An investigation of a portfolio-loss under the CAPM

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Abstract:

We consider a portfolio built according to the Capital Market Line of the Capital-Asset-Pricing Model. The universe of asset classes include marketable shares and bonds only. We investigate losses that emerge when the rate of return of the portfolio is lower than that required to fulfil a defined obligation. We will classify these losses and calculate upper limits for them.

1 Introduction

In Germany, pension liabilities that are financed internally by so called "book reserves", support fund or pension funds are almost entirely underwritten by the Pensionssicherungsverein auf Gegenseitigkeit (PSV aG). This institution levies an insurance premium proportional to the value (measured by the accrued actuarial liability using the Entry Age Normal Valuation Method with fixed assumptions) of the legally vested benefit obligation. The funding method used by the PSV aG can be described as a Pay-As-You-Go method until retirement commencement. For retirees, however, the full present value (on very conservative assumptions) is provided to a consortium of insurers, that account for experience gains and losses in an own profit and loss statement towards the PSV aG (Terminal Funding). The central question to ask therefore is: what percentage of the obligation is used as a premium (over the last 30 years, the percentage has been fairly volatile), and is this premium a good measure of the risk not to fulfil the obligation?

Several reports [1] still suggest a premium which is proportional to the value of the legally vested obligation but that this premium should be modified in accordance with the different pension vehicle (pension fund, support fund, book reserve, etc.) in operation.

In this publication we give a definition of this risk or loss, based on the so called lower partial moments as published by Fishburn[2].

We consider a portfolio consisting of bonds and shares. Portfolios containing derivatives and other instruments such as futures, options, etc. are beyond the scope of this paper. We describe the shares as assets with a variable rate of return. We assume that the capital market can be described with the Capital Asset Pricing Model (CAPM)[3], and that the investment of the portfolio corresponds to the Capital Market Line of the CAPM. Our model is a one periodic model. We give a measure of this loss as a function of expected rate of return, variance, etc. of the assets.

2 Selection of losses

We have selected three different losses: the average loss L_A , the maximum loss L_M and the value-at-risk-loss L_E. We assume that the rate of return R and the probability density $p(R)$ of the portfolio are random variables with real values. From this we conclude that the following integrals which define the losses are well defined.

2.1 The average loss L_A

We define L_A

1) $$L_A = I \int_{-\infty}^{Rnec} (R_{nec} - R)\, p(R)\, dR$$

R_{nec} is the rate of return, which is necessary to fulfil obligations. I is the amount of the investment. The probability to obtain a loss larger than 0 is:

2) $$\int_{-\infty}^{R_{nec}} p(R)\, dR$$

with

3) $$\int_{-\infty}^{\infty} p(R)\, dR = 1 \cdot$$

We use 1) instead of:

4) $$L_A = I\int_{-\infty}^{+\infty}(R_{nec} - R)p(R)dR$$

because the yield above R_{nec},is not of immediate interest. Rather, we focus on the value of the loss.

2.2 The maximum loss L_M

We define L_M

5) $$L_M = I\int_{-\infty}^{R\min}(R_{nec} - R)p(R)dR \cdot$$

We call R_{min} the minimum rate of return. This means that for a rate of return above R_{min} we assume that the loss is not relevant.

2.3 The value-at-risk loss L_V

L_V is the loss which is caused by the very low rates of return from the portfolio. We define it similar to the Value-at-Risk[4]. Therefore we consider the 1-γ-Quantil, which is defined as follows:

6) $$P(R_{nec} - R \geq VaR) = \gamma \,.$$

VaR is the rate of return, for which only γ of all *R* are below. L_V is therefore

7) $$L_V = \int_{-\infty}^{Rnec-VaR}(R_{nec} - R)p(R)d(R)$$

Concerning value-at-risk there is also the following definition[4]:

8) $$VaR = z_\gamma \sigma$$

with z_γ is the γ -Quantil of standard normal distribution.

