A SEMI-EXPLICIT APPROACH TO CANARY SWAPTIONS IN HJM ONE-FACTOR MODEL

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Abstract. Leveraging the explicit formula for European swaptions and coupon-bond options in HJM one-factor model, we develop a semi-explicit formula for 2-Bermudan options (also called Canary options). We first extend the European swaption formula to future times. So equipped, we are able to reduce the valuation of a 2-Bermudan swaption to a single numerical integration at the first expiry date. In that integration the most complex part of the embedded European swaptions valuation has been simplified to performe it only once and not for every point. In a special but very common in practice case, we also provide a semi-explicit formula. Those results lead to a largely faster and more precise implementation of swaption valuation. Those improvements extend even more favorable to sensitivity calculations.

1. Introduction

This article is devoted to Bermudan swaptions, more precisely to 2-Bermudan swaptions (swaptions with 2 exercise dates). Those swaptions are also called Canary swaptions as Canary Islands are halfway between Bermuda and Europe.

We leverage the explicit formula for European swaptions and coupon-bonds in the HJM one-factor model [5]. This is done by first calculating the value of European option at any future point of time. The value of such an European option is a random process which is a function of the fundamental random processes of the problem: the price of zero-coupon bonds. Such a formula is required as we need to compare a swap with the remaining European swaption at the first expiry date.

Using the explicit formula we are able to reduce the valuation of a 2-Bermudan option to a single expectation. This is an improvement to respect to a direct or usual tree approach as, even if there are two expiry dates, the numerical process is done only at one date. This is obtain by imposing a separability condition on the volatility, a condition which is satisfied by the Hull-White volatility [8].

The most time consuming part of the European swaption computation is to solve a non-linear one-dimensional equation. We are able to reduce the computation time by solving it once and then reusing the solution for all the other points of the integration.

Finally we propose a semi-explicit formula for cancelable swaps or options on underlying with similar cash-flows after the second expiry date. The formula is explicit for the valuation of the part corresponding to the exercise at the first expiry date and still written as an expectation for the rest. The size of the interval on which the expectation has to be computed is reduced by the probability of the exercise at the first expiry date. In other words, for first expiry at-the-money options, the number of points in the numerical integration is divided by (around) two.

Those results lead to several possible implementations of valuation formulas. They all perform significantly better than a double integration and standard tree implementation both in term of speed and precision. The precision improvement in particular are striking when sensitivities (delta and gamma) are computed.

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The HJM one-factor model and hypothesis used are described in the next section. Then we present some preliminary results before presenting the main results in Section 4 and the simplified formulas in Section 5. Numerical implementation results are presented in Section 6.

2. Model and hypothesis

The model and main hypothesis used in this paper are the same as in [5].

We use a model for $P(t,u)$, the price at t of the zero-coupon bond paying 1 in u. We will describe this for all $0 \leq t, u \leq T$, where T is some fixed constant.

When the discount curve $P(t,.)$ is absolutely continuous and positive, which is something that is always the case in practice as the curve is constructed from rates and by some kind of interpolation, there exists $f(t,u)$ such that

$$P(t,u) = \exp\left(- \int_t^u f(t,s)ds \right). \tag{1}$$

The idea of Heath-Jarrow-Morton [3] was to exploit this property by modelling f as

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

for some suitable (possibly stochastic) μ and σ.

Here we use a similar model, but we restrict ourself to non-stochastic coefficients. The exact hypothesis on the volatility term σ is described by (H2). We don't need all the technical refinement used in their paper or similar one, like the one described in [10] in the chapter on *dynamical term structure model*. So instead of describing the conditions that lead to such a model, we assume that the *conclusions* of such a model are true. By this we mean we have a model, that we call a *HJM one-factor model*, with the following properties.

Let $A = \{(s,u) \in \mathbb{R}^2 : u \in [0,T] \text{ and } s \in [0,u]\}$. We work in a filtered probability space $(\Omega, F, \mathbb{P}^{\mathsf{real}}, (\mathcal{F}_t))$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\mathsf{real}})_{0\leq t\leq T}$.

H1: There exists $\sigma : [0,T]^2 \to \mathbb{R}^+$ measurable and bounded[1] with $\sigma = 0$ on $[0,T]^2 \setminus A$ such that for some process $(r_s)_{0\leq t\leq T}$, $N_t = \exp(\int_0^t r(s)ds)$ forms, with some measure $\mathbb{N}$, a numeraire pair[2] (with Brownian motion W_t),

$$\begin{aligned} df(t,u) &= \sigma(t,u) \int_t^u \sigma(t,s)ds\, dt - \sigma(t,u)dW_t \\ dP^N(t,u) &= P^N(t,u) \int_t^u \sigma(t,s)ds\, dW_t \end{aligned}$$

and $r(t) = f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Note that ν is increasing in u, measurable and bounded.

