Consensus consumer and intertemporal asset pricing with heterogeneous beliefs*

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Abstract

The aim of the paper is to analyze the impact of heterogeneous beliefs in an otherwise standard competitive complete market economy. The construction of a consensus belief, as well as a consensus consumer, are shown to be valid modulo a finite variation aggregation bias, which takes the form of a discount factor. We then use our construction to rewrite in a simple way the equilibrium characteristics (state price density, market price of risk, risk premium, risk-free rate) in a heterogeneous beliefs framework and to analyze the impact of belief heterogeneity.

We prove that in many cases, the impact of belief heterogeneity on the market price of risk can be easily estimated, with a relatively good precision, by considering the wealth-weighted average belief. The introduction of a discount factor in the aggregation procedure appears to be related to the interpretation of the heterogeneity of beliefs as a source of risk (see Cragg and Malkiel, 1982). However, our results permits us to explain why assets with higher belief dispersion have lower risk premia (Diether et al., 2002). Finally, we show that it is possible to construct specific parametrizations of the heterogeneous beliefs model that lead to globally higher risk premia, lower risk-free rates, and risk premia that are lower for assets with higher belief dispersion.

1. Introduction

The representative agent approach introduced by Negishi (1960) and developed by Rubinstein (1974), Breeden and Litzenberger (1978) and Constantinides (1982) has become a significant cornerstone of theoretical and applied macroeconomics and has been the basis for many developments in finance. Among these developments, the Capital Asset Pricing Model (CAPM, Sharpe, 1964 and Lintner, 1965) and the Consumption based CAPM (CCAPM, Ingersoll, 1987,

*We wish to thank Jean-Michel Grandmont, Hedi Kallal and José Scheinkman for useful comments on earlier drafts. We have benefited from comments from seminar and conference participants at the Institut Henri Poincaré (Bachelier Seminar), London Business School, Princeton (Civitas Foundation Finance Seminar), MIT, French Finance Association Conference (Lyon), Blaise Pascal Conference on Financial Modelling (Paris), Universita della Svizzera Italiana (Lugano) and Bocconi University (Milan) and in particular Harjoat Bhamra, Markus Brunnermeier and Dimitri Vayanos.

Huang and Litzenberger, 1988, Duffie, 1996) play an important role. Given their empirical tractability, these models have generated extensive empirical tests and subsequent theoretical extensions. However, as mentioned by Williams (1977), difficulties remain, significant among which is the restrictive assumption of homogeneous expectations". It has since been repeatedly argued that the diversity of investors' forecasts is an important part of any proper understanding of the workings of asset markets.[1]

The aim of the present paper is to analyze the consequences of the introduction of heterogeneous subjective beliefs in an otherwise standard Arrow-Debreu equilibrium economy. More precisely, we start from a given equilibrium with heterogeneous beliefs in an otherwise standard complete market model and, as in Calvet et al. (2002) in a static setting, we investigate the following issues: 1) Is it possible to define a consensus belief, i.e. a belief which, if held by all individuals, would generate the same equilibrium prices and trade volumes as in the actual heterogeneous economy? 2) Is it still possible in such a context to define a representative agent (or consensus consumer)? 3) What is the impact of belief heterogeneity on the risk premium (or market price of risk)? 4) What is the impact of belief heterogeneity on the risk-free rate? 5) What is the impact of belief heterogeneity on the assets price?

In this paper, we take the different subjective beliefs as given. As in Varian (1985, 1989), Abel (1989) or Harris and Raviv (1993), they reflect difference of opinion among the agents rather than difference of information; indeed, "we assume that investors receive common information, but differ in the way they interpret this information" (Harris and Raviv, 1993). The different subjective beliefs might come from a Bayesian updating of the investors predictive distribution over the uncertain returns on risky securities as in, e.g. Williams (1977), Detemple and Murthy (1994), Zapatero (1998), Gallmeyer (2000), Basak (2000), Gallmeyer and Hollifield (2002), but we do not make such an assumption; we only impose that the subjective probabilities be equivalent to the initial one. Notice that the above-mentioned models with learning are not "more endogenous", since the investors' updating rule and the corresponding probabilities can be determined separately from his/her optimization problem (see e.g. Genotte, 1986).

Since we want to focus on the impact of the presence of heterogeneous beliefs per se, we consider these issues in the context of an otherwise standard model and in particular we do not suppose that there are short sale constraints (except in the additional remarks).

The paper is organized as follows. We present in Section 2 a method to aggregate, in an intertemporal framework, heterogeneous individual subjective beliefs into a single consensus belief. Given an observed equilibrium with heterogeneous probabilities, we look for a consensus belief, which, if held by all investors would lead to an equivalent equilibrium, in the sense that it would leave invariant the equilibrium market prices. We prove the existence of such a consensus belief modulo the introduction of a discount factor. This discount factor might be positive or negative depending on whether the investor is cautious or not. A possible interpretation of this result consists in considering the dispersion of beliefs as a source of risk. When there is more risk involved, depending on whether the investor is cautious or not, it is well known that the investor will reduce or increase current consumption with respect to future consumption, acting as if his/her utility was discounted by a positive or negative discount rate.

[1] See e.g. Lintner (1969), Rubinstein (1975, 1976), Gonedes (1976), Miller (1977), Williams (1977), Jarrow (1980), Mayshar (1981, 1983), Cragg and Malkiel (1982), Varian (1985, 1989), Abel (1989), Harris and Raviv (1993), Detemple and Murthy (1994), Basak (2000), Gallmeyer (2000), Welch (2000), Ziegler (2000), Gallmeyer and Hollifield (2002), Calvet et al. (2002), Diether et al. (2002).

As a consequence of the presence of this discount effect, the heterogeneous beliefs setting cannot in general (in fact, except in the logarithmic case) be simply reduced to a homogeneous beliefs setting, with a (possibly weighted) average belief. Once the consensus belief has been obtained, we construct, as in the standard setting, a consensus consumer, i.e. a consumer, endowed with the market portfolio and the consensus belief, who generates the same equilibrium prices as in the original equilibrium.

We point out two essential effects of the introduction of heterogeneity in the investors' beliefs on the equilibrium (state) price (density). There is first a change of probability effect, the consensus probability being a weighted average of the individual subjective probabilities and second, a (possibly negative) discount effect. We end Section 2 by showing for the class of linear risk-tolerance utility functions how the consensus probability and the discount process can be obtained and we characterize the situations where the discount rate is positive (resp. negative). For this class of utility functions (which includes logarithmic, exponential as well as power utility functions), it clearly appears that the consensus probability is directly related to a weighted average of the individual beliefs and that the discount process is directly related to the dispersion of the individual beliefs. For that class of utility functions and for small belief dispersion, the discount effect is small with respect to the change of probability effect.

In Section 3, we analyze the impact of these two features of the equilibrium under heterogeneous beliefs on the market price of risk and on the risk-free rate. In particular, we derive an adjusted CCAPM formula. We prove that only the change of probability has an impact on the market price of risk. We find that the CCAPM formula under heterogeneous beliefs is given by the CCAPM formula in an economy where all investors would share the same probability belief, namely the consensus belief obtained through the aggregation procedure. The impact of the introduction of heterogeneous beliefs on the market price of risk is then very clear: it leads to an increase (resp. decrease) of the market price of risk of a given asset (with respect to the homogeneous setting) if and only if the consensus probability is pessimistic (resp. optimistic), where pessimistic is meant in the sense that the instantaneous rate of return for that asset (under this probability) is lower than under the objective initial probability. In fact, the equity premium subjectively expected is the same for pessimistic and optimistic agents and the reason that pessimism increases the objective expectation of the equity premium is not that a pessimistic representative agent requires a higher risk premium. He/She requires the same equity premium but his/her pessimism leads him/her to underestimate the average rate of return of equity (leaving unchanged his/her estimation of therisk-free rate). Thus the objective expectation of the equilibrium premium is greater than the representative agent's subjective expectation, hence is greater than the standard equity premium. Our results are consistent with those of Abel (2000), Cecchetti et al. (2000), Hansen et al. (1999), or Anderson et al. (2000), which introduce distorted beliefs associated with cautious/pessimistic individual behavior, but the main difference lies in the fact that in our framework, the optimism/pessimism is relevant at the aggregate level and not at the individual one. In particular, it is possible to have optimism/pessimism at the aggregate level, even in models where the average (equal-weighted) belief is neutral (neither optimistic, nor pessimistic). We provide in Section 4 conditions on the individual subjective probabilities that lead to a pessimistic (resp. optimistic) consensus belief.

Conversely to the market price of risk, both the change of probability and the discount factor have an impact on the risk-free rate. The impact of the change of probability contributes to a lowering of the risk-free rate if and only if the consensus probability is pessimistic. Indeed, if the representative agent is pessimistic about the growth rate of aggregate wealth, then relative to the standard case, he/she will attempt to reduce current consumption and increase current savings. The attempt to increase current savings then puts downward pressure on the interest rate. Besides, the impact of the discount factor contributes to an increase (resp. decrease) of the risk-free rate when the "discount rate" is nonnegative, which has a clear interpretation: a nonnegative "discount rate" means that future consumption is less important for the representative agent, and leads to a higher equilibrium interest rate. For linear risk-tolerance utility functions and for a small belief dispersion, the first effect is quantitatively much more important than the second one.

In Section 4 we show, in the setting with two agents and constant parameters, that for exponential utility functions, the consensus belief (resp. the discount factor) is a risk-tolerance-weighted average (resp. variance) of the individual beliefs, and that for power utility functions, the consensus belief (resp. the discount factor) is, in expected value, approximated by the wealth-weighted average (resp. variance) of the individual beliefs. The analysis of the impact of belief heterogeneity on the risk premium and on the risk-free rate is then very simple. It leads to a lower (resp. higher) expected risk premium if the wealth-weighted average (or the risk-tolerance-weighted average in the exponential case) of the individual beliefs is optimistic (resp. pessimistic). The impact is therefore directly linked to the correlation between risk aversion/tolerance and optimism/pessimism in the exponential case, and between wealth and optimism/pessimism in the power case. There is for instance a bias towards optimism and a lower expected risk premium if we suppose that there is a positive correlation between wealth and optimism, which seems to be supported by empirical studies. For instance, the index of consumer sentiment published by the University of Michigan is systematically higher for families with income above \$50 000 than for families with income under \$50 000. Even though short sale constraints are not part of our model, as in, e.g. Miller (1977), we find that prices reflect the more optimistic view. This means that assets with a higher belief dispersion should yield lower returns. This result is consistent with the findings of Diether et al. (2002), who "provide evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks".

The following example illustrates this result. The volatility of aggregate wealth is taken to be equal to 5%, its drift is equal to 3%, and we assume that the optimists' (resp. pessimists') belief for the drift is equal to 5% (resp. 1%). Let us suppose that the optimists' wealth is three times larger than that of the pessimists. We show that the risk premium is then 3% lower than in the standard setting for an asset whose volatility is equal to 15%.

Moreover, we show that if investors are on average pessimistic, then the assets with a higher belief dispersion still yield lower returns, the risk premium is higher than in the standard setting, and the risk-free rate is lower than in the standard setting, which is interesting in light of the equity premium and the risk-free rate puzzles (Mehra and Prescott, 1985, and Weil, 1989).

Section 5 is devoted to some additional and concluding remarks.

All the proofs are in the Appendix.

2. Consensus belief, consensus consumer

In the classical representative agent approach, all investors are taken as having the same subjective beliefs, the same utility functions and the same opportunity sets. In this section, we analyze to which extent this approach can be extended to heterogeneous subjective beliefs. More precisely, we start from a given equilibrium with heterogeneous beliefs in an otherwise standard complete market model, and we explore to what extent it is possible 1) to define a consensus belief, i.e. a belief, which, if held by all individuals would generate the same equilibrium prices and trading volumes as in the actual heterogeneous economy and 2) to define a representative agent (or a consensus consumer). The starting point of our aggregation procedure is the paper by Calvet et al. (2002).