3 Estimation of loss of the portfolio

We consider a portfolio whose investment is a combination of the bonds and the so called tangency portfolio which consists of assets only. The rate of return of this (pension fund) portfolio R_{PF} is:

9) $$R_{PF} = \alpha R_0 + (1-\alpha) R_{TP}$$

the expected rate of return is E_{PF} (or μ_{PF})

10) $$\mu_{PF} = \alpha R_0 + (1-\alpha)\mu_{TP};$$

and the standard deviation σ_{PF}

11) $$\sigma_{PF} = (1-\alpha)\sigma_{TP};$$

with R_{TP} is the rate of return, μ_{TP} the expected rate of return, σ_{TP} the standard deviation of the Tangency portfolio and R_0 the rate of return of the bonds. Also we assume $0 \leq \alpha \leq 1$. This investment on this tangent is also called an investment on the capital market line[3]. Using 1) we obtain for the loss L_A:

12) $$L_A = I \int_{-\infty}^{R_{nec}} (R_{nec} - \alpha R_0 - (1-\alpha)R_{TP}) p(\alpha R_0 + (1-\alpha)R_{TP}) d(\alpha R_0 + (1-\alpha)R_{TP}).$$

We define

13) $$p_{PF}(R_{TP}) \equiv p(\alpha R_0 + (1-\alpha)R_{TP}).$$

In the following we will use only $p_{PF}(R_{TP})$ and we call this $p(R_{TP})$. From this we conclude:

14) $$L_A = I(1-\alpha) \int_{+\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}} (R_{nec} - \alpha R_0 - (1-\alpha)R_{TP}) p(R_{TP}) dR_{TP}.$$

L_M and L_V we can calculate in a analogue manner.

4 Loss of the portfolio, when the rate of return of the portfolio is normal distributed

We assume, that R_{TP} is "normally" distributed:

15) $$p(R_{TP}) = \frac{\exp\left[-\frac{(R_{TP} - \mu_{TP})^2}{2\sigma_{TP}^2}\right]}{\sqrt{2\pi}\sigma_{TP}}.$$

Calculations with other densities of probability may be done in other publications.

From this we conclude:

16)

$$L_A = I\frac{(1-\alpha)}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}(R_{nec}-\alpha R_0-(1-\alpha)R_{TP})\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}$$

$$= I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha)}-R_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}$$

$$= I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha_1)}-\mu_{TP}+\mu_{TP}-R_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}$$

$$= I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha)}-\mu_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}+$$

$$I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[\mu_{TP}-R_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}=$$

$$= I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha)}-\mu_{TP}\right]\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}-$$

$$I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int\limits_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[R_{TP}-\mu_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]d(R_{TP}-\mu_{TP})$$

using

17) $$\int\limits_{-\infty}^{\infty}\exp\left[-\frac{x^2}{2}\right]dx=\sqrt{2\pi}$$

we obtain

18) $$\frac{1}{\sigma_{TP}}\int\limits_{-\infty}^{\infty}\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}=\int\limits_{-\infty}^{\infty}\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]d(\frac{R_{TP}-\mu_{TP}}{\sigma_{TP}})=\sqrt{2\pi}$$

and

19) $$I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha)}-\mu_{TP}\right]\int_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]dR_{TP}$$
$$\leq I\frac{(1-\alpha)^2}{\sqrt{2\pi}}\left[\frac{R_{nec}-\alpha R_0}{(1-\alpha)}-\mu_{TP}\right]\sqrt{2\pi}=I(1-\alpha)^2\left[\frac{R_{nec}-\alpha R_0-(1-\alpha)\mu_{TP}}{(1-\alpha)}\right].$$

Using

20) $$-\int_{-\infty}^{a}x\exp\left[-x^2\right]dx=\left.\frac{\exp\left[-a^2\right]}{2}\right|_{-\infty}^{a}=\frac{\exp\left[-a^2\right]}{2}$$

we conclude

21) $$I\frac{(1-\alpha)^2}{\sqrt{2\pi}\sigma_{TP}}\int_{-\infty}^{\frac{R_{nec}-\alpha R_0}{1-\alpha}}\left[R_{TP}-\mu_{TP}\right]\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]d(R_{TP}-\mu_{TP})=$$
$$=I\frac{(1-\alpha)^2\sqrt{2}\sigma_{TP}}{\sqrt{\pi}}\int_{-\infty}^{\frac{R_{nec}-\alpha R_0}{(1-\alpha)}}\frac{\left[R_{TP}-\mu_{TP}\right]}{\sqrt{2}\sigma_{TP}}\exp\left[-\frac{(R_{TP}-\mu_{TP})^2}{2\sigma_{TP}^2}\right]d\left(\frac{R_{TP}-\mu_{TP}}{\sqrt{2}\sigma_{TP}}\right)=$$
$$=I\frac{(1-\alpha)^2\sigma_{TP}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{R_{nec}-\alpha R_0-(1-\alpha)\mu_{TP}}{(1-\alpha)\sqrt{2}\sigma_{TP}}\right)^2\right]$$

Using 19) and 21) we can give an upper limit for L_A:

22) $$L_A\leq I(1-\alpha)^2\left[\frac{R_{nec}-\alpha R_0-(1-\alpha)\mu_{TP}}{(1-\alpha)}\right]$$
$$+I\frac{(1-\alpha)^2\sigma_{TP}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{R_{nec}-\alpha R_0-(1-\alpha)\mu_{TP}}{(1-\alpha)\sqrt{2}\sigma_{TP}}\right)^2\right]$$

This can be written as:

23) $$L_A\leq I(1-\alpha)^2 f(\alpha;\mu_{TP})+I\frac{(1-\alpha)^2\sigma_{TP}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{f(\alpha;\mu_{TP})}{\sqrt{2}\sigma_{TP}}\right)^2\right]$$

with

24) $$f(\alpha;\mu_{TP})=\frac{R_{nec}-\alpha R_0-(1-\alpha)\mu_{TP}}{(1-\alpha)}$$

or as

25)

$$L_A \le I(1-\alpha)^2\left(\frac{R_{nec}}{(1-\alpha)} - \frac{\alpha R_0}{(1-\alpha)} - \frac{(1-\alpha)\mu_{TP}}{(1-\alpha)}\right)$$

$$+ I\frac{(1-\alpha)^2\sigma_{TP}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{R_{nec}-\alpha R_0}{(1-\alpha)\sqrt{2}\sigma_{TP}} - \frac{\mu_{TP}}{\sqrt{2}\sigma_{TP}}\right)^2\right) =$$

$$= I\left[\begin{array}{l}(1-\alpha)(R_{nec}-\alpha R_0) - (1-\alpha)^2\mu_{TP} \\ + \dfrac{(1-\alpha)^2\sigma_{TP}}{\sqrt{2\pi}}\exp\left(-\dfrac{1}{2}\left(\dfrac{R_{nec}-\alpha R_0}{(1-\alpha)\sqrt{2}\sigma_{TP}} - \dfrac{\mu_{TP}}{\sqrt{2}\sigma_{TP}}\right)^2\right)\end{array}\right] =$$

We define the function g_A (μ_{TP}, σ_{TP}):

26) $$g_A(\alpha) = \mu_{TP} - \frac{\sigma_{TP}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{R_{nec}-\alpha R_0}{(1-\alpha)\sqrt{2}\sigma_{TP}} - \frac{\mu_{TP}}{\sqrt{2}\sigma_{TP}}\right)^2\right).$$

and therefore follows:

27) $$L_A \le I\left((1-\alpha)(R_{nec}-\alpha R_0) - (1-\alpha)^2 g_A(\alpha)\right).$$

L_M and L_V we can measure similarly.

We want to know how the investment on the Capital Market Line changes from bonds to assets and vice versa when the **investor** demands that the losses should not exceed a value which he defined a priori. We define L, which is an upper limit for L_A, L_M or L_V, as proportional to the investment I, with a rate λ:

28) $$L_A \le L = \lambda I$$

with

29) $$\lambda = \left((1-\alpha)(R_{nec}-\alpha R_0) - (1-\alpha)^2 g_A(\alpha)\right).$$

We have limited α as follows

30) $$0 \le \alpha \le 1.$$

When $\alpha = 1$; i.e. that agent invests only in bonds we obtain:

31) $L_A = 0$.

When $\alpha = 0$, i.e. investor invests only in the tangency portfolio and we obtain:

32) $$L_A \le L = \lambda I = (R_{nec} - g_A)I = I\left(R_{nec} - \mu_{TP} + \frac{\sigma_{TP}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{R_{nec} - \mu_{TP}}{\sqrt{2}\sigma_{TP}}\right)^2\right)\right).$$

Using 22) or 27) we can decide whether a loss which is given a priori can be reached with the characteristics of the portfolio as α, μ_{TP} and σ_{TP}. Assuming, that the tangency portfolio (also μ_{TP} and σ_{TP}) is fixed (e.g. DAX 30) we can decide whether changing α can reduce the loss.

5 Conclusion

Using a one-period model we have defined the loss which an **investor** can suffer when he wants to fulfil obligations by using the returns on his portfolio. We measured these losses as a function of the composition of the portfolio, of the expected rate of return of the assets, of the rate of return of the bonds and of the variance of the asset.

The model prepared in this paper has limitations. In expanding this model the following points should be taken into consideration:

- More than one periodic model.
- A different composition of the portfolio e.g. including futures, options, etc.
- Different distributions of probabilities in rates of return.

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4 Klüppelberg, C., Korn R., Optimale Portfolios mit beschränktem Value-at-Risk, *Solutions* **3**, 23 - 32 (1999).