The Froot-Stein Model Revisited*

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Abstract

We investigate the model of Froot and Stein (1998), a model that has very strong implications for risk management. We argue that their conclusions are too strong and need to be qualified. Also, there are some unusual consequences of their model, which may be linked to the chosen pricing formula.

1 Introduction

A number of models have been developed to characterize the optimal risk selection strategy for financial institutions. One particularly celebrated model is due to Froot, Scharfstein and Stein (1993) and Froot and Stein (1998), henceforth FSS and FS. FS won the best paper award in the Journal of Financial Economics in 1998.

Together these articles present a model using shareholder value as a measure to rate different strategies in a near-perfect market. FSS introduces a model of the dynamics of the hedging position along with future financing and investment opportunities. One basic element of this model is that the company has some technological advantage allowing a higher return on capital than the market

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would offer, although this advantage levels off as a function of capital. FS includes the dynamics of the market into this set-up allowing a valuation of a company in the market. One major conclusion of these articles is that the production technology of a company should not be exposed to financial risk at all if the company operates with the shareholders' interest at heart [see FS, Proposition 1 page 63 and Proposition 2 page 64]. Of course, these papers address more general issues like the overall capital budgeting and allocation decisions of companies, but much of the precise conclusions rely on these earlier results.

The FS result implies that, for example, in the insurance industry underwriting is what creates value, not exposure to financial risks. While this point of view is gaining acceptance in the insurance industry, it certainly was not reflected in the practical strategies observed throughout the 1990's, see Hancock, Huber and Koch (2001) The consequence of FS, i.e., shunning all financial exposure, has generated a lot of interest in the insurance industry due to its huge implications for the practical management of that industry.

The FS approach makes use of the Capital Asset Pricing Model - henceforth the CAPM - for pricing risk, and of the empirically well-documented facts that payoff is marginally decreasing in the capital invested, and that capital does not flow frictionlessly in market. Most financial analysts agree upon the assumptions in the FS approach separately. What Froot and Stein have done is to combine the three assumptions in one model and draw some strong conclusions.

From an academic point of view, the FS conclusion seems too strong. Should companies really never invest in stocks no matter what the relationship is between the risk premium and the volatility of stocks? Larger volatility ceteris paribus makes it less favorable for the firm to invest in stocks relative to what an individual can obtain by himself, because the firm loses out more eventually. On the other hand, a larger risk premium should ceteris paribus make it more favorable for the firm to invest in stocks relative to what an individual can obtain by himself, because of the firm's technological advantage. There is a trade-off between risk and return for the firm as for an individual except there is an extra element of loss aversion to the firm. It is not clear though that this should lead to the total hedging outcome found by FS.

In this note we show that the main results of FS [Proposition 1 and 2] are incorrect as stated. First, an additional condition on market parameters is needed to ensure that the perfect hedging strategy is a local maximum rather than a local minimum in the bank's optimization problem. This condition is that the Sharpe ratio of the market return be less than one, i.e., the risk premium be less than the standard deviation. When that condition is violated, it is optimal to hold some stocks.

Empirically, we perhaps expect to see Sharpe ratios less than one, at least for long horizons and for usual market indices. Using 100 years of data, Dimson, Marsh, and Staunton (2002) obtain numbers in the 0.1-0.4 range. On the other hand, Lo (2002) points to some issues in the construction of the Sharpe ratio and finds some funds producing Sharpe ratios over one. So this case is empirically relevant. Second, we show that even when the necessary condition for perfect hedging to be a local maximum is satisfied, the globally optimal strategy can be quite different, and can be nonsensical. This is best understood as a nasty side effect of the implicit linearization used by the authors in solving the model. Although one can *a posteriori* rule this nasty outcome out, the existence of such a black hole questions the validity of the approach.