To be able to use the explicit formula for the valuation of the European swaptions, we will also use the following hypothesis.

H2: The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive function g and h.

[1] Bounded is too strong for the proof we use, some L^1 and L^2 conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

[2] See [10] for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair $(N, \mathbb{N})$.

Note that this condition is essentially equivalent to the condition (H2) of [5] but written on σ instead of on ν. The condition on ν was $\nu(s,t_2)-\nu(s,t_1)=f(t_1,t_2)g(s)$.

Example: The Ho and Lee volatility model [7] and the Hull and White volatility model [8] satisfy the condition (H2). For Ho and Lee one has $\nu(s,t)=\sigma(t-s)$ and $\sigma(s,t)=\sigma$; for Hull and White one has $\nu(s,t)=(1-\exp(-a(t-s)))\sigma/a$ and $\sigma(s,t)=\sigma\exp(-a(t-s))$. The volatility time-dependent versions of the models also satisfy the conditions.

3. Preliminary results

We want to price some option in this model. For this we recall the generic pricing theorem [10, Theorem 7.33-7.34].

Theorem 1. *Let V_T be some $\mathcal{F}_T$-measurable random variable. If V_T is attainable, then the time-t value of the derivative is given by $V_t^N=V_0^N+\int_0^t\phi_s dP_s^N$ where ϕ_t is the strategy and*

$$V_t=N_t\,\mathrm{E}^{\mathbb{N}}\left[\left.V_TN_T^{-1}\right|\mathcal{F}_t\right].$$

We now state two technical lemmas that generalize the lemmas presented in [5]. Similar formulas can be found in [2] in a different framework.

Lemma 1. *Let $0\leq t\leq u\leq v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v)=\frac{P(t,v)}{P(t,u)}\exp\left(-\frac{1}{2}\int_t^u\left(\nu^2(s,v)-\nu^2(s,u)\right)ds+\int_t^u\left(\nu(s,v)-\nu(s,u)\right)dW_s\right).$$

Proof. By definition of the forward rate and its equation,

$$\begin{aligned}P(u,v) &= \exp\left(-\int_u^v f(u,\tau)d\tau\right)\\ &= \exp\left(-\int_u^v\left[f(t,\tau)+\int_t^u\nu(s,\tau)D_2\nu(s,\tau)ds-\int_t^u D_2\nu(s,\tau)dW_s\right]d\tau\right).\end{aligned}$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$\begin{aligned}P(u,v) &= \frac{P(t,v)}{P(t,u)}\exp\left(-\int_t^u\int_u^v\nu(s,\tau)D_2\nu(s,\tau)d\tau ds+\int_t^u\int_u^v D_2\nu(s,\tau)d\tau dW_s\right)\\ &= \frac{P(t,v)}{P(t,u)}\exp\left(-\frac{1}{2}\int_t^u\left(\nu^2(s,v)-\nu^2(s,u)\right)ds+\int_t^u\nu(s,v)-\nu(s,u)dW_s\right).\end{aligned}$$

□

Lemma 2. *In the HJM one factor model, we have*

$$N_uN_v^{-1}=\exp\left(-\int_u^v r_s ds\right)=P(u,v)\exp\left(\int_u^v\nu(s,v)dW_s-\frac{1}{2}\int_u^v\nu^2(s,v)ds\right).$$

Proof. By definition of r,

$$\begin{aligned}r_\tau &= f(\tau,\tau)=f(t,\tau)+\int_t^\tau df(s,\tau)ds\\ &= f(t,\tau)+\int_t^\tau\nu(s,\tau)D_2\nu(s,\tau)ds+\int_t^\tau D_2\nu(s,\tau)dW_s.\end{aligned}$$

Then using Fubini, we have

$$\begin{aligned}\int_u^v r(\tau)d\tau &= \int_u^v f(t,\tau)d\tau+\int_u^v\int_s^v\nu(s,\tau)D_2\nu(s,\tau)d\tau ds-\int_u^v\int_s^v D_2\nu(s,\tau)d\tau dW_s\\ &= \int_u^v f(t,\tau)d\tau+\frac{1}{2}\int_u^v\nu^2(s,v)ds+\int_u^v\nu(s,v)dW_s.\end{aligned}$$