The model is standard, except that we allow the agents to have distinct subjective probabilities. We fix a finite time horizon T on which we are going to treat our problem. We consider a filtrated probability space $\left(\Omega, (F_t)_{t\in[0,T]}, P\right)$, where the filtration $(F_t)_{t\in[0,T]}$ satisfies the usual conditions. Each investor, indexed by $i = 1, ..., N$, solves a standard dynamic utility maximization problem. He/She has a current income at date t denoted by $e_t^{*^i}$ and a von Neumann-Morgenstern utility function for consumption of the form $E^{Q^i}\left[\int_0^T u_i(t, c_t)dt\right]$, where Q^i is a probability measure equivalent to P which corresponds to the subjective belief of individual i. If we denote by $\left(M_t^i\right)_{t\in[0,T]}$ the positive density process of Q^i with respect to P, then the utility function can be rewritten as $E^P\left[\int_0^T M_t^i u_i(t, c_t)dt\right]$.

We make the following classical assumptions.

Assumption

- For all $t \in \mathbf{T}$, $u_i(t, \cdot) : [k_i, \infty) \to \mathbb{R} \cup \{-\infty\}$ is of class C^1 on (k_i, ∞), strictly increasing and strictly concave,[2]
- $u_i(., c)$ and $u_i'(t, .) = \frac{\partial u_i}{\partial c}(t, .)$ are continuous on $[0, T]$,
- for $i = 1, ..., N$, $P \otimes dt\left\{e^i > k_i\right\} > 0$ and $P \otimes dt\left\{e^i \geq k_i\right\} = 1$,
- there exists $\varepsilon > 0$ such that $e^* > \sum_{i=1}^N k_i + \varepsilon$, $P \otimes dt$ a.s.,
- $E^P\left[\int_0^T e_t^* dt\right] < \infty$,
- the density process M^i is uniformly bounded for $i = 1, ..., N$.

The first two conditions are classical regularity conditions. If we interpret k_i as a minimum subsistence level, the third condition can be interpreted as a survival condition for each agent.[3] The fourth condition is a survival condition for the whole economy. The last two conditions are technical ones and are directly linked to the choice of L^1 as a consumption space for the agents.[4]

We do not specify the utility functions u_i, although we shall focus on the classical cases of linear risk-tolerance utility functions (which include logarithmic, power as well as exponential

[2] Note that we could easily generalize all the following results to the case where k_i is a function of t.

[3] In fact, this condition can be weakened. For instance, if the equilibrium price is known, we only need to impose that $E^P\left[\int_0^T q_t^* e_t^i dt\right] > E^P\left[\int_0^T q_t^* k_i dt\right]$, which permits us, through trade, to reach allocations $\left(y^{*^i}\right) > k_i$ $P \otimes dt$ a.s.

[4] This pair of conditions can easily be replaced by $E^P\left[\int_0^T \left|e_t^i\right|^p dt\right] < \infty$ and $E^P\left[\int_0^T \left|M_t^i\right|^q dt\right] < \infty$ where p and q are such that $\frac{1}{p} + \frac{1}{q} = 1$.

utility functions). We take the different subjective probabilities as given. As in Varian (1985, 1989), Abel (1989) or Harris and Raviv (1993), they reflect difference of opinion among the agents rather than difference of information; indeed, "we assume that investors receive common information, but differ in the way they interpret this information" (Harris and Raviv, 1993). They might come from a Bayesian updating of the investors' predictive distribution over the uncertain returns on risky securities as in, e.g. Williams (1977), Detemple and Murthy (1994), Zapatero (1998), Gallmeyer (2000), Basak (2000) and Gallmeyer and Hollifield (2002), but we do not make such an assumption; we only impose that the subjective probabilities be equivalent to the initial one. This mainly leads to assuming that investors differ only in their opinion about means and agree about variances. This hypothesis is reasonable, since as underlined by Gallmeyer and Hollifield (2002), "aggregate consumption is observed continuously, [and] all investors can perfectly estimate its volatility by computing the output process quadratic variation". Notice that the above-mentioned models with learning are not "more endogenous" since the investors' updating rule and the corresponding probabilities can be determined separately from his/her optimization problem (see e.g. Genotte, 1986).

In the remainder of the paper, an admissible consumption plan for agent i is an adapted $[k_i, \infty)$-valued process y^i such that $E^P\left[\int_0^T \left|y_t^i\right| dt\right] < \infty$. We recall that an equilibrium relative to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$ is defined by a positive, uniformly bounded price process q^* and a family of optimal admissible consumption plans $\left(y^{*^i}\right)$ such that markets clear, i.e.

$$\begin{cases} y^{*^i} = y^i(q^*, M^i, e^i) \\ \sum_{i=1}^N y^{*^i} = \sum_{i=1}^N e^i \equiv e^* \end{cases}$$

where

$$y^i(q, M, e) \equiv \arg \max_{E^P\left[\int_0^T q_t\left(y_t^i - e_t\right)dt\right] \leq 0} E^P\left[\int_0^T M_t u_i(t, c_t) dt\right].$$

We start from an equilibrium $\left(q^*, \left(y^{*^i}\right)\right)$ relative to the beliefs $\left(M^i\right)$ and the income processes e^i. Such an equilibrium, when it exists, can be characterized by the first-order necessary conditions for individual optimality and the market clearing condition. These conditions can be written as follows:

$$\begin{cases} M_t^i u_i'(t, y_t^{*^i}) \leq \lambda_i q_t^*, & \text{on } \left\{y^{*^i} = k_i\right\} \\ M_t^i u_i'(t, y_t^{*^i}) = \lambda_i q_t^*, & \text{on } \left\{y^{*^i} > k_i\right\} \\ E^P\left[\int_0^T q_t^* \left(y_t^{*^i} - e_t^i\right) dt\right] = 0 & \\ \sum_{i=1}^N y^{*^i} = e^* & \end{cases} \tag{2.1}$$

for some set of positive Lagrange multipliers (λ_i).

In the sequel, we will say that $(q^*, (y^{*^i}))$ is an interior equilibrium relative to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$ if $y^{*^i} > k_i$, $P \otimes dt$ a.s. for $i = 1, ..., N$. Note that under the following additional condition:

$$u_i'(t, k_i) = \infty \text{ for } t \in [0, T] \text{ and } i = 1, ..., N,$$

all the equilibria are interior ones.

Our first aim is to find an "equivalent equilibrium" in which the heterogeneous subjective beliefs would be aggregated into a common characteristic M. Following the approach of Calvet et al. (2002), we shall define an "equivalent equilibrium of the first kind" by two requirements. First, the "equivalent equilibrium" should generate the same equilibrium trading volumes $\left(y^{*^i} - e^i\right)$ and price process q^* as in the original equilibrium with heterogeneous beliefs. Second, every investor should be indifferent at the margin between investing one additional unit of income in the original equilibrium with heterogeneous beliefs and in the "equivalent equilibrium", so that each asset gets the same marginal valuation by each investor (in terms of his/her marginal utility) in both equilibria.[5] The existence of such an "equivalent equilibrium of the first kind" is given by the following proposition.

Proposition 2.1. *Consider an interior equilibrium* $(q^*, (y^{*^i}))$ *relative to the beliefs* $\left(M^i\right)$ *and the income processes* $\left(e^i\right)$*. There exists a unique positive and adapted process* $(M_t)_{t\in[0,T]}$ *with* $M_0 = 1$*, there exists a unique family of income processes* $\left(\bar{e}^i\right)$ *with* $\sum_{i=1}^N \bar{e}^i = e^*$ *and a unique family of individual consumption processes* $(\bar{y}^i)$ *such that* $\left(q^*, (\bar{y}^i)\right)$ *is an equilibrium relative to the common characteristic* M *and the income processes* $\left(\bar{e}^i\right)$ *and such that trading volumes and individual marginal valuation remain the same, i.e.*

$$\begin{aligned} y^{*^i} - e^i &= \bar{y}^i - \bar{e}^i \qquad i = 1, ..., N \\ M_t^i u_i'(t, y^{*^i}) &= M_t u_i'(t, \bar{y}^i), \qquad t \in [0, T], \, i = 1, ..., N. \end{aligned}$$

This means that $\left(q^*, (\bar{y}^i)\right)$ is an equilibrium with income transfers relative to the common characteristic M and the income processes $\left(e^i\right)$ such that individual marginal valuation is the same as in the original equilibrium with heterogeneous beliefs. In other words, we proved that modulo a feasible modification of the individual incomes (i.e. $\sum_{i=1}^N \bar{e}^i = \sum_{i=1}^N e^i$) the initial equilibrium price process and trading volumes remain equilibrium price process and trading volumes in a homogeneous beliefs setting. The positive process M can then be interpreted as a consensus characteristic. In particular, if there is no heterogeneity, i.e. if all the investors have the same belief represented by $M^i = \tilde{M}$ for all i, we obtain $M = \tilde{M}$ and there is no transfer nor optimal allocation modification (i.e. $\bar{e}^i = e^i$ and $\bar{y}^i = y^{*^i}$ for all i).

As we said, the consensus characteristic that we obtained is such that the associated equilibrium price as well as the individual marginal valuation remain the same as in the heterogeneous framework. The individual marginal valuation invariance property is in fact equivalent to the invariance of the Lagrange multipliers. In other words, the "equivalent equilibrium of the first kind" is characterized by

$$\left\{ \begin{array}{c} M_t u_i'(t, \bar{y}_t^i) = \lambda_i q_t^*, \qquad t \in [0, T] \\ E^P \left[\int_0^T q_t^* \left(\bar{y}_t^i - \bar{e}_t^i \right) dt \right] = 0 \\ \sum_{i=1}^N \bar{y}_t^i = e_t^* \end{array} \right.$$

[5] This requirement is in fact equivalent to the condition that each investor's observed (or initial) demand be larger than (resp. equal to, less than) his/her demand in the "equivalent equilibrium" if and only if he/she attaches a subjective probability that is larger than (resp. equal to, less than) the aggregate common probability, which appears as a natural requirement for an aggregation procedure.

where the multipliers λ_i are the same as in the characterization of the initial equilibrium (Equation (2.1)). The cost we paid in order to maintain the Lagrange multipliers invariant has been to authorize income transfers between agents. Another way to construct an "equivalent equilibrium" and hence a consensus belief would be to prohibit transfers and to allow for individual marginal valuation modification. More precisely we have the following result, which permits us to define the concept of "equivalent equilibrium of the second kind".

Proposition 2.2. *Consider an interior equilibrium $(q^*, (y^{*^i}))$ relative to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$. There exist a positive and adapted process $(M_t)_{t\in[0,T]}$ with $M_0 = 1$ and a family of individual consumption processes $(\bar{y}^i)$, such that $\left(q^*, (\bar{y}^i)\right)$ is an interior equilibrium relative to the common characteristic M and the income processes $\left(e^i\right)$.*

We shall denote by $\left(\lambda_i'\right)$ the corresponding Lagrange multipliers. For both constructions, once the result on belief aggregation has been achieved, it is easy to construct, as in the standard case, a representative agent, i.e. an expected utility maximizing aggregate investor, representing the economy in equilibrium. More precisely, we look for a single aggregate investor, endowed with the market portfolio, who, when maximizing his expected utility under the aggregate characteristic, generates the same equilibrium prices as in the original equilibrium. The next proposition establishes the existence of such a representative agent.

As in the standard case, for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, we introduce the function

$$u_\alpha(t,x) = \max_{\sum_{i=1}^N x_i \le x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t, x_i).$$

Corollary 2.3. *Consider an interior equilibrium $(q^*, (y^{*^i}))$ relative to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$. There exists a consensus investor defined by the normalized von Neumann-Morgenstern utility function u_λ (resp. $u_{\lambda'}$) and the consensus characteristic M of Proposition 2.1 (resp. Proposition 2.2), in the sense that the portfolio e^* maximizes his/her expected utility $E^P\left[\int_0^T M_t u(t, c_t) dt\right]$ under the market budget constraint $E^P\left[\int_0^T q_t^* \left(c_t - e_t^*\right) dt\right] \le 0$.*

The construction of the representative agent is exactly the same as in the standard setting. As a consequence, all classical properties of the representative agent utility function remain valid in our setting (see e.g. Huang and Litzenberger, 1988). Among other properties, if all individual utility functions are state independent, then the aggregate utility function is also state independent,[6] and if all individual utility functions exhibit linear risk-tolerance, i.e. are such that $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$, then the aggregate utility function is also such that $-\frac{u'(t,x)}{u''(t,x)} = \bar{\theta} + \eta x$ where $\bar{\theta} = \sum_{i=1}^N \theta_i$.