2 The Froot and Stein Model

The FS model has three periods. In the first period the firm chooses its capital structure, in the second period it chooses its financial investments/hedging position. In the final period, the firm realizes its financial investments and then invests the proceeds plus some external financing in a subsequent project. This project has concave payoff to the total investment. The external financing carries convex costs. Our objective is to analyze the hedging position, so we don't consider any 'new products' as they did. We focus on just two time periods, 0 and 1, and two decisions: the hedging decision and the external financing decision.[1]

Assumption 1. *Let at time 0 the company's internal capital be K, and let it be exposed to an initial risk, which at time 1 will generate capital w, where w is a non-degenerate random variable with known distribution at time 0.*

One could think of w as the underwriting plus investment result for an insurance company. At time 1 the company receives capital w, now holding the realized internal capital $I = K \cdot (1-\tau) + w$, where τ is a deadweight cost for holding K between time 0 and time 1, which could be, for example, taxes.

In the final period there are investment opportunities

Assumption 2. *Denote by $F(I)$ the NPV at time 1 of present and future cash flows in the company. Let F be given by $F(I) = f(I) - I$, where $f(I)$, a thrice differentiable function, known at time 0, is the PV at time 1 of the present and future cashflows. Let $f' \geq 1$ and $f'' \leq 0$. Also assume that $f(0) = 0$.*

[1] We do not analyze the capital selection issue as do FS.

Funding of investments can be achieved by raising external capital at time 1 to cover a shortfall in the internal capital, i.e., when w comes out low.

ASSUMPTION 3. *Assume that the company at time 1 can raise capital* $e \geq 0$ *by repaying in the future an amount with PV of* $C(e) + e$*, where* C *is a thrice differentiable function with* $C(0) = 0$*,* $C' \geq 0$*, and* $C'' > 0$*.*

This means that it is proportionally more expensive to loan larger amounts than smaller ones, and that it is a negative NPV transaction. This assumption states that external funding is not frictionless for a company. Define

$$W = K \cdot (1 - \tau) + w,$$

then the level of internal capital after external funding I is $I = W + e$. The PV and NPV of the company at time 1 are, respectively: $f(I) + (-C(e) - e)$ and $f(I) - I + (-C(e) - e) + e = F(I) - C(e)$.

The company's problem in the final period is to derive the optimal external funding, i.e., maximize the NPV at time 1 by solving the following optimization problem:

$$\max_e F(I) - C(e) \qquad \text{subject to} \qquad \begin{aligned} W &= K \cdot (1 - \tau) + w, \\ I &= W + e, \ e \geq 0. \end{aligned}$$

There exists a unique solution to this optimization problem, and the solution is described by a value function $P(W)$. It retains the same properties as F, namely it is concave and increasing, as argued in FSS. A proof of this can be found in Høgh (2003, Lemma 6). If P is concave, the marginal return on investments must be decreasing, and the optimal level of investments must be increasing in the level of internal capital W.

In the first period the firm has to decide on its hedging policy based on its valuation of the payoffs at this earlier time. The outcome w is composed of tradable and non-tradable risks

ASSUMPTION 4. *Assume that* w *can be expressed as* $w = w^T + w^N$*, where* w^T *is tradable and* w^N *is non-tradable in the market. Assume that* w^N *is normal distributed containing only non-systematic risk, and that* w^T *is the financial exposure. The trading choice set consists of just the market portfolio with return* r_M *and the risk free asset with return* R *only so that*

$$w^T = w^T(\alpha) = V \cdot [1 + (\alpha \cdot r_M + (1 - \alpha) \cdot R)],$$

where V *is the total value invested at time 0.*

The stochastic variable w^N represents the part of the risk in the company's production technology for which there exists no combination of tradeable assets that fully or partially hedges that risk. Note that $W = W(\alpha) = w^N + w^T(\alpha) + K \cdot (1-\tau)$ is a function of the scalar α. This parameter represents the hedging decision that has to be chosen so as to maximize the present value of the firm at time 0.

