OVERNIGHT INDEXED SWAPS AND FLOORED COMPOUNDED INSTRUMENT IN HJM ONE-FACTOR MODEL

MARC HENRARD

ABSTRACT. Two types of financial instruments including (overnight) compounding are studied in this note. The first one is overnight compounded instruments in the case where the settlement is delayed with respect to the end of the compounding period (floating leg of the OIS). The second is options on the composition. In both cases we study both continuous and discrete composition. We provide explicit formulas within the HJM one-factor models with deterministic volatility together with hedging strategies.

1. INTRODUCTION

Financial products based on composition of an overnight or short-term rate are fairly common. In particular EONIA swaps (EUR) and overnight indexed swaps (OIS) based on Fed Fund Effective rates (USD) are very liquid.

This note considers two types of instruments linked to composition. The first one is overnight composition (floating leg of the OIS). This can appear obvious as the product is linear and can be priced using the forward rates. This is true except that usually the payment of the amount due is done one or two business days after the end of the composition. This extra time is necessary to insure a smooth settlement of the transaction. From a valuation point of view, it means that there is a so called *convexity adjustment* (the payment of interests does not take place at the end of the period for which they are valid). We propose an explicit formula for the valuation of the instrument in the case of continuous (Section 4) and discrete composition (Section 5). We also estimate the difference between this option-like valuation and the standard valuation using forward rates (Section 6).

The second subject we study about compounded products is the options on the composition. The typical product we want to price is a bond paying the compounded average rate with a *minimal* fixed rate. As the composition is some kind of average of the short term rates over the composition period, this is an Asian option. We study this problem in the case of continuous (Section 7) and discrete composition (Section 8) without payment lag or with payment lag (Section 9 and 10). For these products we also obtain explicit formulas. The instrument described allow to invest at the very short term rate, and so not taking too much interest rate risk, but with the guaranty of a minimal return over the period, even if the rates reach very low levels. We also propose a version of the formula for the case where the fixing of the rate is done in advance of the compounding period. This is typically the case for Libor linked products in USD and EUR where the spot lag is two days. The formulas of Section 7 and 8 can also be used to value option on the one day Brazilian Inter-financial Deposits Index as negotiated on the BMF in Sao Paulo. Some characteristics of this last instrument together with a valuation formula for the Hull-White volatility structure are presented in Viera and Valls [8].

The most practically relevant sections are Section 9 for the options indexed on overnight rates and Section 11 for Libor related products.

The framework of this article is a one-factor Heath-Jarrow-Morton model with deterministic volatility. The exact description of what we mean by that is done in the next section.

2. Model and hypothesis

We use a model for $P(t, u)$, the price in t of the zero-coupon bond paying 1 in u. We will describe this for all $0 \leq t, u \leq T$, where T is some fixed constant.

When the discount curve $P(t, .)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t, u)$ such that

(1)
$$P(t, u) = \exp\left(-\int_t^u f(t, s)ds\right).$$

The idea of Heath-Jarrow-Morton [1] was to exploit this property by modeling f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u)dW_t$$

for some suitable (stochastic) μ and σ and deducing the behavior of P from there.

Here we use a similar model, but we restrict ourself to *deterministic* coefficients. We don't need all the technical refinement to create such a model (see for example the chapter on *dynamical term structure model* in [4]). So instead of describing the conditions that lead to such a model, we suppose that the *conclusion* of such a model are true. By this we mean we have a model, that we call a *HJM one-factor model*, with the following property.

Let $A = \{(s, u) \in \mathbb{R}^2 : u \in [0, T] \text{ and } s \in [0, u]\}$. We work in a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}^{\text{real}})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\text{real}})_{0 \leq t \leq T}$.

H: There exists $\sigma : [0, T]^2 \to \mathbb{R}^+$ measurable and bounded[1] with $\sigma = 0$ on $[0, T]^2 \setminus A$ such that for some process $(r_s)_{0 \leq t \leq T}$, $N_t = \exp(\int_0^t r_s ds)$ forms with some measure $\mathbb{N}$ a numeraire pair[2] (with Brownian motion W_t),

$$df(t, u) = \sigma(t, u)\int_t^u \sigma(t, s)ds\, dt - \sigma(t, u)dW_t$$

$$dP^N(t, u) = P^N(t, u)\int_t^u \sigma(t, s)ds\, dW_t$$

and $r_t = f(t, t)$.

The notation $P^N(t, s)$ designates the numeraire rebased value of P, i.e. $P^N(t, s) = N_t^{-1}P(t, s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t, u) = \int_t^u \sigma(t, s)ds.$$

Note that ν is increasing in u, measurable and bounded. Moreover for $t > u$, $\nu(t, u) = 0$.

The following equations satisfied by the numeraire and the bonds will be useful in the rest of the note:

$$dP(t, s) = P(t, s)r_t dt + P(t, s)\nu(t, s)dW_t, \quad dN_t = N_t r_t dt \quad \text{and} \quad dN_t^{-1} = -N_t^{-1}r_t dt.$$

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[1]Bounded is too strong for the proof we use, some L^1 and L^2 conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

[2]See [4] for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair $(N, \mathbb{N})$.

3. PRELIMINARY RESULTS

We want to price some option in this model. For this we recall the generic pricing theorem [4, Theorem 7.33-7.34].

Theorem 1. *Let V_T be some $\mathcal{F}_T$-measurable random variable. If V_T is attainable, then the time-t value of the derivative is given by $V_t^N = V_0^N + \int_0^t \phi_s dP_s^N$ where ϕ_t is a strategy and*

$$V_t = N_t \, \mathrm{E}_{\mathbb{N}} \left[V_T N_T^{-1} \big| \mathcal{F}_t \right].$$

We now state two technical lemmas that generalize the lemmas presented in [2].