[6] Note that our aggregation procedure applies to a framework where agents have common beliefs but possibly different state-dependent utility functions of the following "separable" form:

$$U_i(t,\omega,x) = v_i(t,\omega)\, u_i(t,x).$$

In that case (note that even if the v_i's are not martingales, our results still apply), we obtain a representative agent utility function of the same form $U(t,\omega,x) = v(t,\omega)\, u(t,x)$, where u is obtained from the u_i's as in the standard framework, and where v is an average of the v_i's (note that even if the v_i's are not martingales, our results still apply).

Example 2.4. *1. If the individual utility functions are of exponential type, i.e. if* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$*, then* $u'(t,x) = ae^{-x/\bar{\theta}}$ *for* $a = e^{e_0^*/\bar{\theta}}$ *and*

$$M = \prod_{i=1}^{N} \left(M^i\right)^{\theta_i/\bar{\theta}} .$$

2. If the individual utility functions are of power type, i.e. such that $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ *for* $\eta \neq 0$*, then* $u'(e) = b(\bar{\theta} + \eta e)^{-\frac{1}{\eta}}$ *for* $b = (\bar{\theta} + \eta e_0)^{\frac{1}{\eta}}$ *and*

$$M = \left[\sum_{i=1}^{N} \gamma_i (M_i)^{\eta}\right]^{\frac{1}{\eta}}$$

for $\gamma_i = \frac{\lambda_i^{-\eta}}{\sum_{j=1}^{N} \lambda_j^{-\eta}}$ *(we have* $\sum_{i=1}^{N} \gamma_i = 1$*).*

The consensus characteristic M *is a martingale (i.e. the density process of a given probability) only when* $\eta = 1$ *(logarithmic case). It is a supermartingale when* $\eta < 1$*, and a submartingale when* $\eta > 1$.

Notice that it is not possible (except in the exponential case) to construct M and to obtain q^* as functions of the aggregate characteristic (e.g. consumption) of the economy and not of the individual ones. However, as we shall see in the next section, the formulation $q_t^* = M_t u'(t, e_t^*)$ will enable us to compare the equilibrium under heterogeneous beliefs with the equilibrium in the standard setting.

It is interesting to notice in Example 2.4 that for all utility functions in the classical class of linear risk-tolerance utility functions, the consensus characteristic is obtained as a weighted average of the individual subjective beliefs. It is easy to see that in the general case, the consensus characteristic can still be considered as an average of the individual beliefs.[7]

The process M represents a consensus characteristic; however, as seen above, except in the logarithmic case, it fails to be a martingale. Consequently, it cannot be interpreted as a belief, i.e. the density process of a given probability measure. It is easy to see that it is not possible in general to recover the consensus characteristic as a martingale, as soon as we want the

[7] Since for all i, we have

$$\frac{1}{\lambda_i} M_t^i u_i'(t, {y_t^*}^i) = M u'(t, e_t^*)$$

we obtain

$$\begin{aligned} e_t^* &= \sum_{i=1}^{N} I_{u_i}\left(t, \lambda_i \frac{M_t}{M_t^i} u'(t, e^*)\right) \\ &= \sum_{i=1}^{N} I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right) . \end{aligned}$$

It is clear then that we cannot have $M_t > M_t^i$ (resp. $M_t < M_t^i$), for all i, with a positive probability. Indeed, this would lead to $\sum_{i=1}^{N} I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right) < \sum_{i=1}^{N} \bar{y}_t^i = e_t^*$ (resp. $\sum_{i=1}^{N} I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right) > \sum_{i=1}^{N} \bar{y}_t^i = e_t^*$) with a positive probability which contradicts the equations above.

equilibrium price to remain the same and the optimal allocations in the equivalent equilibrium to be feasible, in the sense that they still add up to e^* (even if we do not impose the invariance of individual marginal valuation).[8]

This means that in the general case, there is a bias induced by the aggregation of the individual probabilities into a consensus probability. We shall see in the following propositions that the utility function of the representative agent is not an expectation of the future utility from consumption but an expectation of a discounted future utility from consumption. The (possibly negative) discount rate will be on average nonnegative (resp. zero, resp. nonpositive) when M is a supermartingale (resp. martingale, resp. submartingale).

In order to further specify our model, let us assume that $(F_t)_{t\in[0,T]}$ is the $P-$augmentation of the natural filtration generated by a one-dimensional Brownian motion W on (Ω, F, P), and that e^* and the M^i's satisfy the following stochastic differential equations:[9]

$$\begin{aligned} de_t^* &= \alpha_t e_t^* dt + \beta_t e_t^* dW_t, \qquad \beta > 0 \\ dM_t^i &= \delta_t^i M_t^i dW_t, \qquad M_0^i = 1 \end{aligned}$$

If we further assume that the utility functions are of class $C^{1,3}$, the first-order conditions for an interior equilibrium and Itô's Lemma give us that the equilibrium price q^*, the equilibrium allocations $\left(y^{*^i}\right)$ as well as M also satisfy stochastic differential equations of the form

$$\begin{aligned} dq_t^* &= \mu_{q^*}(t) q_t^* dt + \sigma_{q^*}(t) q_t^* dW_t \\ dy_t^{*^i} &= \mu_{y^*}^i(t) y_t^{*^i} dt + \sigma_{y^*}^i(t) y_t^{*^i} dW_t \\ dM_t &= \mu_M(t) M_t dt + \delta_M(t) M_t dW_t \end{aligned}$$

In the sequel, we will assume that δ_M satisfies the so-called Novikov condition,[10] i.e.

$$E\left[\exp\left(\frac{1}{2}\int_0^T \delta_M^2(t)\,dt\right)\right] < \infty.$$

[8] Let us consider again the exponential utility functions case $-\frac{u_i'(t,x)}{u_i''(t,x)} = 1$. Let us consider the "equivalent interior equilibrium" allocations $(\bar{y}_t^i)$ associated with the same equilibrium price q^* as in the initial equilibrium and with a common characteristic M. The first-order conditions then lead to

$$M_t u_i'(t, \bar{y}_t^i) = \varsigma_i M_t^i u_i'(t, y_t^{*^i})$$

where M is the candidate consensus characteristics and where the ς_i's are given positive multipliers. This leads to

$$M_t = \left(\prod_{i=1}^N \varsigma_i\right)^{1/N} \left(\prod_{i=1}^N M_t^i\right)^{1/N}$$

and M is a martingale only if all the M^i's are equal.

[9] We assume that the coefficients of these SDEs satisfy the classical global Lipschitz and linear growth conditions that ensure existence and uniqueness of a strong solution (see Karatzas and Shreve (1988) for more details and weaker conditions).

[10] This condition ensures that $\int \delta_M dW$ is a martingale. For linear risk tolerance utility functions, it is easy to check that this condition is in particular satisfied if, for $i = 1, ..., N$, we have $E\left[\exp\left(\frac{1}{2}\sum_{i=1}^N \int \delta_i^2 dW\right)\right] < \infty$.

Proposition 2.5. *Consider an interior equilibrium price process q^* relative to the beliefs (M^i), and the income processes (e^i). There exist a positive martingale process $\bar{M}$ with $\bar{M}_0 = 1$, and a finite variation positive process B with $B_0 = 1$ such that, with the notations of the previous propositions*

$$\bar{M}_t B_t u'(t, e_t^*) = q_t^*.$$

These processes are given by

$$\begin{aligned} B_t &= \exp \int_0^t \mu_M(s) ds \\ \bar{M}_t &= \exp \left(\int_0^t \delta_M(s) dW_s - \frac{1}{2} \int_0^t \delta_M^2(s) ds \right). \end{aligned}$$

The process B measures the default of martingality of the consensus characteristic M and leads to a (possibly negative) discount of utility from future consumption through the "discount rate" $(-\mu_M)$. The adjustment process B then measures the aggregation bias induced by the heterogeneity of individual beliefs.

There are mainly two cases where there is no such adjustment effect, i.e. B is constant and equal to 1:

- If all investors share the same belief. The consensus belief $\bar{M}$ is then equal to that common belief.
- If all the utility functions are logarithmic. This property makes the case of logarithmic utility functions (which is often considered in the literature, see e.g. Rubinstein, 1976, Detemple and Murthy, 1994, Zapatero, 1998) very specific.

When B fails to be constant, it is a natural concern to determine whether it is greater or smaller than 1, increasing or decreasing. This will permit to analyze the nature of this aggregation bias and its impact on the equilibrium state price density. For linear risk-tolerance utility functions, the processes $\bar{M}$ and B can be computed. We shall denote the risk-tolerance by $T_i(t) \equiv \frac{u_i'(t, y_t^{*i})}{u_i''(t, y_t^{*i})}$.

Proposition 2.6. *For linear risk-tolerance utility functions, we have*

$$\delta_M = \sum_{i=1}^N \kappa^i \delta^i = E^\kappa [\delta]$$

and

$$\mu_M = \frac{1}{2}(\eta - 1) \left[\sum_{i=1}^N \kappa^i \left(\delta^i\right)^2 - \left(\sum_{i=1}^N \kappa^i \delta^i \right)^2 \right] = \frac{1}{2}(\eta - 1) Var^\kappa [\delta]$$

where $\kappa^i = \frac{T_i}{\sum_{i=1}^N T_i}$ satisfy $\sum_{i=1}^N \kappa^i = 1$ and E^κ (resp. Var^κ) denote the expected value (resp. variance) of δ across agents with a weight κ^i for agent i.

Notice that $\mu_M \leq 0$ if and only if $\eta \leq 1$ (this encompasses the exponential case).

For this class of utility functions, δ_M is a risk-tolerance-weighted average of the individual δ^i's and μ_M is proportional to the variance of the δ^i's with respect to the same weights. It appears then that the equilibrium price can be represented as an equilibrium price in an equivalent economy where the individual beliefs are replaced by a risk-tolerance-weighted average belief and where an additional effect is introduced in order to take the initial heterogeneity into account. This effect is measured by B or equivalently by μ_M, which is directly related to the belief dispersion.

Furthermore, it is easy to see that B is nondecreasing, greater than 1 (resp. nonincreasing, lower than 1) if $\eta \geq 1$ (resp. $\eta \leq 1$). The technical reason for this result is the following. Depending on whether $\alpha < 1$ or $\alpha > 1$, the function x^α is convex or concave. Hence the supermartingale result for $\eta < 1$, the submartingale result for $\eta > 1$ and the martingale result for $\eta = 1$. A possible interpretation could be the following. The parameter η is a cautiousness parameter. When there is more risk involved, depending on whether the investor is cautious or not, that is to say, depending on whether $\eta < 1$ or $\eta > 1$, it can be shown that the investor will reduce or increase current consumption with respect to future consumption. For instance, for $\eta < 1$, the investor is cautious and increases current consumption acting as if his/her utility was discounted by a positive discount rate. The converse reasoning leads to a negative discount rate if $\eta > 1$. Now in our context with heterogeneous beliefs, a possible interpretation consists in considering the dispersion of beliefs as a source of risk, thereby leading for the representative agent to a discount factor associated with a positive or negative discount rate depending on whether $\eta < 1$ or $\eta > 1$.

To summarize, we have pointed out through previous propositions two distinct effects of the introduction of some belief heterogeneity on the equilibrium price.

There is first a change of probability effect from P to the new common probability Q, whose density is given by $\bar{M}$. This aggregate probability Q can be seen (at least in the classical utility functions cases) as a weighted average of the individual subjective probabilities. The weights of this average are given by the individual risk-tolerances exactly as in Rubinstein (1976) or Detemple and Murthy (1994), where the authors focused on logarithmic utility functions.

The second effect is represented by an "aggregation bias" of the market portfolio or of the equilibrium (state) price (density), which is of finite variation and takes the form of a discount factor. We are able, for linear risk-tolerance utility functions, to determine if it is associated with a positive or negative discount rate. Moreover, the adjustment process can be seen (at least in classical cases) as a measure of dispersion of individual beliefs.

