

# European Option Pricing and Hedging with both Fixed and Proportional Transaction Costs

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## Abstract

In this paper we extend the utility based option pricing and hedging approach, pioneered by Hodges and Neuberger (1989) and further developed by Davis, Panas, and Zariphopoulou (1993), for the market where each transaction has a fixed cost component. We present a model, where investors have a CARA utility, and derive some properties of reservation option prices. We suggest and implement discretization schemes for computing the reservation option prices. The numerical results of option pricing and hedging are presented for the case of European call options and the investors with different levels of ARA. We also try to reconcile our findings with such empirical pricing bias as the volatility smile.

**Key words:** option pricing, transaction costs, stochastic control, Markov chain approximation.

**JEL classification:** C61, G11, G13.

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# 1 Introduction

The break-through in option valuation theory starts with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). In both papers authors introduced a continuous time model of a complete friction-free market where a price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicates the payoffs of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

In the presence of transaction costs in capital markets the absence of arbitrage argument is no longer valid, since perfect hedging is impossible. Due to the infinite variation of the geometric Brownian motion, the continuous replication policy incurs an infinite amount of transaction costs over any trading interval no matter how small it might be. A variety of approaches have been suggested to deal with the problem of option pricing and hedging with transaction costs. We maintain that the utility based approach, pioneered by Hodges and Neuberger (1989), produces the most “optimal” policies. The rationale under this approach is as follows: Since entering an option contract involves an unavoidable element of risk, in pricing and hedging options one must consider the investor’s attitude toward risk. The other alternative approaches are mainly preference-free and concerned with the “financial engineering” problem of either replicating or super-replicating option payoffs. These approaches are generally valid only in a discrete-time model with a relatively small number of time intervals.

The key idea behind the utility based approach is the indifference argument. The writing price of an option is defined as the amount of money that makes the investor indifferent, in terms of expected utility, between trading in the market with and without writing the option. In a similar way, the purchase price of an option is defined as the amount of money that makes the investor indifferent between trading in the market with and without buying the option. These two prices are also referred to as the investor’s *reservation write price* and the investor’s *reservation purchase price*. In many respects a reservation option price is determined in a similar manner to a *certainty equivalent* within the expected utility framework, which is a well grounded pricing principle in economics.

The utility based option pricing approach is perhaps not entirely satisfactory due to some apparent drawbacks: First, the method does not price options within a general equilibrium framework, and, hence, instead of a unique price one gets two price bounds that depend on the investor's utility function, which is largely unknown. Second, the linear pricing rule from the complete and frictionless market does not apply to the reservation option prices. Generally, the unit reservation purchase price decreases in the number of options, and the unit reservation write price increases in the number of options. Third, the reservation option prices are, to some extent, sensitive to the investor's initial holdings in the stock. Nevertheless, the method is well-defined in contrast to ad-hoc delta hedging in the presence of transaction costs, and, moreover, it yields a narrow price band which is much more interesting than the extreme bounds of a super-replicating strategy<sup>1</sup>. Some attractive features of these bounds are as follows: It can be proved that in a friction-free market the two reservation prices coincide with the Black-Scholes price. The bounds are robust with respect to the choice of utility function since the level of absolute risk aversion seems to be the only important determinant. Judging against the best possible trade-off between the risk and the costs of a hedging strategy, the utility based approach seems to achieve excellent empirical performance (see Martellini and Priaulet (2002), Clewlow and Hodges (1997), and Mohamed (1994)). Quite often one points out that the numerical calculations of reservation option prices are very time-consuming. Considering the exploding development within the computer industry this problem gradually becomes less and less important. All these suggest that the utility based approach is a very reasonable and applicable option pricing method.

The starting point for the utility based option pricing approach is to consider the optimal portfolio selection problem of an investor who faces transaction costs and maximizes expected utility of his terminal wealth. The introduction of transaction costs adds considerable complexity to the utility maximization problem as opposed to the case with no transaction costs. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no

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<sup>1</sup> Shreve, Soner, and Cvitanic (1995) proved, in particular, that in a continuous time model with proportional transaction costs the costs of buying one share of stock is the cheapest super-replicating policy.

transaction costs on trades in the riskless asset. In this case the problem amounts to a *stochastic singular control* problem that was solved by Davis and Norman (1990). Shreve and Soner (1994) studied this problem applying the theory of viscosity solutions to Hamilton-Jacobi-Bellman (HJB) equations (see, for example, Fleming and Soner (1993) for that theory).

The problem is further simplified if the investor's utility function exhibits a constant absolute risk aversion (CARA investor). In this case the option price and hedging strategy are independent of the investor's total wealth and the computational complexity is considerably reduced.

In the presence of proportional transaction costs the solution indicates that the portfolio space is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the no-transaction (NT) region. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the boundary between the Buy region and the NT region, while if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the boundary between the Sell region and the NT region. If a portfolio lies in the NT region, it is not adjusted at that time.

Hodges and Neuberger (1989) introduced the approach and calculated numerically optimal hedging strategies and reservation prices of European call options using a binomial lattice, without really proving the convergence of the numerical method. For simplicity they chose the drift of the risky asset equal to the risk-free rate of return. Davis et al. (1993) rigorously analyzed the same model, showed that the value function of the problem is a unique viscosity solution of a fully nonlinear variational inequality. They proved the convergence of discretization schemes based on the binomial approximation of the stock price, and presented computational results for the reservation write price of an option. Whalley and Wilmott (1997) did an asymptotic analysis of the model of Hodges and Neuberger (1989) and Davis et al. (1993) assuming that transaction costs are small. They showed that the optimal hedging strategy is to hedge to a particular bandwidth<sup>2</sup>. Clewlow and Hodges (1997) extended the earlier work of Hodges and Neuberger (1989) by presenting a more efficient computational method, and a deeper study of the optimal hedging strategy. The further contributions to the study of

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<sup>2</sup>That is, the optimal strategy is not to re hedge until the position moves out of the line with the perfect hedge position by a certain amount.

the utility based option pricing approach in the market with proportional transaction costs was made by Constantinides and Zariphopoulou (1999), Andersen and Damgaard (1999), and some others.

To the best of our knowledge, no one has calculated reservation option prices and hedging strategies in the market with a fixed cost component<sup>3</sup>. The solution to the utility maximization/optimal portfolio selection problem where each transaction has a fixed cost component is more complicated and is based on the theory of *stochastic impulse controls* (see, for example, Bensoussan and Lions (1984) for that theory). The first application of this theory to a consumption-investment problem was done by Eastham and Hastings (1988). They developed a general theory and showed that solving this general problem requires the solution of a system of so-called quasi-variational inequalities (QVI). This initial work was extended by Hastings (1992) and Korn (1998), and was further developed by Øksendal and Sulem (2002) and Chancelier, Øksendal, and Sulem (2000).

In this paper we extend the works of Hodges and Neuberger (1989), Davis et al. (1993), and Clewlow and Hodges (1997), who computed reservation option prices in the model with a CARA investor and the presence of proportional transaction costs only. First, we formulate the option pricing and hedging problem for the CARA investor in the market with both fixed and proportional transaction costs and derive some properties of reservation option prices. Then we numerically solve the problem for the case of European call options applying the method of the Markov chain approximation. The solution indicates that in the presence of both fixed and proportional transaction costs, most of the time, the portfolio space can again be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. The Buy and the NT regions are divided by the lower no-transaction boundary, and the Sell and the NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary.

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<sup>3</sup> Clewlow and Hodges (1997) made computations for a 3-period model in the market with both fixed and proportional transaction costs, without really presenting a continuous-time model for this case.

We examine the effects on the reservation option prices and the corresponding optimal hedging strategies of varying the investor's level of absolute risk aversion (ARA). We find that there are two basic patterns of option pricing and hedging in relation to the investor's level of ARA: For the investors with low ARA, both the reservation option prices are above the corresponding BS-price, and they are very close to each other. For the investors with high ARA, the reservation purchase price is generally below the BS-price, and the reservation write price is above the BS-price. Here the difference between the two prices depends on the level of ARA and the level of transaction costs. Judging against the BS-strategy, the investors with high ARA underhedge out-of-the-money and overhedge in-the-money long option positions. When the investors with high ARA write options, their strategy is quite the opposite. They overhedge out-of-the-money and underhedge in-the-money short option positions. The remarkable features of these strategies are jumps to zero in target amounts in the stock when the stock price decreases below some certain levels. And at these levels the NT region widens.

We point out on two possible resolutions of the question: Under what circumstances will a writer and a buyer agree on a common price for an option? In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. The other possibility arises when a writer and a buyer, both of them being investors with low ARA, face different transaction costs in the market. We also try to reconcile our findings with such empirical pricing bias as the volatility smile. Our general conclusion here is that this empirical phenomenon could not be accounted for solely by the presence of transaction costs.

The rest of the paper is organized as follows. Section 2 presents the continuous-time model and the basic definitions. In Section 3 we derive some important properties of the reservation option prices. Section 4 is concerned with the construction of a discrete time approximation of the continuous time price processes used in Section 2, and the solution method. The numerical results for European call options are presented in Section 5. Section 6 concludes the paper and discusses some possible extensions.

## 2 The Formulation of the Model

We consider a continuous-time economy, similar to that of Øksendal and Sulem (2002), with one risky and one risk-free asset. Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a given filtration  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ . The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of  $r \geq 0$ , and, consequently, the evolution of the amount invested in the bank,  $x_t$ , is given by the ordinary differential equation

$$dx_t = rx_t dt. \quad (1)$$

We will refer to the risky asset to as the stock, and assume that the price of the stock,  $S_t$ , evolves according to a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad (2)$$

where  $\mu$  and  $\sigma$  are constants, and  $B_t$  is a one-dimensional  $\mathcal{F}_t$ -Brownian motion.

The investor holds  $x_t$  in the bank account and the amount  $y_t$  in the stock at time  $t$ . We assume that a purchase or sale of stocks of the amount  $\xi$  incurs a transaction costs consisting of a sum of a fixed cost  $k \geq 0$  (independent of the size of transaction) plus a cost  $\lambda|\xi|$  proportional to the transaction ( $\lambda \geq 0$ ). These costs are drawn from the bank account.

