# Pricing and Hedging Options in Incomplete Markets: Idiosyncratic Risk, Systematic Risk and Stochastic Volatility*

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**Abstract**

*Starting from the European option valuation framework of Chauveau & Gatfaoui (2002), we establish the link with stochastic volatility models. And, we propose both a new vision and a general framework for valuing European options in the light of systematic and idiosyncratic risks affecting risky assets in the financial market. Therefore, we account for the well-known volatility smile in the light of the literature adressing the determinants of the smile effect among which stochastic volatility and market risk. We further discuss briefly the hedging of European options along with the local risk minimization principle. Specifically, we attempt to find a strategy which dominates the usual partial hedging technique often imposed by market's incompleteness.*

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# 1 Introduction

Introducing the well-known Capital Asset Pricing Model (i.e., CAPM), Sharpe (1963, 1964, 1970) and Treynor (1961) among others state that financial assets are subject to two sources of risk. Indeed, each risky asset depends both on a systematic or market risk, and an idiosyncratic or specific risk. Such a risk dependence is recently employed by Chauveau & Gatfaoui (2002) to price European calls. Those authors propose a closed-form formula allowing to price European calls in the light of systematic and idiosyncratic risks. They complete asymptotically the market (i.e., quasi-complete market) while building a portfolio replicating the market risk factor. Such a portfolio is composed of all the stocks existing in the financial market as well as the risk free asset. Their valuation is realized in the lens of the following remark. Observing simultaneously the systematic and idiosyncratic risk factors is equivalent to observe simultaneously the stock's price and the systematic risk factor. Results are interesting since Chauveau & Gatfaoui (2002) succeed in accounting for the volatility smile (i.e., U-shaped implied volatility patterns) which is due to the general asymmetric behavior of any stock's return. In a recent study, Äijö (2003) underlines the significance of such a work while focusing on the systematic nature of the smiles exhibited by stock options. This author shows the importance of the market skewness in explaining the skewness of a given stock's return. Moreover, Duque & Lopes (2000) attempt to explain the smile effect according to two dimensions, namely the option's maturity and the variation of the stock's volatility. They show that the evolution of volatility has an asymmetric impact on the shape of the corresponding implied volatility (i.e., the smile) relative to the moneyness of the option.

In the light of such results, we propose a general framework for valuing and hedging European calls while attempting to account for the smile effect. To this end, we consider a stochastic volatility pricing model subject to both systematic risk and idiosyncratic risk.

Our paper is organized as follows. Section 2 generalizes the European call pricing framework proposed by Chauveau & Gatfaoui (2002) and establishes the link with stochastic volatility models (i.e., incomplete market and stochastic diffusion parameters). Section 3 focuses on the pricing methodology to employ in such a setting. Section 4 considers the problem of hedging European calls in our incomplete market case. In the lens of the local risk minimization principle, we attempt to find some strategy which dominates the usual partial hedging technique.

# 2 Framework

In this section, we present the stochastic option valuation framework and state the necessary assumptions to validate such a setting. We first present the basic framework. Then, we introduce the new formulation.

## 2.1 Basic setting

All the assumptions prevailing in the Black & Scholes (1973) option pricing framework are supposed to hold except that the market is incomplete. We also assume the absence of any opportunity of arbitrage and a constant risk free spot rate $r$. The risk free asset $S_t^0$ satisfies the following SDE: $\frac{dS_t^0}{S_t^0} = r\,dt$ with initial condition $S_0^0 = 1$. Consider a primitive probability space $(\Omega, F, P)$ such that the information set available to any investor is represented by a bidimensional Brownian motion $w_t' = \left(W_t, W_t^i\right)$ where $W_t$ and $W_t^i$ are two independent $P$-Brownian motions. Let $\mathbb{F} = (F_t)_{t\in[0,T]}$ be the $P$-augmentation of the natural filtration $F_t = \left\{\sigma\left(W_s\right) \cup \sigma\left(W_s^i\right), 0 \leq s \leq t\right\}$ generated by $w_t$ with $F = F_T$ and $T < \infty$. Given the framework of Chauveau & Gatfaoui (2002), we consider the diffusion processes describing the levels of the systematic risk factor $X_t$ and idiosyncratic risk factor $I_t^i$ at current time $t$:

$$dX_t = X_t\left[\mu\left(t, X_t\right)\, dt + \sigma\left(t, X_t\right)\, dW_t\right] \tag{1}$$

$$dI_t^i = I_t^i\left[\nu_i\left(t, I_t^i\right)\, dt + \tau_i\left(t, I_t^i\right)\, dW_t^i\right] \tag{2}$$

where $\mu\left(t, X_t\right)$, $\sigma\left(t, X_t\right)$, $\nu_i\left(t, I_t^i\right)$ and $\tau_i\left(t, I_t^i\right)$ are continuous $F_t$-measurable functions on $[0, T] \times \mathbb{R}$. Moreover, we assume that such functions are bounded to ensure strong solutions to their respective SDE (see Karatzas & Shreve [1991] for example). Whatever $t \in [0, T]$ and $X_t, I_t^i \in \mathbb{R}$, we therefore impose:

$$\begin{array}{lcl} \mu_l & < & \mu\left(t, X_t\right) < \mu_u \quad \text{and} \quad \sigma_l < \sigma\left(t, X_t\right) < \sigma_u \\ \nu_l & < & \nu_i\left(t, I_t^i\right) < \nu_u \quad \text{and} \quad \tau_l < \tau_i\left(t, I_t^i\right) < \tau_u \end{array} \tag{3}$$

with $\mu_l$, $\mu_u$, $\nu_l$, $\nu_u$, $\sigma_l$, $\sigma_u$, $\tau_l$, $\tau_u$ constant parameters, and $\sigma_l > 0$ and $\tau_l > 0$.

Consider $S_t^i$ the $F_t$-adapted process of the price of stock $i$ at current time $t$ whose dependence *vis-à-vis* the systematic and idiosyncratic risk factors is as follows:

$$S_t^i = \Lambda_i X_t^{\beta_i} I_t^i \tag{4}$$

where $\Lambda_i$ is a deterministic constant, $\beta_i$ is the beta of the stock (i.e., deterministic constant representing the correlation of the stock with the market). Since Chauveau & Gatfaoui (2002) established the equivalence between observing simultaneously $X_t$ and $I_t^i$, and observing simultaneously $S_t^i$ and $X_t$, the stochastic system above-mentioned could be written:

$$\frac{dS_t^i}{S_t^i} = \Upsilon_i\left(t, X_t, I_t^i\right)\, dt + \left[\beta_i \sigma\left(t, X_t\right)\, dW_t + \tau_i\left(t, I_t^i\right)\, dW_t^i\right] \tag{5}$$

$$dX_t = X_t\left[\mu\left(t, X_t\right)\, dt + \sigma\left(t, X_t\right)\, dW_t\right] \tag{6}$$

with

$$\begin{aligned} \Upsilon_i\left(t, X_t, I_t^i\right) &= \Upsilon_i\left(t, X_t, \frac{S_t^i}{\Lambda_i X_t^{\beta_i}}\right) \\ &= \beta_i \mu\left(t, X_t\right) + \nu_i\left(t, I_t^i\right) + \frac{1}{2}\beta_i\left(\beta_i - 1\right)\sigma^2\left(t, X_t\right) \end{aligned} \tag{7}$$

Such a specification is consistent with a stochastic volatility model insofar as we observe the stock's price, but we do not observe the systematic risk factor.

## 2.2 Stochastic volatility model

Historically, the stochastic volatility pattern is motivated by the observed smile effect (i.e., the U-shaped implied volatility relative to the moneyness of options). Stochastic volatility models usually state two dynamics corresponding to the diffusion process of the stock's price and the diffusion process of some given stochastic variable. Generally, the stock's volatility depends on the stochastic variable which is not observable (See Scott [1987], Hull & White [1987], Stein & Stein [1991], Heston [1993], Bakshi *et al.* [1997] among others). This setting leads therefore to an incomplete market (i.e., there is no unique equivalent martingale measure). Moreover, the volatility is not perfectly correlated with the Brownian motion describing the stock's evolution and encompasses a component which is independent of this Brownian motion (refer to Fouque *et al.* (2000) for details). For example, Bakshi *et al.* (1997) and Nandi (1998) show that a non zero correlation[1] between the stock's volatility and its return allows to account for the smile effect.