Lemma 1. *Let $0 \le t \le u \le v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp \left(\int_t^u (\nu(s,v) - \nu(s,u)) \, dW_s - \frac{1}{2} \int_t^u \left(\nu^2(s,v) - \nu^2(s,u) \right) ds \right).$$

Proof. By definition of the forward rate and its equation,

$$
\begin{aligned}
P(u,v) &= \exp \left(- \int_u^v f(u,\tau) d\tau \right) \\
&= \exp \left(- \int_u^v \left[f(t,\tau) + \int_t^u \nu(s,\tau) D_2\nu(s,\tau) ds - \int_t^u D_2\nu(s,\tau) dW_s \right] d\tau \right).
\end{aligned}
$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$
\begin{aligned}
P(u,v) &= \frac{P(t,v)}{P(t,u)} \exp \left(- \int_t^u \int_u^v \nu(s,\tau) D_2\nu(s,\tau) d\tau ds + \int_t^u \int_u^v D_2\nu(s,\tau) d\tau dW_s \right) \\
&= \frac{P(t,v)}{P(t,u)} \exp \left(- \frac{1}{2} \int_t^u \left(\nu^2(s,v) - \nu^2(s,u) \right) ds + \int_t^u \nu(s,v) - \nu(s,u) dW_s \right).
\end{aligned}
$$

$\square$

Lemma 2. *Let $0 \le u \le v$. In the HJM one factor model, we have*

$$N_u N_v^{-1} = \exp \left(- \int_u^v r_s ds \right) = P(u,v) \exp \left(\int_u^v \nu(s,v) dW_s - \frac{1}{2} \int_u^v \nu^2(s,v) ds \right).$$

Proof. By definition of r,

$$
\begin{aligned}
r_\tau &= f(\tau,\tau) = f(t,\tau) + \int_t^\tau df(s,\tau) ds \\
&= f(t,\tau) + \int_t^\tau \nu(s,\tau) D_2\nu(s,\tau) ds + \int_t^\tau D_2\nu(s,\tau) dW_s.
\end{aligned}
$$

Then using Fubini, we have

$$
\begin{aligned}
\int_u^v r(\tau) d\tau &= \int_u^v f(t,\tau) d\tau + \int_u^v \int_s^v \nu(s,\tau) D_2\nu(s,\tau) d\tau ds - \int_u^v \int_s^v D_2\nu(s,\tau) d\tau dW_s \\
&= \int_u^v f(t,\tau) d\tau + \frac{1}{2} \int_u^v \nu^2(s,v) ds + \int_u^v \nu(s,v) dW_s.
\end{aligned}
$$

$\square$

4. OVERNIGHT INDEXED NOTES WITH CONTINUOUS COMPOUNDING

The first instrument we study is a note that pays, with a lag, the compounded short-term interest, which is equivalent to the floating leg of an OIS.

Theorem 2. *Let $0 \leq t_1 \leq t_2 \leq t_3$. In the a HJM one-factor model, the price of an instruments that pays in t_3 a principal of 1 and the short-term rate continuously compounded between t_1 and t_2 $(N_{t_1}^{-1} N_{t_2})$ is given at 0 by*

$$P(0, t_3) \frac{P(0, t_1)}{P(0, t_2)} \beta$$

where

$$\beta = \exp\left(\int_0^{t_2} (\nu(s, t_2) - \nu(s, t_1))(\nu(s, t_2) - \nu(s, t_3)) ds \right).$$

Remark: For the Hull and White volatility model [3] with $\nu(s, t) = (1 - \exp(-a(t - s)))\sigma/a$, the parameters β is given through

$$\ln \beta = \frac{\sigma^2}{a^3} \left(\exp(-at_3) - \exp(-at_2) \right) \left(\cosh(at_2) - \cosh(at_1) \right).$$

Proof. Using the generic pricing Theorem 1 and the notation

$$\alpha^2 = \int_t^{t_2} (\nu(s, t_1) + \nu(s, t_3) - \nu(s, t_2))^2 ds.$$

we have

$$V_0 = \mathrm{E}_{\mathbb{N}} \left(\exp\left(\int_{t_1}^{t_2} r_s ds \right) P(t_2, t_3) N_{t_2}^{-1} \right) = \mathrm{E}_{\mathbb{N}} \left(N_{t_1}^{-1} P(t_2, t_3) \right).$$

Using Lemma 1 with $u = t_2$ and $v = t_3$ and Lemma 2 with $u = 0$ and $v = t_1$, we have

$$V_0 = P(0, t_1) \frac{P(0, t_3)}{P(0, t_2)} \mathrm{E}_{\mathbb{N}} \left(\exp\left(-\frac{1}{2}\alpha^2 + \alpha X \right) \right) \beta$$

where X is a standard normal distribution. Note that we use the fact that $\nu(s, t_1) = 0$ for $s > t_1$ to write the integral between 0 and t_1 as an integral between 0 and t_2.

Due to the fact that the expected value of the exponential of a normal distribution is given by $\exp(\mu + \frac{1}{2}\sigma^2)$ where μ is its average and σ its standard deviation, we have the result. $\square$

Remark: If the payment takes place at the end of the interest period ($t_2 = t_3$), there is no convexity adjustment and $\beta = 1$.

Remark: A payer OIS pays at settlement date (t_3) a fixed interest against receiving the interest continuously compounded between the start date (t_1) and the end date (t_2). The payment at settlement is $(N_{t_1}^{-1} N_{t_2} - 1) - R$ where R is the fixed interest amount. The value at 0 of the instrument is

$$P(0, t_3) \left(\frac{P(0, t_1)}{P(0, t_2)} \beta - 1 - R \right).$$

5. Notes with discrete compounding

We now study the same instrument but with a discrete composition.

Theorem 3. *Let $t_0 = 0 \leq t_1 < t_2 < \cdots < t_{n-1} \leq t_n$. In the a HJM one-factor model, the price of an instruments that pays in t_n a principal of 1 and the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ $(i = 1, \ldots, n-2)$ (i.e. $\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1})$) is given in 0 by*

$$P(0, t_n) \frac{P(0, t_1)}{P(0, t_{n-1})} \beta_d$$

where

$$\beta_d = \exp\left(-\sum_{i=1}^{n-2} \int_{t_{i-1}}^{t_i} (\nu(s, t_{n-1}) - \nu(s, t_i))(\nu(s, t_n) - \nu(s, t_{n-1})) ds \right).$$

Remark: For the Hull and White volatility model [3], one has

$$\ln \beta_d = \frac{\sigma^2}{a^3} \left(\exp(-at_n) - \exp(-at_{n-1})\right) \sum_{i=1}^{n-2} \left(\exp(-at_i) - \exp(-at_{n-1})\right) \left(\exp(2at_i) - \exp(2at_{i-1})\right).$$

Proof. Using the generic pricing theorem 1 we have

$$V_0 = \mathrm{E}_{\mathbb{N}} \left(\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}) N_{t_n}^{-1} \right).$$

By Lemma 1,

$$\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}) = \frac{P(0, t_1)}{P(0, t_{n-1})}$$

$$\exp \left(\frac{1}{2} \sum_{i=1}^{n-2} \int_0^{t_i} \nu^2(s, t_{i+1}) - \nu^2(s, t_i) ds - \sum_{i=1}^{n-2} \int_0^{t_i} \nu(s, t_{i+1}) - \nu(s, t_i) dW_s \right)$$

By splitting the integrals on the different sub-intervals $[t_i, t_{i+1}]$ and rearranging the terms, we have that

$$\sum_{i=1}^{n-2} \int_0^{t_i} \nu(s, t_{i+1}) - \nu(s, t_i) dW_s = \sum_{i=1}^{n-2} \int_{t_{i-1}}^{t_i} \nu(s, t_{n-1}) - \nu(s, t_i) dW_s$$

and a similar result for the other sum.