If the investor has the amount  $x_t$  in the bank account, and the amount  $y_t$  in the stock at time  $t$ , his *net wealth* is defined as the holdings in the bank account after either selling of all shares of the stock (if the proceeds are positive after transaction costs) or closing of the short position in the stock and is given by

$$X_t(x, y) = \begin{cases} \max\{x_t + y_t(1 - \lambda) - k, x_t\} & \text{if } y_t \geq 0, \\ x_t + y_t(1 + \lambda) - k & \text{if } y_t < 0. \end{cases} \quad (3)$$

We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. The control of the investor is a pure *impulse control*  $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$ . Here  $0 \leq \tau_1 < \tau_2 < \dots$  are  $\mathcal{F}_t$ -stopping times giving the times when the investor decides to change his portfolio, and  $\xi_j$  are  $\mathcal{F}_{\tau_j}$ -measurable random variables giving

the sizes of the transactions at these times. If such a control is applied to the system  $(x_t, y_t)$ , it gets the form

$$\begin{aligned}
dx_t &= rx_t dt; & \tau_i \leq t < \tau_{i+1} \\
dy_t &= \mu y_t dt + \sigma y_t dB_t; & \tau_i \leq t < \tau_{i+1} \\
x_{\tau_{i+1}} &= x_{\tau_{i+1}}^- - k - \xi_{i+1} - \lambda |\xi_{i+1}|, \\
y_{\tau_{i+1}} &= y_{\tau_{i+1}}^- + \xi_{i+1}.
\end{aligned} \tag{4}$$

We consider an investor with a finite horizon  $[0, T]$  who has utility only of terminal wealth. It is assumed that the investor has a constant absolute risk aversion. In this case his utility function is of the form

$$U(\gamma, W) = -\exp(-\gamma W); \quad \gamma > 0, \tag{5}$$

where  $\gamma$  is a measure of the investor's absolute risk aversion (ARA), which is independent of the investor's wealth.

## 2.1 Utility Maximization Problem without Options

The investor's problem is to choose an admissible trading strategy to maximize  $E_t[U(\gamma, X_T)]$ , i.e., the expected utility of his net terminal wealth, subject to (4). We define the value function at time  $t$  as

$$V(t, x, y) = \sup_{v \in \mathcal{A}(x, y)} E_t^{x, y}[U(\gamma, X_T)], \tag{6}$$

where  $\mathcal{A}(x, y)$  denotes the set of admissible controls available to the investor who starts at time  $t$  with an amount of  $x$  in the bank and  $y$  holdings in the stock. We assume that the investor's portfolio space is divided into two disjoint regions: a *continuation region* and an *intervention region*. The intervention region is the region where it is optimal to make a transaction. We define the *intervention operator* (or the maximum utility operator)  $\mathcal{M}$  by

$$\mathcal{M}V(t, x, y) = \sup_{(x', y') \in \mathcal{A}(x, y)} V(t, x', y'), \tag{7}$$

where  $x'$  and  $y'$  are the new values<sup>4</sup> of  $x$  and  $y$ . In other words,  $\mathcal{M}V(t, x, y)$  represents the value of the strategy that consists in choosing the best trans-

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<sup>4</sup>That is,  $y' = y + \Delta y$  and  $x' = x - k - \Delta y - \lambda |\Delta y|$ , where  $\Delta y$  is the size of transaction.

action. The continuation region is the region where it is not optimal to rebalance the investor's portfolio. We define the continuation region  $D$  by

$$D = \{(x, y); V(t, x, y) > \mathcal{M}V(t, x, y)\}. \quad (8)$$

Now, by giving heuristic arguments, we intend to characterize the value function and the associated optimal strategy: If for some initial point  $(t, x, y)$  the optimal strategy is to not transact, the utility associated with this strategy is  $V(t, x, y)$ . Choosing the best transaction and then following the optimal strategy gives the utility  $\mathcal{M}V(t, x, y)$ . The necessary condition for the optimality of the first strategy is  $V(t, x, y) \geq \mathcal{M}V(t, x, y)$ . This inequality holds with equality when it is optimal to rebalance the portfolio. Moreover, in the continuation region, the application of the dynamic programming principle gives  $\mathcal{L}V(t, x, y) = 0$ , where the operator  $\mathcal{L}$  is defined by

$$\mathcal{L}V(t, x, y) = \frac{\partial V}{\partial t} + rx \frac{\partial V}{\partial x} + \mu y \frac{\partial V}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 V}{\partial y^2}. \quad (9)$$

The subsequent theorem formalizes this intuition.

**Theorem 1.** *The value function  $V$  defined by (6) is the unique viscosity solution of the quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI, or just QVI):*

$$\max \left\{ \mathcal{L}V, \quad \mathcal{M}V - V \right\} = 0, \quad (10)$$

*with the boundary condition*

$$V(T, x, y) = U(\gamma, X_T).$$

The proof of the existence of the solution can be made by following along the lines of the proof of Theorem 3.7 in Øksendal and Sulem (2002), with corrections for no consumption and our finite horizon. In addition, the uniqueness of the solution can be proved in the same manner as in Theorem 3.8 (Comparison Theorem with subsequent Corollary) in Øksendal and Sulem (2002).

It is easy to see from (4) that the amount  $x$  in the bank account at time

$T$  is given by

$$x_T = \frac{x}{\delta(T, t)} - \sum_{i=0}^n \frac{(k + \xi_i + \lambda|\xi_i|)}{\delta(T, \tau_i)}, \quad (11)$$

where  $\delta(T, t)$  is the discount factor defined by

$$\delta(T, t) = \exp(-r(T - t)), \quad (12)$$

$n$  is a random number of transactions in  $[t, T)$ , and  $t \leq \tau_1 < \tau_2 < \dots < \tau_n < T$ . Note from (3) that the net wealth at time  $T$  could be written as

$$X_T = x_T + h(y_T), \quad (13)$$

where  $h(\cdot)$  is some function. Therefore, taking into consideration the investor's utility function defined by (5), we can write

$$\begin{aligned} V(t, x, y) &= \sup_{\mathcal{A}(x, y)} E_t^{x, y}[-\exp(-\gamma X_T)] = \sup_{\mathcal{A}(x, y)} E_t^{x, y}[-\exp(-\gamma(x_T + h(y_T)))] \\ &= \exp(-\gamma \frac{x}{\delta(T, t)}) \sup_{\mathcal{A}(y)} E_t^y \left[ -\exp \left( -\gamma \left( h(y_T) - \sum_{i=0}^n \frac{(k + \xi_i + \lambda|\xi_i|)}{\delta(T, \tau_i)} \right) \right) \right] \\ &= \exp(-\gamma \frac{x}{\delta(T, t)}) Q(t, y), \end{aligned} \quad (14)$$

where  $Q(t, y)$  is defined by  $Q(t, y) = V(t, 0, y)$ . It means that the dynamics of  $y$  through time is independent of the total wealth. In other words, the choice in  $y$  is independent of  $x$ . This representation suggests transformation of (10) into the following QVI for the value function  $Q(t, y)$ :

$$\max \left\{ \mathcal{D}Q(t, y), \sup_{y' \in \mathcal{A}(y)} \exp \left( \gamma \frac{k - (y - y') + \lambda|y - y'|}{\delta(T, t)} \right) Q(t, y') - Q(t, y) \right\} = 0, \quad (15)$$

where  $y'$  is the new value of  $y$ ,  $\mathcal{A}(y)$  denotes the set of admissible controls available to the investor who starts at time  $t$  with  $y$  holdings in the stock, and the operator  $\mathcal{D}$  is defined by

$$\mathcal{D}Q(t, y) = \frac{\partial Q}{\partial t} + \mu y \frac{\partial Q}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 Q}{\partial y^2}. \quad (16)$$

This is an important simplification that reduces the dimensionality of the

problem. Note that the function  $Q(t, y)$  is evaluated in the two-dimensional space  $[0, T] \times \mathbb{R}$ .

In the absence of any transaction costs the solution for the optimal trading strategy is given by (see equation (4.30) in Davis et al. (1993))

$$y^*(t) = \frac{\delta(T, t)}{\gamma} \frac{(\mu - r)}{\sigma^2}. \quad (17)$$

The numerical calculations show that in the presence of both fixed and proportional transaction costs, in most cases, the portfolio space<sup>5</sup> can be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. The Buy and NT regions are divided by the lower no-transaction boundary, and the Sell and NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary.

However, there is generally a time interval, say  $[t_1, t_2]$ , which is close to the terminal date, when the NT region consists of two disjoint sub-regions which, in their turn, divide either the Buy region (when  $\mu > r$ ) or the Sell region (when  $\mu < r$ ) into two parts. Nevertheless, as in the former case, the target boundaries are unique. The rationale for the existence of a second (minor) NT sub-region can be explained in terms of fixed transaction costs. Recall how we define the investor's net wealth (see equation (3)). If the investor's holdings in the stock are positive, he will sell all his shares of the stock on the terminal date only if the proceeds are positive after transaction costs. Putting it another way, the rational investor will not sell his shares of the stock if  $y(1 - \lambda) < k$ . Suppose for the moment that  $y_t \rightarrow 0^+$  for some  $t \in [t_1, t_2]$ . Consider the two alternatives: (i) No trade at  $t$  and thereafter up to the terminal date, and (ii) buy a certain number of shares of the stock at  $t$  in order to move closer to the optimal level of holdings in the model with no transaction costs. In the former case it is almost sure that at the terminal date the holdings in the stock will not exceed the fixed transaction fee  $k$ . That is,  $y_T(1 - \lambda) < k$  *a.s.*, and, thus, it is not optimal to sell shares of the stock. Hence, following the first alternative the investor does not pay

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<sup>5</sup>Here we mean the two dimensional space  $(t, y)$ .

any transaction costs. In the second alternative the investor pays at least round trip transaction costs equal to  $2k$  (we ignore the time value of money). It turns out that the first alternative is better than the second one when  $t$  is close to the terminal date.

For fixed values of  $\mu, \sigma, r, \gamma, \lambda$ , and  $k$  all the NT and target boundaries are functions of the investor's horizon only and do not depend on the investor's holdings in the bank account, so that a possible description of the optimal policy for  $t \in (0, t_1) \cup (t_2, \infty)$  may be given by

$$\begin{aligned} y &= y_u(t) \\ y &= y_u^*(t) \\ y &= y_l^*(t) \\ y &= y_l(t), \end{aligned} \tag{18}$$

where the first and the fourth equations describe the upper and the lower NT boundaries respectively, and the second and the third equations describe the target boundaries. For  $t \in [t_1, t_2]$  a possible description of the optimal policy may be given by

$$\begin{aligned} y &= y_u(t) \\ y &= y_u^*(t) \\ y &= y_l^*(t) \\ y &= y_l(t) \\ y &= y_{2u}(t) \\ y &= y_{2l}(t) = 0. \end{aligned} \tag{19}$$

The first and the fourth equations describe the upper and the lower boundaries of the main NT sub-region. The second and the third equations describe the target boundaries. The last two equations characterize the minor NT sub-region which lies in between  $y = y_{2u}(t) < k$  and  $y = y_{2l}(t) = 0$ . It is always the case that  $y_l < y_l^* < y_u^* < y_u$  and  $y_{2l} < y_{2u}$ . The minor NT region is largely insignificant<sup>6</sup>. Further we will not pay any attention to it in order to keep focus on more important issues.