□

We give the pricing formula for swaptions for a future time. This is essentially the Theorem 3.1 of [5] but written for any future time $t \geq 0$. Jamshidian [11] also provides an *exact* solution for European swaption. His approach requires to solve a non-linear equation with respect to the instantaneous short rate r. Even if it is also based on the one-factor model, its approach is less *explicit* and as such more difficult to implement

Theorem 2. *Suppose we work in the HJM one-factor model with a volatility term of the form (H2). Let $\theta \leq t_0 < \cdots < t_n$, $c_0 < 0$ and $c_i \geq 0$ $(1 \leq i \leq n)$. The price of a European receiver swaption, with expiry θ on a swap with cash-flows c_i and cash-flow dates t_i is given at time t by the $\mathcal{F}_t$-measurable random variable*

$$\sum_{i=0}^{n} c_i P(t, t_i) N(\kappa + \alpha_i)$$

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^{n} c_i P(t, t_i) \exp\left(-\frac{1}{2}{\alpha_i}^2 - \alpha_i \kappa\right) = 0 \tag{2}$$

and

$${\alpha_i}^2 = \int_t^\theta (\nu(s, t_i) - \nu(s, \theta))^2 \, ds.$$

The price of the payer swaption is

$$-\sum_{i=0}^{n} c_i P(t, t_i) N(-\kappa - \alpha_i)$$

Proof. Let $\mu(s, \theta) = \nu(s, \theta)$ if $s \geq t$ and 0 if $s < t$. We define $W_\tau^\# = W_\tau - \int_0^\tau \mu(s, \theta) ds$. By Girsanov's theorem ([12, Section 4.2.2, p. 72]), the process $W^\#$ is a standard Brownian motion with respect to the probability $\mathbb{P}^\#$ of density

$$L_\theta = \exp\left(\int_0^\theta \mu(s, \theta) dW_s - \frac{1}{2}\int_0^\theta \mu^2(s, \theta) ds\right).$$

Using Lemma 1 and rewriting $\nu^2(s, t_i) - \nu^2(s, \theta)$ as $(\nu(s, t_i) - \nu(s, \theta))^2 + 2\nu(s, \theta)(\nu(s, t_i) - \nu(s, \theta))$, we have

$$P(\theta, t_i) = \frac{P(t, t_i)}{P(t, \theta)} \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i X\right)$$

where $-\alpha_i X = \int_t^\theta \nu(s, t_i) - \nu(s, \theta) dW_s^\#$ and X is a standard normally distributed with respect to $\mathbb{P}^\#$. The hypothesis (H2) is used here to prove that the random variable X is the same for all i.

Using Lemma 2, we have $N_t N_\theta^{-1} = P(t, \theta) L_\theta$. By the generic pricing theorem 1, the price of the option is

$$V_t = \mathrm{E}^\#\left[\max\left(\sum_{i=0}^{n} c_i P(t, t_i) \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i X\right), 0\right) \middle| \mathcal{F}_t\right].$$

Note that $P(t, t_i)$ is $\mathcal{F}_t$-measurable and X is independent of $\mathcal{F}_t$. Using a property of the conditional expectation ([12, Proposition A.2.5, p. 166]), we can do this computation in two parts.

Let's fix $P(t, t_i) = P_i$. Like in the proof for $t = 0$, we have $\sum c_i P_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i y) > 0$ if and only if $y < \kappa$ where κ is the unique solution of $\sum c_i P_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i y) = 0$.

So we have $V_t = \phi(P)$ where $\phi(p) = \sum c_i p_i N(\kappa + \alpha_i)$. Or more explicitly

$$V_t = \sum c_i P(t, t_i) N(\kappa + \alpha_i)$$

where $P(t, t_i)$ and κ are $\mathcal{F}_t$-measurable and κ is implicitly defined by the equation (2) □

4. 2-Bermudan swaption

We are now in a position to state and prove the main result of this article concerning 2-Bermudan swaptions.