On the other hand, using the Lemma 2, we have

$$N_{t_n}^{-1} = P(0, t_n) \exp \left(\int_0^{t_n} \nu(s, t_n) dW_s - \frac{1}{2} \int_0^{t_n} \nu^2(s, t_n) ds \right).$$

Let μ be defined on $[0, T]$ by

$$\mu(s) = \begin{cases} \nu(s, t_n) - \nu(s, t_{n-1}) + \nu(s, t_i) & \text{if} \quad s \in [t_{i-1}, t_i) \quad i = 1, \ldots, n-1 \\ \nu(s, t_n) & \text{if} \quad s \in [t_{n-1}, t_n] \end{cases}$$

Using those results and notations, we obtain

$$V_0 = P(0, t_n) \frac{P(0, t_1)}{P(0, t_{n-1})} \mathrm{E}_{\mathbb{N}} \left(\exp \left(\int_0^{t_n} \mu(s) dW_s - \frac{1}{2} \int_0^{t_n} \mu^2(s) ds \right) \right) \beta_d.$$

$\square$

Remark: Like for the continuous composition, if the payment takes place at the end of the interest period $(t_{n-1} = t_n)$, there is no convexity adjustment and $\beta_d = 1$.

Remark: Suppose that ν is continuous in (t, t) uniformly for $t \in [0, t_n]$. Let β be the number defined in Theorem 2 on continuous compounding. Then

$$\lim_{\max_{i=0,\ldots,n-2}(t_{i+1}-t_i) \to 0} |\beta_d - \beta| = 0$$

This means that if all the intervals $[t_i, t_{i+1}]$ are small, one can use the continuous version of the instrument as an approximation of the discretely compounded version of the valuation. Numerical estimates in the case of daily composition is given in the next section.

6. Comparison between discounted, continuously compounded and discretely compounded HJM valuation of OIS

The difference between the option continuously compounded and forward approaches for an overnight index swap is

$$P(0, t_3) \frac{P(0, t_1)}{P(0, t_2)} (\beta - 1).$$

We compute this difference in the case of the Hull-White volatility structure. Note that as $t_1 \leq t_2 \leq t_3$ and ν is increasing in the second variable, $\beta < 1$ and the difference is negative.

This is done with a flat curve (1.25%, ACT/360) which is more or less the shape of the USD curve at the time of writing (March 2003).

Note that the yield curve play a role only through the discount factor between t_2 and t_3 $(P(0, t_3)/P(0, t_2))$ and the discount factor between 0 and t_1. So its role is minimal.

Usually, $t_i < 1$ and a is small (typically between 0.01 and 0.10). We can then approximate the exp and the cosh with their second order Taylor approximation. This gives

$$\ln \beta \approx \frac{1}{2}\sigma^2 \left((t_2 - t_3) - \frac{a}{2}(t_2^2 - t_3^2)\right)(t_2^2 - t_1^2).$$

As $a(t_2^2 - t_3^2)$ is very small, the factor which has the largest influence on the adjustment is the volatility factor σ.

Using the Taylor approximation of the exp and removing the factor in a we obtain

$$\beta - 1 \approx \frac{1}{2}\sigma^2 (t_2 - t_3)(t_2^2 - t_1^2).$$

This means that the adjustment is in square of σ. Using standard market conventions, t_1 is small and we obtain that $\beta - 1 \approx \frac{1}{2}\sigma^2(t_3 - t_2)t_2^2$.

The difference, using a mean reversion factor $a = 0.01$ and a volatility factor $\sigma = 0.01$ are given in Table 1. We take $t_1 = 0$, t_2 equal 1w, 1m, 3m, 6m and 12m and t_3 be 1, 2 or 7 days after t_2. The difference is computed on a nominal of USD 1bn.

		Compounding period				
		1w	1m	3m	6m	12m
lag	1d	-0.15	-0.99	-9.06	-35.01	-137.08
	2d	-0.20	-1.97	-18.12	-70.01	-274.10
	1w	-0.35	-6.91	-63.40	-244.88	-958.65

TABLE 1. Difference between the option and forward valuations of the OIS.

We now look at the difference between the continuous and the discrete daily composition. For this comparison, we use 31 December 2002 as value date. With that date, the spot date (t_1) is 3 January 2003 and the 6 months maturity is 3 July 2003. So the payment dates are respectively 7, 8 and 10 July. This increases the difference between the end of compounding period and payment date. Even with this the difference in valuation is very small. The results are given in Table 2

		Compounding period				
		1w	1m	3m	6m	12m
lag	1d	0.04	0.07	0.18	1.46	0.75
	2d	0.05	0.14	0.74	1.82	1.49
	1w	0.09	0.48	1.29	2.55	5.22

TABLE 2. Difference between the daily discrete composition valuation and the continuous one of the OIS.

As a conclusion on this section we can say that, for practical purposes, the difference between the forward valuation of OIS and its valuation using option approach, in its continuous compounding version or daily discretely compounding one, can be neglected. For example with the standard settlement of two days, the maximal difference is around 250 for a one year OIS with a notional of 1 bn. This is approximatively the error coming from a discrepancy of 1/400 bp on the one year rate for the fixed leg of the swap. Even if the volatility is doubled and the adjustment quadrupled, it stays extremely small.

7. Floored and capped continuously compounded instruments

In this section, we analyze options on compounded instruments in the absence of spot-lag ($t_1 = 0$, $t_2 = t_3$). We give the pricing formula at any time between the start and the end of the composition. This allows us to describe the hedging strategy in the second theorem of the section.

Theorem 4. *Let $0 \leq t < T$ and $K > 0$. In the a HJM one-factor model with deterministic volatility, the price of an instrument paying in T the* maximum *of a fixed amount K and the sum of a principal of 1 and the short-term rate continuously compounded between 0 and T ($\max(K, N_T)$) (floored instrument) is given in t by*

$$F_t = N_t N(\kappa) + K P(t, T) N(\sigma - \kappa)$$

where

$$\kappa = \frac{1}{\sigma} \left(\ln \left(\frac{N_t}{KP(t,T)} \right) + \frac{1}{2}\sigma^2 \right)$$

and

$$\sigma^2 = \int_t^T \nu^2(s, T) ds.$$

The price of the instrument paying the minimum *of K and the compounding (capped instrument) ($\min(K, N_T)$) is given in t by*

$$C_t = N_t N(-\kappa) + K P(t, T) N(\kappa - \sigma).$$

Remark: For the Hull and White volatility structure,

$$\sigma^2 = \frac{\sigma^2}{a^2} \left(T - t - \frac{3}{2a} + \frac{2}{a} \exp(-a(T-t)) - \frac{1}{2a} \exp(-2a(T-t)) \right).$$

Proof. By Lemma 2, we have

$$N_T = N_t \exp \left(\int_t^T r_s ds \right) = N_t P^{-1}(t, T) \exp \left(-\int_t^T \nu(s, T) dW_s + \frac{1}{2} \int_t^T \nu^2(s, T) ds \right).$$

As the stochastic integral of a non-stochastic function is normal ([5, Section 3.6, p 65] or [7, Theorem 3.1, p. 60]) we can write

$$N_T = N_t P^{-1}(t, T) \exp \left(\frac{1}{2}\sigma^2 - \sigma X \right)$$

with X a standard normal distribution.