In the spirit of such a dependence, we are going to show that (5) and (1) are equivalent to a stochastic volatility model. Our non observable random variable corresponds here to the systematic or market risk factor $X_t$. Let $\rho$ be the instantaneous correlation between the stock's price and the systematic risk factor such that:

$$d\left\langle S_t^i, X_t\right\rangle = \rho\left(t, S_t^i, X_t\right)\, dt \tag{8}$$

Therefore, brief computations allow to deduce:

$$\rho\left(t, S_t^i, X_t\right) = \frac{\beta_i \sigma\left(t, X_t\right)}{\Sigma\left(t, S_t^i, X_t\right)} \tag{9}$$

with

$$\Sigma\left(t, S_t^i, X_t\right) = \sqrt{\beta_i^2 \sigma^2\left(t, X_t\right) + \tau_i^2\left(t, \frac{S_t^i}{\Lambda_i X_t^{\beta_i}}\right)} \tag{10}$$

After lengthy simple calculations, we finally find that:

$$\frac{dS_t^i}{S_t^i} = b\left(t, S_t^i, X_t\right)\, dt + \Sigma\left(t, S_t^i, X_t\right)\, \left[\rho\left(t, S_t^i, X_t\right)\, dW_t + \sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, dW_t^i\right] \tag{11}$$

where

$$b\left(t, S_t^i, X_t\right) = \Upsilon_i\left(t, X_t, \frac{S_t^i}{\Lambda_i X_t^{\beta_i}}\right) \tag{12}$$

[1] See also the work of Schöbel & Zhu (1999) who deduce a closed-form formula for option pricing in the presence of a stochastic volatility correlated with the stock's return.

given that

$$dX_t = X_t \left[\mu\left(t, X_t\right) \, dt + \sigma\left(t, X_t\right) \, dW_t\right] \tag{13}$$

In our system, $\Sigma\left(t, S_t^i, X_t\right)$ represents the stochastic volatility of the stock's return.[2] Therefore, we lie in a stochastic volatility framework with a little more complex structure. Moreover, the assumptions imposed on the parameters of diffusions (1) and (2) in the previous section ensure the existence of strong solutions to SDE (11) and (1) provided that $\Sigma\left(t, S_t^i, X_t\right)$ is non zero and given a starting condition for this stochastic volatility. Specifically, the diffusion parameters of (11) are continuous and bounded[3] by $b_l$, $b_u$, $\Sigma_l$, $\Sigma_u$, $\rho_l$ and $\rho_u$.

Analogously to Hofmann et al. (1992), we get a state diffusion model with stochastic volatility, which raises some questions. Indeed, we have more state variables (i.e., two risk sources) than primitive assets (i.e., a single stock). Consequently, the market is incomplete and there is no mean to give a unique price to any contingent claim on the stock $i$. The next section will address this problem and propose a pricing for European calls.

# 3 Pricing

In a stochastic volatility framework, there exists two ways to achieve the pricing of a contingent claim written on the stock $S_t^i$ such as a European call for example. First, we could deduce the price of the European call from standard no-arbitrage arguments. In this case, the price of the call satisfies a PDE (i.e., partial differential equation) which depends on our two state variables. Along with the no-arbitrage principle, this PDE[4] is obtained while building a hedging portfolio. Since the market is incomplete, we attempt to hedge the European call with the underlying stock, the risk free asset and another European call with same features except that its maturity is different.[5] Second, the contingent claim's pricing could be achieved while using the equivalent martingale measure

[2] Notice that having $S_t^i = \Lambda_i X_t^{\beta_i} I_t^i$ and knowing dynamics of market risk factor $X_t$ and idiosyncratic risk factor $I_t^i$ is equivalent to having $I_t^i = S_t^i / \Lambda_i X_t^{\beta_i}$ and knowing dynamics of both stock $i$, $S_t^i$ and market factor $X_t$. Moreover, the global volatility's dynamic could be inferred while using Ito's lemma under given regularity assumptions. Such assumptions have to ensure that volatility parameters $\sigma\left(t, X_t\right)$ and $\tau_i\left(t, I_t^i\right)$ are both once derivable relative to time and twice derivable relative to their second argument. Then, the stock's global volatility becomes once derivable relative to time and twice derivable relative to its respective second and third arguments, and Ito's lemma can therefore be applied.

[3] Expressions of bounds are given in the appendix.

[4] This technique is also called Feynman-Kac pricing equation. An equivalent valuation method has been recently proposed in the stochastic volatility literature, namely the Fourier inversion approach based on characteristic functions. Whereas the PDE does not necessarily have an analytic solution, characteristic functions allow to get closed-form solutions in many cases. Indeed, the probabilities intervening in the pricing formula can be expressed as functions of Fourier inversion of the related characteristic functions. Moreover, the characteristic functions describing an option pricing model depend on the characteristic functions related to the concerned independent stochastic factors. For example, this technique allowed Heston (1993), Bates (1996), Scott (1997), Bakshi *et al.* (1997) and Schöbel & Zhu (1999) to get closed-form solutions to their stochastic volatility models.

[5] Refer to Fouque *et al.* (2000) for explanations.

principle. The incompleteness leads to the existence of an infinity of equivalent martingale measures provided that some regularity conditions are satisfied. In this setting, the European call's pricing raises the question of the choice of an equivalent martingale measure.

We are going to address the second method to price a European call on $S_t^i$ with current price $C\left(t, S_t^i, X_t\right)$, strike $K$ and maturity $T$. At maturity, the terminal payoff of this contingent claim is $C\left(T, S_T^i, X_T\right) = \max\left(0, S_T^i - K\right) = \left(S_T^i - K\right)^+$. Given our assumptions, $C\left(t, S_t^i, X_t\right)$ is a $F_t$-adapted process and also a $C^{1,2}\left([0,T] \times \mathbb{R}^2\right)$ function.

## 3.1 Mathematical background

We introduce here all the definitions allowing to achieve a valuation[6] under a given martingale measure (i.e., under which discount prices of assets become martingales). The stake is to choose a martingale measure among all the existing equivalent martingale measures. Our assumptions ensure the existence of such measures, which is studied in the chapter 3 of Mele & Fornari (2000) for example.

In our incomplete market, we only observe the risky asset $S_t^i$ and the risk free asset $S_t^0$, which gives rise to an infinity of equivalent martingale measures $Q$. Such measures are defined by their Girsanov densities $\frac{dQ}{dP} \mid_{F_t} = L(t)$ such as:

$$L(t) = \exp\left\{ -\int_0^t \gamma_1(u)\, dW_u^i - \int_0^t \gamma_2(u)\, dW_u - \frac{1}{2}\int_0^t \left[\gamma_1^2(u) + \gamma_2^2(u)\right] du \right\} \tag{14}$$

where $\gamma_1(.)$ and $\gamma_2(.)$ represent the risk premia related to our two risk factors, and depend on these stochastic factors.[7] Moreover, we consider as a numeraire the risk free asset $S_t^0$ and then note the discount price of the stock as $\tilde{S}_t^i = \frac{S_t^i}{S_t^0}$. In this case, the discount price of the stock becomes a semi-martingale under the historical probability measure $P$. Given the no-opportunity arbitrage principle, we need to state the following equality to make $\tilde{S}_t^i$ become a $Q$-martingale (see Karatzas & Shreve [1991] and Karatzas[1996] for explanations):

$$b\left(t, S_t^i, X_t\right) - r = \Sigma\left(t, S_t^i, X_t\right) \left[\rho\left(t, S_t^i, X_t\right) \gamma_2(t) + \sqrt{1 - \rho^2\left(t, S_t^i, X_t\right)}\, \gamma_1(t)\right] \tag{15}$$

Therefore, the choice of the relevant equivalent martingale measure to apply to our pricing will be defined by the characterization of our risk premia $\gamma_1(.)$ and $\gamma_2(.)$.

According to Föllmer & Schweizer (1991) and given our assumptions, there exists an equivalent martingale measure $\hat{P}$ among all the existing equivalent martingale measures $Q$, which minimizes the relative entropy, or equivalently, the uncertainty surrounding the stock's evolution (see also Delbaen & Schachermayer [1996], Musiela & Rutkowski [1998] and Gouriéroux *et al.* [1998] for tech-

[6] Such a pricing is a risk-neutral valuation relative to the observed risk factor only.

[7] For simplicity, we write $\gamma(u)$ instead of $\gamma\left(u, S_u^i, X_u\right)$.

nical details). $\hat{P}$ is called the minimal equivalent martingale measure[8] and is uniquely defined by its density as follows:[9]

$$\hat{L}(t) = \exp\left\{ \begin{array}{c} -\int_0^t \frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)} \sqrt{1-\rho^2\left(u,S_u^i,X_u\right)}\, dW_u^i \\ -\int_0^t \frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)} \rho\left(u,S_u^i,X_u\right)\, dW_u - \frac{1}{2}\int_0^t \left(\frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)}\right)^2 du \end{array} \right\} \quad (16)$$

with

$$\begin{aligned} \gamma_1(u) &= \frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)} \sqrt{1-\rho^2\left(u,S_u^i,X_u\right)} && (17) \\ \gamma_2(u) &= \frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)} \rho\left(u,S_u^i,X_u\right) \end{aligned}$$

and $\gamma(u) = \frac{b\left(u,S_u^i,X_u\right)-r}{\Sigma\left(u,S_u^i,X_u\right)}$ represents the excess return-to-risk ratio (i.e., the market price of the global risk borne by the stock $i$). In this case, the intrinsic or residual risk depends on the systematic risk factor (i.e., non observable). Moreover, $\hat{P}$ is technically called the minimal relative entropy martingale measure (see Schweizer [1999b]) and corresponds to a special case of $q$-optimal measures. The set of $q$-optimal measures corresponds to a class of equivalent martingale measures designed to be the closest to the historical (i.e., real) world measure $P$ in terms of the $q^{th}$ moment (see Grandits & Rheinländer [2002], Hobson [2002] and Henderson [2002] for example). In particular, Henderson *et al.* (2003) price a correlated stochastic volatility model and compare three $q$-optimal measures, namely the minimal entropy martingale measure (i.e., $q = 0$, see Fritelli [2000]), the variance-optimal martingale measure (i.e., $q = 2$, see Föllmer & Sondermann [1986]) and the minimal reverse entropy martingale measure (i.e., $q = 1$, see Schweizer [1999b]).

