and Delianedis & Geske [2001] for example).

On an other hand, our credit pricing model is equivalent to a stochastic volatility Merton type pricing model whose interests are presented therein. For example, Kealhofer & Kurbat (2001) show that Merton's approach outperforms both Moody's credit ratings and well-known accounting ratios in predicting default. Indeed, it contains all information embedded in such ratings and ratios. In the same way, Phoa (2003) underlines the coherency of Merton type structural models with observed risky debt market data. This author points out the usefulness of such models as risk management tools. In contrast, Eom *et al.* (2002) find that new simplified structural models misestimate credit risk. Therefore, our stochastic volatility framework can help solving this problem by fitting better to empirical behavior of credit spreads, and then reconcile all points of view. In order to check this issue, the further step for future research consists therefore to estimate our model using risky debt data, and then to test its performance.

The significance of our current work and its future implementation are of major importance for a sound assessment of credit risk in the lens of the following points of view. First, credit spreads and default rates are key determinants for the pricing and hedging of credit instruments along with dynamic credit portfolio management. Second, given that we can diversify idiosyncratic risk, systematic risk remains more important at a portfolio level (see Jarrow et al. [2000], Frey & McNeil [2001], Lucas et al. [2001], and Giesecke & Weber [2003] for example). However, Goetzmann & Kumar (2001) show the existence of many under-diversified portfolios. Such portfolios are usually naively diversified and bear an important idiosyncratic risk. Consequently, credit portfolio management has to be envisionned in the light of idiosyncratic and systematic risks' tradeoff. Such a consideration is mostly important at an individual firm level.

6 Appendix

We give here some computational as well as modeling explanations and details about Merton's pricing with stochastic volatility.

6.1 Minimal martingale measure theorem

The minimal martingale measure notion is a key tool to price contingent claims under markets' incompleteness assumption. We proceed in two steps for our explanation. First, we recall a fundamental theorem justifying the pricing methodology. Then, we highlight the pricing implication in our setting.

6.1.1 The theorem

Consider the risk free asset $\left(B_t^0\right)$ which is assumed to follow the diffusion below:

$$dB_t^0 = rB_t^0 dt \tag{56}$$

and let $\tilde{V}_t = V_t / B_t^0$ be the discount value of the firm. Under our assumptions, relation (10) and according to Föllmer & Schweizer (1991), there is a unique martingale decomposition[23] M of $\tilde{V}_t$ such that:

$$d\tilde{V}_t = \tilde{V}_t \left[\mu_V\left(t, V_t, I_t\right) - r\right] dt + dM_t \tag{57}$$

where

$$dM_t = \sigma_V\left(t, V_t, I_t\right) \tilde{V}_t \left[\varsigma_\beta \sqrt{1 - \rho^2\left(t, V_t, I_t\right)}\, dW_t^X + \rho\left(t, V_t, I_t\right) dW_t^I\right] \tag{58}$$

and

$$d\left\langle M\right\rangle_t = \sigma_V^2\left(t, V_t, I_t\right) \tilde{V}_t^2\, dt \tag{59}$$

If we let $\gamma\left(t, V_t, I_t\right) = \frac{\mu_V(t, V_t, I_t) - r}{\sigma_V^2(t, V_t, I_t)\tilde{V}_t}$ such that we get $\gamma\left(t, V_t, I_t\right) d\left\langle M\right\rangle_t = \tilde{V}_t$ $\left[\mu_V(t, V_t, I_t) - r\right] dt$, we can then employ the theorem underneath.

Theorem 1 *(i) The minimal martingale measure $\hat{P}$ is uniquely determined.*
(ii) $\hat{P}$ exists if and only if for all $t \in [0, T]$

$$\hat{L}\left(t\right) = \exp\left\{-\int_0^t \gamma\left(u\right) dM_u - \frac{1}{2}\int_0^t \gamma^2\left(u\right) d\left\langle M\right\rangle_u\right\} \tag{60}$$

is a square-integrable martingale under P. In that case, $\hat{P}$ is given by $\frac{d\hat{P}}{P} = \hat{L}\left(T\right)$.
(iii) The minimal martingale measure preserves orthogonality: Any square-integrable martingale N with $\left\langle N, M\right\rangle = 0$ under P satisfies $\left\langle N, V\right\rangle = 0$ under $\hat{P}$.

Hence, the discount firm value becomes a martingale under $\hat{P}$ whereas it remains just a semi-martingale under the historical probability P. This feature is true for any discount value of F_t-adapted financial assets, and therefore for the discount value of the idiosyncratic risk factor.

