Arbitrage With Fixed Costs and Interest Rate Models

PRELIMINARY VERSION

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Abstract

In this paper, we start by considering market models with fixed costs; in such a context, we characterize the absence of arbitrage opportunity and we provide pricing rules. We then apply these results to extend some classical interest rate and option pricing models.
In particular, we prove that the quite surprising result obtained by Dybvig-Ingersoll-Ross (1996), which asserts that, under the assumption of absence of arbitrage, long zero-coupon rates can never fall, is no longer true in models with fixed costs. Models where the long rate follows a diffusion process as in Brennan-Schwartz (1979) are no more to be rejected for arbitrage considerations.

Keywords arbitrage - fixed costs - contingent claims pricing - interest rates models - long zero-coupon rates - Dybvig, Ingersoll and Ross - Brennan and Schwartz - barrier models.

1 Introduction

In this paper, we start by considering market models with fixed costs; in such a context, we characterize the absence of arbitrage opportunity and we provide pricing rules. We then apply these results to extend some classical interest rate and option pricing models.

More generally, in this paper we analyze the impact of transaction costs that are incurred if and only if a particular asset is traded. These transaction costs are said to be fixed if they are constant regardless of the size of the transaction. In fact, it is possible to consider more general structures where the transaction cost is bounded above and bounded away from zero independently of the quantity of the asset that is traded. We will see in the next that we can even consider a weaker form of "fixed costs" and we will deal with structures where the transaction costs are bounded below by a given positive minimum cost and such that the marginal costs goes to zero in the limit of large quantities. As underlined by Brennan (1975), such a fixed costs structure captures reasonably well the basic features of the commission structure of many stock exchanges which involve a stated amount plus a declining percentage of the value of the transaction. In fact, investors incur many different fees and charges that depend on the traded volume, traded amount, traded asset,... and the total fee is in general bounded below by a given amount that corresponds to the ticket charges that have to be paid for each transaction reagrdless to the size of that transaction and bounded above like, for instance, on the GLOBEX clearing system where fees are capped[1] at \$50 per day per product, per operator. We refer to Jones (2002) for a general description of the trading costs structure on the NYSE accross the 20th century and to Figure 1 for a presentation of the clearing fees structure on Euronext.

However, the introduction of a fixed costs structure is not limited to the modelization of such clearing or trading fees imposed by a given stock market. Indeed, as underlined by Leape (1987),

> acquiring assets virtually always incurs financial charges such as brokerage fees, which in general have a fixed costs component (...). Even in the cases in which such costs appear proportional to the amount invested, there are typically minimum investment requirements which are equivalent to fixed costs of trading. The costs of acquiring assets also include the opportunity costs and information costs associated with analyzing new assets. Such

[1] for trades executed for members trading within their division for their own account.

> costs are truly costs of participating in particular markets and are independent of the amount invested

Our setting includes then brokerage fees, holding costs, fixed investment taxes to gain access to a market (such as a foreign market), operational and processing costs that typically exhibit strong economies of scale (e.g. through automation), fixed costs involved in setting up an office and obtaining access to information, the opportunity cost of looking at a market or of doing a specific trade and brokerage arrangements where marginal fees go to zero beyond a given volume that is reset periodically.

Other frictions studied in the literature include costs that are proportional to the quantity of asset traded. Without denying the potential importance of the other frictions, the fixed costs specification is of importance for threee major reasons developed by Viard (1995) :

> First as Leape (1987) argued, a number of costs associated with purchasing and holding assets approximately satisfy the fixed-costs specification (...)
>
> Second, such costs are more analytically challenging than proportional costs. The latter can be easily accomodated in the standard CAPM framework with little additional analysis, because such costs simply reduce the net rate of return.
>
> Third, fixed costs offer a simple and plausible explanation for the observation that many investors hold exactly zero of most available assets; although proportional costs would alter an investor's holding of any particular asset, they generally would not drive such holdings to zero.

The second point merits further discussion. The choice of a fixed costs structure confronts us from the analytical point of view with strong nonlinearities. This is why those costs can not be incorporated in the asset price. When the costs are constant regardless of the traded quantity and/or amount, these nonlinearities can be easily tackled and it can be seen that such costs lead to CAPM conditions, utility maximization equations as well as arbitrage characterization, which are similar to those obtained in a frictionless setting. Indeed, as Levy (1978) noted, if an investor chooses to incur the fixed cost and hold a particular asset, he will still obey the well known CAPM first order condition with respect to that asset and this is due to the fact that marginal trades are frictionless. However, in the case of general fixed costs structures as defined above, this is not true anymore. The CAPM as well as the individual utility maximization equations are modified by the introduction of these costs. Nevertheless, the declining transaction costs marginal rate makes these costs asymptotically negligible and leads then to no-arbitrage characterizations and pricing rules that are very similar to those obtained in the frictionless setting.

More precisely, we find that the absence of arbitrage opportunity in models with fixed trading costs is equivalent to the existence of a family of nonnegative state price deflators compatible with the asset prices. The only one difference with the frictionless setting lies in the fact that these state-price deflators are nonnegative instead of positive. Furthermore, we define admissible pricing rules on the set of attainable contingent claims as the price functionals that are arbitrage free and are lower than or equal to the surreplication cost (i.e. the lowest cost of dominating a given payoff). Indeed, no rational agent would pay more than its surreplication cost for a contingent claim since there is a cheaper way to achieve at least the same payoff using a trading strategy. We then show[2] that the only admissible pricing rules on the set of attainable contingent claims are those that are equal to the sum of a bounded fixed cost functional and of a linear pricing rule associated to one of the nonnegative state price deflators given by the absence of arbitrage.

The main impact of the introduction of fixed costs appears then as the enlargement of the set of possible state-price deflators from the positive ones to the nonnegative ones. The absence of arbitrage opportunity the fixed costs framework is then weaker than the analogous one in a frictionless model and some models that contained arbitrage opportunities in the perfect classical framework might become arbitrage free with the introduction of fixed costs. This enables us in Section 3 to rationalize some behaviors that are usually rejected for arbitrage considerations.

In particular, we prove that the quite surprising result obtained by Dybvig-Ingersoll-Ross (1996), which asserts that, under the assumption of absence of arbitrage, long zero-coupon rates can never fall, is no longer true in models with fixed costs.

We then consider discrete time versions of the classical Brennan-Schwartz (1979) model. Based on the pure expectation theory, this model assumes a diffusion behavior for long rates[3], and is therefore incompatible with Dybvig-

[2] This study of market models with fixed transaction costs specifies in a discrete time framework previous results of Jouini *et al.* (2001) obtained under the assumption of no-free-lunch (instead of no-arbitrage) in a continuous time setting.

[3] defined as the limit zero-coupon rate when maturity goes to infinity. It does not correspond to what is usually called long rate on the markets, namely the rate associated

Ingersoll-Ross (1996)'s result. We show that such a model is now compatible with the absence of arbitrage (with fixed costs). This enables to reconcile arbitrage conditions and the empirical observations that don't seem to reject Brennan-Schwartz (1979) model (see Chan et al. 1992).

This "new flexibility" on the long rate is more in accordance with the Expectations Hypothesis condition that suggests that the long rate should be more or less the expectation today of what the long rate will be tomorrow.

We provide examples adapted from Ho-Lee (1986) where the long zero-coupon rates can increase and decrease.

The presence of fixed cots permits also to consider models where the asset prices reach at some dates and in some states of the world their upper bound. Such models with upper (resp. lower) bounds on the asset price processes in frictionless markets obviously contain arbitrage, hence they do not admit any compatible pricing rule. We study pricing issues in such market models with fixed transaction costs and we show, in particular, that the price of a classical call option in a model where the asset prices can not go beyond an a priori given barrier is equal to the price of a barrier option in an unrestricted model.

There is an existing body of literature that studies transaction costs and other market frictions. As far as fixed transaction costs are concerned Duffie-Sun (1990), Grossman-Laroque (1990) and Morton-Pliska (1995), among others, study the optimal portfolio problem with transaction fees that are proportional to the size of the overall portfolio (as opposed to the size of the specific transaction). In the same spirit, Luttmer (1999) adresses the empirical problem of the required level of transaction costs for observations on consumption choices to be consistent with data on asset returns. Brennan (1975), Goldsmith (1976), Levy (1978), Mayshar (1979, 1981), Leape (1987) and Viard (1995) consider a CAPM model with fixed costs in order to propose an explanation to the empirical evidence of limited diversification: households hold a very limited set of assets. Jouini *et al.* (2001) study the characterization of the assumption of no-free-lunch as well as viability issues in market models with fixed costs.

The paper is organized as follows. Section 2 presents the arbitrage characterization in models with fixed costs, and provides a general description of pricing rules. Section 3 provides applications. All proofs are in the Appendix.

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to the longest marketed maturities(10 years and over).

2 Models with fixed costs

2.1 Arbitrage in models with fixed costs

2.1.1 Formulation of the market model

The model to be studied here is the so-called event-tree model and the reader, familiar with such a framework, may skip the first paragraph.
A finite set Ω of states of the world is specified and fixed. Also specified is a time horizon T which is the terminal date for all economic activity under consideration. The information arrival is given by a family of increasingly finer partitions $\{F_0, ..., F_T\}$ of Ω such that $F_0 = \{\Omega\}$ and $F_T = \{\{\omega\} : \omega \in \Omega\}$. A date $t-$node is an element of F_t. We denote by Σ_t the set of all date $t-$nodes and for $\sigma_t \in \Sigma_t$, $f(\sigma_t) = \{\sigma_{t+1} \in \Sigma_{t+1} : \sigma_{t+1} \subset \sigma_t\}$ can be interpreted as the set of the immediate successors of the date $t-$node σ_t.