Consequently, we are able to price any contingent claim written on the stock under $\hat{P}$. Specifically, under the minimal equivalent martingale measure, we are able to apply Girsanov's theorem and realize the following change of variable:

$$\begin{aligned} d\hat{W}_t &= \gamma_2(t)\, dt + dW_t = \frac{b\left(t,S_t^i,X_t\right)-r}{\Sigma\left(t,S_t^i,X_t\right)} \rho\left(t,S_t^i,X_t\right) dt + dW_t && (18) \\ d\hat{W}_t^i &= \gamma_1(t)\, dt + dW_t^i = \frac{b\left(t,S_t^i,X_t\right)-r}{\Sigma\left(t,S_t^i,X_t\right)} \sqrt{1-\rho^2\left(t,S_t^i,X_t\right)} dt + dW_t^i \end{aligned}$$

where $\left(\hat{W}_t\right)$ and $\left(\hat{W}_t^i\right)$ are two independent standard Brownian motions under $\hat{P}$. This transformation leads to the following dynamics for the stock and the

[8] The existence of $\hat{P}$ means that there is no compensation for the risk due to the fluctuation of stochastic volatility. This point is in accordance with the basic assumption of Hull & White (1987). Differently, the historical probability $P$, or equivalently, the historical universe allows to consider the dynamic of the non insurable risk.

[9] For explanations about existence and unicity, refer to the appendix.

systematic risk factor under $\hat{P}$:

$$\frac{dS_t^i}{S_t^i} = r\,dt + \Sigma\left(t, S_t^i, X_t\right)\left[\rho\left(t, S_t^i, X_t\right)\,d\hat{W}_t + \sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\,d\hat{W}_t^i\right] \quad (19)$$

$$\frac{dX_t}{X_t} = \left[\mu\left(t, X_t\right) - \sigma\left(t, X_t\right)\frac{b\left(t, S_t^i, X_t\right) - r}{\Sigma\left(t, S_t^i, X_t\right)}\rho\left(t, S_t^i, X_t\right)\right]dt + \sigma\left(t, X_t\right)\,d\hat{W}_t \quad (20)$$

since the risk free rate is constant and our numeraire is the risk free asset. Moreover, the respective discount values $\tilde{S}_t^i$ of stock $i$ and market risk factor $\tilde{X}_t$ are both $\hat{P}$-martingales and semi-martingales under $P$.

Applying the generalized Ito's lemma to function $F\left(t, S_t^i\right) = \ln\left(S_t^i\right)$, we get the following SDE for the stock $i$:

$$d\ln\left(S_t^i\right) = \left[r - \frac{\Sigma^2\left(t, S_t^i, X_t\right)}{2}\right]dt + \Sigma\left(t, S_t^i, X_t\right)\left[\begin{array}{c}\rho\left(t, S_t^i, X_t\right)\,d\hat{W}_t + \\ \sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\,d\hat{W}_t^i\end{array}\right] \quad (21)$$

since we have

$$Var\left(\left.\frac{dS_t^i}{S_t^i}\right|_{F_t}\right) = \Sigma^2\left(t, S_t^i, X_t\right)dt \quad (22)$$

Finally, integrating this equation between times $t$ and $T$, we find that under $\hat{P}$:

$$\begin{aligned}\ln\left(\frac{S_T^i}{S_t^i}\right) &= \int_t^T\left[r - \frac{\Sigma^2\left(u, S_u^i, X_u\right)}{2}\right]dt \\ &+ \int_t^T \Sigma\left(u, S_u^i, X_u\right)\rho\left(u, S_u^i, X_u\right)\,d\hat{W}_u \\ &+ \int_t^T \Sigma\left(u, S_u^i, X_u\right)\sqrt{1-\rho^2\left(u, S_u^i, X_u\right)}\,d\hat{W}_u^i\end{aligned} \quad (23)$$

Once we know the dynamic of the stock under the minimal equivalent martingale measure, we are able to process to the European call's pricing along with the martingale measure principle.

## 3.2 Valuation: principle and application

Given the no-arbitrage principle and the minimal equivalent martingale measure previously introduced, we are able to give a price to the European call written on stock $S_t^i$ although we are in an incomplete market. Indeed, the current price of the European call is the expected discount value of its terminal payoff under $\hat{P}$. Consequently, we can write:

$$C\left(t, S_t^i, X_t\right) = E^{\hat{P}}\left[\left.\frac{S_t^0}{S_T^0}\left(S_T^i - K\right)^+\right| F_t\right] = E^{\hat{P}}\left[\left.e^{-r(T-t)}\left(S_T^i - K\right)^+\right| F_t\right] \quad (24)$$

From relations (23) and (1), we know the evolution of stock $S_t^i$ and we are able to compute (24) while using Monte Carlo simulation techniques. It is more evident if we make the following assumption. Assume that the volatility $\tau_i$ of the idiosyncratic risk factor is deterministic such that $\tau_i\left(t, I_t^i\right) = \tau_i(t)$. Then, we have:

$$d\ln\left(S_t^i\right) = \left[r - \frac{\Sigma^2(t, X_t)}{2}\right] dt + \Sigma(t, X_t) \left[ \begin{array}{c} \rho(t, X_t)\, d\hat{W}_t + \\ \sqrt{1-\rho^2(t, X_t)}\, d\hat{W}_t^i \end{array} \right] \tag{25}$$

with

$$\rho(t, X_t) = \frac{\beta_i \sigma(t, X_t)}{\Sigma(t, X_t)} \tag{26}$$

and

$$\Sigma(t, X_t) = \sqrt{\beta_i^2 \sigma^2(t, X_t) + \tau_i^2(t)} \tag{27}$$

Therefore, the dynamic of the stock rewrites:

$$\begin{aligned} \ln\left(\frac{S_T^i}{S_t^i}\right) = \int_t^T \left[r - \frac{\Sigma^2(u, X_u)}{2}\right] dt \quad & + \int_t^T \Sigma(u, X_u)\, \rho(u, X_u)\, d\hat{W}_u \\ & + \int_t^T \Sigma(u, X_u) \sqrt{1-\rho^2(u, X_u)}\, d\hat{W}_u^i \end{aligned} \tag{28}$$

And applying the iterated expectations theorem, formulation (24) writes as follows:

$$\begin{aligned} C\left(t, S_t^i, X_t\right) &= E^{\hat{P}}\left[E^{\hat{P}}\left[e^{-r(T-t)}\left(S_T^i - K\right)^+ \middle| F_t, X_u, t \leq u \leq T\right] \middle| F_t\right] \\ &= E^{\hat{P}}\left[E^{\hat{P}}\left[e^{-r(T-t)}\left(S_T^i - K\right)^+ \middle| G_t\right] \middle| F_t\right] \end{aligned} \tag{29}$$

with

$$G_t = F_t \cup \{X_u, t \leq u \leq T\} \tag{30}$$

This setting implies that:

$$E^{\hat{P}}\left[\ln\left(\frac{S_T^i}{S_t^i}\right) \middle| G_t\right] = \left(r - \frac{\bar{\Sigma}^2}{2}\right)(T-t) \tag{31}$$

$$Var^{\hat{P}}\left[\ln\left(\frac{S_T^i}{S_t^i}\right) \middle| G_t\right] = \bar{\Sigma}^2(T-t) \tag{32}$$

with

$$\bar{\Sigma}^2 = \frac{1}{T-t} \int_t^T \Sigma^2(u, X_u)\, du \tag{33}$$

$\bar{\Sigma}^2$ is the average variance of the stock over the remaining path until the call's maturity for each realization. Consequently, conditionally on $G_t$ and under $\hat{P}$, $\ln\left(\frac{S_T^i}{S_t^i}\right)$ follows a lognormal distribution which allows us to write (29) under the new form:

$$C\left(t, S_t^i, X_t\right) = E^{\hat{P}}\left[C_{BS}\left(t, S_t^i, K, T, \sqrt{\bar{\Sigma}^2}\right) \middle| F_t\right] \tag{34}$$

where $C_{BS}\left(t, S_t^i, K, T, \sqrt{\bar{\Sigma}^2}\right)$ is the Black & Scholes[10] price with an average time-dependent volatility. The current price of the European call corresponds to the average Black & Scholes price over all the possible average volatility paths. Thus, the deterministic $\tau_i$ assumption and the iterated expectations theorem allow to ease the computation of the Monte Carlo simulation of European call prices since it requires only to generate one Brownian motion path.[11]

Such a setting raises three fundamental remarks. First, since each diffusion parameter is bounded (i.e., the volatility of the stock is bounded), we can show that the price of the European call is also bounded for each time $t$ in the interval $[0, T]$ (see for example, Frey & Sin [1999] and Pham [1998]). Indeed, Frey & Sin (1999) show that the European call's price is at least bounded by $C_{BS}\left(t, S_t^i, K, T, \Sigma_l\right) \leq C\left(t, S_t^i, X_t\right) \leq C_{BS}\left(t, S_t^i, K, T, \Sigma_u\right)$.