The FS analysis is divided in two main arguments. First, FS say: "without any real loss of generality, we can assume that prices are determined by a simple one-factor model", i.e., the CAPM can be invoked to imply that the present value PV of P, where P is the NPV function at time 1, is

$$PV = Q(\alpha), \quad \text{where} \tag{1}$$
$$Q(\alpha) = \frac{E(P(W)) - \gamma \cdot \mathrm{cov}(P(W), r_M)}{1+R},$$

and $W = \hat{I} - \hat{e}$, where $\hat{e}$ is the optimal external funding and $\hat{I}$ is the optimal level of investment, while

$$\gamma = \frac{E(r_M) - R}{\mathrm{var}(r_M)}.$$

Second, they conclude that Q has a single optimum at $\alpha = 0$. They interpret this as implying the following [FS, pp63-64]:

Proposition FS. *The firm will always wish to fully hedge its exposure to any tradeable risks.*

This is perhaps the main mathematical result of their paper, and is extremely strong and far reaching.[2] In Theorem 1 below we show that this proposition is not correct. We show that the optimal value for Q is not always $\alpha = 0$. This value can be a local maximum when certain conditions on the distribution of the risky asset holds, but when those conditions do not hold it can even be a (local) minimum. In such cases there can be a local maximum at some positive α. In addition, Sometimes one even gets the implausible outcome that the global maximum occurs at $\alpha = -\infty$.

3 The Main Result

Define the Sharpe ratio on the market portfolio

$$s_M = \frac{E(r_M) - R}{\mathrm{std}(r_M)}.$$

We can now state the following result, which is proved in the appendix.

Proposition 1. *Suppose that P is a thrice differentiable function. Then the first derivative of Q with respect to α is given by*

$$\frac{\partial}{\partial\alpha} Q(\alpha) = \alpha V^2 \frac{E[P''(W)]}{1+R}[\mathrm{var}(r_M) - [E(r_M) - R]^2] - \alpha^2 V^3 \frac{E[P'''(W)]}{1+R} \mathrm{var}(r_M) E(r_M - R). \tag{2}$$

[2] The remaining analysis of the FS paper, like choice of capital and 'new products' relies on this result quite heavily.

(i) If $E[P''(W)] < 0$, then $\alpha = 0$ is a local maximum for Q if $s_M < 1$ or a local minimum for Q if $s_M > 1$.

(ii) Let $\widehat{\alpha}$ be a critical points different from zero for Q. Then $\widehat{\alpha}$ is found by solving

$$\widehat{\alpha} = \frac{E[P''(W)]|_{\alpha=\widehat{\alpha}}}{E[P'''(W)]|_{\alpha=\widehat{\alpha}}} \cdot \frac{1 - s_M^2}{V \cdot E(r_M - R)}. \tag{3}$$

Proposition 1 says that the critical point $\alpha = 0$, i.e., the point where W contains no systematic risk, is not always even a local maximum. Whether $\alpha = 0$ is a maximum or a minimum does not depend on the technology (i.e., P) at all, but does depend on the prospect of the market, specifically on whether $s_M < 1$ or $s_M > 1$. When $s_M > 1$, the value $\alpha = 0$ is actually a minimum contrary to FS. In that case, there can be a local maximum at some $\widehat{\alpha} > 0$, provided $E[P'''(W)]|_{\alpha=\widehat{\alpha}} > 0$, which can be expected for an increasing and concave function. The precise location of the solution $\widehat{\alpha}$, when $\widehat{\alpha} \neq 0$, depends on the technology through P. A risk averse individual would always seek some positive investment in risky assets, whereas the FS firm sometimes does and sometimes does not invest in risky assets, with the decision hinging on the prospect of the market. Finally, even if $s_M < 1$ and $\alpha = 0$ is a local maximum, $\alpha = 0$ may still not be the global maximum as we show in the following example.

Example 1. Suppose that $P(W)$ is given by

$$P(W) = -\beta_1 \cdot e^{-\beta_2 \cdot W} + \beta_3, \quad \beta_1 > 0, \quad \beta_2 > 0,$$

which is consistent with the model of section 2. Then the relation between $E[P''(W)]$ and $E[P'''(W)]$ is especially simple, since $P''(W) = -\beta_1 \cdot \beta_2^2 \cdot e^{-\beta_2 \cdot W}$ and $P'''(W) = \beta_1 \cdot \beta_2^3 \cdot e^{-\beta_2 \cdot W}$. This implies that $-\beta_2 \cdot E[P''(W)] = E[P'''(W)] \neq 0$. Therefore, for $s_M < 1$, $\alpha = 0$ is a local minimum for $Q(\alpha)$, and for $s_M > 1 > 0$, $\alpha = 0$ is a local maximum. The solution to equation (3), i.e., $\widehat{\alpha} \neq 0$, is determined by V, β_2 and the market parameters only, since equation (3) in this case is equal to

$$\widehat{\alpha} = -\frac{1}{V \cdot \beta_2} \cdot \frac{\mathrm{var}(r_M) - [E(r_M) - R]^2}{\mathrm{var}(r_M) \cdot E(r_M - R)} =: a.$$