We shall now analyze the impact of these two features on the equilibrium properties.

3. Asset pricing with heterogeneous beliefs

In this section, we use our construction of a representative consumer (Section 2) to study the impact of heterogeneity of beliefs on asset pricing. We first explore the impact on the equilibrium (state) price (density). We then turn to the impact on the CCAPM formula (or more precisely on the market price of risk (MPR)) and on therisk-free rate.

We have obtained in the setting with heterogeneous beliefs the following expression for the equilibrium (state) price (density) $q^* = Mu'(e^*) = \overline{M}Bu'(e^*)$ which we want to compare with the expression obtained in the standard setting, which is given by $q = u'(e^*)$. We consider as the standard setting an equilibrium under homogeneous beliefs (given by the objective probability

P), for which the representative agent utility function is given by[11] u_λ with the same (λ_i) as in our heterogeneous beliefs setting. This is in particular the case when the standard setting equilibrium has the same Lagrange multipliers (or equivalently the same marginal valuations) as in our framework, or when investors have linear risk-tolerance utility functions, since in that case (see e.g. Huang and Litzenberger, 1988), the representative agent utility function does not depend upon the individual Lagrange multipliers (or initial allocations).

We recall that we denote by μ_M the drift of the adjustment process B (or equivalently of the consensus characteristic M), which means that $-\mu_M$ is the "discount rate" and by δ_M the volatility of the consensus belief $\overline{M}$ (or equivalently of the consensus characteristic M). We easily obtain the following result.

Proposition 3.1. *The drift and volatility of the equilibrium state price density q^* with heterogeneous beliefs are given by*

$$\begin{cases} \mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \\ \sigma_{q^*} = \sigma_q + \delta_M \end{cases}$$

where μ_q and σ_q denote the drift and volatility of the equilibrium state price density in the standard setting.

Since the expressions of the market price of risk (MPR) and of therisk-free rate are directly related to the drift and volatility of the equilibrium state price density, we shall now analyze the impact of the introduction of belief heterogeneity on these two quantities.

3.1. Adjusted CCAPM and market price of risk

We suppose the existence of a riskless asset with price process S^0 such that $dS_t^0 = r_t^f S_t^0 dt$ and of a risky asset with "cum dividend" price process

$$dS_t = S_t \mu_R(t)\, dt + S_t \sigma_R(t)\, dW_t \qquad \sigma_R > 0.$$

Since q^* is a state price density, the price process S must be such that $q^* S$ is a P-martingale so that, as in the classical case (see, e.g. Duffie, 1996, or Huang and Litzenberger, 1988), we obtain

$$\mu_R - r^f = -\sigma_{q^*} \sigma_R. \tag{3.1}$$

Now, since $q_t^* = \overline{M}_t B_t u'(t, e_t^*)$, with B of finite variation, we have seen in Proposition 3.1 that

$$\sigma_{q^*} = \sigma_q + \delta_M. \tag{3.2}$$

We easily obtain the following expression for the market price of risk $\left(\frac{\mu_R - r^f}{\sigma_R}\right)$ in the heterogeneous beliefs setting.

[11] We recall that for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, $u_\alpha(t,x) \equiv \max_{\sum_{i=1}^N x_i \leq x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t, x_i)$.

Proposition 3.2. *The market price of risk with heterogeneous beliefs is given by*

$$\begin{aligned} MPR\,[heterogeneous] &= -\sigma_q - \delta_M && (3.3)\\ &= MPR\,[standard] - \delta_M \\ &= MPR\,[homogeneous\ under\ Q] && (3.4)\end{aligned}$$

The MPR with heterogeneous beliefs is greater (resp. lower) than in the standard setting if and only if $\delta_M \leq 0$ (resp. $\delta_M \geq 0$).

Equation (3.3) is the CCAPM formula with heterogeneous beliefs. This adjusted formula differs from the classical one only through the change of probability from P to the consensus probability Q and the MPR in the heterogeneous beliefs setting is given by the MPR in an economy where all investors would share the same belief Q. The adjustment process B plays no role.

The introduction of heterogeneity in the investors' beliefs leads to a higher (resp. lower) market price of risk if and only if $\delta_M \leq 0$ (resp. $\delta_M \geq 0$). A nonpositive (resp. nonnegative) δ_M corresponds to a pessimistic (resp. optimistic) consensus probability. Indeed, letting $W_t^Q = W_t - \int_0^t \delta_M(s)\,ds$, we know by Girsanov's theorem that W_t^Q is a Q-Brownian motion, and the dynamics of the aggregate wealth e^* under Q is given by $de_t^* = [\alpha_t + \delta_M(t)\,\beta_t]\,e_t^*dt + \beta_t e_t^* dW_t^Q$, hence a nonpositive δ_M decreases the instantaneous aggregate wealth growth rate. Notice that a pessimistic consensus probability will also systematically decrease the instantaneous rate of return of any asset that is positively correlated with aggregate wealth. We have then obtained that the MPR with heterogeneous beliefs is greater than in the standard setting if and only if the consensus probability is pessimistic. In fact, the market price of risk subjectively expected is not modified by the introduction of some belief dispersion or of some pessimism. In other words, the reason that pessimism increases the objective expectation of the MPR is not that pessimistic consumers require a higher risk premium. A pessimistic representative agent requires the same MPR but his/her pessimism leads him/her to underestimate the average rate of return of equity (leaving unchanged his/her estimation of the risk-free rate). Thus the objective expectation of the MPR is greater than the representative agent's subjective expectation, hence is greater than the standard MPR.

Our results are consistent with those of Abel (2000), Cechetti et al. (2000), Epstein and Wang (1994), Hansen et al. (1999), or Anderson et al. (2000), which introduce distorted beliefs associated with cautious/pessimistic individual behavior. Since we have obtained that the MPR in the heterogeneous beliefs setting is in fact given by the MPR under the homogeneous belief Q, we are actually led to face the same issue as in, e.g. Abel (2000), where investors all have the same subjective probability, different from the initial objective probability. Abel (2000) shows in a discrete time setting, and for power utility functions, that "uniform pessimism" on the agents' (common) subjective probability[12] leads to a higher risk premium. Unlike in the setting of Abel's paper, in our setting with heterogeneous beliefs, there is no need for all investors to be pessimistic, but pessimism at the aggregate level is sufficient in order to ensure an increase in the market price of risk. More precisely, in the case of linear risk-tolerance utility functions, there is no need for all δ_i to be nonpositive, it suffices that some average of the δ_i, namely

[12] Defined as a leftward translation of the objective distribution of the aggregate consumption.

$\delta_M = \sum_{i=1}^{N} \kappa_i \delta_i$, be nonpositive. We shall in Section 4 explore conditions on the individual beliefs under which the consensus probability is pessimistic (resp. optimistic).

3.2. Risk-free rate

We have just seen that the heterogeneity of beliefs leads to a higher market price of risk (resp. lower) if and only if the consensus belief is pessimistic (resp. optimistic). We now turn to the analysis of the expression of the risk-free rate with heterogeneous beliefs, and in particular, we explore if the possible increase of the MPR (induced by a pessimistic aggregate probability) can be associated with a lowering of the risk-free rate.

Since q^*S is a P-martingale, we easily get, as in the classical case (see Duffie, 1996, or Huang and Litzenberger, 1988) that

$$r^f = -\mu_{q^*}. \tag{3.5}$$

Now, since $q^* = \overline{M}Bu'(e^*)$, with B of finite variation, we have seen in Proposition 3.1 that

$$\mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \tag{3.6}$$

hence the following expression of the risk-free rate r^f in the heterogeneous beliefs setting.

Proposition 3.3. *The risk-free rate under heterogeneous beliefs is given by*

$$\begin{aligned} r^f\,[\textit{heterogeneous}] &= -\mu_q - \mu_M - \delta_M \sigma_q & (3.7)\\ &= r^f\,[\textit{standard}] - \mu_M + \delta_M \left(-\frac{u''(e^*)}{u'(e^*)} \beta e^* \right) & \\ &= r^f\,[\textit{homogeneous under } Q] - \mu_M & (3.8) \end{aligned}$$

Conversely to the MPR, both the change of probability and the discount factor have an impact on the risk-free rate. The impact of the discount factor is represented by μ_M. If B is nondecreasing (resp. nonincreasing), then μ_M is nonnegative (resp. nonpositive) and contributes to a decrease (resp. increase) of the risk-free rate. This effect has a clear interpretation; considering $(-\mu_M)$ as a discount rate, a nonpositive μ_M means that future consumption is less important for the representative agent, and leads to a higher equilibrium interest rate. We have seen that for power utility functions with $\eta \leq 1$, as well as for exponential utility functions, μ_M is nonpositive, so that the effect of the aggregation bias is towards an increase of the interest rate. For power utility functions with $\eta \geq 1$, we obtain a nonpositive discount rate $(-\mu_M)$. This is interesting in light of the risk-free rate puzzle. As underlined by Weil (1989), "a value of β above 1 (which corresponds in our setting to a nonpositive discount rate) is a computer's solution of the risk-free rate puzzle".[13]

The impact of the change of probability from P to the consensus probability Q is represented by the covariance $\delta_M \left(-\frac{u''(e^*)}{u'(e^*)} \beta e^* \right)$ and contributes to a lowering of the risk-free rate if and only if Q is pessimistic, i.e. $\delta_M \leq 0$. The interpretation is the following: if consumers are pessimistic

[13] In his setting, a value of β above 1 corresponds to an exogeneous negative discount of future consumption and does not correspond to an economically meaningful behavior. In our setting, the possibly negative discount rate results endogeneously from belief heterogeneity.

about the growth rate of aggregate wealth, then relative to the standard case, they will attempt to reduce current consumption and increase current savings. The attempt to increase current savings puts downward pressure on the interest rate.

Combining both effects, we obtain that the impact of heterogeneity of investors' beliefs is towards a lower (resp. higher) risk-free rate if the aggregate probability is pessimistic (resp. optimistic) and if B is nondecreasing (resp. nonincreasing). The effect may remain, for instance, downwards, if the aggregate probability is pessimistic, and if B is nonincreasing as long as $|\mu_M|$ is "small", which is associated in classical examples with a small dispersion of beliefs.

3.3. Asset price

Adopting the same approach as in the standard setting, we easily obtain that for a risky asset with price process S and dividend yield process e^*,

$$q_t^* S_t = E_t \left[\int_t^T q_s^* e_s^* ds \right]$$

with $q_t^* = \overline{M}_t B_t u'(t, e_t^*)$, so that the asset price in a heterogeneous beliefs setting is given by

$$S_t \,[\text{heterogeneous}] = \frac{1}{u'(t, e_t^*)} E_t^Q \left[\int_t^T \frac{B_s}{B_t} u'(s, e_s^*) e_s^* ds \right]$$

which is to be compared with the following asset price formula in the standard setting:

$$S_t \,[\text{standard}] = \frac{1}{u'(t, e_t^*)} E_t^P \left[\int_t^T u'(s, e_s^*) e_s^* ds \right].$$

As in the risk-free rate analysis, both effects of the change of probability and of the discount factor are to be noticed.

If B is nonincreasing, which is the case for linear risk-tolerance utility functions when $\eta \leq 1$, the effect of the discount factor is towards a lowering of the asset price, and in that case,

$$S_t \,[\text{heterogeneous}] \leq \frac{1}{u'(t, e_t^*)} E_t^Q \left[\int_t^T u'(s, e_s^*) e_s^* ds \right]$$

which means that the asset price under heterogeneous beliefs is lower than the asset price in a model where all investors share the same probability Q. The converse effect occurs for general power utility functions with $\eta \geq 1$. The interpretation here again is clear since B represents a discount factor.