Second, the application of such a valuation requires to estimate the parameters of our diffusion processes (23) and (1). Let $\theta$ be the set of parameters[12] to estimate in our pricing model (including initial conditions on volatility). Provided that the volatility of the stock is an invertible function of parameters $\theta$, the maximum likelihood technique and the method of moments allow to achieve this estimation on observed stock's and calls' prices data respectively. Common practice employs the cross-sectional fitting method, namely it estimates the parameters from market prices of at-the-money European options. Such a technique leads to realize the following minimization of the sum of squared observed errors:

$$\min_{\theta} \sum_{K,T} \left\{ C\left(t, S_t^i, X_t, K, T, \theta\right) - C^{Obs}\left(t, S_t^i, K, T\right) \right\}^2 \tag{35}$$

where $C^{Obs}\left(t, S_t^i, K, T\right)$ are European call market prices for various strikes $K$ and maturity dates $T$, and $C\left(t, S_t^i, X_t, K, T, \theta\right)$ are our related theoretical call prices. Depending on the complexity of the model, such a technique can be computationally slow.

Finally, this drawback comes from the fact that stochastic volatility models usually have no closed-form solution and have then to be solved numerically to achieve the European call's pricing. However, some authors introduced the use of Fourier inversion to describe the characteristic functions associated to the stochastic volatility pricing model.[13] The Fourier transformation technique leads to closed-form solution in many cases. For example, Zhu (2000) proposes a good survey about this topic and introduces a new stochastic volatility model with a closed-form solution.

The natural extension to contingent claim valuation being their hedging, we address this question in the following section.

[10] Refer to the appendix for details.

[11] There is an important computational gain relative to relation (24) which requires to generate two Brownian motion paths.

[12] These parameters characterize each diffusion component (i.e., drifts and volatilities) and therefore both the stock's volatility as well as the related risk premia.

[13] Along with this technique, Heston (1993), Bates (1994), Scott (1997), Bakshi *et al.* (1997) and Schöbel & Zhu (1999) find closed-form pricing formulae.

# 4 Hedging

According to the Föllmer-Schweizer decomposition, any contingent claim on the stock $i$ (e.g., European call) can be seen as the cumulative sum of some trading gains and a martingale which is orthogonal to such gains. The martingale component of the contingent claim is usually considered as the cumulative cost of the hedge. Therefore, any hedging's aim is to reduce such a cost as much as possible.

We focus on partial hedging strategies which minimize the risk borne when only existing primitive assets are used to hedge the contingent claim. There exist two methods to achieve this risk minimization, or equivalently, this cost reduction. The first method considers that the imperfect replication of the claim's value comes from the risk which generates market incompleteness. In this case, the cost is a quadratic functional and some mean-variance criterion has to be satisfied along with self-financing strategies (see Schweizer [1995, 1996]). The second method generally uses non self-financing strategies to replicate the value of the contingent claim. In this case, the risk arising from market incompleteness generates the randomness of the hedging cost implied by such strategies (see Föllmer & Schweizer [1991], Hofmann *et al.* [1992]). This method is called risk-minimizing hedging and distinguishes between two cases. If the stock is a $P$-martingale, the remaining risk (i.e., remaining cost) is minimized. If the stock is only a $P$-semi-martingale, then some local-risk minimization criterion has to be applied. The existence and uniqueness of an optimal strategy is then equivalent to the existence and uniqueness of the Föllmer-Schweizer decomposition.

## 4.1 Partial hedging

In our incomplete market setting, the European call's hedging cannot be realized exactly. Indeed, any strategy replicating the European call's value can only offset one of the risk factors affecting the stock. In fact, the stock's volatility can only be partially hedged with a strategy based on existing primary financial assets (e.g., the stock itself). On the other hand, the part of the volatility which is not hedged describes the intrinsic risk peculiar to incomplete market. Specifically, the intrinsic risk represents the specific risk which cannot be hedged while using the other financial assets (i.e., stocks) since it's peculiar to the underlying risky asset (i.e., the stock $i$).

Since the stock's price is a semi-martingale under the historical probability $P$, we focus on the local risk-minimization method under the minimal martingale measure $\hat{P}$. To this end, we deduce the European call's SDE relative to the two measures $P$ and $\hat{P}$. We introduce the notation $C_x\left(t, S_t^i, X_t\right) = \frac{\partial C\left(t, S_t^i, X_t\right)}{\partial x}$. Applying multivariate Ito's lemma leads to the following call's dynamic under $P$:

$$dC\left(t, S_t^i, X_t\right) = D\left(t, S_t^i, X_t\right) dt + A\left(t, S_t^i, X_t\right) dW_t + B\left(t, S_t^i, X_t\right) dW_t^i \quad (36)$$

with

$$D\left(t,S_t^i,X_t\right) = C_t\left(t,S_t^i,X_t\right)+C_S\left(t,S_t^i,X_t\right)S_t^ib\left(t,S_t^i,X_t\right) \qquad (37)$$
$$+C_X\left(t,S_t^i,X_t\right)\mu\left(t,X_t\right)X_t+\frac{1}{2}C_{SS}\left(t,S_t^i,X_t\right)\Sigma^2\left(t,S_t^i,X_t\right)S_t^{i2}$$
$$+\frac{1}{2}C_{XX}\left(t,S_t^i,X_t\right)\sigma^2\left(t,X_t\right)X_t^2$$
$$+C_{SX}\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_t\Sigma\left(t,S_t^i,X_t\right)S_t^i\rho\left(t,S_t^i,X_t\right)$$

$$A\left(t,S_t^i,X_t\right) = C_S\left(t,S_t^i,X_t\right)S_t^i\Sigma\left(t,S_t^i,X_t\right)\rho\left(t,S_t^i,X_t\right) \qquad (38)$$
$$+C_X\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_t$$

$$B\left(t,S_t^i,X_t\right)=C_S\left(t,S_t^i,X_t\right)S_t^i\Sigma\left(t,S_t^i,X_t\right)\sqrt{1-\rho^2\left(t,S_t^i,X_t\right)} \qquad (39)$$

This relation also rewrites:

$$dC\left(t,S_t^i,X_t\right) = D^*\left(t,S_t^i,X_t\right)dt+C_S\left(t,S_t^i,X_t\right)dS_t^i \qquad (40)$$
$$+C_X\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_tdW$$

with

$$D^*\left(t,S_t^i,X_t\right)=D\left(t,S_t^i,X_t\right)-C_S\left(t,S_t^i,X_t\right)S_t^ib\left(t,S_t^i,X_t\right) \qquad (41)$$

Therefore, we get the next SDE under the minimal martingale measure $\hat{P}$:

$$dC\left(t,S_t^i,X_t\right)=\hat{D}\left(t,S_t^i,X_t\right)dt+A\left(t,S_t^i,X_t\right)d\hat{W}_t+B\left(t,S_t^i,X_t\right)d\hat{W}_t^i \qquad (42)$$

where

$$\hat{D}\left(t,S_t^i,X_t\right) = C_t\left(t,S_t^i,X_t\right)+C_S\left(t,S_t^i,X_t\right)S_t^ir+C_X\left(t,S_t^i,X_t\right)\hat{\mu}\left(t,X_t\right)X_t$$
$$+\frac{1}{2}C_{SS}\left(t,S_t^i,X_t\right)\Sigma^2\left(t,S_t^i,X_t\right)S_t^{i2} \qquad (43)$$
$$+\frac{1}{2}C_{XX}\left(t,S_t^i,X_t\right)\sigma^2\left(t,X_t\right)X_t^2$$
$$+C_{SX}\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_t\Sigma\left(t,S_t^i,X_t\right)S_t^i\rho\left(t,S_t^i,X_t\right)$$

and

$$\hat{\mu}\left(t,X_t\right)=\mu\left(t,X_t\right)-\sigma\left(t,X_t\right)\frac{b\left(t,S_t^i,X_t\right)-r}{\Sigma\left(t,S_t^i,X_t\right)}\rho\left(t,S_t^i,X_t\right) \qquad (44)$$