There are $n + 1$ assets and for a date $t-$node σ_t, we denote by $Z^k(\sigma_t)$ the price at time t and in the node σ_t of asset k. We assume that $Z^0 \equiv 1$: this amounts to assuming the existence of an asset called a numéraire, i.e. with positive prices at every time, and considering discounted processes.

In this framework, a portfolio is described by a vector $\theta \in \mathbb{R}^{n+1}$ where θ_k is the quantity of asset k held by the investor. A portfolio strategy is then an adapted process $(\theta_t)_{t=0,\cdots,T}$ where $\theta(\sigma_t)$ is the portfolio held at date t in the event σ_t.

Each time investors revise their portfolio of risky assets, they pay a given transaction cost. The paid amount might depend on the traded assets, the quantity of assets that are traded, the traded amount as well as on the portfolio composition and on previous trades. For a given portfolio strategy $(\theta_t)_{t=0,\cdots,T}$, the transaction cost paid by the investors at date t and in the node σ_t is given by $c^\theta(\sigma_t)$. We also denote by $C^\theta(\sigma_t)$, the total transaction costs paid by the investor up to time t and we only impose the following conditions :

- For a given strategy θ, the process c^θ is nonnegative and adapted to the information structure $\{F_0, ..., F_T\}$,
- if there is a trade at date t and in the state of the world σ_t (i.e. $\theta_t(\sigma_t) \neq \theta_{t-1}(\sigma_t)$) then $C_t^\theta(\sigma_t) > \mathbf{c}$ where $\mathbf{c}$ is a given positive minimum transaction cost,
- for any strategy θ, any date t and any date-t node σ_t, we have $\lim_{\lambda\to\infty} \frac{c^{\lambda\theta}(\sigma_t)}{\lambda} \to 0$.

The first condition means that the fixed costs depend only on present and past information, the second condition means that investors have to pay a given minimum amount at least once before to trade. The third condition reflects the declining marginal rate of the transaction costs structure.

Many transaction costs structures are taken into account with this modelization

- clearing fees and brokerage arrangements where the amout to be paid is a function of the traded amount and where marginal fees go to zero beyond a given amount. We have then $c_t^\theta = \sum_{k=1}^n f^k(\left(\theta_t^k - \theta_{t-1}^k\right) Z_t^k)$ where f^k is the fees structure associated with asset k and satisfies $f^k(x) > \mathbf{c}$ for $x \neq 0$ and $\lim_{\lambda\to\infty} \frac{f^k(\lambda x)}{\lambda} \to 0$,

- trading fees and brokerage arrangements where the amount to be paid is a function of the quantity of assets that are traded and where marginal fees go to zero beyond a given volume. We have then $c_t^\theta = \sum_{k=1}^n f^k(\theta_t^k - \theta_{t-1}^k)$ where f^k is the fees structure associated with asset k and satisfies the same conditions as abov,.

- holding costs where the amount to be paid is a function of the quantity of assets in the portfolio. We have then $c_t^\theta = \sum_{k=1}^n f^k(\theta_t^k)$ where f^k still satisfies the same conditions as above,

- fixed investment taxes to gain access to a market or fixed costs involved in setting up an office and obtaining access to information where the amount is paid once before the first access to that market. The amount to be paid is given by $c_t^\theta = C^k$ if the investor decides at date t to pay C^kin order to have a later access to the $k-$th market. We have then $C^\theta(\sigma_t) \geq C^k$ at any date where $\theta_t^k(\sigma_t) \neq \theta_{t-1}^k(\sigma_t)$.

- operational and processing costs that exhibit strong economies to scale where the paid amount might be a function of the traded amount or the traded quantity with a possible fixed initial amount. The costs structure is then a combination of the structures presented above.

In all the examples above we have assumed that there are no costs associated to transactions on asset 0. In the case of transaction costs, this assumption is made without any loss of genearlity since any trade on the riskless asset implies a trade on at least another asset. In the case of holding costs, this assumption means that there are no holding fees on the riskless asset. Such an assumption is made in order to make the arguments clearer and be relaxed without difficulties.

2.1.2 Arbitrage opportunities

We can now introduce the notion of arbitrage. As usual, an arbitrage opportunity is a plan that yields through some combination of buying and selling, a positive gain in some circumstances without a countervailing threat of loss in other circumstances. More precisely,

Definition 1 *An arbitrage opportunity with fixed costs consists of a portfolio*[4] θ *such that*

$$\sum_{k=0}^{n} \theta_k Z^k(\sigma_t) + c^\theta(\sigma_t) \leq 0$$

and

$$\sum_{k=0}^{n} \theta_k Z^k(\sigma_{t+1}) - c^\theta(\sigma_{t+1}) \geq 0 \text{ for all } \sigma_{t+1} \in f(\sigma_t)$$

with at least one strict inequality, for some t *and some* σ_t *in* Σ_t.

Before going any further, we can consider a simple example that gives the intuition of how things work and illustrates our results; we consider the following financial market where two securities, thereafter denoted by A and B, can be traded at two dates 0 and 1 and in two possible states of the world s_1 and s_2 at date 1: security A has a value of 1 at date 0 and a value of 1 or 2 at date 1 in state s_1 or s_2 respectively and security B (the bond) is normalized to be always worth one unit of account. In the frictionless market case, this model yields an arbitrage opportunity which consists in simply buying one unit of A and selling one unit of B at date 0 and closing the position at date 1. If we now introduce fixed costs, the preceding arbitrage opportunity vanishes, since the investment required at date 0 by the strategy is not zero anymore but is equal to the fixed cost. Hence, the assumption of absence of arbitrage in the frictionless case is not equivalent to the assumption of absence of arbitrage in the case with fixed costs.

In order to clarify the link between the concepts of arbitrage with fixed costs and of frictionless arbitrage, we may remark that we have the following:

[4] Strictly speaking, the transaction costs process c^θ is only defined for strategies θ and not for portfolios. However, for a give date$-t$ node σ_t a portfolio θ can obviously be identified with the strategy that consists in buying that portfolio at date t in the event σ_t and selling it at date $t+1$.

Proposition 1 *There is an arbitrage opportunity with fixed costs if and only if there exists a portfolio θ such that*

$$\theta \cdot Z(\sigma_t) < 0$$

and

$$\theta \cdot Z(\sigma_{t+1}) \geq 0 \text{ for all } \sigma_{t+1} \in f(\sigma_t)$$

for some t and some σ_t in Σ_t.

The existence of an arbitrage opportunity with fixed costs appears then as equivalent to the existence of a strong form of a frictionless arbitrage, i.e. a portfolio with negative initial cost and nonnegative payoffs.

In order to illustrate this result, consider a two dates binomial model, i.e. a model consisting of one risky security with a value of 1 at date 0 and a value of u in state up and of d in state down at date 1; and one riskless security, with a value of 1 at date 0 and a value of $(1+r)$ in both states of the world at date 1.
It is easy to see that if $d < (1+r) < u$, then the model is arbitrage free, in both cases, with and without fixed transaction costs; if $d = (1+r)$ or $u = (1+r)$, then it is arbitrage free in the model with fixed transaction costs but not in the frictionless model anymore; if $d > (1+r)$ or $u < (1+r)$, then it contains arbitrage opportunities in both cases.

A characterization of the absence of arbitrage with fixed costs follows then:

Proposition 2 *There is no arbitrage with fixed costs if for each date t and each date $t-$node σ_t there exists a nonnegative state price density on $f(\sigma_t)$ compatible with the asset prices i.e. there exists a family of nonnegative numbers $(\pi_\rho)_{\rho \in f(\sigma_t)}$ such that*

$$Z^k(\sigma_t) = \sum_{\sigma_{t+1} \in f(\sigma_t)} \pi_{\sigma_{t+1}} Z^k(\sigma_{t+1}) \qquad k = 0, ..., n.$$

Let us consider again the simple example where two securities denoted by A and B can be traded at two dates 0 and 1 and in two possible states of the world s_1 and s_2 at date 1 (security A has a value of 1 at date 0 and a value of 1 or 2 at date 1 in state s_1 or s_2 respectively and security B is always worth one unit of account). According to the Fundamental Theorem of Asset Pricing, since there is an arbitrage opportunity for the frictionless case, there cannot exist a positive compatible state price density.

Nevertheless, the state price density defined on the set $S = \{s_1, s_2\}$ of the possible states of the world at date 1 by $\pi_{s_1} = 1$ and $\pi_{s_2} = 0$ is compatible with (A, B).

Remark that the previous example does not correspond to an economically feasible situation when the transaction costs are constant regardless of the number of traded assets or of the traded amount. Indeed, in such a framework, it is easy to see that the individual utility maximization problem when an investor is endowed with an initial wealth w can be solved by comparing the utility level obtained with no trade and the utility level obtained in an otherwise identical frictionless model with an initial wealth $w - c$. If $w - c > 0$ and if the utility function is not bounded above, it is easy to see that the no-trade strategy is dominated by some strategy with trade and that any candidate optimum with trade can be dominated by increasing consumption in state s_2. Consequently there is no solution to the utility maximization problem. More generally, when the transaction costs are constant and if we know that any optimizing monotone agent with a large enough initial wealth choses to incur the fixed cost and trade the risky asset, then there must be no arbitrage with respect to the marginal trade which is frictionless. In such a situation we should have no difference between the no arbitrage characterization in the frictionless setting and in the transaction costs setting. This fact has already been underlined by Levy (1978) who noted that if an investor chooses to incur the fixed cost and hold a particular asset then he will still obey the well known first order utility maximization condition with respect to that asset.