There is one critical point different from $\alpha = 0$ for $Q(\alpha)$. For $V > 0$, $E(r_M) > R$, and $P(W) = -e^{-W} + 1$, the function Q has the appearance shown in Figure 1

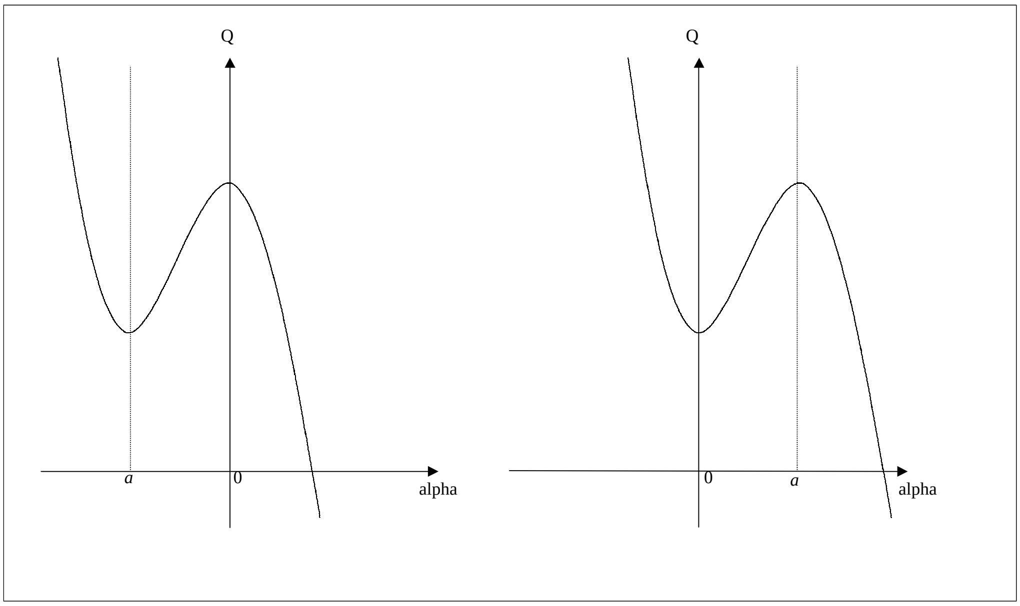

Figure 1. The function $Q(\alpha)$ for Example 1. In the first case, $s_M < 1$, while the second case has $s_M > 1$.

The Q function attains its global maximum at $\alpha = -\infty$. If one also assumes that (1) holds as do FS, i.e., that $Q = PV$, this says that the firm should borrow an infinite amount at the rate of return of the market portfolio and invest in the rate of return of the risk free asset. If the company chooses to borrow a nearly infinite amount at the rate of return of the market portfolio and lend a nearly infinite amount at the rate of return of the risk free asset, then the resulting expected return is nearly $-\infty$, since $E(r_M) \geq R$. This seems like an unrealistic consequence. Why does this occur? We think that a possible explanation of this anomaly lies with the assumption that $PV(P) = Q$, which leads to a logical inconsistency. When we transform a symmetric random variable W by a concave transformation P, the random variable $P(W)$ can't be symmetrically distributed. The CAPM can't be applied to both payoffs W and $P(W)$.