This equation is equivalent to the coming formulation:

$$dC\left(t,S_t^i,X_t\right) = \hat{D}^*\left(t,S_t^i,X_t\right)dt+C_S\left(t,S_t^i,X_t\right)dS_t^i \qquad (45)$$
$$+C_X\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_td\hat{W}_t$$

with

$$\hat{D}^*\left(t,S_t^i,X_t\right)=\hat{D}\left(t,S_t^i,X_t\right)-C_S\left(t,S_t^i,X_t\right)S_t^ir \qquad (46)$$

Consider a self-financing strategy $V\left(t,S_t^i\right)$ (i.e., hedging portfolio) composed of a portion $\Delta$ of stock $i$ such that its value is $V\left(t,S_t^i\right)=\Delta S_t^i$ with $\Delta$ deterministic if not constant. This strategy is designed to replicate partially the European call's value. Let $\pi\left(t,S_t^i,X_t\right)$ be the value of the portfolio corresponding to the difference between the hedging portfolio and the European call's values. $\pi\left(t,S_t^i,X_t\right)$ is the value of the hedging (i.e., tracking) error. Therefore, we have $\pi\left(t,S_t^i,X_t\right)=V\left(t,S_t^i\right)-C\left(t,S_t^i,X_t\right)=\Delta S_t^i-C\left(t,S_t^i,X_t\right)$. Since we focus on local-risk minimization, we consider infinitesimal variations of our portfolios' values. Our goal is to minimize the volatility (i.e., the variance) of the tracking error on any infinitesimal time subset. On any time subset $[t,t+dt]$, we then have under $\hat{P}$:

$$d\pi\left(t,S_t^i,X_t\right)=\Delta dS_t^i-dC\left(t,S_t^i,X_t\right) \tag{47}$$

which gives

$$\begin{aligned} d\pi\left(t,S_t^i,X_t\right) &= \left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)dS_t^i-C_X\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_td\hat{W}_t \\ &\quad -\hat{D}^*\left(t,S_t^i,X_t\right)dt \end{aligned} \tag{48}$$

Computing the conditional expectation of $d\pi\left(t,S_t^i,X_t\right)$, we find:

$$E^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right]=\left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)S_t^irdt-\hat{D}^*\left(t,S_t^i,X_t\right)dt \tag{49}$$

and deduce

$$\begin{aligned} d\pi\left(t,S_t^i,X_t\right)-E^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right] &= \left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)dS_t^i \\ &\quad -C_X\left(t,S_t^i,X_t\right)\sigma\left(t,X_t\right)X_td\hat{W}_t \\ &\quad -\left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)S_t^irdt \end{aligned} \tag{50}$$

Therefore, the instantaneous tracking error's variance is:

$$Var^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right]=E^{\hat{P}}\left[\left(d\pi\left(t,S_t^i,X_t\right)-E^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right]\right)^2\Big|_{F_t}\right] \tag{51}$$

that is

$$\begin{aligned} \frac{1}{dt}Var^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right] &= \left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)^2\Sigma^2\left(t,S_t^i,X_t\right)S_t^{i2} \\ &\quad +C_X^2\left(t,S_t^i,X_t\right)\sigma^2\left(t,X_t\right)X_t^2 \\ &\quad -2\left\{\begin{array}{c}\left(\Delta-C_S\left(t,S_t^i,X_t\right)\right)C_X\left(t,S_t^i,X_t\right) \\ \sigma\left(t,X_t\right)X_t\Sigma\left(t,S_t^i,X_t\right)S_t^i\rho\left(t,S_t^i,X_t\right)\end{array}\right\} \end{aligned} \tag{52}$$

And, stating $Var^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right]=f\left(\Delta\right)dt$, we achieve our local-risk minimization by solving the program:

$$\min_{\Delta}\frac{1}{dt}Var^{\hat{P}}\left[d\pi\left(t,S_t^i,X_t\right)\big|_{F_t}\right]=\min_{\Delta}f\left(\Delta\right) \tag{53}$$

Computing the first and second derivatives of function $f$, we find that:

$$\begin{aligned} f^{(1)}(\Delta) &= 2\left(\Delta - C_S\left(t, S_t^i, X_t\right)\right)\Sigma^2\left(t, S_t^i, X_t\right)S_t^{i2} \\ &\quad -2C_X\left(t, S_t^i, X_t\right)\sigma\left(t, X_t\right)X_t\Sigma\left(t, S_t^i, X_t\right)S_t^i\rho\left(t, S_t^i, X_t\right) \end{aligned} \tag{54}$$

$$f^{(2)}(\Delta) = 2\Sigma^2\left(t, S_t^i, X_t\right)S_t^{i2} > 0 \tag{55}$$

Since $f$ is a convex function of $\Delta$, the solution of (53) is achieved for $f^{(1)}(\Delta) = 0$ which leads to the following optimal hedging strategy:

$$\Delta = \Delta^* = C_S\left(t, S_t^i, X_t\right) + \frac{C_X\left(t, S_t^i, X_t\right)\sigma\left(t, X_t\right)X_t\rho\left(t, S_t^i, X_t\right)}{\Sigma\left(t, S_t^i, X_t\right)S_t^i} \tag{56}$$

In this case, we have:

$$\frac{1}{dt}Var_{\Delta^*}^{\hat{P}}\left[d\pi\left(t, S_t^i, X_t\right)\big|_{F_t}\right] = C_X^2\left(t, S_t^i, X_t\right)\sigma^2\left(t, X_t\right)X_t^2\left(1 - \rho^2\left(t, S_t^i, X_t\right)\right) \tag{57}$$

which is the minimum variance attainable (i.e., optimal variance) given a strategy only composed of the stock $i$. Moreover, since the market risk factor $X$ is unknown and non observable, one could choose to state $\Delta = C_S\left(t, S_t^i, X_t\right)$ such that $V\left(t, S_t^i\right) = C_S\left(t, S_t^i, X_t\right)S_t^i$. Consequently, the tracking error's variance becomes $Var^{\hat{P}}\left[d\pi\left(t, S_t^i, X_t\right)\big|_{F_t}\right] = C_X^2\left(t, S_t^i, X_t\right)\sigma^2\left(t, X_t\right)X_t^2\,dt$ whose value is superior or equal to $Var_{\Delta^*}^{\hat{P}}\left[d\pi\left(t, S_t^i, X_t\right)\big|_{F_t}\right]$ with an equality only for $\rho\left(t, S_t^i, X_t\right) = 0$.[14] Thus, the subsequent 'hedging error' is expressed as $\varepsilon = \left|V\left(t, S_t^i\right) - V_{\Delta^*}\left(t, S_t^i\right)\right| = \left|S_t^i\left(\Delta^* - C_S\left(t, S_t^i, X_t\right)\right)\right|$, and finally:

$$\varepsilon = \frac{C_X\left(t, S_t^i, X_t\right)\sigma\left(t, X_t\right)X_t\left|\rho\left(t, S_t^i, X_t\right)\right|}{\Sigma\left(t, S_t^i, X_t\right)} \tag{58}$$

In practice, investors build strategies replicating the European call by encompassing the stock and another European call with the same features except that its maturity date is superior to $T$. This process allows to get a delta-sigma hedge which presents some drawbacks. First, transaction costs are higher for such strategies. Second, trading the second European call (i.e., derivative) exposes investors to a lower liquidity. Such problems explain why we try to hedge 'at best' using only the stock as a hedge portfolio. Are we able to improve this process?

## 4.2 An implementation

We try here to propose one way to lower the residual variance of the previous partial hedging of the European call on stock $i$. One way to reach such a

[14] The equality is reached only when the stock $i$ and the market risk factor are not instantaneously correlated.

goal consists of finding the missing information relative to the previously non insurable risk. Specifically, we consider two risk factors whereas we can only observe the stock's value. We therefore need to find some additional information about one of our risk factors in order to be able to characterize completely our stock's evolution. For this purpose, we need to modify slightly our framework and to extend our probability space. We assume that there exist $(n+1)$ assets in the market with $\left(S_t^0\right)$ being the risk free asset and $\left(S_t^i\right)_{i=1,\dots,n}$ being the risky assets (i.e., stocks). Therefore, our multivariate Brownian motion becomes $w_t' = \left(W_t, W_t^1, ..., W_t^n\right)$ with $(W_t)$ and $\left(W_t^i\right)_{i=1,\dots,n}$ mutually independent Brownian motions. In the same way, the natural filtration becomes $F_t = \sigma\left(w_s, 0 \leq s \leq t\right)$. Each asset satisfies the diffusion dynamics introduced in sections 1 and 2. We lie in an incomplete market since we only observe $n$ risky assets or stocks and we have $(n+1)$ risk factors.