However, in the real world situation of bounded below fees with a declining (but nonzero) marginal structure, the previous argument does not permit to eliminate zero state-prices. Indeed, the marginal trade is not frictionless anymore even if it involves less and less frictions when we increase the traded quantities. The frictionless setting is obtained only asymptotically (in the limit of large quantities) and the investor may take arbitrage profit from a zero state-price only if he is infinitely rich.

More precisely, let us consider in the context of the previous example a utility maximizing agent with a constant absolute risk aversion coefficient $\alpha = 1$, an initial endowment w and a probability $\left(\frac{1}{2}, \frac{1}{2}\right)$ on the states of the world. Furthermore, let us assume that a transaction fee $c^\theta = \frac{1+2|\theta|}{2+2|\theta|}$ is paid at date 0 only if the agent trades a quantity $\theta \neq 0$ of asset A at date 0. Clearly, the transaction costs increases with the quantity traded and is bounded above and bounded away from zero for $\theta \neq 0$. It is easy to see that an optimal trade should satisfy the condition $\theta \geq 0$ and that it leads then to a terminal wealth equal to[5] $(w-c^\theta, w+\theta-c^\theta)$. If θ is large enough, we have $w-c^\theta \approx w-1$, $w+\theta-c^\theta \approx w+\theta-1$ and any marginal increase of the traded quantity from θ to $\theta+\delta\theta$ would lead to a move of the utility level from u to $u+\delta u$ with $\delta u = -\frac{1}{4}\exp(-w+c^\theta)\frac{1}{(1+\theta)^2} + \frac{1}{2}\exp(-w-\theta+c^\theta)(1-\frac{1}{2(1+\theta)^2}) \approx -\frac{\exp(-w+1)}{4(1+\theta)^2} + \frac{1}{2}\exp(-w-\theta-1) < 0$ for θ large enough. Furthermore, it is easy to see that for $\theta > -\ln(2\exp\left(-c^\theta\right)-1)$, the utility level associated to that trade is higher than the utility level associated with no trade. In particular, for $\theta > -\ln(2\exp\left(-\frac{1}{2}\right)-1) \approx 1.55$, it is better to trade. Hence, in such a framework our agent choses to trade at a given optimal level θ^* such that $0 < \theta^* < \infty$.

It is easy to check that this example is not a singular one and does not depend on the size of the transaction cost. In fact we have the following general result. Even if there is zero state-price densities, a large class of agents will find a positive optimal solution to a consumption ivestment problem, namely those agents who have a utility function decreasing faster than the fixed cost function. In particular, this condition is always satisfied if we assume that the transaction costs exhibit strong economies to scale but are not bounded above and that the utility functions are in the HARA class (i.e. $-\frac{u'}{u''}(x) = \alpha + \eta x$) and satisfies the empirically reasonable condition $\eta < 1$.

The same results can be obtained in a dynamic setting. Furthermore, in such a setting an arbitrage opportunity can be defined as a strategy on the whole tree instead of a strategy between a node and its successors. The next result provides a straightforward extension of the no-arbitrage characterization to such a dynamic framework.

Definition 2 *A dynamic arbitrage opportunity is a strategy described by a nonnull dynamic portfolio θ (i.e. a portfolio $\theta(\sigma_t)$ for all date $t = 0, ..., T-1$ and all date $t-$node σ_t) satisfying for all t, for all $\sigma_t \in \Sigma_t$ and all $\sigma_{t+1} \in f(\sigma_t)$ the following self-financing condition*

$$\theta(\sigma_t) \cdot Z(\sigma_{t+1}) = \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) + c^\theta(\sigma_{t+1})$$

and such that

$$\theta(\sigma_0) \cdot Z(\sigma_0) + c_0^\theta \leq 0$$

and, for all $\sigma_T \in \Sigma_T$,

$$\begin{aligned} \theta^k(\sigma_T) &= 0, \qquad k = 1, ..., n, \\ \theta(\sigma_T) \cdot Z(\sigma_T) - c^\theta(\sigma_T) &\geq 0. \end{aligned} \tag{1}$$

[5] If $\theta = 0$ then there is no trade and no transaction costs. We define then c^0 by $c^0 = 0$.

The condition $\theta^k(\sigma_T) = 0$, $k = 1, ..., n$ means that the portfolio is liquidated at the final date.

We can remark that the existence of a dynamic arbitrage opportunity is equivalent to the existence of an arbitrage opportunity as defined previously. Indeed, if there exists an arbitrage portfolio θ as in Definition 1 between a node σ_t and its successors, it suffices to consider the dynamic strategy that consists in buying the portfolio θ at the node σ_t, and investing the "profits" in bonds.
Conversely suppose that there is a dynamic arbitrage opportunity described by a portfolio θ then there exists a node σ_t such that $\theta(\sigma_t) \cdot Z(\sigma_t) < 0$. If $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) \geq 0$ for all $\sigma_{t+1} \in f(\sigma_t)$, then according to Proposition 1 we have an arbitrage portfolio at σ_t; but if $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) < 0$ for some $\sigma_{t+1} \in f(\sigma_t)$, then using the self-financing condition, we have $\theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) < 0$; we start again from the node σ_{t+1} and there must exist a date τ and a node $\sigma \in \Sigma_\tau$ such that $\theta(\sigma) \cdot Z(\sigma) < 0$ and $\theta \cdot Z \geq 0$ on $f(\sigma)$ in order to obtain at the end $\theta \cdot Z \geq 0$.
As a corollary of Proposition 2 the no-arbitrage condition implies the existence of a nonnegative compatible state price deflator π on the whole tree, i.e. $\pi = (\pi(\sigma_t))_{t=0,\cdots,T,\sigma_t \in \Sigma_t}$ such that, at each date $t = 0, \cdots, T-1$ and at each date $t-$node σ_t, we have

$$\pi(\sigma_t) Z(\sigma_t) = \sum_{\sigma_{t+1} \in f(\sigma_t)} \pi(\sigma_{t+1}) Z(\sigma_{t+1}).$$

Indeed, it suffices to multiply the different state price densities obtained at each node between that node and its successors to construct a nonnegative state price deflator compatible with the assets prices.

Such a state price deflator can be seen as a nonnegative (instead of positive as in the frictionless case) subjective probability on the space Ω of the states of the world and the compatibility condition between the state price deflator and the price processes becomes then a martingale condition on the primitive assets prices under this subjective probability.

However the existence of a nonnegative compatible state price deflator is not a sufficient condition to rule out any arbitrage opportunity. We consider the following model for which $T = 2$, $\Omega = \{\omega_1, ..., \omega_4\}$, $F_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, $n = 1$ and the prices are given by:

1 1 1 for the bond

		1.5	
	1		
		0.5	
1			for the risky security.
		1	
	0.5		
		1	

There exists a nonnegative state price deflator given by

		0.5
	1	
		0.5
1		
		0
	0	
		0

Nevertheless, as we have seen earlier, the model is not arbitrage free: we only need at date 1 and in the lower date 1−node $\{\omega_3, \omega_4\}$ to buy one unit of the risky security and to sell one unit of bond and to clear the position at date 2 to obtain an arbitrage opportunity.

In order to truly characterize the assumption of absence of arbitrage, we need to introduce a whole family of nonnegative state price deflators or equivalently of martingale measures with nonnegative weights. Let us first give some definitions.

For a given date $i \in \{0, \cdots, T-1\}$ and a given date $i-$node σ, we define the sub-model S^σ by a set $\Omega^\sigma = \sigma$ of states of the world and a family of increasingly finer partitions $(F_t^\sigma)_{t\in\{0,\ldots,T-i\}}$ with $F_t^\sigma = \{\sigma' \in F_{t+i} : \sigma' \subset \sigma\}$ and asset prices defined for $\sigma' \in F_t^\sigma$, for t in $\{0, ..., T-i\}$, by $Z^\sigma(\sigma') = Z(\sigma')$. In fact, the sub-model S^σ can be described by the sub-tree starting in σ.

Theorem 1 *There is no arbitrage opportunity if and only if for all date i and all date $i-$node σ, there exists a nonnegative state price deflator $\pi^\sigma = (\pi^\sigma(\sigma_t))_{t=0,\cdots,T-i,\sigma_t\in F_t^\sigma}$ compatible with the asset prices in the sub-model S^σ, i.e.for all $t \in \{0, \cdots, T-i\}$ and all $\sigma_t \in F_t^\sigma$*

$$\pi^\sigma(\sigma_t)Z^\sigma(\sigma_t) = \sum_{\sigma_{t+1}\in f(\sigma_t)} \pi^\sigma(\sigma_{t+1})Z^\sigma(\sigma_{t+1}).$$

The absence of arbitrage opportunity is then characterized by the existence of martingale measures with nonnegative weights for every sub-tree.

2.2 Pricing in models with fixed costs

A contingent claim x is defined by its payoffs in each terminal node $\sigma_T \in \Sigma_T$. It can be thought of as a contract or an agreement which pays, at date T, $x(\sigma_T)$ at the node σ_T.