The analysis of the optimal portfolio policy without options for a CARA investor with a finite horizon and a large set of realistic parameters, as well as the illustration of the case where the NT region consists of two disjoint sub-regions, is beyond the scope of this paper. The interested reader may

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<sup>6</sup>It disappears if we assume that there are no transaction costs charged on cashing out the investor's portfolio on the terminal date.

consult Zakamouline (2002) for details.

If the function  $Q(t, y)$  is known in the NT region, then

$$Q(t, y) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)(y-y_u^*)}{\delta(T,t)}\right) Q(t, y_u^*) & \forall y(t) \geq y_u(t), \\ \exp\left(\gamma \frac{k+(1+\lambda)(y_l^*-y)}{\delta(T,t)}\right) Q(t, y_l^*) & \forall y(t) \leq y_l(t). \end{cases} \quad (20)$$

This follows from the optimal transaction policy described above. That is, if a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest target boundary.

## 2.2 Utility Maximization Problem with Options

Now we introduce a new asset, a cash settled European-style option contract with expiration time  $T$  and payoff  $g(S_T)$  at expiration. For the sake of simplicity, we assume that these options may be bought or sold only at (initial) time  $t$ . This means that there is no trade in options thereafter, between times  $t$  and  $T$ .

Consider an investor who trades in the riskless and the risky assets and, in addition, buys  $\theta > 0$  options ( $\theta$  is a constant) at time  $t$ . This investor we will refer to as the buyer of options. The buyer's problem is to choose an admissible trading strategy to maximize  $E_t[U(\gamma, X_T + \theta g(S_T))]$  subject to (4). We define his value function at time  $t$  as

$$J^b(t, x, y, S, \theta) = \sup_{v \in \mathcal{A}_\theta^b(x, y)} E_t^{x, y}[U(\gamma, X_T + \theta g(S_T))], \quad (21)$$

where  $\mathcal{A}_\theta^b(x, y)$  denotes the set of admissible controls available to the buyer who starts at time  $t$  with an amount of  $x$  in the bank and  $y$  holdings in the stock.

**Definition 1.** The unit reservation purchase price of  $\theta$  European-style options is defined as the price  $P_\theta^b$  such that

$$V(t, x, y) = J^b(t, x - \theta P_\theta^b, y, S, \theta). \quad (22)$$

In other words, the reservation purchase price,  $P_\theta^b$ , is the highest price at which the investor is willing to buy options, and where the investor is indifferent between the two alternatives: (i) a utility maximization problem where he trades in the riskless and risky assets only, and (ii) a utility

maximization problem where the investor, in addition, buys options at price  $P_\theta^b$ .

Consider now an investor who trades in the riskless and the risky assets and, in addition, writes  $\theta > 0$  options. This investor we will refer to as the writer of options. The writer's problem is to choose an admissible trading strategy to maximize  $E_t[U(\gamma, X_T - \theta g(S_T))]$  subject to (4). We define his value function at time  $t$  as

$$J^w(t, x, y, S, \theta) = \sup_{v \in \mathcal{A}_\theta^w(x, y)} E_t^{x, y}[U(\gamma, X_T - \theta g(S_T))], \quad (23)$$

where  $\mathcal{A}_\theta^w(x, y)$  denotes the set of admissible controls available to the writer who starts at time  $t$  with an amount of  $x$  in the bank and  $y$  holdings in the stock.

**Definition 2.** The unit reservation write price of  $\theta$  European-style options is defined as the compensation  $P_\theta^w$  such that

$$V(t, x, y) = J^w(t, x + \theta P_\theta^w, y, S, \theta) \quad (24)$$

That is, the reservation write price,  $P_\theta^w$ , is the lowest price at which the investor is willing to sell options, and where the investor is indifferent between the two alternatives: (i) a utility maximization problem where he trades in the riskless and risky assets only, and (ii) a utility maximization problem where the investor, in addition, writes options at price  $P_\theta^w$ .

**Theorem 2.** *The value functions of both problems (21) and (23) are the unique viscosity solutions of the quasi-variational Hamilton-Jacobi-Bellman inequalities:*

$$\max \left\{ \bar{\mathcal{L}}J, \quad \mathcal{M}J - J \right\} = 0 \quad (25)$$

with the boundary conditions

$$\begin{aligned} J^b(T, x, y, S, \theta) &= U(\gamma, X_T + \theta g(S_T)), \\ J^w(T, x, y, S, \theta) &= U(\gamma, X_T - \theta g(S_T)), \end{aligned}$$

where the operator  $\bar{\mathcal{L}}$  given by

$$\bar{\mathcal{L}}J = \frac{\partial J}{\partial t} + rx \frac{\partial J}{\partial x} + \mu y \frac{\partial J}{\partial y} + \mu S \frac{\partial J}{\partial S} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 J}{\partial y^2} + \sigma^2 y S \frac{\partial^2 J}{\partial y \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 J}{\partial S^2}. \quad (26)$$

The proof can be carried out by following along the lines of the proof of Theorem 1.

As in the case of the optimal portfolio selection problem without options, we can show that the dynamics of  $y$  through time is independent of the total wealth. Therefore

$$\begin{aligned} J^b(t, x, y, S, \theta) &= \exp(-\gamma \frac{x}{\delta(T, t)}) H^b(t, y, S, \theta), \\ J^w(t, x, y, S, \theta) &= \exp(-\gamma \frac{x}{\delta(T, t)}) H^w(t, y, S, \theta), \end{aligned} \quad (27)$$

where  $H^b(t, y, S, \theta)$  and  $H^w(t, y, S, \theta)$  are defined by

$$\begin{aligned} H^b(t, y, S, \theta) &= J^b(t, 0, y, S, \theta), \\ H^w(t, y, S, \theta) &= J^w(t, 0, y, S, \theta). \end{aligned}$$

This suggests transformation of (25) into the following QVI for the value function  $H(t, y, S, \theta)$ :

$$\max \left\{ \bar{\mathcal{D}}H, \sup_{y' \in \mathcal{A}(y)} \exp \left( \gamma \frac{k - (y - y') + \lambda |y - y'|}{\delta(T, t)} \right) H(t, y', S, \theta) - H(t, y, S, \theta) \right\} = 0, \quad (28)$$

where  $y'$  is the new value of  $y$ ,  $\mathcal{A}(y)$  denotes the set of admissible controls available to the investor who starts at time  $t$  with  $y$  holdings in the stock, and the operator  $\bar{\mathcal{D}}$  is defined by

$$\bar{\mathcal{D}}H = \frac{\partial H}{\partial t} + \mu y \frac{\partial H}{\partial y} + \mu S \frac{\partial H}{\partial S} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 H}{\partial y^2} + \sigma^2 y S \frac{\partial^2 H}{\partial y \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 H}{\partial S^2}. \quad (29)$$

We have reduced the dimensionality of the problem by one. Note that the function  $H(t, y, S, \theta)$  is evaluated in the three-dimensional space  $[0, T] \times \mathbb{R} \times \mathbb{R}^+$ . Consequently, after all the simplifications, the unit reservation purchase price of  $\theta$  options is given by (follows from (22), (14), and (27))

$$P_\theta^b(t, S) = \frac{\delta(T, t)}{\theta \gamma} \ln \left( \frac{Q(t, y)}{H^b(t, y, S, \theta)} \right), \quad (30)$$

and the unit reservation write price is given by (follows from (24), (14), and (27))

$$P_\theta^w(t, S) = \frac{\delta(T, t)}{\theta \gamma} \ln \left( \frac{H^w(t, y, S, \theta)}{Q(t, y)} \right). \quad (31)$$

*Remark 1.* Note from (30) and (31) that the unit reservation purchase price, as well as the unit reservation write price, is a function of many parameters,

that is

$$P_\theta^{(\cdot)} = P_\theta^{(\cdot)}(t, \gamma, y, k, S, \theta).$$

*Remark 2.* The solutions to problems (6), (21), and (23) provide the unique reservation option prices and the optimal strategies. We interpret the difference in the two trading strategies, with and without options, as “hedging” the options.

*Remark 3.* Note in particular, that a reservation option price is, to some extent, sensitive to the investor’s initial holdings in the stock (this phenomenon was closely studied in Monoyios (2003)). To avoid ambiguity, in practical applications one usually assumes that the investor has zero holdings in the stock at the initial time  $t$ .

In the absence of any transaction costs the solution for the optimal trading strategy for the writer of options in the  $(y, S)$ -plane is given by (see, for example, equation (4.31) in Davis et al. (1993))

$$y_w^*(t, S, \theta) = \theta S \frac{\partial P_{BS}(t, S)}{\partial S} + \frac{\delta(T, t)}{\gamma} \frac{(\mu - r)}{\sigma^2}, \quad (32)$$

and the solution for the optimal trading strategy for the buyer of options is given by

$$y_b^*(t, S, \theta) = -\theta S \frac{\partial P_{BS}(t, S)}{\partial S} + \frac{\delta(T, t)}{\gamma} \frac{(\mu - r)}{\sigma^2}, \quad (33)$$

where  $P_{BS}(t, S)$  is the price of one option in a market with no transaction costs (i.e., the Black-Scholes price).

As in the case without options, in the presence of both fixed and proportional transaction costs the option holder’s portfolio space can again be divided into three disjoint regions<sup>7</sup> (Buy, Sell, and NT). In a similar manner as before, a possible description of the optimal policy may be given by

$$\begin{aligned} y &= y_u(t, S) \\ y &= y_l^*(t, S) \\ y &= y_u^*(t, S) \\ y &= y_l(t, S). \end{aligned} \quad (34)$$

Note that all these boundaries are now functions of the underlying stock

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<sup>7</sup>To put it more precisely, some of them may have sub-regions. Recall our stipulation that during the presentation we do not pay any attention to the minor NT sub-region.

price. Section 5 of this paper provides illustrations of the optimal portfolio strategy with options.

If the function  $H(t, y, S, \theta)$  (here we suppress the superscripts  $w$  and  $b$ ) is known in the NT region, then

$$H(t, y, S, \theta) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)(y-y_u^*)}{\delta(T,t)}\right) H(t, y_u^*, S, \theta) & \forall y(t, S) \geq y_u(t, S), \\ \exp\left(\gamma \frac{k+(1+\lambda)(y_l^*-y)}{\delta(T,t)}\right) H(t, y_l^*, S, \theta) & \forall y(t, S) \leq y_l(t, S). \end{cases} \quad (35)$$

That is, according to the optimal transaction policy, if a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest target boundary.

### 3 No-Arbitrage Bounds and Properties of Reservation Option Prices

#### 3.1 No-Arbitrage Bounds in Presence of Transaction Costs

In this subsection we want to derive upper and lower bounds for option prices that do not depend on any particular assumptions about the investor's utility function<sup>8</sup>. To put it more precisely, we want to adjust the no-arbitrage pricing bounds derived in Merton (1973) for the presence of both fixed and proportional transaction costs. We will consider cash settled call and put options with exercise price  $K$ .