4 Conclusion

In the late 1990s' large returns on financial investments resulted in the insurance industry exposing itself to large financial risk and to cash flow underwriting. The economic approach to the valuation of companies introduced by FS apparently suggests a clear direction the insurance industry needs to go to improve the shareholder outcome, see Hancock, Huber and Koch (2001). It is therefore unfortunate that the analysis in the innovative paper FS is not fully correct. We suggest two possible paths to follow. One could extend the two-moments CAPM to a three moments CAPM, thereby eliminating the systematic error made when using the two moments CAPM together with the $P(W)$, since there no longer are any restrictions distribution-wise on P. Or one could use arbitrage pricing theory like Black-Scholes to find the present value of P instead. Either way, the conclusions are likely to be much less strong, and whether some net investment in stocks is the optimal strategy is likely to depend on model parameters.

5 Appendix

PROOF OF PROPOSITION 1. By the CAPM applied to W, Stein's Lemma [Cochrane (2001, pp164-165)], and interchanging differentiation and integration, the first derivative of $(1+R) \cdot PV\{P(W)\}$

with respect to α can be written as:

$$
\begin{aligned}
(1+R)\frac{\partial}{\partial \alpha}Q(\alpha) &= \frac{\partial}{\partial \alpha}[E[P(W)] - \gamma \cdot \mathrm{cov}(P(W), r_M)] \\
&= \frac{\partial}{\partial \alpha}[E[P(W)] - \gamma \cdot E[P'(W)] \cdot \mathrm{cov}(W, r_M)] \\
&= \frac{\partial}{\partial \alpha}[E[P(W)] - E[P'(W)] \cdot V \cdot \alpha \cdot E(r_M - R)] \\
&= E[P'(W) \cdot V \cdot (r_M - R)] \\
&\quad -E[P''(W) \cdot V \cdot (r_M - R)] \cdot V \cdot \alpha \cdot E(r_M - R) \\
&\quad -E[P'(W)] \cdot V \cdot E(r_M - R) \\
&= V^2 \cdot \alpha \cdot E[P''(W)] \cdot \mathrm{var}(r_M) \\
&\quad -V^2 \cdot E[P''(W) \cdot (r_M - R)] \cdot \alpha \cdot E(r_M - R) \\
&= V^2 \cdot \alpha \cdot E[P''(W)] \cdot \mathrm{var}(r_M) \\
&\quad -V^2 \cdot \alpha \cdot E(r_M - R) \\
&\quad \cdot [E[P'''(W)] \cdot \mathrm{cov}(W, r_M) + E[P''(W)] \cdot E(r_M - R)] \\
&= V^2 \cdot \alpha \cdot E[P''(W)] \cdot (\mathrm{var}(r_M) - [E(r_M) - R]^2) \\
&\quad -V^3 \cdot \alpha^2 \cdot E[P'''(W)] \cdot \mathrm{var}(r_M) \cdot E(r_M - R).
\end{aligned}
$$

This is as stated in (2). Clearly, $\alpha = 0$ is a critical point.

Since P is concave, $P'' \leq 0$ everywhere. Therefore

$$
\begin{aligned}
\frac{\partial^2}{\partial \alpha^2}Q(\alpha)|_{\alpha=0} &= \frac{E[P''(W)]}{1+R} \cdot V^2 \cdot (\mathrm{var}(r_M) - [E(r_M) - R]^2) \\
&\quad \begin{cases} > 0 & \text{if } \Delta < 0 \text{ and } E[P''(W)]|_{\alpha=0} < 0 \\ = 0 & \text{if } \Delta = 0 \text{ and } E[P''(W)]|_{\alpha=0} = 0 \\ < 0 & \text{if } \Delta > 0 \text{ and } E[P''(W)]|_{\alpha=0} < 0, \end{cases}
\end{aligned} \tag{4}
$$

where $\Delta = \mathrm{var}(r_M) - [E(r_M) - R]^2$. Critical points different from zero solve the equation

$$\widehat{\alpha} = \frac{E[P''(W)]|_{\alpha=\widehat{\alpha}}}{V \cdot E[P'''(W)]|_{\alpha=\widehat{\alpha}}} \cdot \frac{\mathrm{var}(r_M) - [E(r_M) - R]^2}{\mathrm{var}(r_M) \cdot E(r_M - R)},$$

which is as stated in (3) on dividing through. ■

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