Considering the European call on stock $i$, our aim is to find means to offset at least asymptotically the variance of the tracking error. To this end, we propose to build a portfolio allowing us to attain the market risk factor $X$ asymptotically and to improve therefore the previous hedging strategy.

Let $P_t$ be the value of such a portfolio which is constituted of the $n$ existing stocks. This portfolio's value $P_t$ depends on $t$, $X_t$ and $\left(S_t^i\right)_{i=1,\dots,n}$ in the following way:

$$P_t = \sum_{i=1}^{n} a_i S_t^i \tag{59}$$

with $\sum\limits_{i=1}^{n} a_i = 1$ and then $dP_t = \sum\limits_{i=1}^{n} a_i dS_t^i$. Such a setting leads to the following expression for $dP_t$ under the minimal martingale measure:

$$\begin{aligned} dP_t &= r\left(\sum_{i=1}^{n} a_i S_t^i\right) dt + \left(\sum_{i=1}^{n} a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \rho\left(t, S_t^i, X_t\right)\right) d\hat{W}_t \\ &\quad + \sum_{i=1}^{n}\left(a_i S_t^i \Sigma\left(t, S_t^i, X_t\right)\sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, d\hat{W}_t^i\right) \end{aligned} \tag{60}$$

Applying Ito's lemma to function $f\left(t, P_t, X_t\right) = \ln\left(P_t / X_t\right)$, we get:

$$d\ln\left(\frac{P_t}{X_t}\right) = Drift\, dt + Vol_1\, d\hat{W}_t + \frac{\sum\limits_{i=1}^{n}\left(a_i S_t^i \Sigma\left(t, S_t^i, X_t\right)\sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, d\hat{W}_t^i\right)}{P_t} \tag{61}$$

with

$$\begin{aligned} Drift &= r + \frac{1}{2}\sigma^2\left(t, X_t\right) - \frac{\left(\sum\limits_{i=1}^{n} a_i S_t^i \Sigma\left(t, S_t^i, X_t\right)\rho\left(t, S_t^i, X_t\right)\right)^2}{2P_t^2} \\ &\quad - \frac{1}{2P_t^2}\sum_{i=1}^{n}\left(a_i^2 S_t^{i2} \Sigma^2\left(t, S_t^i, X_t\right)\left(1-\rho^2\left(t, S_t^i, X_t\right)\right)\right) \end{aligned} \tag{62}$$

$$Vol_1 = \frac{1}{P_t} \sum_{i=1}^{n} \left\{ a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \rho\left(t, S_t^i, X_t\right) \right\} - \sigma\left(t, X_t\right) \tag{63}$$

When $n$ tends towards infinity, we further assume that first the next quantity $\frac{1}{P_t} \sum_{i=1}^{n} a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \rho\left(t, S_t^i, X_t\right)$ converges towards $\sigma\left(t, X_t\right)$ $\hat{P}$-a.s., then the following expression $\frac{1}{P_t^2} \sum_{i=1}^{n} \left(a_i^2 S_t^{i2} \Sigma^2\left(t, S_t^i, X_t\right)\left(1 - \rho^2\left(t, S_t^i, X_t\right)\right)\right)$ converges towards a deterministic positive value $\bar{m}$ $\hat{P}$-a.s., and finally the stochastic sum $\frac{1}{P_t} \sum_{i=1}^{n} \left(a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \sqrt{1 - \rho^2\left(t, S_t^i, X_t\right)}\, d\hat{W}_t^i\right)$ converges towards zero in the quadratic sense of $L^2\left(\hat{P}\right)$. Therefore, we have for $n$ in the neighbourhood of infinity:

$$d\ln\left(\frac{P_t}{X_t}\right) = \left[r - \frac{\bar{m}}{2}\right] dt + \frac{1}{P_t} \sum_{i=1}^{n} \left(a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \sqrt{1 - \rho^2\left(t, S_t^i, X_t\right)}\, d\hat{W}_t^i\right) \tag{64}$$

Moreover, the first two conditional moments of $d\ln\left(\frac{P_t}{X_t}\right)$ are as follows:

$$E^{\hat{P}}\left[ d\ln\left(\frac{P_t}{X_t}\right)\bigg|_{F_t}\right] = \left[r - \frac{\bar{m}}{2}\right] dt \tag{65}$$

$$Var^{\hat{P}}\left[ d\ln\left(\frac{P_t}{X_t}\right)\bigg|_{F_t}\right] = E_t^{\hat{P}}\left[ \frac{\left(\sum_{i=1}^{n} a_i S_t^i \Sigma\left(t, S_t^i, X_t\right) \sqrt{1 - \rho^2\left(t, S_t^i, X_t\right)}\, d\hat{W}_t^i\right)^2}{P_t^2} \right] \tag{66}$$

Consequently, when the number of existing stocks tends towards infinity, we can make the following approximation:

$$d\ln\left(\frac{P_t}{X_t}\right) = \left[r - \frac{\bar{m}}{2}\right] dt \tag{67}$$

which implies that $P_t = X_t \exp\left\{\left(r - \frac{\bar{m}}{2}\right) t\right\} = X_t g\left(t\right)$ provided that $P_0/X_0 = 1$. In this case, portfolio $P_t$ proxies asymptotically the market risk factor with a deterministic link. We can easily link our framework to the one of Chauveau & Gatfaoui (2002) (i.e., CG2002) by first setting $\alpha^0 = 0$ in CG2002 since we have no risk free asset in our portfolio. Then, given our framework, we have to set constant diffusion parameters such that $\mu\left(t, X_t\right) = \mu$, $\sigma\left(t, X_t\right) = \sigma$, $\nu_i\left(t, I_t^i\right) = \mu_i$ and finally $\tau_i\left(t, I_t^i\right) = \sigma_i$ analogously to CG2002. Hence, our minimal martingale measure $\hat{P}$ coincides with the risk-neutral measure $Q$ of CG2002 since we lie consequently in a complete market.[15] Recall that CG2002

[15] We are then able to hedge against all the existing risks, which allows us to set the tracking error portfolio's local risk at zero.

portfolio's weights are such that $a_i = a_i^* = \frac{P_t}{n\beta_i S_t^i}$ for each $i$ in $\{1, ..., n\}$. In such a case, our framework is equivalent to impose the following limit condition, namely $\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{r}{\beta_i} + \frac{1}{2}(\beta_i - 1)\sigma^2\right\} = r$ (i.e., convergent hamonic and arithmetic means of the risky assets' betas). We expose in the appendix the risk-neutral dynamics of both portfolio's value $P_t$ and market risk factor $X_t$ allowing us to draw such conclusions. Notice that the later limit condition rewrites under the more general form $\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{a_i S_t^i}{P_t}\left\{r + \frac{1}{2}\beta_i(\beta_i - 1)\sigma^2\right\} = r$ for any portfolio's weight $a_i \in \mathbb{R}$.

Given this asymptotic deterministic feature, we use this portfolio to build our European call's replicating strategy in addition to the stock $i$. The value $V_t$ of such a strategy is then $V_t = \Delta S_t^i + \delta P_t$ with $\Delta$ and $\delta$ deterministic if not constant. The difference between the value of our strategy and the value of the call (i.e., instantaneous tracking error) becomes $\pi_t = V_t - C\left(t, S_t^i, X_t\right) = \Delta S_t^i + \delta P_t - C\left(t, S_t^i, X_t\right)$ so that $d\pi_t = \Delta dS_t^i + \delta dP_t - dC\left(t, S_t^i, X_t\right)$. Therefore, the variation of our tracking error on an infinitesimal time interval expresses:

$$d\pi_t = Jdt + Kd\hat{W}_t + Ld\hat{W}_t^i + \sum_{j=1}^{n}\left(a_j S_t^j \Sigma\left(t, S_t^j, X_t\right)\sqrt{1-\rho^2\left(t, S_t^j, X_t\right)}\, d\hat{W}_t^j\right) \tag{68}$$

where

$$J = r\Delta S_t^i + r\delta P_t - \hat{D}\left(t, S_t^i, X_t\right) \tag{69}$$

$$\begin{aligned} K &= \Delta S_t^i \Sigma\left(t, S_t^i, X_t\right)\rho\left(t, S_t^i, X_t\right) + \delta\left(\sum_{j=1}^{n} a_j S_t^j \Sigma\left(t, S_t^j, X_t\right)\rho\left(t, S_t^j, X_t\right)\right) \\ &\quad - A\left(t, S_t^i, X_t\right) \end{aligned} \tag{70}$$

$$L = \Delta S_t^i \Sigma\left(t, S_t^i, X_t\right)\sqrt{1-\rho^2\left(t, S_t^i, X_t\right)} - B\left(t, S_t^i, X_t\right) \tag{71}$$

for $A\left(t, S_t^i, X_t\right)$ and $B\left(t, S_t^i, X_t\right)$ being defined as in the previous subsection. Given our previous assumptions, the first two conditional moments of $d\pi_t$ then satisfy:

$$E^{\hat{P}}\left[d\pi_t|_{F_t}\right] = Jdt \tag{72}$$

$$\begin{aligned} \frac{1}{dt}Var^{\hat{P}}\left[d\pi_t|_{F_t}\right] = & \left(\Delta - C_S\left(t, S_t^i, X_t\right)\right)\Sigma^2\left(t, S_t^i, X_t\right)S_t^{i2} \\ & \left[\Delta - C_S\left(t, S_t^i, X_t\right) + 2a_i\left(1-\rho^2\left(t, S_t^i, X_t\right)\right)\right] \\ & +\sigma^2\left(t, X_t\right)\left[\delta P_t - C_X\left(t, S_t^i, X_t\right)X_t\right]^2 \\ & +2S_t^j\Sigma\left(t, S_t^j, X_t\right)\rho\left(t, S_t^i, X_t\right)\sigma\left(t, X_t\right) \\ & \left(\Delta - C_S\left(t, S_t^i, X_t\right)\right)\left[\delta P_t - C_X\left(t, S_t^i, X_t\right)X_t\right] \end{aligned} \tag{73}$$