The asset N_T is larger than K if and only if $X < \kappa$, so for the floored instrument we have

$$
\begin{aligned}
F_t &= N_t \, \mathrm{E}_{\mathbb{N}} \left[\max(N_T, K) N_T^{-1} \middle| \mathcal{F}_t \right] \\
&= N_t \, \mathrm{P}_{\mathbb{N}} \left[X < \kappa \middle| \mathcal{F}_t \right] + K P(t, T) \, \mathrm{E}_{\mathbb{N}} \left[\exp \left(-\frac{1}{2}\sigma^2 + \sigma X \right) \mathbb{1}(X \geq \kappa) \middle| \mathcal{F}_t \right].
\end{aligned}
$$

In the case of the capped instrument, we have

$$C_t = N_t \, \mathrm{P}_{\mathbb{N}} \left[X > \kappa \middle| \mathcal{F}_t \right] + K P(t, T) \, \mathrm{E}_{\mathbb{N}} \left[\exp \left(-\frac{1}{2}\sigma^2 + \sigma X \right) \mathbb{1}(X \leq \kappa) \middle| \mathcal{F}_t \right].$$

As X is independent of $\mathcal{F}_t$ and κ is $\mathcal{F}_t$-measurable, using a property of the conditional expectation [5, Proposition A.2.5], we have that $V_t^F = N_t \phi_1(\kappa) + K P(t, T) \phi_2(\kappa)$ where

$$\phi_1(y) = P(X < y) \quad \text{and} \quad \phi_2(y) = \mathrm{E}_{\mathbb{N}} \left(\exp \left(-\frac{1}{2}\sigma^2 + \sigma X \right) \mathbb{1}(X \geq y) \right).$$

So we obtain

$$F_t = N_t N(\kappa) + K P(t, T) N(\sigma - \kappa).$$

Similarly

$$C_t = N_t N(-\kappa) + K P(t, T) N(\kappa - \sigma).$$

□

We turn now to the hedging strategy.

Theorem 5. *Under the hypothesis of Theorem 4, a hedging strategy for the floored instrument is to hold a nominal*

$$\Delta_t = KN(\sigma - \kappa)$$

of the bond $P(t,T)$ and an amount of $N_t N(\kappa)$ in the cash account ($N(\kappa)$ units of the numeraire).

Proof. Using the results of the previous Theorem, we know that the value can be written as

$$F_t^N = G(t, P(t,T), N_t)N_t^{-1}.$$

Using the multidimensional Itô formula and writing $X_s^1 = P(s,T)$ and $X_s^2 = N_s$, we have

$$
\begin{aligned}
F_t^N &= F_0^N + \int_0^t G(s, P(s,T), N_s)dN_s^{-1} + \int_0^t N_s^{-1}D_1 G(s, P(s,T), N_s)ds \\
&\quad + \int_0^t N_s^{-1}D_2 G(s, P(s,T), N_s)dP(s,T) + \int_0^t N_s^{-1}D_3 G(s, P(s,T), N_s)dN_s \\
&\quad + \frac{1}{2}\int_0^t N_s^{-1}\sum_{i,j=1}^2 D_{i+1,j+1}^2 G(s, P(s,T), N_s)d\left\langle X^i, X^j\right\rangle_s \\
&= F_0^N + \int_0^t D_2 F(s, P(s,T), N_s)dP^N(s,T) + \int_0^t H_s ds.
\end{aligned}
$$

As F_t^N is a martingale under $\mathbb{N}$ we have $H_s = 0$. So we obtain

$$F_t^N = F_0^N + \int_0^t D_2 G(s, P(s,T), N_s)dP^N(s,T).$$

By Theorem 1, we have that the hedging quantity in the bond of maturity T is

$$\Delta_t = D_2 G(t, P(t,T), N_t).$$

We now compute the value of this derivative. For this remember that

$$G(t, P(s,T), N_t) = N_t N(\kappa) + KP(t,T)N(\sigma - \kappa)$$

where κ is implicitly defined by

$$g(P, N, \kappa) = N_t P^{-1}(t,T)\exp\left(\frac{1}{2}\sigma^2 - \sigma\kappa\right) - K = 0.$$

We use the implicit function theorem [6]. As the derivative of g with respect to κ and P exist and the first one is non zero, the derivative of κ with respect to P exists. So we have

$$
\begin{aligned}
D_2 G(t, P(t,T), N_t) &= N_t N'(\kappa)D_1\kappa - KP(t,T)N'(\sigma - \kappa)D_1\kappa + KN(\sigma - \kappa) \\
&= D_1\kappa\frac{1}{\sqrt{2\pi}}\exp\left(\left(-\frac{1}{2}\kappa^2\right)\right)\left(N_t - KP\exp\left(\sigma\kappa - \frac{1}{2}\sigma^2\right)\right) + KN(\sigma - \kappa) \\
&= KN(\sigma - \kappa).
\end{aligned}
$$

□

8. Floored and capped discretely compounded instruments

This section is devoted to the pricing and hedging of instruments with discrete composition.

Theorem 6. *Let $0 = t_0 \le t < t_1 < t_2 < \cdots < t_n$ and $K > 0$. In the a HJM one factor model, the price of an instrument paying in t_n the maximum of a fixed amount K and the sum of a principal of 1 and the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ ($i = 0, \ldots, n-1$) (i.e. $\prod_{i=0}^{n-1} P^{-1}(t_i, t_{i+1})$) is given in t by*

$$F_t = \frac{P(t, t_1)}{P(0, t_1)}N(\kappa + \sigma) + KP(t, t_n)N(-\kappa)$$

where

$$\sigma_d^2 = \sum_{i=1}^{n-1} \int_{t \vee t_{i-1}}^{t_i} \left(\nu(s, t_n) - \nu(s, t_i) \right)^2 ds$$

and

$$\kappa_d = \frac{1}{\sigma_d} \left(\ln \left(\frac{P(t, t_1)}{K P(0, t_1) P(t, t_n)} \right) - \frac{1}{2} \sigma_d^2 \right).$$