From both the definition of an option and the absence of arbitrage condition, we have that

$$P_\theta(t, S) \geq 0, \quad (36)$$

where  $P_\theta(t, S)$  is a unit option price of a position of  $\theta$  options for both the buyer and the writer.

**Proposition 1.** *The upper bound for the price of a call option is given by*

$$P_\theta(t, S) \leq S(t) \frac{1+\lambda}{1-\lambda} + \frac{2k}{\theta}. \quad (37)$$

Here we use the condition that the option can never be worth more than the stock. If this relationship is not true, an arbitrageur can make a riskless

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<sup>8</sup>The only requirement that the investor prefers more to less.

profit by buying  $\frac{\theta}{1-\lambda}$  stocks and selling  $\theta$  call options. The upper bound for the price of a put option is the same as in the case of no transaction costs, i.e.,  $K$ .

For European call and put options we can derive tighter lower bounds than relationship (36).

**Proposition 2.** *A lower bound for the price of a European call option is*

$$P_{\theta}(t, S) \geq \max \left[ 0, S(t) \frac{1-\lambda}{1+\lambda} - \frac{k}{\theta} (1 + \delta(t, T)) - K\delta(t, T) \right]. \quad (38)$$

This proposition is an extension of Theorem 1 in Merton (1973) in the presence of both fixed and proportional transaction costs. If this relationship is not true, an arbitrageur can make a riskless profit by shorting  $\frac{\theta}{1+\lambda}$  stocks, buying  $\theta$  call options, and investing the proceeds risk-free.

**Proposition 3.** *A lower bound for the price of a European put option is*

$$P_{\theta}(t, S) \geq \max \left[ 0, K\delta(t, T) - S(t) \frac{1+\lambda}{1-\lambda} - \frac{k}{\theta} (1 + \delta(t, T)) \right]. \quad (39)$$

If this relationship is not true, an arbitrageur can make a riskless profit by borrowing  $\theta K\delta(t, T)$  at the risk-free rate, and buying  $\frac{\theta}{1-\lambda}$  stocks and  $\theta$  put options.

Note that in all the relationships, due to the presence of a fixed transaction fee, the bounds depend on the number of options. These bounds converge to the bounds in the market with only proportional transaction costs when the number of options goes to infinity.

### 3.2 Properties of the Reservation Option Prices

The purpose of this section is to derive some properties of the reservation option prices.

**Theorem 3.** *In a complete and friction-free market the reservation option prices coincide with the no-arbitrage price.*

If it is not true, an arbitrage opportunity is available in the market.

Let us for the moment write the investor's value function of the utility maximization problem without options as  $V(t, \gamma, x, y, k)$ , and the corresponding value function of the utility maximization problem with options

as  $J(t, \gamma, x, y, k, S, \theta)$ . By this we want to emphasize that both the value functions depend on the investor's coefficient of absolute risk aversion and the fixed transaction fee.

**Theorem 4.** *For an investor with exponential utility function and an initial endowment  $(x, y)$  we have*

$$V(t, \gamma, x, y, k) = V(t, \theta\gamma, \frac{x}{\theta}, \frac{y}{\theta}, \frac{k}{\theta}), \quad (40)$$

$$J(t, \gamma, x, y, k, S, \theta) = J(t, \theta\gamma, \frac{x}{\theta}, \frac{y}{\theta}, \frac{k}{\theta}, S, 1). \quad (41)$$

**Proof.** Both these relationships can be easily established from the form of the exponential utility function. In particular, the portfolio process  $\{\frac{x_s}{\theta}, \frac{y_s}{\theta}; s > t\}$  is admissible given the initial portfolio  $(\frac{x_t}{\theta}, \frac{y_t}{\theta})$  and fixed transaction cost fee  $\frac{k}{\theta}$  if and only if  $\{x_s, y_s; s > t\}$  is admissible given the initial portfolio  $(x_t, y_t)$  and fixed transaction cost fee  $k$ . Furthermore,  $U(\gamma, X_T) = U(\theta\gamma, \frac{X_T}{\theta})$  and  $U(\gamma, X_T \pm \theta P_\theta) = U(\theta\gamma, \frac{X_T}{\theta} \pm P_\theta)$ .  $\square$

**Corollary 5.** *For an investor with the exponential utility function and an initial holding in the stock  $y$  we have*

$$Q(t, \gamma, y, k) = Q(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta}), \quad (42)$$

$$H(t, \gamma, y, k, S, \theta) = H(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta}, S, 1). \quad (43)$$

**Proof.** This follows from Theorem 4 and the definitions of the value functions  $Q$  and  $H$ .

**Theorem 6.** *For an investor with exponential utility function we have that*

1. *An investor with an initial holding in the stock  $y$ , ARA coefficient  $\gamma$ , and the fixed transaction fee  $k$  has a unit reservation purchase price of  $\theta$  options equal to his reservation purchase price of one option in the case where he has an initial holding in the stock  $\frac{y}{\theta}$ , ARA coefficient  $\theta\gamma$  and the fixed transaction fee  $\frac{k}{\theta}$ . That is,*

$$P_\theta^b(t, S) = \frac{\delta(T, t)}{\gamma} \ln \left( \frac{Q(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta})}{H^b(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta}, S, 1)} \right). \quad (44)$$

2. An investor with an initial holding in the stock  $y$ , ARA coefficient  $\gamma$ , and the fixed transaction fee  $k$  has a unit reservation write price of  $\theta$  options equal to his reservation write price of one option in the case where he has an initial holding in the stock  $\frac{y}{\theta}$ , ARA coefficient  $\theta\gamma$  and the fixed transaction fee  $\frac{k}{\theta}$ . That is,

$$P_{\theta}^w(t, S) = \frac{\delta(T, t)}{\gamma} \ln \left( \frac{H^w(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta}, S, 1)}{Q(t, \theta\gamma, \frac{y}{\theta}, \frac{k}{\theta})} \right). \quad (45)$$

**Proof.** This follows from Theorem 4, the definitions of the value functions  $Q$  and  $H$ , Corollary 5, and equations (22) and (24).  $\square$

As mentioned above, in the practical applications of the utility based option pricing method one assumes that the investor has zero holdings in the stock at the initial time  $t$ , i.e.,  $y = 0$ , hence  $\frac{y}{\theta} = 0$  as well. In this case Theorem 6 says that the resulting unit reservation option price and the corresponding optimal hedging strategy<sup>9</sup> in the model with the triple of parameters  $(\gamma, k, \theta)$  will be the same as in the model with  $(\theta\gamma, \frac{k}{\theta}, 1)$ . That is, instead of calculating a model with  $\theta$  options we can calculate a model with 1 option only. All we need is adjusting the two parameters for  $\theta$ : the absolute risk aversion from  $\gamma$  to  $\theta\gamma$ , and the fixed transaction fee from  $k$  to  $\frac{k}{\theta}$ .

**Conjecture 1.** For an investor with exponential utility function, an initial holding in the stock  $y = 0$ , and the fixed transaction fee  $k = 0$  we have that

1. The unit reservation purchase price,  $P_{\theta}^b(t, S)$ , is decreasing in the number of options  $\theta$ .
2. The unit reservation write price,  $P_{\theta}^w(t, S)$ , is increasing in the number of options  $\theta$ .

The above conjecture is quite intuitive. When there are transaction costs in the market, holding options involves an unavoidable element of risk. Therefore, the greater number of options the investor holds, the more risk he takes. When, in particular, there are only proportional transaction costs, according to the pricing formulas in Theorem 6 an increase in  $\theta$  corresponds

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<sup>9</sup>Here, the hedging strategy per option. For  $\theta$  options the strategy must be re-scaled accordingly.

only to an increase in the investor’s “pseudo”  $ARA = \theta\gamma$ . Consequently, the more options the risk averse investor has to buy, the less he is willing to pay per option. Similarly, the seller of options will demand a unit price which is increasing in the number of options. When the fixed transaction fee  $k \neq 0$ , the dependence of the unit reservation price on the number of options is not obvious. The unit reservation write price can, for example, first decrease<sup>10</sup> and then increase when the number of options increases. Note, in particular, that the linear pricing rule from the complete and frictionless market does not apply to the reservation option prices.

## 4 A Markov Chain Approximation of the Continuous Time Problem

The main objective of this section is to present numerical procedures for computing the investor’s value functions and the corresponding optimal trading policies. To find the solutions of continuous-time continuous-space stochastic control problems described by (10) and (25), we apply the method of the Markov chain approximation suggested by Kushner (see, for example, Kushner and Martins (1991) and Kushner and Dupuis (1992)). The basic idea involves a consistent approximation of the problem under consideration by a Markov chain, and then the solution of an appropriate optimization problem for the Markov chain model.

First, according to the Markov chain approximation method, we construct discrete time approximations of the continuous time price processes used in the continuous time model presented in Section 2. Then the discrete time program is solved by using the discrete time dynamic programming algorithm (i.e., the backward recursion algorithm).

Consider the partition  $0 = t_0 < t_1 < \dots < t_n = T$  of the time interval  $[0, T]$  and assume that  $t_i = i\Delta t$  for  $i = 0, 1, \dots, n$  where  $\Delta t = \frac{T}{n}$ . Let  $\varepsilon$  be a stochastic variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1 - p. \end{cases}$$

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<sup>10</sup>Note that the fixed transaction fee per option is decreasing in the number of options.

We define the discrete time stochastic process of the stock as

$$S_{t_{i+1}} = S_{t_i} \varepsilon, \quad (46)$$

and the discrete time process of the risk-free asset as

$$x_{t_{i+1}} = x_{t_i} \rho. \quad (47)$$

If we choose  $u = e^{\sigma\sqrt{\Delta t}}$ ,  $d = e^{-\sigma\sqrt{\Delta t}}$ ,  $\rho = e^{r\Delta t}$ , and  $p = \frac{1}{2} \left[ 1 + \frac{\mu}{\sigma} \sqrt{\Delta t} \right]$ , we obtain the binomial model proposed by Cox, Ross, and Rubinstein (1979). An alternative choice is  $u = e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}$ ,  $d = e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$ ,  $\rho = e^{r\Delta t}$ , and  $p = \frac{1}{2}$ , which was proposed by He (1990). As  $n$  goes to infinity, the discrete time processes (46) and (47) converge in distribution to their continuous counterparts (2) and (1). This is what is called the *local consistency conditions* for a Markov chain.