For proportions such that $\Delta = \Delta^* = C_S\left(t, S_t^i, X_t\right)$ and $\delta = \delta^* = \frac{C_X\left(t, S_t^i, X_t\right) X_t}{P_t}$ then $Var^{\hat{P}}\left[d\pi_t|_{F_t}\right] = 0$ which leads to an optimal strategy offsetting the risks caused by the existing factors.[16] Moreover, assuming that $n$ tends towards infinity and given diffusion (67), we find that:

$$\frac{dP_t}{P_t} = \left(r - \frac{\bar{m}}{2} + \hat{\mu}\left(t, X_t\right)\right) dt + \sigma\left(t, X_t\right) d\hat{W}_t = \left(r - \frac{\bar{m}}{2}\right) dt + \frac{dX_t}{X_t} \tag{74}$$

which implies that $C_X\left(t, S_t^i, X_t\right) = \frac{C_P\left(t, S_t^i, \frac{P_t}{g(t)}\right) P_t}{X_t}$. Consequently, we have $\delta^* = C_P\left(t, S_t^i, \frac{P_t}{g(t)}\right)$ and the optimal replicating strategy writes $V_t = \Delta^* S_t^i + \delta^* P_t = C_S\left(t, S_t^i, \frac{P_t}{g(t)}\right) S_t^i + C_P\left(t, S_t^i, \frac{P_t}{g(t)}\right) P_t$ (i.e., strategy with a local risk equal to zero).[17] As a conclusion, thanks to portfolio of value $P_t$ which serves as an asymptotic proxy of the market risk factor, we are able to hedge the European call on stock $i$ and succeed in setting asymptotically the risks to zero. The next step would be to verify (and find the related required conditions if possible) whether such a strategy is indeed locally and asymptotically risk free. If this is the case, our minimal martingale measure would therefore coincide asymptotically with the risk-neutral measure. Such a process would contribute to show that our portfolio of value $P_t$ helps to complete asymptotically the market, and we could then say that we have an 'almost complete' market.

[16] Here, our result is analogous to that of CAPM theory stating that a diversified portfolio needs an *infinite* number of stocks. Many empirical works show that for a given *finite* number of stocks, we are able to achieve a convenient level of diversification (i.e., setting the idiosyncratic part of the portfolio's global risk to an extremely small level). In the practical viewpoint of our setting, a convenient choice of weights $(a_i)_{i=1,\ldots,n}$ of our $P_t$-valued portfolio will allow to set our tracking error's local variance $Var\left[d\pi_t|_{F_t}\right]$ nearly to zero for a given *finite* number $n$ of stocks.

[17] In CG2002 case, the tracking error related to the optimal replicating strategy is shown to have a minimum local risk and therefore a zero asymptotic local risk as long as our portfolio's weights satisfy $a_i = \frac{P_t\, c_i}{n S_t^i}$ for finite constant coefficients $c_i \in \mathbb{R}$ whatever risky asset $i$. It is sufficient to consider the minimization problem of the tracking error's residual variance relative to weights $a = (a_i)_{1 \leq i \leq n}$ of our portfolio whose current value is $P_t$. Namely, we just have to consider $\min_{a \in \mathbb{R}^n} Var\left[Sto\left(d\pi_t^*\right)\right] = \min_{a \in \mathbb{R}^n} \sum_{i=1}^{n} \frac{a_i^2 S_t^{i2}}{P_t^2} \sigma_i^2$ when $\delta_t = \delta_t^* = \frac{C_X\left(t, S_t^i, X_t\right) X_t}{P_t}$. The study of the corresponding gradient vector and Hessian matrix leads to the previous conclusion. Specifically, the functional form of each $a_i$ relative to $c_i$ implies that the Hessian matrix is semi-definite positive and the gradient can be asymptotically set to zero. Our quadratic minimization can thus be achieved asymptotically. Moreover, we observe the following feature for any given $n \in \mathbb{N}^*$, namely $Var_{a_i = \frac{P_t\, c_i}{n S_t^i}}\left[Sto\left(d\pi_t^*\right)\right] < Var_{a_i^* = \frac{P_t}{n \beta_i S_t^i}}\left[Sto\left(d\pi_t^*\right)\right]$ whatever $c_i \in \mathbb{R}$ such that $|c_i| < \frac{1}{|\beta_i|}$, or equivalently, $Var_{a_i}\left[Sto\left(d\pi_t^*\right)\right] < Var_{a_i^*}\left[Sto\left(d\pi_t^*\right)\right]$ whatever $a_i \in \mathbb{R}$ such that $|a_i| < \left|a_i^*\right|$.

# 5 Conclusion

In this paper, we considered the study of Chauveau & Gatfaoui (2002) about a European option pricing model subject both to systematic and idiosyncratic risk factors. We showed the analogy with option pricing models based on stochastic volatility. Indeed, such a framework led to an incomplete market insofar as we observe only the underlying stock while having two risk factors. Such a modeling is motivated by recent literature which shows that determinants of the smile effect encompass the market's influence, the stock's volatility, and the correlation between the stock's return and its volatility among others. Hence, such a framework raised two major questions that we considered, namely option valuation and hedging.

First, in an incomplete market, the stock bears an intrinsic risk which enables us only to reduce the remaining risk to its minimal component. Therefore, we have to divide the space of assets consistent with the stock in two subspaces, namely traded assets and non tradable assets. Such a typology is achieved with the minimal equivalent martingale measure wich minimizes the uncertainty. Consequently, we used this measure to price a European stock call along with the no-arbitrage principle. The current European call's price was priced as the discount expected value of its terminal payoff under the minimal martingale measure. Under some regularity conditions (e.g., the specific risk factor's volatility is deterministic), the call's current price is the expected value of the Black & Scholes price applied to an average time-dependent volatility.

Second, incompleteness implies that the call's risk exposure cannot usually be totally offset with some appropriate hedging strategy. However, an optimal strategy has to achieve the quadratic minimization of the volatility of the tracking error or, equivalently the losses due to the impossibility of a perfect hedge in incomplete market (see Schweizer [1999a] for example). Specifically, we attempted to find a local risk-minimizing replicating strategy under the minimal martingale measure. The set of replicating strategies considered were stock-based hedging strategies. Since the minimal risk measure $\hat{P}$ is unique given some regularity assumptions (i.e., a unique Föllmer-Schweizer decomposition), the optimal strategy exists and is unique. Recall that the current call's price valued under $\hat{P}$ corresponds to the local risk-minimizing value of the call.

Finally, we addressed the question of implementing a partial hedging strategy based on only one stock. For this purpose, we constructed a portfolio composed of all the existing stocks and designed to proxy the market risk factor asymptotically. In particular, the link between such a portfolio and the systematic risk factor is deterministic under some appropriate conditions. Then, using a replicating hedging strategy based on the underlying stock and the portfolio to offset the global risk of the European stock call, we succeeded in cancelling the variance of the tracking error and then to offset asymptotically the risks carried by the call. However, some work has to be done to study whether this strategy is asymptotically risk free. The significance of this inquiry is important insofar as we could asymptotically reach some 'almost complete' market and then perfect the European call's hedge.

To conclude, some extensions of our work may be achieved. First, we should apply this framework to some specific pricing model such as Stein & Stein (1991), Heston (1993), Bakshi *et al.* (1997) for example, and realize the related simulation study (with given parameters value) or empirical calibration study. Second, we could encompass the well-known interest rate risk while introducing stochastic interest rates. In particular, we could thus focus on some stochastic volatility HJM (i.e., Heath-Jarrow-Morton [1992]) interest rate modeling as pointed by Chiarella *et al.* (2004). Then, we should finally relax some assumptions to extend our framework to an unbounded stochastic volatility (i.e., unbounded functional parameters in diffusions).

# 6 Appendix

In this part, we give some computational details or information linked to our option pricing setting.