The price of an instrument paying in t_n the minimum *of K and the discrete composition (capped instrument) is given in 0 by*

$$C_0 = \frac{P(t, t_1)}{P(0, t_1)} N(-\kappa - \sigma) + K P(t, t_n) N(\kappa).$$

Remark: For the Hull and White volatility structure,

$$\sigma_d^2 = \frac{\sigma^2}{a^2} \left(\frac{1}{2a} \exp(-2at_n)(\exp(-2at_{n-1}) - \exp(-2at)) + \frac{1}{2a} \sum_{i=1}^{n-1} (1 - \exp(-2a(t_i - t \vee t_{i-1}))) \right.$$

$$\left. -2 \exp(-at_n) \sum_{i=1}^{n-1} \exp(-at_i)(t_i - t \vee t_{i-1}) \right).$$

Proof. The price of the instrument is

$$F_t = N_t \, \mathrm{E}_{\mathbb{N}} \left(\max \left\{ \prod_{i=0}^{n-1} P^{-1}(t_i, t_{i+1}), K \right\} N_{t_n}^{-1} \right).$$

Using Lemma 1, we have

$$\prod_{i=0}^{n-1} P^{-1}(t_i, t_{i+1}) = \frac{P(t, t_1)}{P(0, t_1) P(t, t_n)}$$

$$\exp \left(\frac{1}{2} \sum_{i=1}^{n-1} \int_t^{t_i} \nu^2(s, t_{i+1}) - \nu^2(s, t_i) ds - \sum_{i=1}^{n-1} \int_t^{t_i} \nu(s, t_{i+1}) - \nu(s, t_i) dW_s \right)$$

By splitting the integrals on the different sub-intervals $[t_i, t_{i+1}]$ and rearranging the terms, we have that

$$\sum_{i=1}^{n-1} \int_t^{t_i} \nu(s, t_{i+1}) - \nu(s, t_i) dW_s = \sum_{i=1}^{n-1} \int_{t \vee t_{i-1}}^{t_i} \nu(s, t_n) - \nu(s, t_i) dW_s$$

(where $t \vee s$ is the maximum between t and s) and a similar result for the other sum.

By Lemma 2,

$$N_t N_{t_n}^{-1} = P(t, t_n) \exp \left(\int_t^{t_n} \nu(s, t_n) dW_s - \frac{1}{2} \int_t^{t_n} \nu^2(s, t_n) ds \right).$$

We denote this last exponential by L_{t_n}. Let $W_s^{\#} = W_s - \int_0^s \nu(\tau, t_n) d\tau$. By the Girsanov's theorem ([5, Section 4.2.2, p. 72]), $W_t^{\#}$ is a standard Brownian motion with respect to the probability $\mathbb{P}^{\#}$ of density L_{t_n} with respect to $\mathbb{N}$.

The value of the instrument can now be written as

$$F_t = \mathrm{E}^{\#} \left(P(t, t_n) \max \left(\frac{P(t, t_1)}{P(0, t_1) P(t, t_n)} \exp \left(-\frac{1}{2} \sigma_d^2 - \sigma_d X^{\#} \right), K \right) \right)$$

where $X^{\#}$ is a random variable with a standard normal distribution with respect to $\mathbb{P}^{\#}$. The first term of the maximum operator is the actual maximum when $X^{\#} < \kappa_d$. So we obtain

$$F_t = \frac{P(t, t_1)}{P(0, t_1)} \mathrm{E}^{\#} \left(\exp \left(-\sigma_d X^{\#} - \frac{1}{2} \sigma_d^2 \right) \mathbb{1}(X^{\#} < \kappa_d) \right) + K P(t, t_n) \mathbb{P}^{\#}(X^{\#} \geq \kappa_d)$$

which by standard manipulation on the expectation and on the normal distribution lead to the result.

The proof for the capped instrument is similar. □

We turn now to the hedging strategy.

Theorem 7. *Under the hypothesis of Theorem 6, a hedging strategy for the floored instrument is to hold a nominal of Δ_t^1 of the bond $P(t, t_1)$ and a nominal Δ_t^2 of the bond $P(t, t_n)$ with*

$$\Delta_t^1 = \frac{1}{P(0, t_1)} N(\kappa + \sigma)$$

and

$$\Delta_t^2 = KN(-\kappa)$$

(and no cash).

Proof. Using the results of the previous Theorem, we know that the option value can be written as

$$F_t^N = G(P(t, t_1), P(t, t_n))N_t^{-1}.$$

Using the multidimensional Itô formula and writing $X_s^1 = P(s, t_1)$ and $X_s^2 = P(s, t_n)$, we have

$$
\begin{aligned}
F_t^N &= F_0^N + \int_0^t G(X_s^1, X_s^2)dN_s^{-1} + \int_0^t N_s^{-1} D_1 G(X_s^1, X_s^2)dP(s, t_1) \\
&\quad + \int_0^t N_s^{-1} D_2 G(X_s^1, X_s^2)dP(s, t_n) + \frac{1}{2}\int_0^t N_s^{-1} \sum_{i,j=1}^{2} D_{i,j}^2 G(X_s^1, X_s^2)d\left\langle X^i, X^j \right\rangle_s \\
&= F_0^N + \int_0^t D_1 G(X_s^1, X_s^2)dP^N(s, t_1) + \int_0^t D_2 G(X_s^1, X_s^2)dP^N(s, t_n) + \int_0^t H_s ds.
\end{aligned}
$$

As F_t^N is a martingale under $\mathbb{N}$ we have $H_s = 0$. By Theorem 1, we have that the hedging quantity in the bond of maturity t_1 and t_n are

$$\Delta_t^1 = D_1 G(X_t^1, X_t^2) \quad \text{and} \quad \Delta_t^2 = D_2 G(X_t^1, X_t^2).$$

We now compute the value of this derivative. For this remember that

$$G(P(t, t_1), P(t, t_n)) = \frac{P(t, t_1)}{P(0, t_1)} N(\kappa + \sigma) + KP(t, t_n)N(-\kappa)$$

where κ is implicitly defined by

$$g(X^1, X^2, \kappa) = P^{-1}(0, t_1)X^1 \exp\left(-\frac{1}{2}\sigma^2 - \sigma\kappa\right) - KX^2 = 0.$$

We use the implicit function theorem [6]. As the derivative of g with respect to κ, X^1 and X^2 exist and the first one is non zero, κ can be written locally as a function of X^1 and X^2 and the derivatives of the implicit function exist. So we have

$$
\begin{aligned}
D_2 G(X_t^1, X_t^2) &= KN(-\kappa) + \frac{X^1}{P(0, t_1)} N'(\kappa + \sigma)D_2\kappa - KX^2 N'(\kappa)D_2\kappa \\
&= KN(-\kappa) + D_2\kappa \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\kappa^2)\left(\frac{X^1}{P(0, t_1)}\exp\left(-\frac{1}{2}(2\kappa\sigma + \sigma^2)\right) - KX^2\right) \\
&= KN(-\kappa),
\end{aligned}
$$

and similarly

$$D_1(X_t^1, X_t^2) = \frac{1}{P(0, t_1)} N(\kappa + \sigma).$$

□

9. Floored and capped continuously compounded instruments with payment lag

We come back to the pricing of options on compounded instrument, but this time in the presence of payment lag.