On the other hand, the change of probability effect can be measured through the comparison between $E_t^Q \left[\int_t^T u'(s, e_s^*) e_s^* ds \right]$ and $E_t^P \left[\int_t^T u'(s, e_s^*) e_s^* ds \right]$. The effect is towards a raising (resp. lowering of the asset price) when $E_t^Q [u'(s, e_s^*) e_s^*] \geq E_t^P [u'(s, e_s^*) e_s^*]$, which again is related to the pessimism/optimism of the consensus belief Q. Indeed, consider for instance the case of power utility functions, $u'(x) = (\eta x)^{-1/\eta}$, and suppose that for all t, α_t and β_t are deterministic.

Then $E_t^Q\left[u'\left(s,e_s^*\right)e_s^*\right] = E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right]$, and it is easy to see (see the appendix, Proof of Inequality (3.9) that if the consensus probability is pessimistic, i.e. if $\delta_M \leq 0$, and if $\eta \leq 1$, then

$$E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right] \geq E_t^P\left[\left(e_s^*\right)^{1-1/\eta}\right]. \tag{3.9}$$

4. Applications

We have proved in Section 2 the existence of a consensus belief $\overline{M}$ modulo the introduction of a discount factor B, and analyzed in Section 3 the impact of heterogeneous beliefs on asset pricing through μ_M (the drift of B), and δ_M (the volatility of the consensus belief $\overline{M}$). For instance, we have seen that the introduction of heterogeneity in the investors' beliefs leads to a higher (resp. lower) market price of risk if and only if the aggregate investor is pessimistic (resp. optimistic), i.e. if and only if $\delta_M \leq 0$ (resp. $\delta_M \geq 0$). The problem is now to identify the situations where $\delta_M \leq 0$ or more generally, to determine M, μ_M and δ_M.

We shall focus in this section on the case of linear risk-tolerance utility functions, and in order to disentangle the initial wealth effect and the pure risk aversion effect, we will analyze separately the case of exponential utility functions and the case of power utility functions. In particular, we will see that the optimism/pessimism at the aggregate level is directly related to the correlation between individual optimism/pessimism and risk aversion in the first case and to the correlation between individual optimism/pessimism and initial wealth in the second case.

4.1. Exponential utility functions and the risk aversion effect

In the case of exponential utility functions, the parameters involved in the expressions of M, μ_M and δ_M are exogenously determined. Indeed, we have, for utility functions of the form $u_i'(x) = a_i \exp\left(-\frac{x}{\theta_i}\right)$,

$$\begin{aligned}
M &= \prod_{i=1}^{N}\left(M^i\right)^{\theta_i/\overline{\theta}}, \\
\delta_M &= \sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\delta^i, \\
\mu_M &= -\frac{1}{2}\left[\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\left(\delta^i\right)^2 - \left(\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\delta^i\right)^2\right], \\
\Delta\left(MPR\right) &= -\delta_M, \Delta\left(\text{Risk Premium}\right) = -\delta_M\sigma_R \\
\Delta\left(r^f\right) &= -\mu_M + \beta e^*\delta_M
\end{aligned}$$

where for a given equilibrium characteristic x (risk premium, market price of risk, risk-free rate, etc.) $\Delta(x)$ stands for the difference between the value of x in the heterogeneous beliefs setting and in the standard setting. The aggregate characteristic M (resp. the aggregate belief volatility δ_M) overweights the individual beliefs M^i (resp. the individual belief volatilities δ^i) for which θ_i is greater than the average and underweights the beliefs M^i (resp. the volatilities δ^i) for which θ_i is smaller than the average.

If all θ_i are equal, then $M = \prod_{i=1}^{N} \left(M^i\right)^{1/N}$ and $\delta_M = \frac{1}{N}\sum_{i=1}^{N} \delta^i$, which means that the consensus characteristic M is an equal-weighted average of the individual beliefs and corresponds to the notion of a consensus characteristic in the common sense. The impact on the risk premium is simply given by the pessimism/optimism of the "equal-weighted average" investor. If the investors are on average optimistic (resp. pessimistic), then the risk premium is lower (resp. higher) than in the standard setting. In particular, if we suppose that there is no systematic bias in the beliefs, i.e. that $E_a\left[\delta^i\right] \equiv \frac{1}{N}\sum_{i=1}^{N} \delta^i = 0$, then we get that the market price of risk remains unchanged with respect to the standard setting. Furthermore, the impact of belief heterogeneity on the interest rate is moderate and towards raising of the interest rate.

Consider now the setting with different risk-tolerance parameters θ_i. In that case, $\delta_M = E_a\left[\delta^i\right] + cov_a\left[\frac{\theta_i}{\theta}, \delta^i\right]$ where E_a and cov_a denote the agents' equal-weighted expectation and covariance. The first effect on the volatility δ_M (hence on the risk premium) is given as in the previous case by the average level of optimism/pessimism $E_a\left[\delta^i\right]$. The second effect is given by the covariance between risk-tolerance/aversion and optimism/pessimism. If we suppose that there is no systematic bias in the beliefs and that there is no correlation between risk-tolerance/aversion, then the belief heterogeneity has no impact on the market price of risk. If we assume that there is some correlation between risk-tolerance/aversion and optimism/pessimism, by for instance assuming that the more risk tolerant investors are pessimistic (resp. optimistic) and the less risk tolerant investors are optimistic (resp. pessimistic), then the second effect corresponds to an increase (resp. decrease) of the market price of risk and to a decrease (resp. increase) of the interest rate (as soon as the dispersion remains moderate).[14] Conversely, if risk-tolerance and optimism are positively correlated, this induces a lower risk premium for assets with a higher dispersion. It remains to analyze the validity of such a positive/negative correlation. This could be done through behavioral or psychological empirical studies, and to our knowledge, this question is still open. This could also be done through the introduction in our model of a specific learning process which would lead to such a correlation, and this is left for future research.[15]

4.2. Power utility functions and initial wealth effect

We have seen that in the case of power utility functions, i.e. for utility functions of the form $u_i'(x) = b_i x^{-\frac{1}{\eta}}$, we have

$$\begin{aligned} M^\eta &= \sum_{i=1}^{N} \frac{(\lambda_i)^{-\eta}}{\sum_{i=1}^{N} (\lambda_i)^{-\eta}} \left(M^i\right)^\eta, \\ \delta_M &= \sum_{i=1}^{N} \frac{{y^*}^i}{e^*} \delta^i = \sum_{i=1}^{N} \frac{(\lambda_i)^{-\eta} \left(M^i\right)^\eta}{\sum_{i=1}^{N} (\lambda_i)^{-\eta} \left(M^i\right)^\eta} \delta^i, \\ \mu_M &= \frac{1}{2}(\eta - 1)\left[\sum_{i=1}^{N} \frac{{y^*}^i}{e^*} \left(\delta^i\right)^2 - \left(\sum_{i=1}^{N} \frac{{y^*}^i}{e^*} \delta^i\right)^2\right], \end{aligned}$$

[14] We obtain analogous results on the expected value of the market price of risk if we define a pessimistic (resp. optimistic) investor by $E[\delta_i] \leq 0$ (resp. $E[\delta_i] \geq 0$) or more generally if we suppose that $\sum_{i=1}^{N} \theta_i E\left[\delta^i\right] \leq 0$ (resp. $\sum_{i=1}^{N} \theta_i E\left[\delta^i\right] \geq 0$).

[15] Once again, notice that the introduction of such a learning process is consistent with our framework.

$$\Delta(MPR) = -\delta_M,\ \Delta(\text{Risk Premium}) = -\delta_M \sigma_R,$$
$$\Delta\left(r^f\right) = -\mu_M + \frac{\beta}{\eta}\delta_M.$$

Let us analyze these formulas in the specific framework of two agents, with constant volatility parameters δ^i. The aggregate characteristic M (resp. the aggregate belief volatility δ_M) overweights the individual belief M^i (resp. the individual belief volatility δ^i) for which $(1/\lambda_i)^\eta$ (resp. $\left(M^i/\lambda_i\right)^\eta$) is greater. The problem is that the parameters $(1/\lambda_i)^\eta$ and $\left(M^i/\lambda_i\right)^\eta$ are endogenously specified.

In the case of logarithmic utility functions ($\eta = 1$), we easily obtain that $1/\lambda_i = w_i$, where $w_i = E\left[\int_0^T q_t e_t^{i*} dt\right]$ denotes investor i's initial wealth and the consensus belief is a wealth-weighted average of the individual beliefs (see Rubinstein, 1976). We prove in the sequel that this last result remains true, in an approximate way, for power utility functions.

We start by proving that $\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta}$ is approximately equal to $\frac{w_1}{w_2}$. More precisely:

Lemma 4.1. *When the parameters $(\delta_1, \delta_2, \alpha, \beta)$ are constant, we have*

$$\frac{w_1}{w_2}\exp k_2\varepsilon\eta \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta} \leq \frac{w_1}{w_2}\exp k_1\varepsilon\eta$$

where ε is given by $\frac{|\delta_1-\delta_2|}{2}$ and $k_1 = \max\left(2\left(1-\frac{1}{\eta}\right)(\varepsilon\eta-\beta)_+, -2\left(1-\frac{1}{\eta}\right)(\varepsilon\eta+\beta)\right)$, $k_2 = \min\left(2\left(1-\frac{1}{\eta}\right)(\varepsilon\eta-\beta)_+, -2\left(1-\frac{1}{\eta}\right)(\varepsilon\eta+\beta)\right)$.

This means that for a small dispersion of beliefs (i.e. for small ε), the aggregate characteristic M can be approximated by the wealth-weighted average of the individual beliefs, given by $\left[\left(\frac{w_1}{w_1+w_2}\right)\left(M^1\right)^\eta + \left(\frac{w_2}{w_1+w_2}\right)\left(M^2\right)^\eta\right]^{1/\eta}$. The consensus characteristic is not in general an equal-weighted average of the individual beliefs, but it reflects the optimism/pessimism of the "wealthier" agent. Depending on the relative wealth of the agents, there is a bias towards optimism or pessimism.

Note that in the case of two agents with the same initial wealth, we prove (see the appendix) that there is a bias towards optimism (resp. pessimism) when $\eta < 1$ (resp. $\eta > 1$), even if this effect is negligible, since as stated in Lemma 4.1 the relative weight $\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta}$ is near 1.

As a direct application of Lemma 4.1, we provide in Table 1 numerical results for the upper bound of the relative error that we make when we approximate the parameter $\left(\frac{\lambda_1^{-\eta}}{\lambda_1^{-\eta}+\lambda_2^{-\eta}}\right)$ by $\left(\frac{w_1}{w_1+w_2}\right)$.

TABLE 1

The consensus characteristic M is a weighted average of the individual beliefs M^1 and M^2. For a belief dispersion ε equal to 0.2 and 0.33 and for aggregate wealth volatility $\beta = 0.15$, the approximation of the weight of M^1 (given by $\frac{\lambda_1^{-\eta}}{\lambda_1^{-\eta}+\lambda_2^{-\eta}}$) by the relative wealth $\left(\frac{w_1}{w_1+w_2}\right)$ leads to the following upper bounds for the relative errors.

	$\frac{w_1}{w_2}=1$		$\frac{w_1}{w_2}=3$		$\frac{w_1}{w_2}=9$	
	$\varepsilon=0.2$	$\varepsilon=0.33$	$\varepsilon=0.2$	$\varepsilon=0.33$	$\varepsilon=0.2$	$\varepsilon=0.33$
$\eta=0.2$	3.04%	5.70%	1.47%	2.69%	0.59%	1.08%
$\eta=0.5$	2.50%	5.19%	1.22%	2.47%	0.49%	0.99%
$\eta=0.9$	0.66%	1.47%	0.33%	0.73%	0.13%	0.29%
$\eta=1$	0	0	0	0	0	0
$\eta=10/9$	0.83%	1.89%	0.42%	0.96%	0.17%	0.38%
$\eta=3/2$	4.50%	10.60%	2.35%	5.93%	0.94%	2.37%

It appears from Table 1 that for a large range of possible values for η (namely $\eta\in\,]0.2;1.5[$), there is a simple way to determine the aggregate characteristic with a reasonable precision: it suffices to approximate it by the wealth-weighted average of the individual beliefs.[16]

As far as the risk premium is concerned, we have seen, in Section 3, that the impact of the heterogeneity of beliefs is measured by $\sigma_R\delta_M$, where σ_R denotes the volatility of the asset under consideration. We prove that the expected value of the consensus belief volatility δ_M is approximately given by the wealth-weighted average of the individual belief volatilities δ_i. The following lemma gives an upper bound for the error induced by this approximation in the specific case of no systematic bias, i.e. when $E^a[\delta_i]=0, \delta_1=-\delta_2=\varepsilon$. A similar approximation can be obtained in the general case.