The following discretization scheme is proposed to find the value function  $V(t, x, y)$  defined by QVI (10):

$$V^{\Delta t}(t_i, x, y) = \max \left\{ \begin{aligned} &\max_m V^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y), \\ &\max_m V^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y), \\ &E\{V^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)\} \end{aligned} \right\}, \quad (48)$$

where  $m$  runs through the positive integer numbers ( $m = 0, 1, 2, \dots$ ), and

$$\begin{aligned} &V^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y) \\ &= E \left\{ V^{\Delta t}(t_{i+1}, (x - k - (1 + \lambda)m\delta y)\rho, (y + m\delta y)\varepsilon) \right\}, \end{aligned} \quad (49)$$

$$\begin{aligned} &V^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y) \\ &= E \left\{ V^{\Delta t}(t_{i+1}, (x - k + (1 - \lambda)m\delta y)\rho, (y - m\delta y)\varepsilon) \right\}, \end{aligned} \quad (50)$$

as at time  $t_i$  we do not know yet the value function. In this case we use the known values at the next time instant,  $t_{i+1}$ . Here we have discretized the  $y$ -space in a lattice with grid size  $\delta y$ , and the  $x$ -space in a lattice with grid size  $\delta x$ <sup>11</sup>. This scheme is a dynamic programming formulation of the

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<sup>11</sup>It is supposed that  $\lim_{\Delta t \rightarrow 0} \delta y \rightarrow 0$ , and  $\lim_{\Delta t \rightarrow 0} \delta x \rightarrow 0$ , that is,  $\delta y = c_y \Delta t$ , and

discrete time problem. The solution procedure is as follows. Start at the terminal date and give the value function values by using the boundary conditions as for the continuous value function over the discrete state space. Then work backwards in time. That is, at every time instant  $t_i$  and every particular state  $(x, y)$ , by knowing the value function for all the states in the next time instant,  $t_{i+1}$ , find the investor's optimal policy. This is carried out by comparing maximum attainable utilities from buying, selling, or doing nothing.

**Theorem 7.** *The solution  $V^{\Delta t}$  of (48) converges weakly to the unique continuous viscosity solution of (10) as  $\Delta t \rightarrow 0$ .*

For a rigorous treatment of a proof of this type of convergence theorems, we refer the reader to, for example, Kushner and Martins (1991), Davis et al. (1993), and Davis and Panas (1994). Instead of presenting a cumbersome proof of Theorem 7, we want to prove a simple proposition:

**Proposition 4.** *Assuming  $\lim_{\Delta t \rightarrow 0} V^{\Delta t} = V$ , the solution of discrete time program (48) converges to the solution of continuous time quasi-variational inequalities (10) as  $\Delta t \rightarrow 0$ .*

**Proof.** We choose the choice of  $u$ ,  $d$ ,  $\rho$ , and  $p$  which was proposed by He (1990). This choice clearly satisfies the local consistency conditions. In this case we can approximate the dynamics of the controlled processes as

$$\begin{aligned} y(t + \Delta t) - y(t) &= y(t)\mu\Delta t \pm y(t)\sigma\sqrt{\Delta t}, \\ x(t + \Delta t) - x(t) &= x(t)r\Delta t. \end{aligned}$$

Consider the term  $E \{V^{\Delta t}(t + \Delta t, x(t + \Delta t), y(t + \Delta t))\}$ . Assuming that  $V^{\Delta t}$  is differentiable (in the viscosity sense), using the Taylor expansion of  $V^{\Delta t}$  around  $(t, x, y)$ , and taking the expectation we get

$$\begin{aligned} &E \{V^{\Delta t}(t + \Delta t, x(t + \Delta t), y(t + \Delta t))\} \\ &= V^{\Delta t}(t, x, y) + (V_t^{\Delta t} + rxV_x^{\Delta t} + \mu yV_y^{\Delta t} + \frac{1}{2}\sigma^2 y^2 V_{yy}^{\Delta t})\Delta t + o(\Delta t), \end{aligned}$$

where  $o(\Delta t)$  are error terms containing  $\Delta t$  of order higher than one. Allow-  


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 $\delta x = c_x \Delta t$  for some constants  $c_y$  and  $c_x$ .

ing  $\Delta t \rightarrow 0$  we obtain

$$\lim_{\Delta t \rightarrow 0} E \{V^{\Delta t}(t + \Delta t, x(t + \Delta t), y(t + \Delta t))\} = V(t, x, y) + \mathcal{L}V(t, x, t)dt. \quad (51)$$

Moreover, as  $\Delta t \rightarrow 0$ ,

$$\lim_{\Delta t \rightarrow 0} \max_m V^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y) = \sup_{\Delta y \geq 0} V(t, x - k - (1 + \lambda)\Delta y, y + \Delta y), \quad (52)$$

$$\lim_{\Delta t \rightarrow 0} \max_m V^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y) = \sup_{\Delta y \geq 0} V(t, x - k + (1 - \lambda)\Delta y, y - \Delta y). \quad (53)$$

Note here that  $\Delta y$  can take any positive real value. Allowing  $\Delta y$  to take both positive and negative values, we can combine (52) and (53) and, thus, get

$$\begin{aligned} & \sup_{\Delta y} V(t, x - k - \Delta y - \lambda|\Delta y|, y + \Delta y) \\ &= \lim_{\Delta t \rightarrow 0} \max \begin{cases} \max_m V^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y), \\ \max_m V^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y). \end{cases} \end{aligned} \quad (54)$$

By definition (7), the left hand side of (54) is nothing else than the maximum utility operator, that is

$$\sup_{\Delta y} V(t, x - \Delta y - \lambda|\Delta y|, y + \Delta y) = \mathcal{M}V(t, x, y). \quad (55)$$

Therefore in the limit as  $\Delta t \rightarrow 0$  the discrete time program (48) converges to (using (51), (54), and (55))

$$V(t, x, y) = \max\{V(t, x, y) + \mathcal{L}V(t, x, t)dt, \quad \mathcal{M}V(t, x, y)\},$$

which can be rewritten as

$$\max\{\mathcal{L}V(t, x, t), \quad \mathcal{M}V(t, x, y) - V(t, x, y)\} = 0.$$

This completes the proof.  $\square$

The following discretization scheme is proposed to find the value function

$J(t, x, y, S)$  defined by QVI (25):

$$J^{\Delta t}(t_i, x, y, S) = \max \left\{ \begin{aligned} &\max_m J^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y, S), \\ &\max_m J^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y, S), \\ &E\{J^{\Delta t}(t_{i+1}, x\rho, y\varepsilon, S\varepsilon)\} \end{aligned} \right\}, \quad (56)$$

where  $m$  runs through the positive integer numbers, and

$$\begin{aligned} &J^{\Delta t}(t_i, x - k - (1 + \lambda)m\delta y, y + m\delta y, S) \\ &= E \{ J^{\Delta t}(t_{i+1}, (x - k - (1 + \lambda)m\delta y)\rho, (y + m\delta y)\varepsilon, S\varepsilon) \}, \end{aligned} \quad (57)$$

$$\begin{aligned} &J^{\Delta t}(t_i, x - k + (1 - \lambda)m\delta y, y - m\delta y, S) \\ &= E \{ J^{\Delta t}(t_{i+1}, (x - k + (1 - \lambda)m\delta y)\rho, (y - m\delta y)\varepsilon, S\varepsilon) \}. \end{aligned} \quad (58)$$

The principle behind this scheme is the same as for the discretization scheme (48). As before, we have discretized the  $y$ -space in a lattice with grid size  $\delta y$ , and the  $x$ -space in a lattice with grid size  $\delta x$ . In addition, we use a binomial tree for the stock price process.

**Theorem 8.** *The solution  $J^{\Delta t}$  of (56) converges weakly to the unique viscosity solution of the continuous time problem characterized by (25) as  $\Delta t \rightarrow 0$ .*

The proof follows along similar arguments as in Theorem 7.

All the discretization schemes described above are valid for any type of utility function. Note that for a general utility function we need to perform the calculations first in a three-dimensional space  $(t, x, y)$  and then in a four-dimensional space  $(t, x, y, S)$ , and the amount of computations is very high. For the negative exponential utility function the dynamics of  $y$  through time is independent of the total wealth in the continuous time and in the discrete time framework as well. Therefore (14) and (27) can be written as follows:

$$\begin{aligned} V^{\Delta t}(t, x, y) &= \exp(-\gamma \frac{x}{\delta(T,t)}) Q^{\Delta t}(t, y), \\ J^{b,\Delta t}(t, x, y, S, \theta) &= \exp(-\gamma \frac{x}{\delta(T,t)}) H^{b,\Delta t}(t, y, S, \theta), \\ J^{w,\Delta t}(t, x, y, S, \theta) &= \exp(-\gamma \frac{x}{\delta(T,t)}) H^{w,\Delta t}(t, y, S, \theta). \end{aligned} \quad (59)$$

The discretization scheme for the function  $Q^{\Delta t}(t, y)$  is derived from (48) and

(59) to be

$$Q^{\Delta t}(t_i, y) = \max \left\{ \begin{aligned} &\max_m \exp \left( \gamma \frac{k + (1 + \lambda)m\delta y}{\delta(T, t_i)} \right) Q^{\Delta t}(t_i, y + m\delta y), \\ &\max_m \exp \left( \gamma \frac{k - (1 - \lambda)m\delta y}{\delta(T, t_i)} \right) Q^{\Delta t}(t_i, y - m\delta y), \\ &E\{Q^{\Delta t}(t_{i+1}, y\varepsilon)\}. \end{aligned} \right. \quad (60)$$

As in the continuous time case, if the value function  $Q^{\Delta t}(t_i, y)$  is known in the NT region, then it can be calculated in the Buy and Sell region by using the discrete space version of (20):

$$Q^{\Delta t}(t_i, y) = \begin{cases} \exp \left( \gamma \frac{k - (1 - \lambda)(y - y_u^*)}{\delta(T, t_i)} \right) Q^{\Delta t}(t_i, y_u^*) & \forall y(t_i) \geq y_u(t_i), \\ \exp \left( \gamma \frac{k + (1 + \lambda)(y_l^* - y)}{\delta(T, t_i)} \right) Q^{\Delta t}(t_i, y_l^*) & \forall y(t_i) \leq y_l(t_i). \end{cases} \quad (61)$$

In the same manner we can derive from (56) and (59) the discretization schemes for the value functions  $H^{b, \Delta t}(t, y, S, \theta)$  and  $H^{w, \Delta t}(t, y, S, \theta)$ .

Davis et al. (1993) and Damgaard (2000b) used only one discretization scheme analogous to (56) for calculating both the value functions<sup>12</sup>  $Q^{\Delta t}(t, y)$  and  $H^{\Delta t}(t, y, S)$  ( $V^{\Delta t}(t, x, y)$  and  $J^{\Delta t}(t, x, y)$  in the work of Damgaard (2000b), since for the CRRA utility function one cannot reduce the dimensionality of the problem). We propose to use different discretization schemes as the evaluation of the value function without options is a much easier task than the evaluation of the value function with options. Consequently, our method of calculating the value function  $Q^{\Delta t}(t, y)$  is much more efficient.