Theorem 8. *Let* $0 \leq t \leq t_1 \leq t_2 \leq t_3$, $K > 0$,

$$\sigma_1^2 = \int_t^{t_2} \left(\nu(s, t_2) - \nu(s, t_1)\right)^2 ds, \quad \sigma_{12} = \int_t^{t_2} \nu(s, t_3) \left(\nu(s, t_2) - \nu(s, t_1)\right) ds, \quad \sigma_2^2 = \int_t^{t_2} \nu^2(s, t_3) ds$$

and the matrix Σ defined by

$$\Sigma = \left(\begin{array}{cc} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{array} \right).$$

In the a HJM one-factor model with deterministic volatility, if the matrix Σ is invertible, the price of an instrument paying in t_3 the maximum *of a fixed amount K and the principal Q gross-up by the short-term rate continuously compounded between t_1 and t_2 $(QN_{t_1}^{-1} N_{t_2})$ (floored instrument with spot lag) is given in t by*

$$F_t = QP(t, t_3) \frac{P(t, t_1)}{P(t, t_2)} \beta N \left(\kappa + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1} \right) + KP(t, t_3) N \left(\frac{\sigma_{12}}{\sigma_1} - \kappa \right)$$

where

$$\kappa = \frac{1}{\sigma_1} \left(\ln \left(Q \frac{P(t, t_1)}{KP(t, t_2)} \right) - \frac{1}{2}\sigma_1^2 + \alpha \right),$$

$$\alpha = \int_t^{t_2} \nu(s, t_2)(\nu(s, t_2) - \nu(s, t_1)) ds$$

and

$$\beta = \exp \left(\int_t^{t_2} (\nu(s, t_2) - \nu(s, t_1))(\nu(s, t_2) - \nu(s, t_3)) ds \right).$$

The price of an instrument paying in t_3 the minimum *of K and the composition between t_1 and t_2 (capped instrument with spot lag) is given in 0 by*

$$C_t = QP(t, t_3) \frac{P(t, t_1)}{P(t, t_2)} \beta N \left(-\kappa - \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1} \right) + KP(t, t_3) N \left(\kappa - \frac{\sigma_{12}}{\sigma_1} \right)$$

Proof. Using the generic pricing theorem 1 we have

$$F_t = N_t \, E_{\mathbb{N}} \left[\max \left\{ QN_{t_1}^{-1} N_{t_2}, K \right\} P(t_2, t_3) N_{t_2}^{-1} \big| \mathcal{F}_t \right].$$

This expected value can be computed explicitly using standard decomposition and computation of normal distribution. Here those computation are a little bit more involved and required some extra notations. Let

$$X_1 = \int_t^{t_2} \nu(s, t_2) - \nu(s, t_1) dW_s, \quad X_2 = \int_t^{t_2} \nu(s, t_3) dW_s.$$

The random variables X_1 and X_2 are jointly normally distributed ([7, Theorem 3.1, p. 60]) with covariance matrix Σ.

With those notations, using Lemma 2 twice and the fact that $\nu(s, t_1) = 0$ for $s > t_1$, we have

$$N_{t_1}^{-1} N_{t_2} = \frac{P(t, t_1)}{P(t, t_2)} \exp \left(-X_1 - \frac{1}{2}\sigma_1^2 + \alpha \right)$$

and so $QN_{t_1}^{-1} N_{t_2} > K$ when $X_1 < \sigma_1 \kappa$. Moreover, using Lemmas 1 and 2,

$$N_t N_{t_2}^{-1} P(t_2, t_3) = P(t, t_3) \exp \left(X_2 - \frac{1}{2}\sigma_2^2 \right)$$

and

$$N_t N_{t_1}^{-1} P(t_2, t_3) = P(t, t_1) \frac{P(t, t_3)}{P(t, t_2)} \exp \left(X_2 - X_1 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_1^2 + \alpha \right).$$

So the expected value is obtain by

$$
\begin{aligned}
F_t \;=\;& A \int_{x_1 < \sigma_1 \kappa} QP(t,t_1)\frac{P(t,t_3)}{P(t,t_2)} \exp\left(x_2 - x_1 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_1^2 + \alpha \right) \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x \right) dx \\
& + A \int_{x_1 \ge \sigma_1 \kappa} KP(t,t_3) \exp\left(-\frac{1}{2}\sigma_2^2 + x_2 \right) \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x \right) dx
\end{aligned}
$$

where $A = \frac{1}{2\pi}\frac{1}{\sqrt{|\Sigma|}}$.

To obtain an explicit solution, we will need to compute the following integral

$$
I = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(x_2 - \frac{1}{2}x^T\Sigma^{-1}x \right) dx_2.
$$

For that we use the notation

$$
T = \Sigma^{-1} = \begin{pmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{pmatrix}.
$$

and we obtain

$$
I = \frac{1}{\tau_2} \exp\left(-\frac{1}{2}\left(\frac{|T|}{\tau_2^2}x_1^2 + 2x_1\frac{\tau_{12}}{\tau_2^2} - \frac{1}{\tau_2^2} \right) \right).
$$

So we are now left with one dimensional integrals in the computation of the price

$$
\begin{aligned}
F_t \;=\;& QP(t,t_1)\frac{P(t,t_3)}{P(t,t_2)}\exp(\alpha)\frac{1}{\sqrt{2\pi}} \int_{x_1 < \sigma_1 \kappa} \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_1}x_1 + \sigma_1\left(\frac{\tau_{12}}{\tau_2^2} + 1 \right) \right)^2 - \sigma_{12} \right)\frac{1}{\sigma_1}dx_1 \\
& + KP(t,t_3)\frac{1}{\sqrt{2\pi}} \int_{x_1 \ge \sigma_1 \kappa} \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_1}x_1 - \frac{\sigma_{12}}{\sigma_1} \right)^2 \right)\frac{1}{\sigma_1}dx_1.
\end{aligned}
$$

A straightforward but tedious computation gives then the result. In particular we use the fact that $\exp(\alpha - \sigma_{12}) = \beta$.

The proof for the capped instrument is similar. $\square$

Remark: The coefficient β is the same as the one defined in Theorem 2.