Lemma 4.2. *If $\delta_1=-\delta_2=\varepsilon\in\mathbb{R}_+^*$, then*

$$\eta^2\varepsilon^3k_5+k_6\varepsilon\left(\exp k_2\varepsilon\eta-1\right)\leq E\left[\delta_M\right]-\left(\frac{w_1}{w_1+w_2}\delta_1+\frac{w_2}{w_1+w_2}\delta_2\right)\leq\eta^2\varepsilon^3k_3+k_4\varepsilon\left(\exp k_1\varepsilon\eta-1\right)$$

$$k_3=\max\left(0,\frac{1}{3}\frac{1-\frac{w_1}{w_2}\exp k_2\varepsilon\eta}{1+\frac{w_1}{w_2}\exp k_2\varepsilon\eta},\frac{1-\frac{w_1}{w_2}\exp k_1\varepsilon\eta}{1+\frac{w_1}{w_2}\exp k_1\varepsilon\eta}\frac{4}{2+\left(\frac{w_1}{w_2}+\frac{w_2}{w_1}\right)\exp k_1\varepsilon\eta}\right),k_4=\frac{2}{\left(1+\frac{w_2}{w_1}\right)\left(1+\frac{w_1}{w_2}\exp k_1\varepsilon\eta\right)},$$
$$k_5=\min\left(0,\frac{1}{3}\frac{1-\frac{w_1}{w_2}\exp k_1\varepsilon\eta}{1+\frac{w_1}{w_2}\exp k_1\varepsilon\eta},\frac{1-\frac{w_1}{w_2}\exp k_2\varepsilon\eta}{1+\frac{w_1}{w_2}\exp k_2\varepsilon\eta}\frac{4}{2+\left(\frac{w_1}{w_2}+\frac{w_2}{w_1}\right)\exp k_2\varepsilon\eta}\right),k_6=\frac{2}{\left(1+\frac{w_2}{w_1}\right)\left(1+\frac{w_1}{w_2}\exp k_2\varepsilon\eta\right)}.$$

As a direct application of this lemma, we provide in Table 2 for different values of belief dispersion ε, cautiousness parameter η, and wealth ratio $\frac{w_1}{w_2}$, an estimation of the impact of belief heterogeneity on the market price of risk, as well as the error made.

TABLE 2

The impact of the belief heterogeneity on the market price of risk $\Delta(MPR)$ is approximately given by the wealth-weighted average of the individual forecast errors (difference between the actual drift and the subjective drift). In this table, we provide for different values of the belief dispersion ε and different values of the wealth dispersion $\frac{w_1}{w_2}$, the value of this approximate impact. This value does not depend on the cautiousness parameter η. However, the exact value of this impact depends on η and an upper

[16] Notice that empirical studies support a value of η near 1 and below 1.

bound for the approximation error is provided for different values of η. Note that this upper bound is almost always lower than 0.01, which means that we have a good approximation of $\Delta(MPR)$.

	$\varepsilon = 0.2$			$\varepsilon = 0.3$		
$\frac{w_1}{w_2}$	9	3	1	9	3	1
$\Delta(MPR) \sim$	0.16	0.1	0	0.264	0.15	0
	Upper bound on the error of approximation$\times 10^3$					
$\eta = 0.2$	2.13	4.49	6.07	5.22	11.0	15.1
$\eta = 0.5$	1.76	3.70	4.99	4.68	9.89	13.4
$\eta = 0.9$	1.91	2.54	1.32	6.76	9.19	3.85
$\eta = 1$	2.30	3.00	0	7.77	11.0	0
$\eta = \frac{10}{9}$	3.48	4.98	16.7	11.6	16.4	5.03

The wealth-weighted average of the individual belief volatilities is then a good approximation of the expected value of the aggregate belief volatility. This permits a simple analysis of the impact of the introduction of belief heterogeneity on the risk premium.

This impact is directly related to the relative wealth distribution among optimists and pessimists. The consensus belief volatility reflects the optimism/pessimism of the "wealthier" agent. If wealth is positively correlated with optimism, which seems to be supported by empirical studies,[17] we obtain a lower expected risk premium for assets with high belief dispersion. Prices will reflect the optimistic view, even though short sale constraints are not part of our model. This result is consistent with the empirical study of Diether et al. (2002), who "provide evidence that stocks with higher dispersion in analysts' earnings forecasts earn lower future returns than otherwise similar stocks".

For example, let us suppose that the volatility β of aggregate wealth e^* is equal to 5%, and that its drift α is equal to 3%, and let us assume that the optimists' (resp. pessimists') belief for this drift denoted by α_1(resp. α_2) is equal to 5% (resp. 1%).[18] Let us also suppose that the optimists' wealth is three times larger than that of pessimists. The risk premium is then 3% lower than in the standard setting for an asset whose volatility is equal to 15%.[19]

If we now introduce some bias in the investors' beliefs, and suppose as in, e.g. Abel (2000) that the investors are on average pessimistic (i.e. that $\overline{\delta} = E_a\left[\delta^i\right] < 0$), then we obtain that the consensus belief volatility is in expected value approximated by $\overline{\delta} + \left(\frac{w_1 - w_2}{w_1 + w_2}\right)\varepsilon$, which remains negative if the dispersion of beliefs ε or of wealth is small. As above, the risk premium is lower for stocks with higher belief dispersion. But in this situation, the risk premium is higher and the risk-free rate lower than in the standard setting, which is interesting in light of the risk premium and risk-free rate puzzles.

In order to calibrate the model with market data, let us assume that $\sigma_e = 3.6\%$, $\mu_e = 1.8\%$ and $\sigma_R = 16.8\%$. These figures correspond to the observed volatility and drift for the US consumption during the last century and to the market portfolio volatility during the same period (Grossman and Shiller, 1981). In Table 3, we provide for these specifications and for

[17] For instance, the index of consumer sentiment published by the University of Michigan is systematically higher for families with income above \$50 000 than for families with income under \$50 000.

[18] This corresponds to a nonsystematic bias situation ($\overline{\delta} = 0$) with $\Delta = \frac{\alpha_1 - \alpha_2}{2\beta} = 0.4$.

[19] Indeed, it is immediate that the impact $-\sigma_R \delta_M$ on the risk premium is equal to -$\frac{\sigma_R}{\beta}\left(\frac{\alpha_1 w_1 + \alpha_2 w_2}{w_1 + w_2} - \alpha\right)$.

different values of the wealth distribution parameter $\frac{w_1}{w_1+w_2}$, of the cautiousness parameter η, of the average belief $\overline{\delta}$ and of belief dispersion ε, the equilibrium risk premium and risk-free rate. If $\overline{\delta}=\varepsilon=0$, there is no belief heterogeneity and the common belief corresponds to the true probability (rational expectations). In that case, the estimated risk premium is too low. When η decreases, it is possible to obtain higher values for the risk premium; however, the risk-free rate becomes then too high. These observations correspond to the classical risk premium and risk-free rate puzzles (see Kocherlakota, 1996, for a survey on these "puzzles").

If we now introduce some belief dispersion without aggregate bias ($\bar{\delta}=0$), we obtain a decrease of the risk premium and a possible increase of the risk-free rate. If we introduce some pessimism ($\bar{\delta}<0$) without belief dispersion, we obtain higher risk premia and lower risk-free rates: the introduction of some pessimism might be a possible partial explanation of the puzzles (see e.g. Abel, 2000).

Finally, if we introduce some pessimism ($\bar{\delta}<0$) as well as some belief dispersion, we see first that we obtain a higher risk premium and a lower risk-free rate than in the standard setting, and second that assets with high belief dispersion have a lower risk premium than those with low belief dispersion. This permits us then to propose an explanation of Diether et al. (2002) findings (at least in this simple two-agent model and for some parameter specification), that is compatible with a "resolution" of the risk premium and risk-free rate puzzles.

TABLE 3

In this table, for σ_e= 3.6%, μ_e= 1.8% and σ_R= 16.8%, we report the values of the risk premium and of the risk-free rate for different values of the cautiousness parameter η, the average belief $\bar{\delta}$ and the belief dispersion ε.

$(RP\%; r\%)$		$\delta=0$		$\delta=-0.25$			
η	$\frac{w_1}{w_1+w_2}$	$\varepsilon=0$	$\varepsilon=0.10$	$\varepsilon=0$	$\varepsilon=0.10$	$\varepsilon=0.20$	$\varepsilon=0.30$
0.5	0.5	(2.1; 3.2)	(2.1; 2.9)	(9.6; 1.4)	(9.6; 1.1)	(9.6; 0.4)	(9.6; −0.8)
	0.9	(2.1; 3.2)	(−0.2; 3.7)	(9.6; 1.4)	(7.2; 1.9)	(4.8; 2.2)	(2.4; 2.3)
0.9	0.5	(1.2; 1.8)	(1.2; 1.8)	(8.7; 0.8)	(8.7; 0.8)	(8.7; 0.6)	(8.7; 0.4)
	0.9	(1.2; 1.8)	(−1.2; 2.1)	(8.7; 0.8)	(6.3; 1.1)	(3.9; 1.4)	(1.5; 1.6)
1.1	0.5	(0.9; 1.5)	(0.9; 1.5)	(8.4; 0.7)	(8.4; 0.7)	(8.4; 0.9)	(8.4; 1.1)
	0.9	(0.9; 1.5)	(−1.4; 1.8)	(8.4; 0.7)	(6.0; 0.9)	(3.6; 1.3)	(1.2; 1.6)

5. Concluding and additional remarks

In this paper, we provide an aggregation procedure which permits us to rewrite in a simple way the equilibrium characteristics (state price density, market price of risk, risk premium, risk-free rate) in a heterogeneous beliefs framework and to compare them with an otherwise similar standard setting. We prove that in many cases, the impact of belief heterogeneity on the market price of risk can be easily approximated with a relatively good precision by considering the wealth-weighted average belief. Our results permit us to explain the findings of Diether et al. (2002). Furthermore, it seems possible to construct specific parametrizations of the heterogeneous beliefs model, that lead to globally higher risk premia, lower risk-free rates, and risk premia that are lower for assets with higher belief dispersion.

Our aggregation procedure can also be used for the study of models with constraints. Indeed, let us consider a model where investors share common beliefs but are subjected to possible short sale constraints. The equilibrium prices and allocations in such a model are the same as in a model without short sale constraint, where the initially unconstrained investors' beliefs remain unchanged and where the initially constrained investors' beliefs are replaced by well-chosen, more optimistic ones. The consensus consumer is then more optimistic (or more precisely, his/her consensus belief is more optimistic) and the impact on the MPR and risk-free rate follows: the short sales constraints lead to a lower market price of risk and to a higher risk-free rate.

In Basak and Cuoco (1998), the authors consider a two-agent equilibrium model with restricted market participation. The first agent does not have access to the risky asset market and the second one is not restricted. Without market restrictions, the first agent would buy shares of the risky asset. It is then easy to see that the equilibrium prices and allocations in such a model are the same as in a model without market restrictions, where the second agent's belief remains unchanged and where the first agent's belief is replaced by a well-chosen, more pessimistic one. The aggregate probability is then pessimistic, which leads to a higher risk premium and a lower risk-free rate.

The case of borrowing constraints can be analyzed similarly and leads to the same results. Indeed, the beliefs of the constrained agents have to be replaced by more pessimistic ones in order to maintain the equilibrium prices and allocations unchanged. This leads to more pessimistic consensus beliefs and contributes to a higher market price of risk and a lower risk-free rate.