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[23] This results from the application of Ito's lemma to $\tilde{V}_t$.

6.1.2 Implications for pricing

Given this setting, we can apply Girsanov change of measure and express the dynamics of the firm value and its idiosyncratic risk factor under the minimal martingale measure $\hat{P}$:

$$\frac{dV_t}{V_t} = r\,dt + \sigma_V\left(t, V_t, I_t\right)\left[\varsigma_\beta \sqrt{1-\rho^2\left(t, V_t, I_t\right)}\,d\hat{W}_t^X + \rho\left(t, V_t, I_t\right)\,d\hat{W}_t^I\right] \qquad (61)$$

$$\frac{dI_t}{I_t} = \left[\mu_I\left(t, I_t\right) - \sigma_I\left(t, I_t\right)\frac{\mu_V\left(t, V_t, I_t\right) - r}{\sigma_V\left(t, V_t, I_t\right)}\rho\left(t, V_t, I_t\right)\right]dt + \sigma_I\left(t, I_t\right)\,d\hat{W}_t^I \qquad (62)$$

with $Var^{\hat{P}}\left(\left.\frac{dV_t}{V_t}\right| F_t\right) = \sigma_V^2\left(t, V_t, I_t\right)dt$. Hence, given formulation (19) and integrating relative to time on the subset $[t, T]$, we get:

$$\begin{aligned}\ln\left(\frac{V_T}{V_t}\right) &= \int_t^T \left[r - \frac{\sigma_V^2\left(s, V_s, I_s\right)}{2}\right]ds \\ &+ \int_t^T \sigma_V\left(s, V_s, I_s\right)\varsigma_\beta\sqrt{1-\rho^2\left(s, V_s, I_s\right)}d\hat{W}_s^X \qquad (63) \\ &+ \int_t^T \sigma_V\left(s, V_s, I_s\right)\rho\left(s, V_s, I_s\right)d\hat{W}_s^I\end{aligned}$$

This describes the dynamic followed by the firm value under the minimal martingale measure. Then, the no arbitrage principle and the martingale measure valuation allow to price any contingent claim written on the firm value. Intuitively, such a pricing reduces the uncertainty peculiar to the non insurable risk. In our case, the non insurable risk (i.e., non observed and non tradable risk) is the idiosyncratic risk factor affecting the firm value.

6.2 Stochastic variables' simulation

We simulate the behavior of the idiosyncratic risk factor and we compute the related correlation coefficient. Then, we plot the obtained paths for these random variables.

Let state the following values for our simulations which take place on the time subset $[t, T]$ such that the initial time to maturity of the debt is $\tau = T - t = 10$ years:

$$\begin{array}{lll}\alpha = -\frac{1}{4} & \epsilon = 0.5 & \gamma = 1 \\ I_t = 3.5 & \Omega = \sqrt{\frac{0.2}{\epsilon}} & F\left(t, I_t\right) = \beta^2 t^{-\frac{1}{2}} + 0.4\, I_t\end{array} \qquad (64)$$

We simulate daily the value of I_t for different values of beta and lambda parameters (i.e., $\beta = 0, 0.5, 1, 1.5$ and $\lambda = 0.2, 1, 5$). The related plots are displayed underneath.

[Insert figure 5 here]

The computation of the correlation coefficient $\rho(t, I_t) = \frac{\sigma_I(t,I_t)}{\sigma_V(t,I_t)}$ leads to the following graphs. As only $\sigma_V(t, I_t)$ depends on beta parameter, the dependence of $\rho(t, I_t)$ relative to beta will be described by the evolution of $\frac{1}{\sigma_V(t,I_t)}$.

[Insert figures 6, 7 and 8 here]

Recall that when $\beta = 0$, we have $F(t, I_t) = \sigma_I(I_t) = \Omega\sqrt{I_t}$ which implies that $\rho(t, I_t) = 1$ whatever t and I_t.

6.3 Black & Scholes (1973) formula

In 1973, Black & Scholes introduced a closed-form formula for pricing European stock options. The current price of a European option on a given stock depends only on a set of five parameters, namely τ the current time t to expiration date T of the option, the strike price K, the risk free interest rate r and finally the current level S_t as well as the volatility σ of the underlying on the investment horizon. Given their findings, those authors price a European call on the stock S_t as follows:

$$C_{BS}(\tau, r, S_t, K, \sigma) = S_t N(d_1) - K e^{-r\tau} N(d_2) \tag{65}$$

with

- $N(.)$ the cumulative distribution function of the standard normal law;
- $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$;
- $d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$.