A contingent claim x is said to be attainable (in the frictionless model) if there exists some dynamic portfolio θ satisfying for all t, for all $\sigma_t \in \Sigma_t$ and all $\sigma_{t+1} \in f(\sigma_t)$ the self-financing frictionless condition $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) = \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1})$ such that $\theta \cdot Z = x$ for all $\sigma_T \in \Sigma_T$. Let M denote the set of all attainable contingent claims. Notice that M is a linear space.

Definition 3 *An admissible pricing rule on M is a functional p defined on M, such that*

1. *p induces no arbitrage, i.e. it is not possible to find a dynamic portfolio θ satisfying the self-financing condition $\theta(\sigma_t) \cdot Z(\sigma_{t+1}) = \theta(\sigma_{t+1}) \cdot Z(\sigma_{t+1}) + c^\theta(\sigma_{t+1})$ and contingent claims $m_1, ..., m_d$ in M for which $\sum_{i=1}^d p(m_i) + \theta(\sigma_0) \cdot Z(\sigma_0) + c_0^\theta \leq 0$, $\sum_{i=1}^d m_i + \theta \cdot Z - c_T^\theta \geq 0$ on Σ_T and one of the two is nonnull.*

2. *$p(m) \leq p_s(m)$, where*

$$p_s(m) \equiv \inf \left\{ \theta(\sigma_0) \cdot Z(\sigma_0) + c_0^\theta,\ \theta \textit{ is self financing and } \theta \cdot Z - c_T^\theta \geq m \textit{ on } \Sigma_T \right\}$$

Part 1 is the usual no-arbitrage condition; it encompasses the fact that the model is arbitrage free and could be replaced in an arbitrage free model by $\sum_{i=1}^d p(m_i) \leq 0$, $\sum_{i=1}^d m_i \geq 0$ and one of the two is nonnull. Part 2 says that an admissible price for m is smaller than its superreplication price and corresponds to a monotonicity assumption; we impose that if it is possible to obtain a better payoff than m at a cost $p_s(m)$, then no one will accept to pay more than $p_s(m)$. Notice that since m is attainable by a frictionless self financing strategy, and since the total cost incurred for any strategy is bounded, there always exists at least a self financing (inclusive of transaction costs) strategy dominating m.

The following proposition gives us the admissible pricing rules on M through the use of the nonnegative state-price deflators on the whole tree π^Ω introduced in Theorem 1. Notice that the linear functional l given by $l\left[(\theta(\sigma_T) \cdot Z(\sigma_T))_{\sigma_T \in \Sigma_T}\right] = \theta(\sigma_0) \cdot Z(\sigma_0)$ for all frictionless self financing strategy θ is well defined on M: indeed, if two frictionless self financing strategies θ and θ' generate the same contingent claim m, i.e. if their terminal values are both equal to m, then their initial value must also be equal; if not, the strategy given by $\theta - \theta'$ or by $\theta' - \theta$ would be a strong form of frictionless arbitrage opportunity, which would lead to the existence of an arbitrage opportunity with fixed costs as seen in Proposition 1. We find that any admissible pricing rule can be written as the sum of the linear functional

l and of a fixed cost, or expressed in terms of the nonnegative state price deflators found in the preceding section,

Proposition 3 *Any admissible pricing rule* p *on* M *is given by*

$$p(m) = \sum_{\sigma_T \in \Sigma_T} \pi^{\Omega}(\sigma_T)\, m(\sigma_T) + c(m) \qquad \textit{for all } m \textit{ in } M$$

where π^{Ω} *is any nonnegative state-price deflator like in Theorem* 1 *and where* $\frac{c(\lambda m)}{\lambda} \to_{\lambda \to \infty} 0$.
Moreover, if $p(\lambda x) \leq \lambda [p(x)]$*, for* λ *sufficiently large, then the fixed cost functional* $c(\cdot)$ *is nonnegative.*

This means that $\frac{p(\lambda m)}{\lambda} \to_{\lambda \to \infty} \sum_{\sigma_T \in \Sigma_T} \pi^{\Omega}(\sigma_T) m(\sigma_T)$ for any attainable contingent claim m. Therefore, the unit price of any attainable contingent claim m is given by $\sum_{\sigma_T \in \Sigma_T} \pi^{\Omega}(\sigma_T) m(\sigma_T)$ in the limit of large quantities.

As usual, we say that the market is complete (in the frictionless model) if any contingent claim is attainable (in the frictionless model). It is then easy to see that if the market is arbitrage free and complete, there exists a unique nonnegative state price deflator and there exists a unique fair price for any contingent claim in the limit of large quantities.

If we further assume that the frictionless model is arbitrage free then the unique state price deflator is then positive and the unique fair price for a given contingent claim in the model with frictions is equal to the frictionless price in the limit of large quantities. If the market is incomplete, the set of possible prices[6] in the model with frictions for a given contingent claim is equal to the closure of the set of possible frictionless prices.

3 Applications

We start by proving that the quite surprising result obtained by Dybvig-Ingersoll-Ross (1996), which asserts that, under the assumption of absence of arbitrage, long zero-coupon can never fall, is no longer true in models with fixed costs.

3.1 Long rates can fall

3.1.1 A simple example

Let us recall the framework of Dybvig-Ingersoll-Ross (1996). There are traded zero coupon bonds of all integer maturities. For $t < t'$, we define

[6] defined, as usual, as the set of prices given by all the possible admissible pricing rules.

$\nu(t,t')$ to be the discount bond price, i.e. the price at t of receiving a unit payoff at some later date t'. The zero-coupon rate $z(t,t')$ is defined implicitly by

$$\nu\left(t,t'\right) \equiv \frac{1}{\left[1+z\left(t,t'\right)\right]^{t'-t}}.$$

The long zero-coupon rate at time t, denoted by $z_L(t)$ is given by

$$z_L(t) = \lim_{t'\to\infty} z\left(t,t'\right)$$

if the limit exists.

A free lunch is defined as a sequence of net trades such that $i)$ the payoff tends uniformly to a nonnegative random variable that is positive with positive probability and the price tends to zero or $ii)$ the price tends to a negative number but the payoff tends uniformly to a nonnegative random variable.

Then the authors prove that, under the assumption of no-free lunch, if for $s<t$, the long zero coupon rate exists at time s and exists with probability one at time t, then $z_L(s) \leq z_L(t,\omega)$ with probability one. To do so, they use an arbitrage argument of the form $i)$. However, if we transpose their result in our discrete time and finite states of the world framework, the set of attainable payoffs is closed and a free-lunch is in fact an arbitrage.

Their result can then be rephrased in our framework as follows. An arbitrage is a net trade with zero cost and nonnegative, nonzero terminal payoff. Under the assumption of no-arbitrage, if for $s<t$, the long zero coupon rate exists at time s and exists with probability one at time t, then $z_L(s) \leq z_L(t,\omega)$ with probability one.

As a consequence of Dybvig et al. (1996) result, models where the long term rate follows a specified nonmonotone behavior and in particular models where the long rate follows the discrete time analogon of a diffusion are then rejected by the no-arbitrage condition. However this does not mean that models where the long-term yield is a risk factor like in Duffie-Kan (1996) have to be systematically rejected. Indeed, in these models, the long term yield corresponds to a yield associated to a long but finite horizon while in Dybvig et al. (1996), the long term rate is the limit rate when the horizon goes to infinity. In fact, it is possible to construct models where

- for any t, the function $t' \to z(t,t')$ is nonomonotone,
- for any T, the function $t \to z(t,t+T)$ is nonmonotone,
- however, $t \to z_L(t)$ have a monotone nondecreasing behavior.

Such an example is provided in the Appendix.

We have seen that in a model with fixed costs, an arbitrage opportunity consists of a negative price for a nonnegative payoff so that arbitrages may vanish as soon as we introduce fixed trading costs. We show on a simple counterexample that in models with fixed costs, long zero-coupon rates can fall. Suppose it is known that the spot rate will be r_1 until s, and that at time s it will be revealed to either shift to $r_2\,(< r_1)$ forever or to remain at r_1 forever, which is illustrated by

$$\begin{array}{llll} & & (1+r_1)^{s+1} \quad \rightarrow ... \rightarrow (1+r_1)^{t} & \rightarrow ... \\ & \nearrow & & \\ 1 \rightarrow ... \rightarrow (1+r_1)^{s} & & & \\ & \searrow & & \\ & & (1+r_1)^{s}(1+r_2) \rightarrow ... \rightarrow (1+r_1)^{s}(1+r_2)^{t-s} & \rightarrow ... \end{array}$$

We assume that the discount bond prices are given by

$$\nu\left(t,t'\right) = \frac{1}{(1+r_1)^{t'-t}} \quad \forall t' \geq t \quad \text{ for } t \leq s$$

and for $t > s$,

$$\begin{aligned} \nu\left(t,t'\right) &= \frac{1}{(1+r_1)^{t'-t}} \quad \forall t' \geq t \quad \text{if } r_1 \\ \nu\left(t,t'\right) &= \frac{1}{(1+r_2)^{t'-t}} \quad \forall t' \geq t \quad \text{if } r_2 \end{aligned}$$

In fact, for $t' \geq t > s$ and for $s \geq t' \geq t$, the interest rate structure is deterministic and the bond prices are uniquely determined without imposing any assumption on the state price defators or on the risk premium. For $t' > s \geq t$, our bond prices correspond to a state price deflator which puts a zero weight on the lower branch of the tree. From Theorem 1 and in a fixed costs framework this model does not contain any arbitrage opportunity in a finite horizon. As far as free lunches are concerned, it is clear that we need not pay attention to strategies starting after date $s+1$. For strategies starting before date s , it is easy to see that any net trade in the state r_1 is equal to 0 if we do not take the fixed cost into account and is smaller than $-\mathbf{c}$ if we take the fixed cost into account. Hence, there is no sequence of net trades, taking the fixed costs into account and converging to a nonnegative nonnull random variable.