Appendix

Proof of Proposition 2.1. Since q^* is an interior equilibrium price process relative to the beliefs $\left(M^i\right)$, and the income processes e^i, we know that $\sum_{i=1}^N y^{*^i} = e^*$ and that there exist positive Lagrange multipliers (λ_i) such that for all i and for all t,

$$M_t^i u_i'\left(t, y_t^{*^i}\right) = \lambda_i q_t^*.$$

We consider the maximization problem

$$\left(\mathcal{P}^\lambda(t,\omega)\right): \max_{\sum_{i=1}^N x_i \leq e_t^*(\omega)} \sum \frac{1}{\lambda_i} u_i(t, x_i).$$

Since all utility functions are increasing and strictly concave and take values in $\mathbb{R}\cup\{-\infty\}$, for each t and each ω this program admits a unique solution $\left(y^{i,\lambda}(t,\omega)\right)$ and we have $\sum_{i=1}^N y^{i,\lambda}(t,\omega) = e^*(t,\omega)$. Furthermore, the process $\left(\frac{1}{\lambda_i} u_i'\left(t, y_t^{i,\lambda}\right)\right)_t$ is independent of i. We denote this process by $p^{(\lambda)}$. Letting $M^{(\lambda)} \equiv \frac{q^*}{p^{(\lambda)}}$, we then have for all i and for all t

$$M_t^{(\lambda)} u_i'\left(t, y_t^{i,(\lambda)}\right) = M_t^i u_i'\left(t, y_t^{*^i}\right).$$

Let us take $\overline{M} = M^{(\lambda)}$, $\overline{y}^i = y^{i,(\lambda)}$, and $\overline{e}^i = e^{*^i} - y^{*^i} + \overline{y}^i$. The process $\overline{M}$ is adapted and positive. Moreover, at date $t = 0$, we have for all i, $M_0^i = 1$, and $\sum_{i=1}^N \overline{y_0}^i = \sum_{i=1}^N y_0^{*^i} = e_0^*$, so that $\overline{M_0} = 1$.

As far as uniqueness is concerned, notice that any process y^i such that $\sum_{i=1}^N y^i = e^*$ and

$$M_t u_i'\left(t, y_t^i\right) = M_t^i u_i'\left(t, y_t^{*^i}\right)$$

for some positive process M is a solution of the maximization problem $\left(\mathcal{P}^\lambda\right)$, which, as seen above, admits a unique solution. ■

Proof of Proposition 2.2. With the same notation as in the previous proof, we construct the following application:

$$\begin{array}{cccc} \Phi & \Sigma & \rightarrow & T(\Sigma) \\ & (\alpha_i) & \rightarrow & E^P\left[\int_0^T q_t^*\left(y_t^{i,\alpha} - e_t^i\right) dt\right] \end{array}$$

where $\left(y^{i,\alpha}(t,\omega)\right)$ is the solution of $(\mathcal{P}^\alpha(t,\omega))$, where Σ is the simplex of $\mathbb{R}^N$, i.e. $\Sigma = \left\{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 1, x_i \geq 0, i = 1, ..., N\right\}$ and $T(\Sigma)$ is the tangent space to Σ, i.e. $T(\Sigma) = \left\{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0\right\}$.

The application Φ is well defined. Furthermore, $\alpha \rightarrow y^{i,\alpha}$ is continuous, $k_i \leq y^{i,\alpha} \leq e^* - \sum_{j\neq i} k_j$ and q^* is uniformly bounded. By the dominated convergence theorem, the application Φ is then continuous. On the other hand, if for some i we have $\alpha_i = 0$ then $y^{i,\alpha} = k_i$ and

$\Phi_i(\alpha) < 0$. By the classical equilibrium for outward applications theorem there exists then an interior zero for Φ. Let us denote by (β_i) this zero and by $(\bar{y}^i)$ the associated $\left(y^{i,\beta}\right)$. We have then

$$\begin{aligned} \sum_{i=1}^{N} \bar{y}^i &= e^* \\ \beta_i u_i'(t, \bar{y}^i) &= q \\ E^P \left[\int_0^T q_t \left(\bar{y}_t^i - e_t^i \right) dt \right] &= 0 \end{aligned}$$

where q is a given process. By construction, q and q^* are positive processes and it suffices to define then M by $M = \frac{q^*}{q} \times \frac{q_0}{q_0^*}$ and λ_i' by $\lambda_i' = \frac{1}{\beta_i} \frac{q_0}{q_0^*}$, $i = 1, ..., N$ in order to conclude. ■

Proof of Corollary 2.3. Similar to the proof of the analogous result in a standard setting. ■

Proof of Example 2.4. Since, as seen in the proof of Corollary 2.3, the representative utility function u is given by

$$u_\lambda(t, x) = \max_{\sum_{i=1}^{N} x_i \leq x} \sum_{i=1}^{N} \frac{1}{\lambda_i} u_i(t, x_i)$$

the expression for u_λ in the specific setting of linear risk-tolerance utility functions is obtained as in the standard case (see e.g. Huang and Litzenberger, 1988).

The expression for M is obtained by using $M_t u_i' \left(t, \overline{y}_t^i\right) = M_t^i u_i' \left(t, y_t^{*^i}\right)$, as well as

$$\sum_{i=1}^{N} y^{*^i} = \sum_{i=1}^{N} \overline{y}^i = e^*.$$

Indeed, in the case of exponential utility functions, we have for all i,

$$M^i \exp\left(-\frac{y^{*^i}}{\theta_i}\right) = M \exp\left(-\frac{\overline{y}^i}{\theta_i}\right)$$

hence

$$\prod_{i=1}^{N} \left(M^i\right)^{\theta_i} \exp\left(-\sum_{i=1}^{N} y^{*^i}\right) = M^{\overline{\theta}} \exp\left(-\sum_{i=1}^{N} \overline{y}^i\right),$$

or equivalently

$$M = \prod_{i=1}^{N} \left(M^i\right)^{\frac{\theta_i}{\overline{\theta}}}.$$

In the case of power utility functions, we get for all i,

$$M^i \left(\theta_i + \eta y^{*^i}\right)^{-1/\eta} = M \left(\theta_i + \eta \overline{y}^i\right)^{-1/\eta} = \lambda_i M \left(\overline{\theta} + \eta e^*\right)^{-1/\eta} b$$

hence

$$\left(M^i\right)^\eta \lambda_i^{-\eta} = M^\eta \left(\overline{\theta} + \eta e^*\right)^{-1} b^\eta \left(\theta_i + \eta {y^*}^i\right)$$

and

$$M = \left[\sum_{i=1}^N \frac{\lambda_i^{-\eta}}{\sum_{i=1}^N \lambda_i^{-\eta}} \left(M^i\right)^\eta\right]^{1/\eta}.$$

■

Proof of Proposition 2.5. We know by Corollary 2.3 that there exists a consensus consumer with consensus characteristic M and utility function u such that $M_t u'(t, e_t^*) = q_t^*$. Since $dM_t = M_t [\mu_M(t) dt + \delta_M(t) dW_t]$ with $M_0 = 1$, we have

$$M_t = \exp\left[\int_0^t \delta_M(u) dW_u + \int_0^t \left(\mu_M(u) - \frac{1}{2}\delta_M^2(u)\right) du\right],$$

hence with the notation of the proposition, $M_t = B_t \overline{M}_t$. ■

Proof of Proposition 2.6. When $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$, then we know from Example 2.4 that $M_t = \prod_{i=1}^N \left(M_t^i\right)^{\theta_i/\bar{\theta}}$. Since $M_t^i = \exp\left[\int_0^t \delta^i(u) dW_u - \frac{1}{2}\int_0^t \left(\delta^i\right)^2 (u) du\right]$, we have

$$\prod_{i=1}^N \left(M_t^i\right)^{\theta_i/\bar{\theta}} = \exp\left[\int_0^t \sum_{i=1}^N \frac{\theta_i}{\overline{\theta}}\delta^i(u) dW_u - \frac{1}{2}\int_0^t \sum_{i=1}^N \frac{\theta_i}{\overline{\theta}}\left(\delta^i\right)^2 (u) du\right]$$

hence by Itô's lemma,

$$\begin{cases} \delta_M = \sum_{i=1}^N \frac{\theta_i}{\overline{\theta}}\delta^i \\ \mu_M = -\frac{1}{2}\left[\sum_{i=1}^N \frac{\theta_i}{\overline{\theta}}\left(\delta^i\right)^2 - \left(\sum_{i=1}^N \frac{\theta_i}{\overline{\theta}}\delta^i\right)^2\right] \end{cases}$$

When $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ for $\eta \neq 0$, the result is obtained in the same way, by applying Itô's lemma to the expression of the consensus characteristic $M = \left[\sum_{i=1}^N \gamma_i (M^i)^\eta\right]^{\frac{1}{\eta}}$ obtained in Example 2.4. ■

Proof of Proposition 3.1. Since $q^* = Mq$ with

$$\begin{aligned} dM_t &= M_t [\mu_M(t) dt + \delta_M(t) dW_t] \\ dq_t &= q_t \left[\mu_q(t) dt + \sigma_q(t) dW_t\right] \end{aligned}$$

we easily get, through Itô's lemma, that

$$dq_t^* = q_t^* \left[\left(\mu_q + \mu_M + \delta_M \sigma_q\right)(t) dt + \left(\sigma_q + \delta_M\right)(t) dW_t\right]$$

hence

$$\begin{cases} \mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \\ \sigma_{q^*} = \sigma_q + \delta_M \end{cases}$$

■

Proof of Proposition 3.2. Immediate, using Equations (3.1) and (3.2). ■

Proof of Proposition 3.3. Immediate, using (3.5) and (3.6). ■

Proof of Inequality 3.9. Since $de_t^* = e_t^* \left(\alpha_t dt + \beta_t dW_t\right)$, we have

$$e_t^* = e_0^* \exp\left[\int_0^t \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_0^t \beta_u dW_u\right].$$

Letting $W_t^Q \equiv W_t - \int_0^t \delta_M(u)\, du$, we know, by Girsanov's Lemma, that W^Q is a Q-Brownian motion, and

$$\left(\frac{e_s^*}{e_t^*}\right)^{1-1/\eta} = \exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u^Q\right] \exp\left(1-1/\eta\right) \int_t^s \beta_u \delta_M(u)\, du.$$

If $\delta_M \leq 0$, and $\eta \leq 1$, then $\exp\left(1-1/\eta\right) \int_t^s \beta_u \delta_M(u)\, du \geq 1$, hence,

$$\begin{aligned}
E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right] &= \left(e_t^*\right)^{1-1/\eta} E_t^Q\left[\left(\frac{e_s^*}{e_t^*}\right)^{1-1/\eta}\right] \\
&\geq \left(e_t^*\right)^{1-1/\eta} E_t^Q\left[\exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u^Q\right]\right] \\
&\geq \left(e_t^*\right)^{1-1/\eta} E_t^P\left[\exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u\right]\right] \\
&\geq E_t^P\left[\left(e_s^*\right)^{1-1/\eta}\right].
\end{aligned}$$

The same approach leads to the same result (resp. the opposite result, $E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right] \leq E_t^P\left[\left(e_s^*\right)^{1-1/\eta}\right]$) when $\delta_M \geq 0$ and $\eta \geq 1$ (resp. $\delta_M \geq 0$ and $\eta \leq 1$, $\delta_M \leq 0$ and $\eta \geq 1$). ■

Proof of Lemma 4.1. In equilibrium, we have

$$\frac{1}{\lambda_1} M_t^1 u_1'\left(t, y_t^{*^1}\right) = \frac{1}{\lambda_2} M_t^2 u_2'\left(t, y_t^{*^2}\right) = q_t \tag{5.1}$$

$$y_t^{*^1} + y_t^{*^2} = e^* \tag{5.2}$$

$$w_2 E\left[\int_0^T q_t y_t^{*^1} dt\right] = w_1 E\left[\int_0^T q_t y_t^{*^2} dt\right] \tag{5.3}$$