The current price of the European call has two distinct components. First, $KN(d_2)$ is the risk neutral expectation of the payment realized by the option's buyer at time T when the call is exercised. Second, $e^{r\tau} S_t N(d_1)$ is the risk neutral expectation of the stock's price at time T conditional on the call being in the money.

6.4 Diffusions when β is zero

In this subsection, we express the pertinent diffusions when β is zero under the minimal martingale measure. We focus on the evolutions of the firm assets value and the idiosyncratic risk factor.

When β parameter is zero, we have the following implications:

$$\sigma_V(t, I_t) = \sigma_I(t, I_t) = \Omega\sqrt{I_t} \tag{66a}$$

$$\mu_V(t, I_t) = \mu_I(t, I_t) = \lambda\left(\frac{\epsilon}{I_t} - 1\right) \tag{66b}$$

$$\rho(t, I_t) = 1 \tag{66c}$$

Therefore, the diffusions of the firm assets value and the idiosyncratic factor under the minimal martingale measure become:

$$dV_t = rV_t\,dt + \Omega V_t\sqrt{I_t}\,d\hat{W}_t^I \tag{67a}$$

$$dI_t = rI_t\,dt + \Omega I_t\sqrt{I_t}\,d\hat{W}_t^I \tag{67b}$$

Consequently, the evolutions of both the firm assets value and the idiosyncratic factor are independent of λ when β parameter is zero. On the other hand, equity and debt are independent of λ as functions of V_t and I_t.

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Plots

Figure 1: Volatility simulation for different values of beta parameter when lambda is equal to 0.2

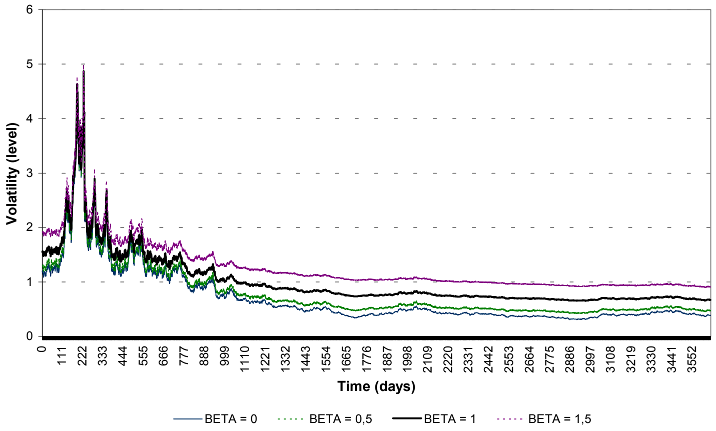

Figure 2: Volatility simulation for different values of beta parameter when lamda is equal to 1

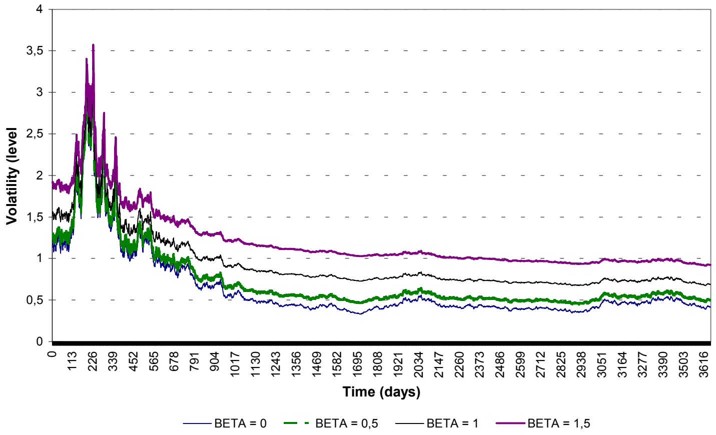

Figure 3: Volatility simulation for different values of beta parameter when lambda is equal to 5

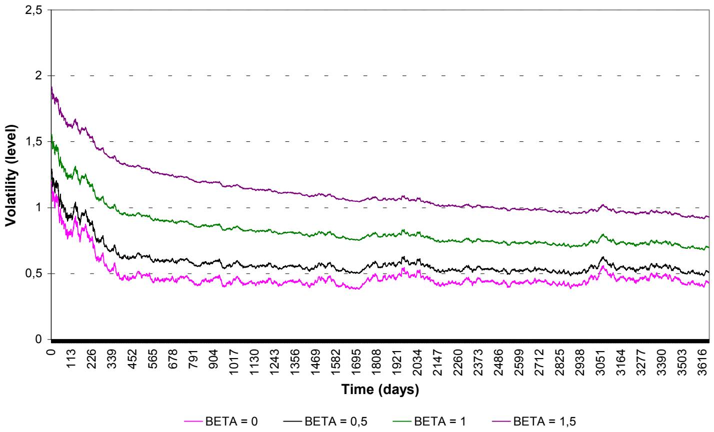

Figure 4: Credit spread simulation for different values of beta parameter when lambda is equal to 5