If $t_1 = 0$ and $t_2 = t_3$ the matrix Σ is not invertible and the proof does not hold, but the formula we obtain is the one of Theorem 4 and is still valid. For ν continuous, the value of the option is continuous in t_1, t_2 and t_3 including in $t_1 = 0$ and $t_2 = t_3$.

Remark: If the composition period has already started ($0 \le \bar{t}_1 < t$), one can use the formula but with $t_1 = t$ and a modified notional. The notional is modified by the interests already compounded: $Q = \bar{Q}N_{\bar{t}_1}^{-1}N_t$. The formula is similar with only Q replaced by $\bar{Q}N_{\bar{t}_1}^{-1}N_t$. With this adjustment one can price "aged" instruments for which the composition period has already started.

We turn now to the hedging strategy.

Theorem 9. *Under the hypothesis of Theorem 8, a hedging strategy for the floored instrument is to hold a nominal of Δ_t^i of the bonds $P(t,t_i)$ ($i = 1, 2, 3$) with*

$$
\Delta_t^1 = Q\frac{P(t,t_3)}{P(t,t_2)}\beta N\left(\kappa + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1} \right),
$$

$$
\Delta_t^2 = -QP(t,t_3)\frac{P(t,t_1)}{P^2(t,t_2)}\beta N\left(\kappa + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1} \right)
$$

and

$$
\Delta_t^3 = KN\left(\frac{\sigma_{12}}{\sigma_1} - \kappa \right).
$$

(and some cash...).

Proof. Like in the proof of Theorem 7, we have that $\Delta_t^i = D_iG(X_s^1, X_s^2)$.

We now compute the value of the derivatives. For this remember that

$$
G(P(t,t_1), P(t,t_n)) = QP(t,t_3)\frac{P(t,t_1)}{P(t,t_2)}\beta N\left(\kappa + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1} \right) + KP(t,t_3)N\left(\frac{\sigma_{12}}{\sigma_1} - \kappa \right)
$$

where κ is implicitly defined by

$$g(X^1, X^2, \kappa) = QP(t, t_1)\beta \exp(-\sigma_1 \kappa - \frac{1}{2}\sigma_1^2 + \alpha) - KX^2 = 0.$$

Using again techniques similar to the one of Theorem 7 we obtain the result. $\qquad \square$

10. FLOORED AND CAPPED DISCRETELY COMPOUNDED INSTRUMENTS WITH PAYMENT LAG

This section is of less practical importance. For overnight indexed instruments, the continuous compounding is close enough to the discrete compounding for practical purposes. Instruments with a longer reset tenor are usually linked to a Libor-like rate that fixes in advance and pays without lag. Those last instruments are the subject of the next section.

Theorem 10. *Let* $0 \le t_1 \le t_2 \le \cdots \dots \le t_{n-1} \le t_n$, $K > 0$,

$$\sigma_1^2 = \sum_{i=0}^{n-3} \int_{t_i}^{t_{i+1}} \left(\nu(s, t_{n-1}) - \nu(s, t_{i+1})\right)^2 ds, \quad \sigma_2^2 = \int_0^{t_{n-1}} \nu^2(s, t_n) ds,$$

$$\sigma_{12} = \sum_{i=0}^{n-3} \int_{t_i}^{t_{i+1}} \nu(s, t_n) \left(\nu(s, t_{n-1}) - \nu(s, t_{i+1})\right) ds$$

and the matrix Σ *be defined by*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

In the a HJM one-factor model with deterministic volatility, if the matrix Σ *is invertible, the price of an instrument paying in* t_n *the* maximum *of a fixed amount* K *and the sum of a principal of 1 and the discrete compounding of interest rates over the periods* $[t_i, t_{i+1}]$ *for* $i = 1, \dots, n-2$ *(i.e* $\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1})$*) is given in 0 by*

$$F_0 = P(0, t_n) \frac{P(0, t_1)}{P(0, t_{n-1})} \beta N \left(\kappa + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1}\right) + KP(0, t_n) N \left(\frac{\sigma_{12}}{\sigma_1} - \kappa\right)$$

where

$$\kappa = \frac{1}{\sigma_1} \left(\ln\left(\frac{P(0, t_1)}{KP(0, t_{n-1})}\right) - \frac{1}{2}\sigma_1^2 + \alpha\right),$$

$$\alpha = \sum_{i=0}^{n-3} \int_{t_i}^{t_{i+1}} \nu(s, t_{n-1})(\nu(s, t_{n-1}) - \nu(s, t_{i+1})) ds$$

and

$$\beta = \exp\left(\sum_{i=0}^{n-3} \int_{t_i}^{t_{i+1}} (\nu(s, t_{n-1}) - \nu(s, t_{i+1}))(\nu(s, t_{n-1}) - \nu(s, t_n)) ds\right).$$

The price of an instrument paying in t_n *the* minimum *of a fixed amount* K *and the sum of a principal of 1 and the discrete compounding of interest rates over the periods* $[t_i, t_{i+1}]$ *(capped discrete instrument with spot lag) is given in 0 by*

$$C_0 = P(0, t_n) \frac{P(0, t_1)}{P(0, t_{n-1})} \beta N \left(-\kappa - \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1}\right) + KP(0, t_n) N \left(\kappa - \frac{\sigma_{12}}{\sigma_1}\right)$$

Proof. Using the generic pricing theorem 1 we have

$$V_0 = E_\mathbb{N} \left(\max\left\{\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}), K\right\} P(t_{n-1}, t_n) N_{t_{n-1}}^{-1}\right).$$

This expected value can be computed explicitly using standard decomposition and computation of normal distribution. Here those computation are a little bit more involved and required some extra notations. Let

$$X_1 = \sum_{i=0}^{n-3} \int_{t_i}^{t_{i+1}} \nu(s, t_{n-1}) - \nu(s, t_{i+1}) dW_s, \quad X_2 = \int_0^{t_{n-1}} \nu(s, t_n) dW_s.$$

The random variables X_1 and X_2 are jointly normally distributed ([7, Theorem 3.1, p. 60]) with covariance matrix Σ.