The practical implementation of the numerical scheme for  $Q^{\Delta t}(t, y)$  is based on the qualitative knowledge of the form of the optimal trading strategy. That is, at every time  $t$  the optimal strategy is completely described by four numbers:  $y_l(t)$  - the lower boundary of the NT region,  $y_u(t)$  - the upper boundary of the NT region,  $y_u^*(t)$  - the Sell target boundary, and  $y_l^*(t)$  - the Buy target boundary. This qualitative knowledge of the optimal portfolio strategy can be exploited to build an efficient practical realization of a Markov chain approximation scheme. The idea is to implement the maximum utility operator  $\mathcal{M}V(t, x, y)$  as a function `MaxUtilityOp(t, x, y, newy)`, which returns `true` when the optimal strategy for  $(t, x, y)$  is to transact, and

<sup>12</sup> Hodges and Neuberger (1989) and Clewlow and Hodges (1997) avoided the evaluation of the value function  $V$  by choosing  $\mu = r$ .

`false` otherwise. In the former case (if the function returns `true`) the variable `newy` contains the target amount in  $y$ . Assuming we know the value function at  $t_{i+1}$  and that  $y_l(t_i) \in (y_{min}, y_{max})$ , a schematic computer program of the *bisection* algorithm for the search of the lower boundary of the NT region and the Buy target boundary at  $t_i$  can be implemented as follows:

```

a=yMin;
b=yMax;
while (b-a) >  $\delta y$ 
    c=(b+a)/2;
    if MaxUtilityOp(t,x,c,newy)
        a=c;
    else
        b=c;
end;

```

It is supposed that the point  $(t_i, x, y_{min})$  lies in the Buy region and the point  $(t_i, x, y_{max})$  lies inside the NT region. Bisection proceeds by evaluating the maximum utility operator at the midpoint of the original interval  $c = (b + a)/2$  and testing to see in which of the subintervals  $[a, c]$  or  $[c, b]$  the boundary of the NT region lies. The procedure is then repeated with the new interval as often as needed to locate the solution with the desired accuracy. A similar algorithm could be used to search for the upper boundary of the NT region and the Sell target boundary. The value function outside the NT region is determined in accordance with (61).

The practical realization of the numerical scheme for  $H^{\Delta t}(t, y, S)$  is analogous to that of  $Q^{\Delta t}(t, y)$  with the correction for an additional discrete space for  $S$ .

## 5 Numerical Results

In this section we present the results of our numerical computations of reservation purchase and write prices and the corresponding hedging strategies for European call options. In most of our calculations we used the following model parameters: the risky asset price at time zero ( $t = 0$ )  $S_0 = 100$ , the strike price  $K = 100$ , the volatility  $\sigma = 20\%$ , the drift  $\mu = 10\%$ , and the

risk-free rate of return  $r = 5\%$  (all in annualized terms). The options expire at  $T = 1$  year. The proportional transaction costs  $\lambda = 1\%$  and the fixed transaction fee  $k = 0.5$ . The discretization parameters of the Markov chain, depending on the investor's ARA, are:  $n \in [100, 150]$  periods of trading, and the grid size  $\delta y \in [0.001, 0.1]$ . For high levels of the investor's ARA we cannot increase the number of periods of trading beyond some threshold as the values of the exponential utility are either overflow or underflow.

The number of options is always 1 in all our calculations. Recall that, according to Theorem 6, the resulting unit reservation option price and the corresponding optimal hedging strategy in the model with the triple of parameters  $(\gamma, k, 1)$  is the same as in the model with  $(\frac{\gamma}{\theta}, \theta k, \theta)$ . This means that if, for example, we choose  $\gamma = 1$ ,  $k = 0.5$ , and  $\theta = 1$ , then we get the same unit reservation option price as in the model with  $\gamma = 0.01$ ,  $k = 50$ , and  $\theta = 100$ . For the sake of comparison, we also present the results of the numerical computations of the corresponding Black-Scholes (BS) prices and the reservation option prices in the model with proportional transaction costs only (that is, when  $k = 0$ ).

We begin our presentation with the study of how reservation option prices depend on the level of the investor's absolute risk aversion  $\gamma$ . Hodges and Neuberger (1989), Davis et al. (1993), and Clewlow and Hodges (1997) operated only with  $\gamma = 1$  and found that the reservation purchase price is below, and the reservation write price is above the corresponding BS-price. Lo, Mamaysky, and Wang (2001) calibrated  $\gamma$  in their model to be between 0.0001 and 5.0. We see that  $\gamma = 1$  lies in the upper end of the interval and corresponds to a very high risk aversion. Damgaard (2000a) and Damgaard (2000b) studied the sensitivity of reservation option prices to the investor's relative risk aversion (RRA) and the level of the investor's initial wealth. He found that the above mentioned pattern, when the reservation purchase price is below, and the reservation write price is above the BS-price, is valid only for either low levels of the investor's initial wealth or high levels of RRA. When either the investor's initial wealth increases or RRA decreases, both the reservation option prices approach the horizontal asymptote located above the BS-price. Either a higher wealth or a lower RRA for a CRRA utility corresponds to a lower ARA<sup>13</sup>. This suggests that it is the level of ARA that influences the reservation option prices.

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<sup>13</sup>Recall that  $ARA = \frac{RRA}{w}$ , where  $ARA \equiv \gamma$  and  $w$  is the investor's wealth.

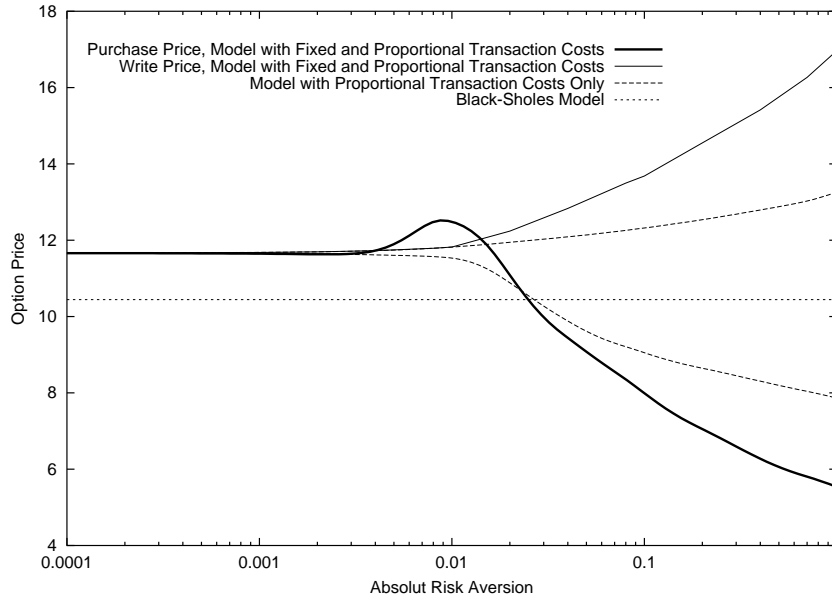


Figure 1: Reservation prices of European call options versus the investor's absolute risk aversion  $\gamma$ .

The results of the numerical computations are presented in Figure (1). Our general findings agree with the findings<sup>14</sup> presented in Damgaard (2000a). In particular, on the basis of studying the Figure, we can make the following observations concerning the reservation option prices: The reservation write price is always above the corresponding BS-price and is an increasing function of  $\gamma$ . The reservation purchase price is a decreasing function of  $\gamma$  in the model with  $k = 0$ , and for the most part the reservation purchase price is a not increasing function of  $\gamma$  in the model with  $k > 0$ . For high values of  $\gamma$  the reservation purchase price is below the BS-price, and for low values of  $\gamma$  the reservation purchase price is above the BS-price. As  $\gamma$  decreases, both the reservation option prices approach a horizontal asymptote located above the BS-price. Here, for low values of  $\gamma$ , the reservation option prices are virtually independent of the choice of  $\gamma$  and are very close to each other.

One can note that for  $k > 0$  and high values of  $\gamma$  the reservation write price is above, and the reservation purchase price is below than the corresponding prices in the model with  $k = 0$  (as one quite logically expects).

<sup>14</sup>Note that Damgaard calculated reservation option prices in the model with proportional transaction costs only.

However, for low value of  $\gamma$  the reservation option prices are practically identical in both the models, with and without a fixed fee component, which seems to be counterintuitive. Moreover, when transaction costs have a fixed fee component, there is an interval  $\gamma \in (0.006, 0.014)$  for which the reservation purchase price is higher than the reservation write price,  $P^b > P^w$ . Note that in the model with proportional transaction costs only, the reservation purchase price seems to be always lower than the reservation write price,  $P^b < P^w$ .

When the stock drift is equal to the risk-free interest rate (we do not present a figure for this case, just give the qualitative description), that is when  $\mu = r = 5\%$ , the reservation write price is always above the BS-price and the reservation purchase price is always below the BS-price. The reservation prices are located more or less symmetrically on each side of the BS-price. The parameter  $\gamma$  seems to influence only the magnitude of the deviation of reservation option prices from the corresponding BS-prices. The higher  $\gamma$  is the more a reservation option price deviates from the corresponding BS-price. The deviation is greater in the model with both fixed and proportional transaction costs than in the model with proportional transaction costs only. When  $\mu > r$  and  $\mu$  rises above some threshold value<sup>15</sup>, we obtain the same dependency as in Figure (1) regardless of how high  $\mu$  we define. Moreover, for high values of  $\gamma$ , for example when  $\gamma = 1$ , the reservation option prices are practically independent of the drift of the underlying stock.

Now we present the intuition behind the observed dependence of the reservation option prices on the parameter  $\gamma$ . The basic idea here is that it is optimal for an investor to take a certain amount of risk, depending on the investor's level of risk aversion, if the risk is properly rewarded (see equation (17)). In the model of Hodges and Neuberger (1989) the investor is "forced" to hold an option contract, which presents some risk to the investor. The question of critical importance is whether or not the risk from an option contract exceeds the optimal amount of risk the investor is willing to take. If the option risk is large, this risk needs to be hedged away by some proper trading strategy. In this case the hedging part of the investor's overall portfolio strategy is prevailing and a larger part of the investor's holdings in the stock is devoted to hedge the risk of options. That is, the

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<sup>15</sup>Threshold is approximately 7% when  $\lambda = 1\%$ .

investor's "net" behavior is hedging. Otherwise, if the option risk is small, the investing part of the investor's overall portfolio strategy is prevailing, and the investor's "net" behavior is investing.

Let us elaborate on this a bit further. Roughly speaking, we suppose that the investor's overall portfolio problem can be separated into an investment problem and a hedging problem<sup>16</sup>. After such a separation it is possible to determine which problem is "bigger" in terms of the funds used to resolve every problem. If the hedging problem prevails, we can call such an investor a "net" hedger. If not, we can call an investor a "net" investor.

In the presence of transaction costs the hedging of options is costly. The amount of hedging transaction costs increases when the option holder's risk aversion increases (a more risk averse option holder hedges options more often). These hedging transaction costs reduce the reservation purchase price and increase the reservation write price. That is, the more risk averse the option buyer is, the less he is willing to pay per an option. Similarly, the writer of an option will demand a unit price which increases as the writer's risk aversion increases.