We obtain

$$z_L(t) = r_1 \quad \text{for } t \leq s$$

and for $t > s$,

$$
\begin{array}{rcl}
z_L(t) & = & r_1 \quad \text{if } r_1 \\
z_L(t) & = & r_2 \quad \text{if } r_2
\end{array}
$$

so that, since $r_2 < r_1$, we don't have $z_L(s) \leq \min_\omega z_L(t,\omega)$ (more precisely, for all $t_1 \leq s$ and all $t_2 > s$, we have $z_L(t_1) \geq z_L(t_2,\omega)$ with probability one and $P[z_L(t_1) > z_L(t_2,\omega)] > 0$).

We have then proved that when there are fixed costs, Dybvig-Ingersoll-Ross (1996) result is no longer true : long rates can fall. In our example they fall with a probability that can be chosen arbitrarily, ,namely the probability for the short rate to shift from r_1 to r_2 at date s.

3.1.2 Ho and Lee-like models

It is shown in Dybvig-Ingersoll-Ross (1996, Section IV), that in the Ho-Lee (1986) model, the long rates can only increase. We prove here that with the introduction of fixed costs, Ho and Lee-like models are compatible with a decrease of long rates. More precisely, we first give an example of an arbitrage free Ho and Lee model in which long rates fall with probability one. We then give an example of a trinomial model, analogous to Ho and Lee, for which long rates can both increase or decrease.
Suppose that $\Sigma_t = \{u,d\}^t$. In such a context, a date$-t$ node σ_t is a sequence $(\rho_1,\cdots,\rho_t)$ of elements of $\{u,d\}$ and, for $\sigma_t = (\rho_1,\cdots,\rho_t)$, $f(\sigma_t) = \{(\sigma_t,u),(\sigma_t,d)\}$ with, by definition, $(\sigma_t,u) = (\rho_1,\cdots,\rho_t,u)$ and $(\sigma_t,d) = (\rho_1,\cdots,\rho_t,d)$.

Assume now as in Ho and Lee (1986) that the value of any portfolio is independent from the chosen path and that

$$
\nu(t,T,(\sigma_{t-1},\rho_t)) = \frac{\nu(t-1,T,\sigma_{t-1})}{\nu(t-1,t,\sigma_{t-1})} H(T-t,\rho_t)
$$

with

$$
H(\cdot,\rho) = \begin{cases} h(\cdot) & \text{if } \rho = u \\ h^*(\cdot) & \text{if } \rho = d \end{cases}
$$

and $h \neq h^*$.

Proposition 4 *If there are fixed costs, the functions h and h^* can be chosen such that h is constant equal to 1 and $h^*(n) = [h^*(1)]^n > 1$ and the model is then arbitrage free. Furthermore, in such a framework we have*

$$
\begin{array}{rcl}
1 + z_L(t,(\sigma_{t-1},u)) & = & 1 + z_L(t-1,\sigma_{t-1}) \\
1 + z_L(t,(\sigma_{t-1},d)) & < & 1 + z_L(t-1,\sigma_{t-1})
\end{array}
$$

We have then proved that Ho and Lee (1986) model can lead to nonincreasing long rates in a fixed costs setting. In fact, in our setting the long rates fall with a probability equal to one.

If we want now a model where the long rates can fluctuate (i.e. increase and decrease depending on the states of the world), we have to extend Ho and Lee (1986) model to a trinomial setting.

In order to construct such a trinomial generalization of Ho and Lee (1986) model, we still assume that the value of any portfolio is independent from the chosen path and that

$$\nu(t,T,(\sigma_{t-1},\rho_t)) = \frac{\nu(t-1,T,\sigma_{t-1})}{\nu(t-1,t,\sigma_{t-1})} H(T-t,\rho_t)$$

but with $\Sigma_t = \{u,m,d\}^t$. In the nexts, such a model is called generalized trinomial Ho and Lee model. We have then the following result

Proposition 5 *In a fixed costs framework, there exists an arbitrage free specification of the generalized trinomial Ho and Lee model such that*

$$\begin{array}{rcl} 1 + z_L\left(t,(\sigma_{t-1},u)\right) & > & 1 + z_L\left(t-1,\sigma_{t-1}\right) \\ 1 + z_L\left(t,(\sigma_{t-1},m)\right) & = & 1 + z_L\left(t-1,\sigma_{t-1}\right) \\ 1 + z_L\left(t,(\sigma_{t-1},d)\right) & < & 1 + z_L\left(t-1,\sigma_{t-1}\right) \end{array}$$

We have then constructed a Ho and Lee like model where long rates can fluctuate. More precisely they increase along the upper branchs of the trinomial tree and they decrease along the lower branches. Consequently, models where long rates have a nonmonotonic specified behaviour are no more to be rejected for arbitrage considerations. It is essentially for this reason that the Brennan and Schwartz model [1979] is usually rejected. We shall now consider a discrete version of this model.

3.1.3 The Brennan and Schwartz model

In this section we consider a discrete time version of Brennan and Schwartz model and we prove that such a model where the long rates follow a diffusion process (or more precisely a discrete time vapproximation of a diffusion process) is compatible with the no arbitrage condition in a fixed costs framework. Consequently, even if in such a model the long rates fluctuate and decrease in some states of the world, we can not reject them for arbitrage considerations if there are fixed costs.

Let us first recall that in Brennan-Schwartz (1979) model, the short rate and the long rate, respectively denoted by l and r, are governed by the following diffusion equation

$$\begin{cases} d\ell_t = \mu\left(\ell_t, r_t, t\right) dt + \sigma_1 \ell_t dW_t^1 \\ dr_t = a\left(\ell_t - r_t\right) dt + \sigma_2 \sqrt{r_t} dW_t^2 \end{cases}$$

where a, σ_1 and σ_2 are given positive constants and where μ is a given bounded function.of the current date and of the long and short rates.

In the next we consider the following model for the long and short rates. It is not recombining but our aim is to show the viability of such a model and as we will see, we obtain simple pricing formulas. Let

$$\ell_{n+1} = \ell_n \begin{cases} \exp\left\{\left(\frac{\mu\left(\ell_n, r_n, n\frac{T}{N}\right)}{\ell_n} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} + \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right\} & \text{with a probability } p_u \\ 1 & \text{with a probability } p_m \\ \exp\left\{\left(\frac{\mu\left(\ell_n, r_n, n\frac{T}{N}\right)}{\ell_n} - \frac{1}{2}\tilde{\sigma}_1^2\right)\frac{T}{N} - \tilde{\sigma}_1\sqrt{\frac{T}{N}}\right\} & \text{with a probability } p_d \end{cases}$$

$$r_{n+1} = r_n \begin{cases} \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N} + \sqrt{2}\frac{\sigma_2}{\sqrt{r_n}}\sqrt{\frac{T}{N}}\right\} & \text{with a probability } q_u \\ \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N}\right\} & \text{with a probability } q_m \\ \exp\left\{\left(\frac{a(\ell_n - r_n)}{r_n} - \frac{1}{2}\frac{\sigma_2^2}{r_n}\right)\frac{T}{N} - \sqrt{2}\frac{\sigma_2}{\sqrt{r_n}}\sqrt{\frac{T}{N}}\right\} & \text{with a probability } q_d \end{cases}$$

where T is a given horizon, $\tilde{\sigma}_1, a, \sigma_2$ are positive constants and where μ is a given bounded function.

This model consists of 9 different states of the world at each date, with a probability $p_i q_j$ for each state $(i,j)_{\substack{i=u,m,d \\ j=u,m,d}}$ and we have the following result

Proposition 6 *There exists a specification of the probabilities* (p_u, p_m, p_d) *and* (q_u, q_m, q_d) *such that the model above is a discrete time approximation of Brennan-Schwartz* (1979) *model. Furthermore, in a fixed costs setting, there exists an aritrage free bond price structure compatible with this model, i.e. such that the short rate is given by* r *and the long rate by* ℓ.

We have then proved that in the presence of fixed costs, Brennan-Schwartz (1979) model (or more precisely a discrete time approximation of this model) is compatible with the no-arbitrage condition. This compatibility is directly linked with the introdution of fixed costs since in such a model the long rates fluctuate and following Dybvig-Ingersoll-Ross (1996) this property leads to arbitrage opprtunities in a frictionless setting. There are obviously many other frictions that might lead to the same result however the introduction of fixed costs is the only one for which arbitrage opportunities vanish and such that by Proposition 3 the bond prices structure remains the same as in the frictionless setting (at least in the limit of large quantities).

3.2 Option pricing with bounds on the underlying securities

The introduction of fixed costs enables us to consider as viable models where the asset price processes reach some bounds and to determine in such a framework a fair pricing rule for contingent claims. Indeed if we assume that the discounted price process of a given asset is submitted to a given upper barrier and reaches this barrier without being absorbed then it is clear that there is no positive state-price density compatible with this asset. This price process leads then to arbitrage opportunities in a frictionless setting. However a state price density which once the barrier reached puts all the weight on the constant discouted price path might be compatible with this asset. By Theorem 1 the presence of upper bounds on the discounted asset prices may then be compatible with the no-arbitrage condition in a fixed costs setting.