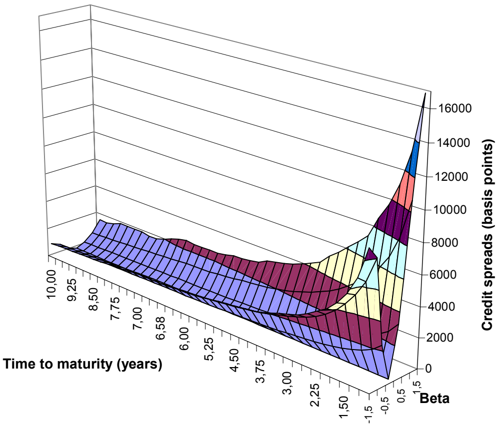

Figure 5: Idiosyncratic factor simulation for different values of lambda parameter

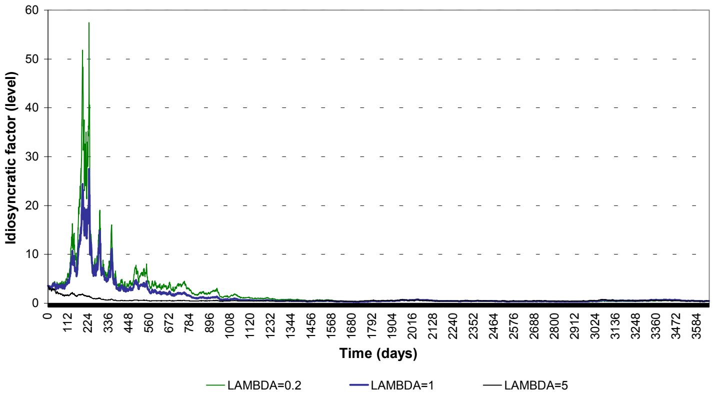

Figure 6: Correlation coefficient for different values of beta parameter and for lambda equal to 0.2

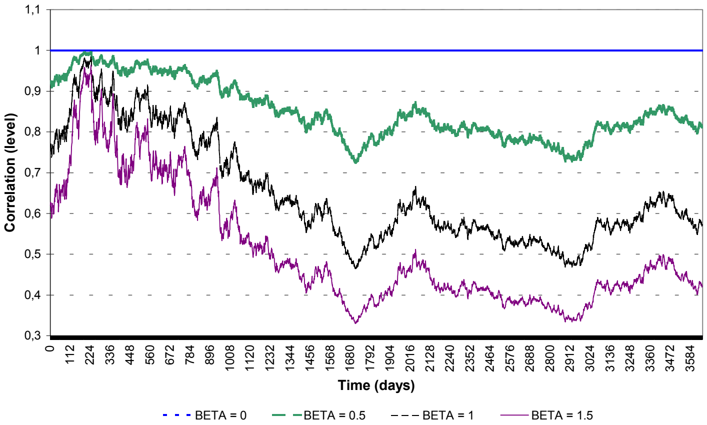

Figure 7: Correlation coefficient for different values of beta parameter and for lambda equal to 1

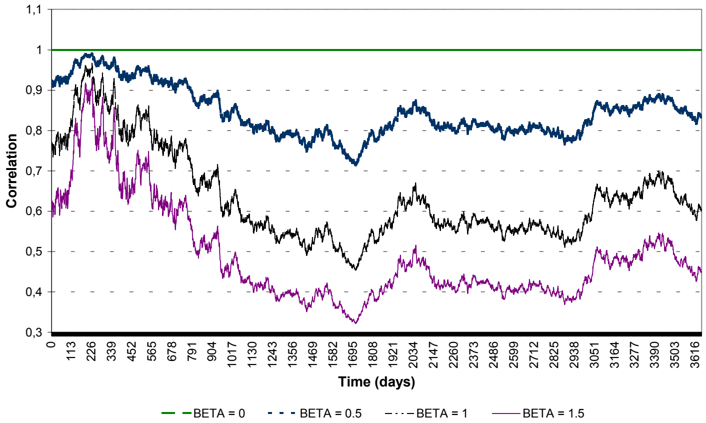

Figure 8: Correlation coefficient for different values of beta parameter and for lambda equal to 5