On the other hand, as the option holder's risk aversion decreases, he invests more wealth in the risky stock and gradually the hedging strategy becomes "absorbed" by the investing strategy. Since that moment a reservation option price does not depend on the measure of the investor's absolute risk aversion. That is why a reservation option price approaches a horizontal asymptote as  $\gamma$  decreases. Now we turn on to explain why this horizontal asymptote located above the option price in the market with no transaction costs.

Consider an investor who invests in the risky and the risk-free assets and, in addition, buys some number of options. Since the payoffs from a call option and the stock are positively correlated, an option serves as a substitute for the stock. Investing in options causes the investor to invest less in the stock in order to maintain the amount of undertaken risk at the optimal level. Thus, it reduces transaction costs payed in the stock market, and these savings increase the reservation purchase price. On the contrary, the investor who writes options needs to buy additional number of shares of the stock at time zero and sell them on the terminal date in order to hedge

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<sup>16</sup>Clearly, the possibility of such a separation is obvious in the no transaction costs case. Just look at equations (17), (32), and (33).

the short option position. These additional transaction costs increase the reservation write price.

As a result, both of the “net” investor’s reservation option prices are above the BS-price, and they are very close to each other. Approximately, the “net” investor’s reservation (either purchase or write) option price can be calculated using the formula<sup>17</sup>

$$P = P_{BS} + 2\Delta_{BS}(0)S_0\lambda, \quad (62)$$

where  $P_{BS}$  and  $\Delta_{BS}(0)$  is the option price and the option delta at time zero, respectively, in the model with no transaction costs (Black-Scholes price and delta). Note that the level of proportional transaction costs fully explains the magnitude of the discrepancy between the option prices with and without transaction costs. Note in addition that the level of the fixed transaction fee does not influence the price of an option, because a “net” investor pays the same fixed costs regardless of the presence of an option.

For the chosen model parameters,  $\gamma \approx 0.01$  can serve as a point of division between the “net” investor’s ( $\gamma < 0.01$ ) and the “net” hedger’s ( $\gamma > 0.01$ ) behavior. Around this division point the option risk is roughly equal to the optimal level of risk an investor is willing to take. In this case it is optimal for the buyer of options not to invest in the stock. We can interpret this situation as follows: The buyer moves his risky investment into options and goes out of the stock market. In this case, buying options on the stock instead of buying the stock saves the buyer from both fixed and proportional transaction costs. Thus, the reservation purchase price goes up, and may exceed, under certain model parameters, the reservation write price.

There has been one unresolved question in the utility based option pricing framework with transaction costs: Under what circumstances will a writer and a buyer agree on a common price for an option? Generally in the model with only proportional transaction costs  $P_\theta^w > P_\theta^b$  if all parameters are the same for all the calculations. In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. Thus, the agreement is possible. We indicate another possibil-

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<sup>17</sup>This is a conjecture which is confirmed by comparison with the numerically calculated reservation option prices.

ity for such an agreement. It exploits the fact that the reservation purchase price for a “net” investor lies above the BS-price. Note that this possibility exists also for the case with proportional transaction costs only.

The other possibility for the agreement might arise in the situation when a writer and a buyer, both of them being “net” investors in the underlying stocks, face different transaction costs in the market. Indeed, in real markets the commissions one pays on purchase, sale, and short borrowing are negotiated and depend on the annual volume of trading, as well as on the investor’s other trading practices. In order to model realistic transaction costs one usually distinguishes between two classes of investors: large and small (see, for example, Dermody and Prisman (1993)). Large investors are defined as those who frequently make large trades in “blocks” (defined as 10,000 shares or more) via the block trading desks or brokerage houses. Large investors usually face transaction costs schedule with no minimum fee specified. In contrast, small investors are defined as those who use retail brokerage firms and often trade in 100-share round lots. For small investors there is a minimum fee on any trade. The main point is that the small investors have higher commission rates than the large ones. In other words, the large investors have lower level of proportional transaction costs than the small ones. In this case the reservation write price might be less than the reservation purchase price <sup>18</sup>, that is, the large investors could sell options to the small investors at an acceptable price.

We now turn to the analysis of how an option is hedged in the market with both fixed and proportional transaction costs. For this analysis we will consider an investor with a high level of ARA, that is, a “net” hedger.

As it was described in Section 2, in the presence of both fixed and proportional transaction costs, most of the time, the investor’s portfolio space can be divided into three disjoint regions (Buy, Sell, and NT), and the optimal policy is described by four boundaries. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary. In Figures (2) and (3) all the four boundaries are converted to control limits comparable to the BS-delta, so the results of our numerical computations may be considered as an extension of the results presented

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<sup>18</sup>This situation could be easily deduced from pricing formula (62).

in Hodges and Neuberger (1989) and Clewlow and Hodges (1997). All the model parameters are approximately the same except for the introduction of a fixed cost component.

We define *moneyness* as the ratio between the strike price and the futures stock price, i.e.,  $M = \frac{K}{Se^{(T-t)r}}$ . We will refer to an option as at-the-money if its  $M = 1$ , out-of-the-money if its  $M > 1$ , and in-the-money if  $M < 1$ . As it is seen from Figures (2) and (3), the target *average*<sup>19</sup> delta for at-the-money options is very close to the Black-Scholes delta. The buyer of options underhedges out-of-the-money options and overhedges in-the-money options as compared to the BS-strategy. On the contrary, the writer of options overhedges out-of-the-money options and underhedges in-the-money options. These observations are also valid for the model with proportional transaction costs only, but in that model the degree, to which the hedger over/underhedges an option, is less. Note that the NT region and the distance between the two target boundaries are larger for the buyer than for the writer of options. This reflects the fact that an option is more risky for the writer than for the buyer. Hence, the writer hedges an option more frequently, and, thereby, charges a greater risk premium<sup>20</sup> than the buyer.

The most remarkable features of the “net” hedger’s strategies are jumps to zero in target deltas when the stock price decreases below some certain levels. At these levels the NT regions widen. Especially the picture is clear for the writer of options (see Figure (3)) at  $S \approx 70$ . This behavior is fairly easy to understand. When transaction costs have a fixed fee component, it is not optimal to transact to some levels below a certain threshold. Instead, it is better to liquidate the stock position, that is, the hedge. The decision “to hedge or not to hedge” is somewhat crucial, so it is, to some extent, better not to hurry with a transaction, but wait and see what happens with the stock price.

Figures (4) and (5) show the bounds on reservation option prices versus the price of the underlying stock for an option holder with  $\gamma = 0.001$  and  $\gamma = 1$  respectively. For the option holder with  $\gamma = 0.001$ , the reservation purchase price and the reservation write price are almost coincide. We observe that the reservation option prices are always above the corresponding

<sup>19</sup>That is, the average between Sell and Buy targets.

<sup>20</sup>We interpret the notion of risk premium as the absolute value of the difference between a reservation option price and the corresponding BS-price.

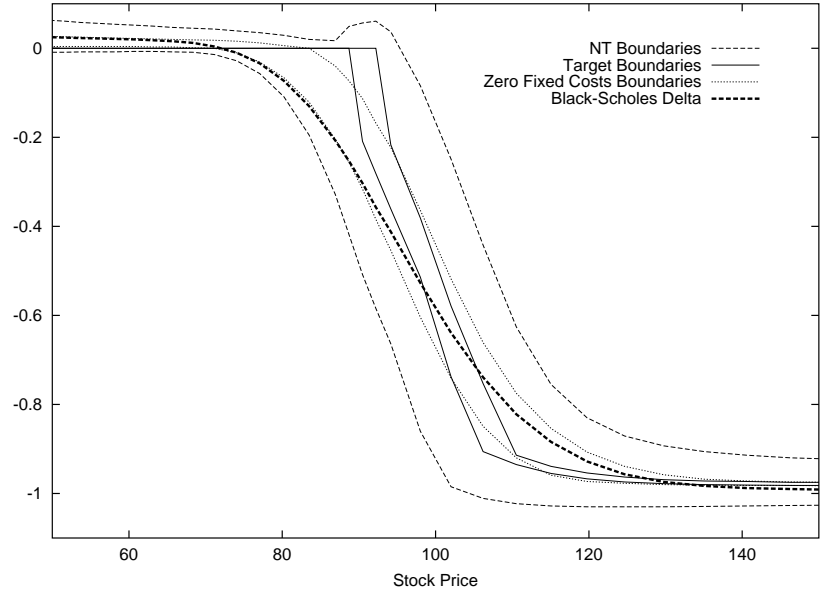


Figure 2: Optimal strategy control limits of a long European call option for a “net” hedger with  $\gamma = 1$ .

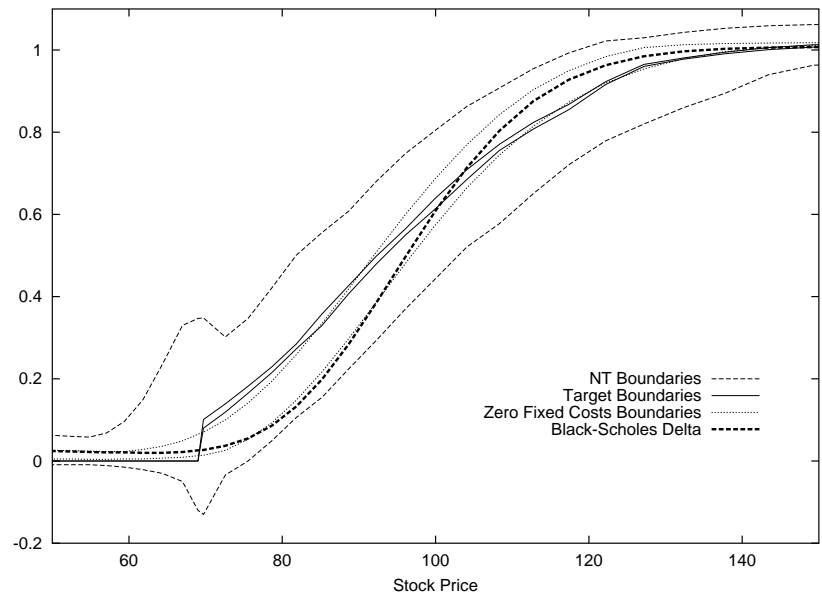


Figure 3: Optimal strategy control limits of a short European call option for a “net” hedger with  $\gamma = 1$ .

prices in the market with no transaction costs. As the underlying stock price increases, the deviation of a reservation option price from the BS-price also increases. For the option holder with  $\gamma = 1$ , the reservation purchase price is below, and the reservation write price is above the corresponding price in the market with no transaction costs.