We will illustrate this property on the specific example of the binomial model of Cox-Ross-Rubinstein (1979) with bounds. This model has been studied in Sondermann (1988)[7], starting from other considerations.

More precisely, the model is the following. Denote by $(W_t)_{t\in\{0,\cdots,T\}}$, the stochastic process given by:

$$W_t = Z_0 \cdot \prod_{n=1}^{t} (1+\rho_n)$$

where (ρ_n) is a family of independent random variables such that

$$\rho_n = \begin{cases} \delta + \frac{1}{2}\delta^2 \overset{def}{=} \delta_u & \text{with probability } p \\ \\ -\delta + \frac{1}{2}\delta^2 \overset{def}{=} \delta_d & \text{with probability } 1-p \end{cases}$$

for fixed $2 > \delta > 0$, $0 < p < 1$. The process $(W_t)_{t\in\{0,\cdots,T\}}$ is a geometric random walk with compensated stochastic drift and corresponds to the security price process in the classical Cox, Ross and Rubinstein model [1979] . Notice that in such a context, a date$-t$ node σ_t is a sequence $(\rho_1, \cdots, \rho_t)$. As we have mentioned, we assume that for some reason, the price process can not

[7] This paper might be difficult to consult. See also Sondermann (1987).

go beyond some a priori given bounds. We shall here fix an upper bound M and we assume for simplicity of exposition that M belongs to the set of values taken by the process $(W_t)_{t\in\{0,\cdots,T\}}$. The price process of our security is given by $(Z_t)_{t\in\{0,\cdots,T\}}$ with $Z_0 = W_0$ and where, for $\sigma_t = (\rho_1, \cdots, \rho_t)$ and $\sigma_{t+1} = \big(\rho_1, \cdots, \rho_{t+1}\big) \in f(\sigma_t)$,

$$
\begin{aligned}
Z(\sigma_{t+1}) &= Z(\sigma_t)\big(1+\rho_{t+1}\big) \quad \text{if } Z(\sigma_t) < M \text{ and, if } Z(\sigma_t) = M, \\
Z(\sigma_{t+1}) &= \begin{cases} M & \text{if } \rho_{t+1} = \delta_u \\ \\ Z(\sigma_t)\big(1+\rho_{t+1}\big) & \text{if } \rho_{t+1} = \delta_d \end{cases} .
\end{aligned}
$$

In the next this model is called the generalized Cox, Ross and Rubinstein model [1979] with bounds. The aim is to valuate a European call $x = (Z_T - K)_+$ for some a priori given bound $M \geq Z_0$.
If we now introduce fixed costs, we have the following result

Proposition 7 *The generalized Cox, Ross and Rubinstein model* [1979] *with bounds is arbitrage free in a fixed costs setting. Furthermore the price of a European call option in this model is equal (in the limit of large quantities) to the price of a barrier option with same strike and maturity in the classical Cox, Ross and Rubinstein model* [1979] .

It is shown in Sondermann (1988) that barrier option pricing formula can be simplified for small grid sizes. Letting the grid size go to zero, by the Central Limit Theorem the analog of the Black and Scholes formula is obtained such options and then for European call options with bounds on the underlying asset. These formulas reduce to the Black and Scholes formula if the upper bound goes to infinity. The same kind of results can be obtained if we introduce a lower bound on the asset prices and the pricing formulas we obtain reduce to the Black and Scholes formula if the lower bound approaches zero.

We have then proved that the introduction of fixed costs permits to enlarge the set of possible models and in particular to take into account models with bounds on some discounted asset price processes. Such bounds can be seen as subjectively introduced by some investors in order to model the fact that, from their point of view, the price of a given asset can not exceed a given level. They can also be seen as objective bounds due to regulatory constraints like in target zones where some bounds are imposed on the exchange rates, like in the previous European Monetary System. They finally can be seen as objective bounds resulting from the choice of a given model. Such situations can be encountered for instance in interest rate models where $r = 0$ is an absorbing barrier like in Longstaff (1992). Indeed, in such models, the price of a given bond when the barrier $r = 0$ is reached is equal to 1 which corresponds to an upper bound for a bond price in models where the short rates are always nonnegative.

Longstaff (1992) introduced such models in order to enlarge the set of possible shapes of the interest rate term structure in a Cox-Ingersoll-Ross model (1985), and in particular to generate possible humps. However the introduction of a totally absorbing barrier is not realistic since it is difficult to accept that when the short rate reaches zero it will stay at that level forever. In the next section we prove that we can obtain the same results as in Longstaff (1992) without imposing the strong absorbing barrier condition.

3.3 The CIR model with fixed transaction costs

Let us consider the following model for the short rate

$$r_{n+1} = r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{with a probability } 1/4 \\ a(\ell - r_n)\frac{T}{N} & \text{with a probability } 1/2 \\ a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{with a probability } 1/4 \end{cases}$$

Following Nelson and Ramaswamy (1990) this model is a discrete time approximation of Cox-Ingersoll-Ross model (1985) model (CIR) for which the short rate is governed by the following diffusion equation

$$dr_t = a(\ell - r_t)\,dt + \sigma\sqrt{r_t}dW_t.$$

For this reason, the discrete time model introduced above is called in the next discrete CIR model.

If $2\sigma^2 < a\ell$ as imposed by Cox-Ingersoll-Ross model (1985) and if $r_0 > 0$ then it is possible to check that both models lead to a (strictly) positive short rate at any date with probability one. Cox-Ingersoll-Ross model (1985) does not explore the case where the condition $2\sigma^2 < a\ell$ is not satisfied and where the short rate might be equal to zero with a positive probability.

However, Longstaff (1992) pointed out that this condition can be relaxed if we impose a boundary condition on the bond prices when the short rate is equal to zero. In particular, he considers an absorbing barrier at $r = 0$ for the short rate and imposes bond prices equal to one once the barrier is reached. In that case, he provides analytical pricing formulas for the bonds and he argues that this model leads to more realistic interest rate term structure.

The discrete time version of Longstaff (1992) absorbing barrier model is given by :

$$\begin{aligned} r_{n+1} &= r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & 1/4 \\ a(\ell - r_n)\frac{T}{N} & 1/2 \\ \left[a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}}\right]_+ & 1/4 \end{cases} \quad \text{if } r_n \neq 0 \\ r_{n+1} &= r_n \text{ if } r_n = 0 \end{aligned}$$

Our aim is to show that Longstaff (1992)'s approach and formulas (or more precisely their discrete time version) are compatible with the no arbitrage condition with fixed costs, even if $r = 0$ is not an absorbing barrier. This permits to enlarge the set of possible shapes of the interest rate term structure, and in as underlined by Longstaff (1992) to generate possible humps, without imposing the strong absorbing barrier condition. This is important feature of the model since an often-cited criticism of the CIR model is its inability to generate humps at longer maturities. In the next the discrete time version of the bond prices formulas obtained by Longstaff (1992) in the absorbing barrier framework will be called Longstaff's bond prices.

Let us modify the discrete CIR model as follows.

$$r_{n+1} = r_n + \begin{cases} a(\ell - r_n)\frac{T}{N} + \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & 1/4 \\ a(\ell - r_n)\frac{T}{N} & 1/2 \\ [\varphi(r_n)]_+ & 1/4 \end{cases}$$

where

$$\varphi(r_n) = \begin{cases} a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}} & \text{if } r_n > 0 \\ 0 & \text{if } r_n = 0 \end{cases}$$

This model clearly corresponds to a generalization of the discrete CIR model with a partially absorbing barrier at $r = 0$. Indeed, when r_n is positive, we define r_{n+1} as in the discrete CIR model with an only one modification : $a(\ell - r_n)\frac{T}{N} - \sqrt{2}\sigma\sqrt{r_n}\sqrt{\frac{T}{N}}$ in the lower branch is replaced by its positive part in order to ensure that the short rate remains positive. When r_n is equal to zero, then there are three possible states of the world at date $n+1$: for one of them the short rate is absorbed and remains equal to zero and for the two others the short rate is reflected. In the next we refer to this model as the partially absorbing CIR model.

Proposition 8 *For λ small enough and N large enough*[8]*, Longstaff's bond prices are compatible with the partially absorbing CIR model with fixed costs (in the sense that these prices can be derived in an arbitrage-free way from the partially absorbing model for the short rate).*

In order to obtain the same term structure as in the absorbing barrier model we chose a risk-neutral probability for which zero is actually an absorbing barrier. This is made possible by that fact that in the presence of fixed costs the risk-neutral probability and the true probability do not have to be equivalent. A given barrier can then be absorbing under one of these probabilities and not for the other one.

Our term structure is then the discrete time analog of the specific example of Longstaff (1992, Section 3). However, the short rate dynamics is not the same: we do not assume that zero is an absorbing barrier for the short rate. We only impose that when reaching zero, there is a nonnull probability of staying in zero, for at least one period. As underlined by Longstaff, the behavior of Treasury-bill yields appears to be more consistent with such a sticky boundary behavior than with totally reflecting behavior: extremely low levels of yields tend to persist rather than immediately increasing back toward higher levels.