## 6.1 Bounds

We give the expressions of the bounds related to the diffusion parameters of the stock's price (11). Bounds are obtained after some lengthy simple calculations and are given whatever $t \in [0,T]$, $S_t^i$ and $X_t \in \mathbb{R}$.

Concerning the drift parameter $b\left(t, S_t^i, X_t\right)$, we have for $\beta_i > 1$:

$$\begin{aligned} b_l &= \beta_i \mu_l + \nu_l + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_l^2 \\ b_u &= \beta_i \mu_u + \nu_u + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_u^2 \end{aligned} \tag{75}$$

which become when $0 < \beta_i < 1$:

$$\begin{aligned} b_l &= \beta_i \mu_l + \nu_l + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_u^2 \\ b_u &= \beta_i \mu_u + \nu_u + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_l^2 \end{aligned} \tag{76}$$

And when $\beta_i < 0$, these bounds translate into:

$$\begin{aligned} b_l &= \beta_i \mu_u + \nu_l + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_l^2 \\ b_u &= \beta_i \mu_l + \nu_u + \frac{\beta_i}{2}\left(\beta_i - 1\right)\sigma_u^2 \end{aligned} \tag{77}$$

For the global volatility parameter, we get the following lower and upper bounds:

$$\begin{aligned} \Sigma_l &= \sqrt{\beta_i^2 \sigma_l^2 + \tau_l^2} \\ \Sigma_u &= \sqrt{\beta_i^2 \sigma_u^2 + \tau_u^2} \end{aligned} \tag{78}$$

Finally, if $\beta_i > 0$, the bounds of the correlation coefficient are as follows:

$$\begin{aligned} \rho_l &= \frac{\beta_i \sigma_l}{\Sigma_u} \\ \rho_u &= \frac{\beta_i \sigma_u}{\Sigma_l} \end{aligned} \tag{79}$$

And if $\beta_i < 0$, we have:

$$\begin{aligned} \rho_l &= \frac{\beta_i \sigma_u}{\Sigma_l} \\ \rho_u &= \frac{\beta_i \sigma_l}{\Sigma_u} \end{aligned} \tag{80}$$

Whatever the value of the stock's beta $\beta_i$, every diffusion parameter of (11) remains bounded.

## 6.2 Minimal martingal measure

In this subsection, we explain how to justify the existence of the minimal martingale measure along with a useful theorem.

Recall that the dynamic of the stock $i$ and the risk free asset are respectively:

$$\frac{dS_t^i}{S_t^i} = b\left(t, S_t^i, X_t\right)\, dt + \Sigma\left(t, S_t^i, X_t\right) \left[\rho\left(t, S_t^i, X_t\right)\, dW_t + \sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, dW_t^i\right] \tag{81}$$

$$\frac{dS_t^0}{S_t^0} = r\, dt \tag{82}$$

Therefore, the discount price of the stock $\tilde{S}_t^i = \frac{S_t^i}{S_t^0}$ has the following dynamic after applying Ito's lemma:

$$\frac{d\tilde{S}_t^i}{\tilde{S}_t^i} = \left[b\left(t, S_t^i, X_t\right) - r\right] dt + \Sigma\left(t, S_t^i, X_t\right) \left[\begin{array}{c} \rho\left(t, S_t^i, X_t\right)\, dW_t \\ +\sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, dW_t^i \end{array}\right] \tag{83}$$

Therefore, the martingale component $M$ of the discount price $\tilde{S}_t^i$ of the stock is given by:

$$dM_t = \Sigma\left(t, S_t^i, X_t\right) \tilde{S}_t^i \left[\rho\left(t, S_t^i, X_t\right)\, dW_t + \sqrt{1-\rho^2\left(t, S_t^i, X_t\right)}\, dW_t^i\right] \tag{84}$$

with

$$S_t^i = S_t^0 \tilde{S}_t^i = \exp(rt)\, \tilde{S}_t^i \tag{85}$$

such that

$$d\langle M\rangle_t = \Sigma^2\left(t, S_t^i, X_t\right) \tilde{S}_t^{i2} dt \tag{86}$$

For $\alpha(t) = \frac{b\left(t, S_t^i, X_t\right) - r}{\Sigma^2\left(t, S_t^i, X_t\right)\tilde{S}_t^i}$, we find that $\alpha(t)\, d\langle M\rangle_t = \tilde{S}_t^i \left[b\left(t, S_t^i, X_t\right) - r\right] dt$ and we can apply the theorem therein.

**Theorem 1** *(i) The minimal martingale measure $\hat{P}$ is uniquely determined.*
*(ii) $\hat{P}$ exists if and only if for all t$\in [0,T]$*

$$\hat{L}(t) = \exp\left\{-\int_0^t \alpha(u)\, dM_u - \frac{1}{2}\int_0^t \alpha^2(u)\, d\langle M\rangle_u\right\} \tag{87}$$

*is a square-integrable martingale under P. In that case, $\hat{P}$ is given by $\frac{d\hat{P}}{P} = \hat{L}(T)$.*
*(iii) The minimal martingale measure preserves orthogonality: Any square-integrable martingale N with $\langle N, M^i\rangle = 0$ for i=1,...,n under P satisfies $\langle N, S^i\rangle = 0$ under $\hat{P}$.*

## 6.3 Black & Scholes pricing formula

We recall briefly the European option pricing formula proposed by Black & Scholes (1973). Those authors assume a complete market such that:

$$\frac{dS_t^i}{S_t^i} = b\, dt + \Sigma dW_t \tag{88}$$

where $b$ and $\Sigma$ are deterministic constants with $\Sigma > 0$. $(W_t)$ is a standard Brownian motion in the historical universe. In this case, the minimal equivalent martingale measure coincides with the risk-neutral martingale measure, and the stock's dynamic in the risk-neutral universe becomes:

$$\frac{dS_t^i}{S_t^i} = r\, dt + \Sigma d\hat{W}_t \tag{89}$$

with

$$d\hat{W}_t = \frac{b-r}{\Sigma} dt + dW_t \tag{90}$$

Therefore, Black & Scholes establish a closed-form formula pricing a European call on the stock as follows:

$$C_{BS}\left(t, S_t^i, K, T, \Sigma\right) = S_t^i N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{91}$$

with

- $N(.)$ the cumulative distribution function of the standard normal law;
- $d_1 = \frac{\ln\left(\frac{S_t^i}{K}\right) + \left(r + \frac{\Sigma^2}{2}\right)(T-t)}{\Sigma\sqrt{T-t}}$;
- $d_2 = d_1 - \Sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t^i}{K}\right) + \left(r - \frac{\Sigma^2}{2}\right)(T-t)}{\Sigma\sqrt{T-t}}$.

Black & Scholes find then a nice analytical and easily tractable formula for valuing European calls written on the stock $S_t^i$.

## 6.4 Link with CG2002

We present therein the respective risk-neutral dynamics of the portfolio and market risk factor in a CG2002 world. Then, we exhibit the link between these two stochastic variables.

Let $\left(\hat{W}_t\right)$ be the risk-neutral Brownian motion characterizing the market risk factor's uncertainty. Then, the market risk factor's dynamic in the risk-neutral universe reads:

$$dX_t = X_t \left[ r\, dt + \sigma\, d\hat{W}_t \right] \tag{92}$$

In the same way, let $\left(\hat{W}_t^i\right)$ for $i \in \{1, ..., n\}$ be the standard risk-neutral Brownian motions characterizing respectively all existing specific risk factors' uncertainty. Therefore, the portfolio's risk-neutral dynamic writes:

$$\frac{dP_t}{P_t} = \sum_{i=1}^{n} \frac{1}{n\beta_i} \left\{ r\left(\beta_i + 1\right) + \frac{1}{2}\beta_i \left(\beta_i - 1\right) \sigma^2 \right\} dt + \sigma\, d\hat{W}_t + \sum_{i=1}^{n} \left( \frac{\sigma_i}{n\beta_i}\, d\hat{W}_t^i \right) \tag{93}$$

Applying Ito's bivariate lemma to function $F\left(t, P_t, X_t\right) = \ln\left(\frac{P_t}{X_t}\right)$, we are able to characterize the link prevailing between the diversified portfolio and the market risk factor:

$$d\ln\left(\frac{P_t}{X_t}\right) = \left\{ \sum_{i=1}^{n} \frac{1}{n\beta_i} \left[ r + \frac{1}{2}\beta_i \left(\beta_i - 1\right)\sigma^2 \right] - \frac{1}{2} \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2 \beta_i^2} \right\} dt + \sum_{i=1}^{n} \left( \frac{\sigma_i}{n\beta_i}\, d\hat{W}_t^i \right) \tag{94}$$

Such a representation allows us to establish easily the link with CG2002 framework. Under our original assumptions, the required limit condition becomes obvious. Such a constraint being aimed at minimizing the local remaining risk given our hedging portfolio. Moreover, this limit condition ensures that the remaining local risk is asymptotically zero.


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