Finally, we try to reconcile our findings with such empirical pricing bias as the volatility smile: Given the assumptions in the Black-Scholes model, all option prices on the same underlying asset with the same expiration date but different exercise prices should have the same implied volatility. However, the shape of implied (from the market prices of traded contracts) volatility resembles either a smile or a skew. A skew is a general pattern for equity options. The implied volatility decreases as the strike price increases. This means, for example, that in-the-money call options are overpriced as compared to the theoretical Black-Scholes option price.

To this end let us assume that the option price in the market is the average of the reservation write and purchase prices. This resulting option price, as a function of the strike price, differs from the BS-price in a way that could be interpreted in terms of the implied volatility. Figure (6) shows some possible forms of the volatility smile. For “net” hedgers the form of the volatility smile is a standard smile. The rationale for this form is the difference in the writer’s and buyer’s behavior. In fact, the writer hedges options more often than the buyer. The difference in frequency of hedging is more substantial for out-of-the-money and in-the-money options. Thus, the reservation write price drives up the average price for these options. For “net” investors the form of the implied volatility is a classical skew. As mentioned above, for the “net” investors-buyers an option serves as a substitute for the stock. Holding an option saves the buyer from some transaction costs. On the contrary, the writer adds extra transaction costs to the option price. Both of the effects increase the option price. The less the strike price, the more the buyer saves and the writer adds.

Unfortunately, the steepness of the theoretical implied volatility smile is not high enough to explain the empirical facts<sup>21</sup>. If we take, for example, “net” investors and calculate the implied volatilities for  $M = 1$  and  $M = 0.9$  we get 22.4% and 24.0% respectively. The difference between them is 1.6%.

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<sup>21</sup>Note, that we use 1% proportional transaction costs rate, which is higher than the realistic rate.

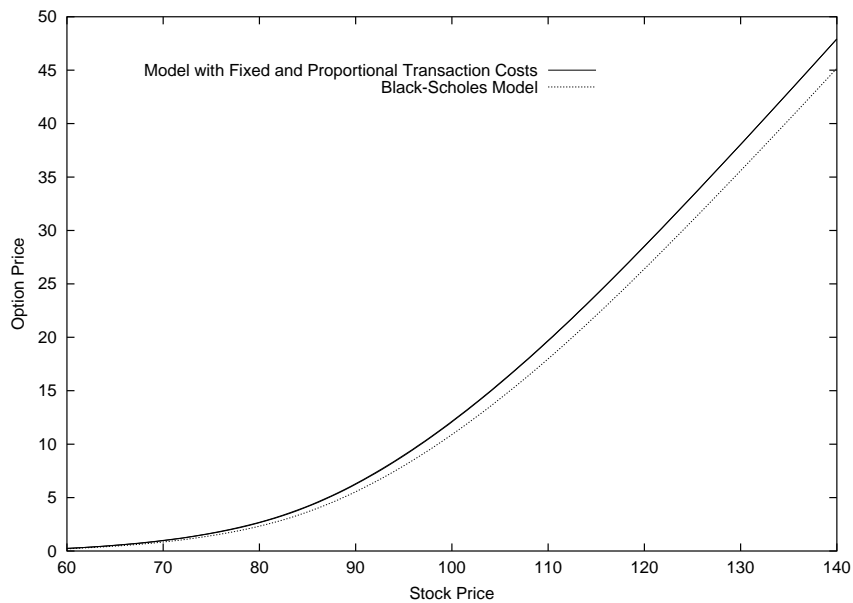


Figure 4: Bounds on reservation prices of European call options versus the price of the underlying stock for an option holder with  $\gamma = 0.001$ .

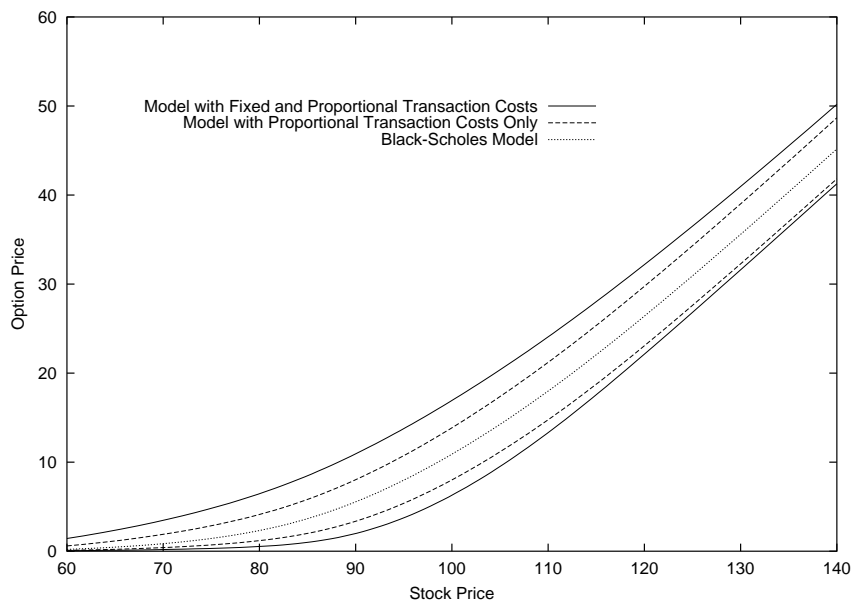


Figure 5: Bounds on reservation prices of European call options versus the price of the underlying stock for an option holder with  $\gamma = 1$ .

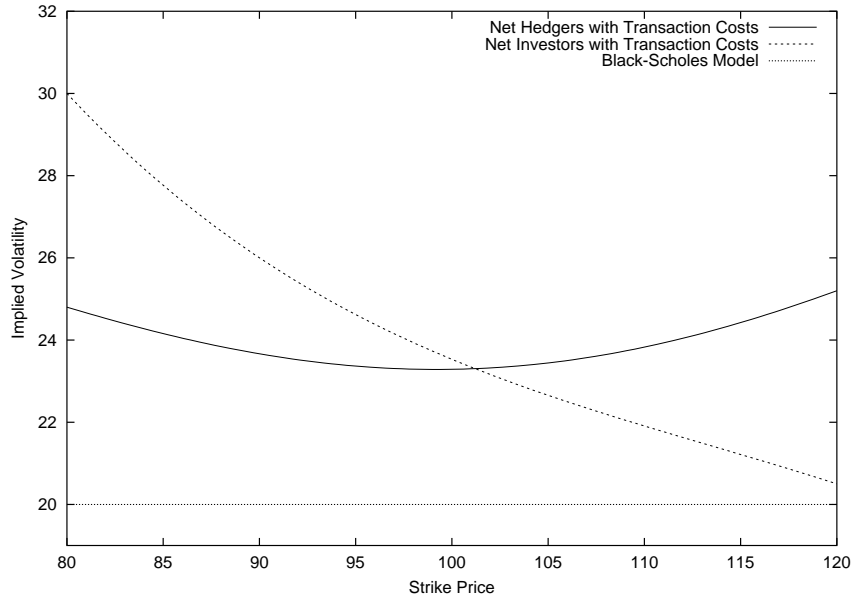


Figure 6: The possible forms of implied volatility as a function of the strike price.

The differences among the empirical implied volatilities, however, are much larger (up to 5 – 10%) to be accounted for by transaction costs.

## 6 Conclusions and Extensions

In this paper we extended the utility based option pricing and hedging approach, pioneered by Hodges and Neuberger (1989), for the market where each transaction has a fixed cost component. We formulated the continuous time option pricing and hedging problem for the CARA investor in the market with both fixed and proportional transaction costs. Then we numerically solved the problem applying the method of the Markov chain approximation for the case of European-style call options.

We examined the effects on the reservation option prices and the corresponding optimal hedging strategies of varying the investor’s level of ARA. We found that there are two basic patterns of option pricing and hedging in relation to the investor’s level of absolute risk aversion: For the investors with low ARA, both the reservation option prices are above the corresponding BS-price, and they are very close to each other. For the investors with high ARA, the reservation purchase price is generally below the BS-price,

and the reservation write price is above the BS-price. Here the difference between the two prices depends on the level of ARA and the level of transaction costs. Judging against the BS-strategy, the investors with high ARA underhedge out-of-the-money and overhedge in-the-money long option positions. When the investors with high ARA write options, their strategy is quite the opposite. They overhedge out-of-the-money and underhedge in-the-money short option positions. The remarkable features of these strategies are jumps to zero in target amounts in the stock when the stock price decreases below some certain levels.

We pointed out on two possible resolutions of the question: Under what circumstances will a writer and a buyer agree on a common price for an option? In the model with both fixed and proportional transaction costs under certain model parameters there occurs a situation when the reservation purchase price is higher than the reservation write price. The other possibility arises when a writer and a buyer, both of them being investors with low ARA, face different transaction costs in the market. We also tried to reconcile our findings with such empirical pricing bias as the volatility smile. Our general conclusion here is that this empirical phenomenon could not be accounted for solely by the presence of transaction costs.

There are several directions in which our work could be extended.

**1. Nonexponential utilities.** There is no issue of principle here, but only of increase of computational load, since the reduction from four to three dimensions is no longer available. However, as it was conjectured by Davis et al. (1993) and showed in Andersen and Damgaard (1999), Damgaard (2000b), the reservation option prices are approximately invariant to the specific form of the investor's utility function, and mainly only the level of absolute risk aversion plays an important role. In particular, we calculated the reservation option prices for the investors with low levels of ARA and with the same parameters as in the paper<sup>22</sup> by Damgaard (2000b) and obtained practically the same values. As a result, it seems to be of a little practical interest to calculate the reservation option prices and optimal hedging strategies using other utility functions besides the exponential one. These calculations will be very time-consuming, and, moreover, the

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<sup>22</sup>Here we mean the reservation option prices in the market with proportional transaction costs only.

optimal hedging strategy will be difficult to interpret because of its four-dimensional  $(t, x, y, S)$ -form.

- 2. American options.** As it was suggested by Davis et al. (1993) and presented in Davis and Zariphopoulou (1995), the utility based option pricing approach could also be applied to the pricing of American-style options. The problem of finding the reservation write price of an American-style option is somewhat tricky, because it is the buyer of option who chooses the optimal exercise policy. Therefore, the writer's problem must be treated from both the writer's and the buyer's perspective simultaneously. The problem of finding the reservation purchase price is simpler, since it suffices to consider the buyer's problem alone. We believe that it is of a great practical interest to calculate the reservation prices of American options and the corresponding optimal hedging/exercise policies in the markets with transaction costs, as the majority of traded option contracts are of American-style.
- 3. Incomplete markets.** The utility based option pricing approach can be generalized to cover the case of an incomplete market with transaction costs. In particular, this approach could be extended to include jumps in the price of the risky asset.
- 4. Several risky assets.** Another interesting extension could be the calculation of reservation option prices in economies with more than one risky asset. We conjecture that for the CARA utility and two risky assets the problem can be solved quite efficiently.

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