Equations (5.1) and (5.2) lead to

$$\begin{cases} y^{*^1} = e^* \frac{\left(M_1/\lambda_1\right)^\eta}{\left(M_1/\lambda_1\right)^\eta + \left(M_2/\lambda_2\right)^\eta} \\ y^{*^2} = e^* \frac{\left(M_2/\lambda_2\right)^\eta}{\left(M_1/\lambda_1\right)^\eta + \left(M_2/\lambda_2\right)^\eta} \end{cases},$$

and Equation (5.3) then implies that

$$E\left[\int_0^T \left(e_t^*\right)^{1-1/\eta} \frac{w_2\left(M_1(t)/\lambda_1\right)^\eta - w_1\left(M_2(t)/\lambda_2\right)^\eta}{\left\{\left(M_1(t)/\lambda_1\right)^\eta + \left(M_2(t)/\lambda_2\right)^\eta\right\}^{1-1/\eta}} dt\right] = 0. \tag{5.4}$$

Let $A \equiv \frac{w_2(M_1/\lambda_1)^\eta - w_1(M_2/\lambda_2)^\eta}{\{(M_1/\lambda_1)^\eta + (M_2/\lambda_2)^\eta\}^{1-1/\eta}}$. Then

$$A = \left(\frac{M_1}{\lambda_1}\right) \frac{w_2 - w_1 X}{\{1+X\}^{1-1/\eta}}$$

for $X = \left(\frac{M_2}{M_1}\frac{\lambda_1}{\lambda_2}\right)^\eta$. We first consider the case $\left(1 - \frac{1}{\eta}\right) \geq 0$. It is easy to verify that

$$\frac{w_2 - w_1 X}{\{1+X\}^{1-1/\eta}} \geq \frac{w_2 - w_1 X}{\left\{1 + \frac{w_2}{w_1}\right\}^{1-1/\eta}}.$$

Now, Equation (5.4) implies that

$$E\left[\int_0^T (e_t^*)^{1-1/\eta} \left(\frac{M_1(t)}{\lambda_1}\right) \left(\frac{w_2 - w_1 X_t}{\left\{1 + \frac{w_2}{w_1}\right\}^{1-1/\eta}}\right) dt\right] \leq 0$$

hence

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta} \frac{w_2}{w_1} \leq \frac{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t) \left(\left(\frac{M_2(t)}{M_1(t)}\right)^\eta\right) dt\right]}{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\, dt\right]}.$$

We obtain analogously that

$$A = \left(\frac{M_2}{\lambda_2}\right) \frac{w_2/X - w_1}{\{1/X+1\}^{1-1/\eta}}$$

with

$$\frac{w_2/X - w_1}{\{1/X_t+1\}^{1-1/\eta}} \leq \frac{w_2/X - w_1}{\left\{1 + \frac{w_1}{w_2}\right\}^{1-1/\eta}}$$

hence

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta} \frac{w_2}{w_1} \geq \frac{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_2(t)\, dt\right]}{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_2(t) \left(\left(\frac{M_1(t)}{M_2(t)}\right)^\eta\right) dt\right]}.$$

Now,

$$\begin{aligned} M_1(t) &= \exp(\delta+\varepsilon) W_t - \frac{1}{2}(\delta+\varepsilon)^2 t \\ M_2(t) &= \exp(\delta-\varepsilon) W_t - \frac{1}{2}(\delta-\varepsilon)^2 t \end{aligned}$$

and

$$\begin{aligned} E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\, dt\right] &= \int_0^T E\left[(e_t^*)^{1-1/\eta} M_1(t)\right] dt \\ &= \int_0^T (\exp at)\, dt \end{aligned}$$

with $a = \frac{1}{2}\left[\delta+\varepsilon+\beta\left(1-\frac{1}{\eta}\right)\right]^2 + \left(\alpha-\frac{1}{2}\beta^2\right)\left(1-\frac{1}{\eta}\right) - \frac{1}{2}(\delta+\varepsilon)^2$. Besides,

$$E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\left(\left(\frac{M_2(t)}{M_1(t)}\right)^\eta\right) dt\right] = \int_0^T (\exp bt)\, dt$$

with $b = a + 2\left(1-\frac{1}{\eta}\right)\varepsilon\eta(\varepsilon\eta-\beta)$. If $b-a \le 0$, then

$$\frac{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\left(\left(\frac{M_2(t)}{M_1(t)}\right)^\eta\right) dt\right]}{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\, dt\right]} \le 1$$

and if $b-a \ge 0$, then

$$\frac{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\left(\left(\frac{M_2(t)}{M_1(t)}\right)^\eta\right) dt\right]}{E\left[\int_0^T (e_t^*)^{1-1/\eta} M_1(t)\, dt\right]} \le \exp(b-a)T \le \exp 2\left(1-\frac{1}{\eta}\right)\varepsilon\eta(\varepsilon\eta-\beta)T$$

hence $\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta} \le \frac{w_1}{w_2}\max\left(1, \exp\left\{2\left(1-\frac{1}{\eta}\right)\varepsilon\eta(\varepsilon\eta-\beta)\right\}\right)$. The other inequality is obtained similarly, and the same approach leads to the result for $\left(1-\frac{1}{\eta}\right) \le 0$. ■

Proof of the following result: If $\delta_1 = -\delta_2 = \varepsilon \in \mathbb{R}_+^*$, if $(\alpha_t, \beta_t) \in \mathbb{R}^2$, and if the investors are endowed with the same initial wealth, then

$$\begin{array}{lll} \lambda_1 < \lambda_2 & \text{for } \eta < 1 \\ \lambda_1 > \lambda_2 & \text{for } \eta > 1 \\ \lambda_1 = \lambda_2 & \text{for } \eta = 1 \end{array}$$

Proof. Proceeding as in the proof of Lemma 4.1, we obtain Equation (5.4) with $w_1 = w_2$, namely

$$A \equiv E\left[\int_0^T (e_t^*)^{1-1/\eta} \frac{(M_1/\lambda_1)^\eta - (M_2/\lambda_2)^\eta}{\{(M_1/\lambda_1)^\eta + (M_2/\lambda_2)^\eta\}^{1-1/\eta}} dt\right] = 0.$$

It is immediate that A can be written in the form $A = \frac{1}{\lambda_1} x^{-1/\eta} g(x)$ with $x^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^\eta$ and

$$g(x) = E\left[\int_0^T (e_t^*)^{1-1/\eta} \frac{x M_1^\eta(t) - \frac{1}{x} M_2^\eta(t)}{\left\{x M_1^\eta(t) + \frac{1}{x} M_2^\eta(t)\right\}^{1-1/\eta}} dt\right].$$

For $\eta < 1$, we show that $\lambda_1 \le \lambda_2$. We prove 1) that $g(1) \le 0$, and 2) that g is increasing with x, which implies that for $\lambda_1 > \lambda_2$ we would have $A < 0$, which is impossible. We have

$$g'(x) = E\left[\int_0^T \frac{x\left(M_1^\eta(t) + M_2^\eta(t)/x^2\right)^2 - x\left(M_1^\eta(t) - M_2^\eta(t)/x^2\right)^2(1-1/\eta)}{\left(x M_1^\eta(t) + M_2^\eta(t)/x\right)^{2-1/\eta}} (e_t^*)^{1-1/\eta}\, dt\right],$$

which is positive for $\eta < 1$ and proves 2). Now, $g(1) = \int_0^T E\left[(e_t^*)^{1-1/\eta} \frac{M_1^\eta - M_2^\eta}{\{M_1^\eta + M_2^\eta\}^{1-1/\eta}}\right] dt$. With deterministic coefficients, $M_1^\eta(t) = R(t)(e^*(t))^{\frac{\eta\varepsilon}{\beta}}$ and $M_2^\eta(t) = R(t)(e^*(t))^{-\frac{\eta\varepsilon}{\beta}}$ and it is easy to see that $\frac{M_1^\eta - M_2^\eta}{\{M_1^\eta + M_2^\eta\}^{1-1/\eta}}$ is increasing in e^*, hence decreasing in $(e^*)^{1-1/\eta}$, leading to

$$E\left[(e_t^*)^{1-1/\eta} \frac{M_1^\eta(t) - M_2^\eta(t)}{\{M_1^\eta(t) + M_2^\eta(t)\}^{1-1/\eta}}\right] \leq E\left[(e_t^*)^{1-1/\eta}\right] E\left[\frac{M_1^\eta(t) - M_2^\eta(t)}{\{M_1^\eta(t) + M_2^\eta(t)\}^{1-1/\eta}}\right].$$

Since $E\left[\frac{M_1^\eta(t) - M_2^\eta(t)}{\{M_1^\eta(t) + M_2^\eta(t)\}^{1-1/\eta}}\right] = 0$, we obtain that $g(1) \leq 0$. ■

Proof of Lemma 4.2. We have

$$E[\delta_M] - \left(\frac{w_1}{w_1 + w_2}\delta_1 + \frac{w_2}{w_1 + w_2}\delta_2\right) = E\left[A_{(\lambda_1^{-\eta}, \lambda_2^{-\eta})}\right] + B_{(\lambda_1^{-\eta}, \lambda_2^{-\eta})}$$

for

$$\begin{aligned} A_a &\equiv \sum_{i=1}^2 \frac{a_i (M_i(t))^\eta}{\sum a_i (M_i(t))^\eta}\delta_i - \sum_{i=1}^2 \frac{a_i}{\sum a_i}\delta_i \\ B_{(\lambda_1^{-\eta}, \lambda_2^{-\eta})} &= \sum_{i=1}^2 \frac{\lambda_i^{-\eta}}{\lambda_1^{-\eta} + \lambda_2^{-\eta}}\delta_i - \sum_{i=1}^2 \frac{w_i}{w_1 + w_2}\delta_i \end{aligned}$$

We easily get that $A_{(a_1,a_2)} = \frac{a_1 a_2}{(a_1 + a_2)}(2\delta)\frac{M_1^\eta(t) - M_2^\eta(t)}{a_1 M_1^\eta(t) + a_2 M_2^\eta(t)}$. In this setting, we know that $M_1^\eta(t) = \exp\left(\eta\delta W_t - \frac{1}{2}\eta\delta^2 t\right)$ and $M_2^\eta(t) = \exp\left(-\eta\delta W_t - \frac{1}{2}\eta\delta^2 t\right)$. Then

$$\begin{aligned} E\left[\frac{M_1^\eta(t) - M_2^\eta(t)}{a_1 M_1^\eta(t) + a_2 M_2^\eta(t)}\right] &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} \frac{\exp(\delta\eta x) - \exp(-\delta\eta x)}{a_1\exp(\delta\eta x) + a_2\exp(-\delta\eta x)}\exp\left(-\frac{x^2}{2t}\right) dt \\ &= \frac{(a_2 - a_1)}{a_1 a_2\sqrt{2\pi t}}\int_0^{+\infty} \frac{\exp(2\delta\eta x) + \exp(-2\delta\eta x) - 2}{\exp(2\delta\eta x) + \exp(-2\delta\eta x) + \frac{a_1}{a_2} + \frac{a_2}{a_1}}\exp\left(-\frac{x^2}{2t}\right) dt. \end{aligned}$$

Since $\frac{\exp^y + \exp^{-y} - 2}{\exp^y + \exp^{-y} + c} \leq \max\left(\frac{1}{2+c}, \frac{1}{12}\right) y^2$, we obtain

$$\begin{aligned} 0 &\leq \frac{E[A_a]}{(a_2 - a_1)} \leq \frac{a_1 a_2}{(a_1 + a_2)}(2\delta)\frac{1}{a_1 a_2}\max\left(\frac{1}{2 + \frac{a_1}{a_2} + \frac{a_2}{a_1}}, \frac{1}{12}\right)(2\delta\eta)^2\frac{1}{\sqrt{2\pi t}}\int_0^{+\infty}\frac{1}{2}x^2\exp\left(-\frac{x^2}{2t}\right) dt \\ &\leq \frac{1}{(a_1 + a_2)}\eta^2\delta^3\max\left(\frac{1}{3}, \frac{4}{2 + \frac{a_1}{a_2} + \frac{a_2}{a_1}}\right) \end{aligned}$$

Now, $B_{(\lambda_1^{-\eta}, \lambda_2^{-\eta})} = \frac{2\delta\left[\left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta}\frac{w_2}{w_1} - 1\right]}{\left(1 + \frac{w_2}{w_1}\right)\left(1 + \left(\frac{\lambda_1}{\lambda_2}\right)^{-\eta}\right)}$. The rest of the proof comes from Lemma 4.1. ■

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