4 Conclusion

We have shown that when transaction costs are taken into account, we enlarge the set of possible arbitrage free models. In particular, models with bounded prices processes reaching their bounds are now arbitrage-free and viable and we have relaxed the monotonicity constraint imposed for arbitrage considerations on the long rates by Dybvig *et al.* (1996).

When the initial frictionless model is arbitrage free we proved that the derivatives prices remain the same in the limit of large quantities after the introduction of the transaction costs. When the initial frictionless model contains arbitrages, we proved that the no arbitrage condition in the mdel with frictions is characterized by the existence of a nonnegative state-price density (or equivalently by a martingale measure with possible zero-weights) and that derivatives prices are given in the limit of large quantities by the expectation of the terminal payoff under that probability. In particular, it

[8] More precisely for $|\lambda| < \frac{a}{\sqrt{2}}$ and $\sigma^2 \geq \lambda^2 \bar{r} \frac{T}{N}$, where $\bar{r} = \sup\left\{ r_0, \left(\frac{\sqrt{2}\sigma + \sqrt{2\sigma^2 + 4a^2 \ell h^2}}{2ah} \right)^2 \right\}$.

becomes possible to price derivative assets in markets where the underlying assets have bounded price processes and reach their bounds. The bound on the price processes can then be interpreted in the pricing formula as a barrier on the derivative asset in a frictionless model without bounds.

Appendix

Proof of Proposition 1 The direct implication is immediate and it suffices to multiply the considered portfolio by a sufficiently large constant in order to establish the converse implication.

Proof of Proposition 2 As usual, this result is a direct application of Farkas-Minkowski Lemma.

Proof of Theorem 1 If there is no arbitrage opportunity it suffices to multiply the different state price densities obtained at each node between that node and its successors starting from σ to construct a nonnegative state price deflator compatible with the asset prices.

Conversely, for a given date $i-$node σ, it suffices to consider the state price deflator $(\pi_\rho)_{\rho\in f(\sigma)}$ defined on the successors of σ by $\pi_\rho = \pi^\sigma(\rho)$. This state price deflator characterizes the absence of arbitrage opportunity between σ and its successors for every σ and then the absence of dynamic arbitrage opportunity.$\square$

Proof of Proposition 3 It is easy to see that for all m in M,

$$\lim_{\lambda\to+\infty} \frac{p_s(\lambda m)}{\lambda} = \lim_{\lambda\to+\infty} \frac{-p_s(-\lambda m)}{\lambda} = l(m)$$

Since there is no arbitrage, we must have $p(m) \geq -p(-m)$ so that

$$-\pi_s(-m) \leq -p(-m) \leq p(m) \leq \pi_s(m),$$

and the price functional p can be written as the sum of a linear functional and of a fixed cost functional, i.e. for all m, $p(m) = l(m) + c(m)$ where $\frac{c(\lambda m)}{\lambda} \to_{\lambda\to\infty} 0$.

If we assume that $p(\lambda m) \leq \lambda[p(m)]$, it is easy to see that the fixed cost functional is nonnegative. Furthermore, for all frictionless self-financing dynamic portfolio θ,

$$l\left(\left(\theta(\sigma_T)\cdot Z(\sigma_T)\right)_{\sigma_T\in\Sigma_T}\right) = \sum_{\sigma_T\in\Sigma_T} \pi^\Omega(\sigma_T)\theta(\sigma_T)\cdot Z(\sigma_T),$$

where π^Ω is any nonnegative state-price deflator like in Theorem 1.

Consequently, the fair price $p(m)$ associated with any attainable contingent claim m is given by

$$p(m) = \sum_{\sigma_T \in \Sigma_T} \pi^{\Omega}(\sigma_T) m(\sigma_T) + \text{a fixed cost}$$

An example of nondecreasing long rate with nonomonotone finite horizon rates Let us assume that the short rate r_n between date $t = n$ and $n+1$ is deterministic but randomly chosen between 0 and 1 with a probability p.independently from the rates at other dates. It is clear that $z(n, n')$ (resp. $z(n, n+N)$) can randomly increase or decrease when n' (resp. n) increases. However $\ln(1 + z_L(n))$ is the limit of the average of the $\ln(1 + r_n)$ and is equal to $p \ln 2$. We have then that $z_L(n)$ is constant, independent of n and equal to $2^p - 1$.

Proof of Proposition 4 Proceeding like in the frictionless case, we obtain that the absence of arbitrage with fixed costs implies the existence of $\pi \in [0, 1]$, independent from n, and such that

$$\begin{aligned} \pi h(n) + (1-\pi) h^*(n) &= 1 \\ \frac{h(n)}{h^*(n)} &= \left[\frac{h(1)}{h^*(1)}\right]^n \end{aligned}$$

Indeed, in that case we get that the discounted process $X^T = \left(\frac{\nu(t,T)}{\beta_t}\right)_t$ satisfies for all $\sigma_t \in \Sigma_t$,

$$X_t^T(\sigma_t) = \pi X_{t+1}^T(\sigma_t, u) + (1-\pi) X_{t+1}^T(\sigma_t, d)$$

and the prices do not depend on the chosen path.

The choice of $\pi = 1$ (which, by Theorem 1, is licit in a fixed costs framework) leads to $h \equiv 1$ and $h^*(n) = [h^*(1)]^n$. As far as long rates are concerned, we have

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, u)) &= \lim_{T \nearrow \infty} [\nu(t, T, (\sigma_{t-1}, u))]^{-\frac{1}{T-t}} \\ &= \lim_{T \nearrow \infty} \left[\frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} h(T-t)\right]^{-\frac{1}{T-t}} \\ &= 1 + z_L(t-1, \sigma_{t-1}) \end{aligned}$$

and

$$\begin{aligned} 1 + z_L(t, (\sigma_{t-1}, d)) &= \lim_{T \nearrow \infty} [\nu(t, T, (\sigma_{t-1}, d))]^{-\frac{1}{T-t}} \\ &= \lim_{T \nearrow \infty} \left[\frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} h^*(T-t)\right]^{-\frac{1}{T-t}} \\ &= \lim_{T \nearrow \infty} \left[\frac{\nu(t-1, T, \sigma_{t-1})}{\nu(t-1, t, \sigma_{t-1})} h^*(1)^{T-t}\right]^{-\frac{1}{T-t}} \\ &= \frac{1}{h^*(1)} [1 + z_L(t-1, \sigma_{t-1})] \\ &< 1 + z_L(t-1, \sigma_{t-1}) \end{aligned}$$

Proof of Proposition 5 It suffices to take

$$H(n,\rho) = \begin{cases} a^n & \text{if } \rho = u \\ 1 & \text{if } \rho = m \\ \frac{1}{a^n} & \text{if } \rho = d \end{cases}$$

where $a < 1$ is a given constant. There exists $\pi = (0,1,0) \in [0,1]^3$, such that $\pi_1 a^n + \pi_2 + \pi_3 (1/a^n) = 1$, and, with the notations of the previous proof, the process $X^T = \left(\frac{\nu(t,T)}{\beta_t}\right)_t$ satisfies

$$X_t^T(\sigma_t) = \pi_1 X_{t+1}^T(\sigma_t, u) + \pi_2 X_{t+1}^T(\sigma_t, m) + \pi_3 X_{t+1}^T(\sigma_t, d).$$

By Theorem 1 this model is arbitrage free in a fixed costs framework. As far as long rates are concerned, we obtain

$$\begin{aligned}
1 + z_L(t,(\sigma_{t-1},u)) &= \lim_{T\nearrow\infty} \left[\nu(t,T,(\sigma_{t-1},u))\right]^{-\frac{1}{T-t}} \\
&= \lim_{T\nearrow\infty} \left[\frac{\nu(t-1,T,\sigma_{t-1})}{\nu(t-1,t,\sigma_{t-1})} a^{T-t}\right]^{-\frac{1}{T-t}} \\
&= \frac{1}{a}\left[1 + z_L(t-1,\sigma_{t-1})\right] \\
&> 1 + z_L(t-1,\sigma_{t-1}) \\
1 + z_L(t,(\sigma_{t-1},u)) &= 1 + z_L(t-1,\sigma_{t-1}) \\
1 + z_L(t,(\sigma_{t-1},d)) &= \lim_{T\nearrow\infty} \left[\nu(t,T,(\sigma_{t-1},d))\right]^{-\frac{1}{T-t}} \\
&= \lim_{T\nearrow\infty} \left[\frac{\nu(t-1,T,\sigma_{t-1})}{\nu(t-1,t,\sigma_{t-1})} \frac{1}{a^{T-t}}\right]^{-\frac{1}{T-t}} \\
&= a\left[1 + z_L(t-1,\sigma_{t-1})\right] \\
&< 1 + z_L(t-1,\sigma_{t-1})
\end{aligned}$$

Proof of Proposition 6 Let us take $p_u = \frac{\sigma_1^2 - \frac{\mu}{l}((\frac{\mu}{l} - \frac{1}{2}\tilde{\sigma}_1^2)\frac{T}{N} - \tilde{\sigma}_1\sqrt{\frac{T}{N}})}{2\tilde{\sigma}_1\left[\tilde{\sigma}_1 + (\frac{\mu}{l} - \frac{1}{2}\tilde{\sigma}_1^2)\sqrt{\frac{T}{N}}\right]}$, $p_m = \frac{\tilde{\sigma}_1^2 - \sigma_1^2 + (\frac{\mu^2}{l^2} - \frac{1}{4}\tilde{\sigma}_1^4)\frac{T}{N}}{\tilde{\sigma}_1^2 - (\frac{\mu}{l} - \frac{1}{2}\tilde{\sigma}_1^2)^2\frac{T}{N}}$, $p_d = \frac{\sigma_1^2 - \frac{\mu}{l}((\frac{\mu}{l} - \frac{1}{2}\tilde{\sigma}_1^2)\frac{T}{N} + \tilde{\sigma}_1\sqrt{\frac{T}{N}})}{2\tilde{\sigma}_1\left[\tilde{\sigma}_1 - (\frac{\mu}{l} - \frac{1}{2}\tilde{\sigma}_1^2)\sqrt{\frac{T}{N}}\right]}$, $q_u = 1/4, q_m = 1/2, q_d = 1/4$, for some positive constant $\sigma_1 < \tilde{\sigma}_1$ and for N sufficiently large. In that case, it is easy to check that $P = (p_u, p_m, p_d)$ defines a probability.and following Nelson, D. Ramaswamy, K. (1990) our model is a discrete approximation of the Brennan-Schwartz (1979) model.