With those notations and using Lemma 1, we have

$$\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}) = \frac{P(0,t_1)}{P(0,t_{n-1})} \exp\left(-X_1 - \frac{1}{2}\sigma_1^2 + \alpha\right)$$

and so $\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}) > K$ when $X_1 < \sigma_1 \kappa$. Moreover, using Lemmas 1 and 2,

$$N_{t_{n-1}}^{-1} P(t_{n-1}, t_n) = P(0,t_n) \exp\left(X_2 - \frac{1}{2}\sigma_2^2\right)$$

and

$$\prod_{i=1}^{n-2} P^{-1}(t_i, t_{i+1}) N_{t_{n-1}}^{-1} P(t_{n-1}, t_n) = P(0,t_n) \frac{P(0,t_1)}{P(0,t_{n-1})} \exp\left(X_2 - X_1 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_1^2 + \alpha\right).$$

So the expected value is obtain exactly like in the previous theorem, except the the constant have slightly different meaning. $\square$

11. FLOORED AND CAPPED DISCRETELY COMPOUNDED INSTRUMENTS WITH FIXING LAG

Theorem 11. *Let $0 \le t_1 < t_2 < \cdots < t_n$, $0 = s_0 \le t < s_1 < s_2 < \cdots < s_{n-1}$ with $s_i \le t_i$ and $K > 0$. In the a HJM one factor model, the price of an instrument paying in t_n the maximum of a fixed amount K and of a principal of Q gross-up by the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ fixed in s_i (i.e. $\prod_{i=1}^{n-1} P(s_i, t_i)/P(s_i, t_{i+1})$) is given in t by*

$$F_t = Q P(t,t_1) N(\kappa + \sigma) + K P(t,t_n) N(-\kappa)$$

where

$$\sigma^2 = \sum_{i=1}^{n-1} \int_{t \vee s_{i-1}}^{s_i} (\nu(s,t_n) - \nu(s,t_i))^2 \, ds$$

and

$$\kappa = \frac{1}{\sigma}\left(\ln\left(\frac{P(t,t_1)}{K P(t,t_n)}\right) - \frac{1}{2}\sigma^2\right).$$

The price of an instrument paying in t_n the minimum of a fixed amount K and of a principal of Q gross-up by the discrete compounding of interest rates over the periods $[t_i, t_{i+1}]$ fixed in s_i is given in t by

$$C_t = Q P(t,t_1) N(-\kappa - \sigma) + K P(t,t_n) N(\kappa)$$

Proof. The price of the instrument is

$$F_t = N_t \, \mathrm{E}_{\mathbb{N}}\left(\max\left\{Q \prod_{i=1}^{n-1} \frac{P(s_i, t_i)}{P(s_i, t_{i+1})}, K\right\} N_{t_n}^{-1}\right).$$

Using Lemma 1, we have

$$\prod_{i=1}^{n-1} \frac{P(s_i, t_i)}{P(s_i, t_{i+1})} = \frac{P(t,t_1)}{P(t,t_n)} \exp\left(\frac{1}{2}\sum_{i=1}^{n-1}\int_t^{s_i} \nu^2(s,t_{i+1}) - \nu^2(s,t_i) ds - \sum_{i=1}^{n-1}\int_t^{s_i} \nu(s,t_{i+1}) - \nu(s,t_i) dW_s\right)$$

By splitting the integrals on the different sub-intervals $[s_i, s_{i+1}]$ and rearranging the terms, we have that

$$\sum_{i=1}^{n-1}\int_t^{s_i} \nu(s,t_{i+1}) - \nu(s,t_i) dW_s = \sum_{i=1}^{n-1}\int_{t \vee s_{i-1}}^{s_i} \nu(s,t_n) - \nu(s,t_i) dW_s$$

and a similar result for the other sum.

By Lemma 2,

$$N_t N_{t_n}^{-1} = P(t,t_n) \exp\left(\int_t^{t_n} \nu(s,t_n) dW_s - \frac{1}{2}\int_t^{t_n} \nu^2(s,t_n) ds\right).$$

We denote this last exponential by L_{t_n}. Let $W_s^{\#} = W_s - \int_0^s \nu(\tau, t_n) d\tau$. By the Girsanov's theorem ([5, Section 4.2.2, p. 72]), $W_t^{\#}$ is a standard Brownian motion with respect to the probability $\mathbb{P}^{\#}$ of density L_{t_n} with respect to $\mathbb{N}$.

The value of the instrument can now be written as

$$F_t = \mathrm{E}^{\#} \left(P(t, t_n) \max \left(Q \frac{P(t, t_1)}{P(t, t_n)} \exp \left(-\frac{1}{2}\sigma^2 - \sigma X^{\#} \right), K \right) \right)$$

where $X^{\#}$ is a random variable with a standard normal distribution with respect to $\mathbb{P}^{\#}$. The first term of the maximum operator is the actual maximum when $X^{\#} < \kappa$. So we obtain

$$F_t = QP(t, t_1) \, \mathrm{E}^{\#} \left(\exp \left(-\sigma X^{\#} - \frac{1}{2}\sigma^2 \right) \mathbb{1}(X^{\#} < \kappa) \right) + KP(t, t_n)\mathbb{P}^{\#} \left(X^{\#} \geq \kappa \right)$$

which by standard manipulation on the expectation and on the normal distribution lead to the result.

The proof for the capped instrument is similar. $\qquad\square$

Remark: The case where t is after the fist fixing ($t \geq s_1$) can be treated in a way similar to Section 9. The formula can be applied with only the remaining fixing but the notional modified by the interest already fixed. If $s_k \leq t < s_{k+1}$, the notional is replaced by $Q \prod_{i=1}^k P(s_i, t_i)/P(s_i, t_{i+1})$.

Theorem 12. *Under the hypothesis of Theorem 11, a hedging strategy for the floored instrument is to hold a nominal of Δ_t^1 of the bond $P(t, t_1)$ and a nominal Δ_t^2 of the bond $P(t, t_n)$ with*

$$\Delta_t^1 = QN(\kappa + \sigma) \quad and \quad \Delta_t^2 = KN(-\kappa)$$

(and no cash).

Proof. Like in the proof of Theorem 7, we have that $\Delta_t^i = D_i G(X_s^1, X_s^2)$. And a similar argument lead to $\Delta_t^1 = QN(\kappa + \sigma)$ and $\Delta_t^2 = KN(-\kappa)$. $\qquad\square$

12. CONCLUSION

For overnight indexed swaps, the convexity adjustment due to the delay between the end of the composition period and the payment is, for all practical purposes, negligible. Also the difference between the continuous composition and the discrete composition can, for daily composition, be neglected.

We propose analytical formulas for floor and cap on the compounded average rate. Those formulas are proposed for continuous and discrete compounding and with and without spot-lag. Like for the overnight indexed swaps, the price with discrete composition tends to the price with continuous composition when the composition interval lengths tend to 0. Also the value with spot-lag tends to the value without when the lag tends to 0.

Nevertheless the exact formula allows to see the exact dependency of the price on the rate of different maturity and the volatility structure. For discrete composition, the price includes the volatility to the start date of the last composition. So when the last interval is long, the difference between continuous and discrete composition can be substantial.

We finally propose results for floored and capped instruments on composition with fixing lag. This apply particularly for Libor related products.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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Contents

DERIVATIVES GROUP, BANKING DEPARTMENT, BANK FOR INTERNATIONAL SETTLEMENTS, CH-4002 BASEL (SWITZERLAND)

E-mail address: Marc.Henrard@bis.org