Let the risk-neutral probability $[\pi(i,j)]_{\substack{i=u,m,d \\ j=u,m,d}}$ be such that $\pi(m,m) = 1$ and let π denote the induced probability measure on the terminal states of the world. We construct then bond prices as the discounted expected value under the probability π of the bond's terminal payoff, and obtain

$$\nu\left(n,n';(\ell,r)\right) = \left[\left(1 + u_n^{\ell,r}\right) \ldots \left(1 + u_{n'-1}^{\ell,r}\right)\right]^{-1}$$

where the sequence $u_s^{l,r}$ is defined inductively by

$$\begin{cases} u_{s+1}^{\ell,r} = u_s^{\ell,r} \exp\left\{ \frac{a\left(\ell - u_s^{l,r}\right)}{u_s^{l,r}} \frac{T}{N} \right\} \\ u_n^{\ell,r} = r \end{cases}$$

By construction and by Theorem 1, the financial market consisting of these bonds is arbitrage free in a fixed costs setting and admits r as short rate. Furthermore, we can verify that the long rate induced by these bond prices is given by ℓ. Indeed, we have

$$\ln(1 + z\left(n, n'; (\ell, r)\right)) = \frac{1}{n' - n} \sum_{k=0}^{n'-n-1} \ln\left(1 + u_{n+k}^{\ell,r}\right)$$

and by a classical Cesaro argument we have

$$\ln(1 + z_L\left(n; (\ell, r)\right)) = \lim_{k \to \infty} \ln\left(1 + u_k^{\ell,r}\right).$$

when this last limit exists. It is then easy to check that the limit of $u_k^{\ell,r}$ is equal to ℓ and we have then $z_L\left(n; (\ell, r)\right) = \ell$.

Proof of Proposition 7 For $M = \infty$, the solution is well known: the process $(Z_t)_{t\in\{0,\cdots,T\}}$ is then equal to the process $(W_t)_{t\in\{0,\cdots,T\}}$; since $\delta > 0$, there is a unique equivalent probability measure that we shall denote by P^* that makes the process $(Z_t)_{t\in\{0,\cdots,T\}}$ a martingale and it is given by the positive transition probability

$$p^* = \frac{\delta - \frac{1}{2}\delta^2}{2\delta} = \frac{1}{2} - \frac{1}{4}\delta.$$

Then, in a frictionless model, any contingent claim admits a unique fair price which is equal to its expected value with respect to P^*. In the model with fixed costs, there exists a unique martingale measure with nonnegative weights which is equal to P^* and by Proposition 3 any admissible pricing rule is given by the sum of a fixed cost functional and of the expected value with respect to P^* which corresponds to the option price in a frictionless setting..

If $M < \infty$, the model, in a frictionless market, contains arbitrage opportunities if there are date-T nodes $\sigma_T = (\rho_1, \cdots, \rho_T)$ for which $\tau\left(\sigma_T\right)$ given by

$$\tau\left(\sigma_T\right) \equiv \inf\left\{ t, Z_0 \prod_{s=1}^{t} (1 + \rho_s) = M \right\}$$

satisfies $\tau(\sigma_T) < T$. In a fixed costs setting we know by Theorem 1 that the model is arbitrage free if and only if there exists for each date t and each date-t node σ_t a nonnegative state price density on $f(\sigma_t)$ compatible with the asset prices.

Let us study what happens between any node and its successors; according to the model adopted for $(Z_t)_{t\in\{0,\cdots,T\}}$, there are in fact two different possible situations at any node σ_t in Σ_t:

- If $Z(\sigma_t)$ is strictly smaller than M, then there is no arbitrage opportunity in the frictionless model and there exists a unique positive state price density on $f(\sigma_t)$ compatible with the asset prices. So, according to our previous remarks, there is no arbitrage opportunity in the model with fixed costs and the state price deflator is the same as in the frictionless case.
- If $Z(\sigma_t)$ is equal to M, then $Z(\sigma_{t+1})$ can take two values for $\sigma_{t+1} \in f(\sigma_t)$ which are M in what we call state up and $M(1+\delta_d)$ in what we call state down. There exists no positive state price density compatible with the asset prices but there exists a unique nonnegative one. So the model is arbitrage free in a context with fixed costs.

We now turn to pricing issues. As we have seen, there is a unique nonnegative state price deflator compatible with (Z_t) or equivalently a unique probability measure P_M^* with nonnnegative weights which makes (Z_t) a martingale. This probability is equal to the above mentioned probability measure (with positive weights) P^* on the paths for which the bound M is not reached and puts a zero weight on the paths for which the bound is reached and left, i.e. P_M^* is such that,

- for all date$-T$ node σ_T such that $\tau(\sigma_T) \geq T$, $P_M^*(\sigma_T) = P^*(\sigma_T)$,
- for all date$-T$ node $\sigma_T = (\rho_1, \cdots, \rho_T)$ such that $\tau(\sigma_T) = t < T$, then $P_M^*(\sigma_T) = P^*\left(\sigma_T^t\right)$ with $\sigma_T^t = (\rho_1, \cdots, \rho_t)$ if $\rho_{t+1} = \cdots = \rho_T = \delta_u$ and $P_M^*(\sigma_T) = 0$ if $\rho_s = \delta_d$ for some $t < s \leq T$.

If we let $\left(\tilde{W}_t\right)$ denote the primitive price process absorbed at the upper bound M, i.e.

$$\tilde{W}_t \equiv W_{t\wedge\tau} \quad \text{for all } t,$$

it appears that, for all t, the random variable Z_t has under P_M^* the same distribution as the random variable $\tilde{W}_t$ under P^* and we obtain that any admissible price for a European call $x = (Z_T - K)_+$ can be written as the sum of a fixed cost and of $\pi_M(x)$, where

$$\begin{aligned}\pi_M(x) &= E^{P^*_M}\left[(Z_T - K)_+\right] \\ &= E^{P^*}\left[\left(\tilde{W}_T - K\right)_+\right]. \end{aligned} \quad (2)$$

This last formula corresponds to the price of a barrier option, where the barrier is set at M, in the classical binomial model where the risk-neutral probability is given by P^*.

Proof of Proposition 8 As in CIR or in Longstaff (1992) we look for a state price deflator π compatible with the no-arbitrage condition in a fixed costs setting such that the risk premium is equal to λ for $r \neq 0$.

More precisely, we look for $\pi = (\pi_1, \pi_2, \pi_3) \in]0,1[^3$, such that under π, the drift is given by the initial drift minus λr (as in CIR), and the variance remains unchanged (and given by $\sigma^2 r \frac{T}{N}$). If $|\lambda| < \frac{a}{\sqrt{2}}, \sigma^2 \geq \lambda^2 \bar{r} \frac{T}{N}$, where $\bar{r} = \sup\left\{r_0, \left(\frac{\sqrt{2}\sigma + \sqrt{2\sigma^2 + 4a^2 l h^2}}{2ah}\right)^2\right\}$, the solution is given by

$$\begin{aligned}\pi_1 &= \frac{1}{2}\left[\frac{\sigma^2 + \frac{\lambda^2 T r}{N}}{2\sigma^2} - \frac{\lambda\sqrt{r}}{\sigma\sqrt{2}}\sqrt{\frac{T}{N}}\right] \\ \pi_3 &= \frac{1}{2}\left[\frac{\sigma^2 + \frac{\lambda^2 T r}{N}}{2\sigma^2} + \frac{\lambda\sqrt{r}}{\sigma\sqrt{2}}\sqrt{\frac{T}{N}}\right] \\ \pi_2 &= 1 - \pi_1 - \pi_3.\end{aligned}$$

Furthermore, we want our model to have an absorbing barrier under the ris-bneutral probability in order to generate the same prices as in Longstaff (1992). It suffices then to take, for $r = 0$, $(\pi_1, \pi_2, \pi_3) = (0, 0, 1)$.

Letting $\nu(r; n, n')$ be given by

$$\nu\left(r; n, n'\right) = E^\pi\left[\frac{1}{\prod_{s=n}^{n'-1}(1 + r_s)} \mid r\right],$$

we obtain an arbitrage free (with fixed costs) term structure which satisfies $\nu(0; n, n') = 1$ (as in Longstaff (1992)) and the market price of interest-rate risk is λ (as in CIR and in Longstaff (1992)).

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