Theorem 3. *Let $\theta_1 < \theta_2$, $t_{i,j}$ $(i = 1, 2,\ j = 0, \dots, n_i)$ be such that $\theta_i \leq t_{i,0} < t_{i,1} < \dots < t_{i,n_i}$ and $c_{i,j}$ $(i = 1, 2,\ j = 0, \dots, n_i)$ be such that $c_{i,0} < 0$ and $c_{i,j} \geq 0$ $(j > 0)$. In the HJM one-factor model, when the volatility term has the form (H2), the price of a 2-Bermudan receiver swaption with expiries θ_i and underlying swaps with cash-flow $c_{i,j}$ and cash-flow dates $t_{i,j}$ is given by*

$$\begin{aligned} V_0 \quad = \quad & \mathrm{E}\left(\max\left(\sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j}^2(0, \theta_1) - \alpha_{1,j}(0, \theta_1) X\right),\right.\right. \\ & \left.\left.\sum_{j=0}^{n_2} c_{2,j} P(0, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_1) - \alpha_{2,j}(0, \theta_1) X\right) N\left(\kappa(X) + \alpha_{2,j}(\theta_1, \theta_2)\right)\right)\right) \end{aligned} \tag{3}$$

where $\kappa(X)$ is the unique solution of

$$\sum_{j=0}^{n_2} c_{2,j} P(0, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_2) - \alpha_{2,j}(0, \theta_1) X - \alpha_{2,j}(\theta_1, \theta_2)\kappa\right) = 0, \tag{4}$$

X is a standard normally distributed random variable with respect to E *and*

$$\alpha_{i,j}^2(u, v) = \int_u^v \left(\nu(s, t_{i,j}) - \nu(s, v)\right)^2 ds.$$

The price of the payer swaption is

$$\begin{aligned} V_0 \quad = \quad & \mathrm{E}\left(\max\left(-\sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j}^2(0, \theta_1) - \alpha_{1,j}(0, \theta_1) X\right),\right.\right. \\ & \left.\left.-\sum_{j=0}^{n_2} c_{2,j} P(0, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_1) - \alpha_{2,j}(0, \theta_1) X\right) N\left(-\kappa(X) - \alpha_{2,j}(\theta_1, \theta_2)\right)\right)\right) \end{aligned} \tag{5}$$

Proof. In θ_1 the price of the swaption is given by the maximum of the price of the first swap and the price of the European swaption on the second swap.

We define $W_t^\# = W_t - \int_0^t \nu(s, \theta_1) ds$. By Girsanov theorem, the process $W^\#$ is a standard Brownian motion with respect to the probability $P^\#$ of density

$$L_{\theta_1} = \exp\left(\int_0^{\theta_1} \nu(s, \theta_1) dW_s - \frac{1}{2}\int_0^{\theta_1} \nu^2(s, \theta_1) ds\right).$$

Using Lemma 1, we have

$$P(\theta, t_{i,j}) = \frac{P(0, t_{i,j})}{P(0, \theta_1)} \exp\left(-\frac{1}{2}\alpha_{i,j}^2(0, \theta_1) - \alpha_{i,j}(0, \theta_1) X\right) \tag{6}$$

where $-\alpha_{i,j}(0, \theta_1) X = \int_0^{\theta_1} \nu(s, t_{i,j}) - \nu(s, \theta_1) dW_s^\#$ and X is a random variable with standard normal distribution with respect to $\mathbb{P}^\#$. Like in Theorem 2, we use the hypothesis (H2) to prove that the random variable X is the same for all i and j.

By Lemma 2, $N_{\theta_1}^{-1} = P(0, \theta_1) L_{\theta_1}$ and so using the generic pricing Theorem 1 and the swaption pricing Theorem 2, we then have the equation (3) where κ is the solution of

$$\sum_{j=0}^{n_2} P(\theta_1, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(\theta_1, \theta_2) - \alpha_{2,j}(\theta_1, \theta_2)\kappa\right) = 0.$$

By using the equation (6) we obtain the described result for the value of κ. □

Remark: The same approach is also valid for 1-payer-1-receiver swaptions, choice swaption (at first expiry the holder has the choice between a payer and a receiver swaption, possibly with different expiry dates) or any combination of swaps and swaptions. For choice swaption where the choice is between different swaps (or set of cash-flows), as indicated in [5], an explicit formula can be obtained.

5. Simplified formulas

Subject to an extra condition, this result can be written in a form easier to compute.

Theorem 4. *Under the conditions of Theorem 3, if the volatility structure satisfies (H2), then the value of the 2-Bermudan receiver swaption is given by the same formula (3) (payer given by (5)) but with* $\kappa(X) = (\Lambda - \sqrt{G(\theta_2) - G(\theta_1)}X)/\sqrt{G(\theta_1) - G(0)}$ *where* Λ *is the unique solution of*

$$\sum_{j=0}^{n_2} c_{2,j} P(0, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_2) + f(t_{2,j})\Lambda\right) = 0, \tag{7}$$

G *is a primitive of* g^2 *and* H, *is a primitive of* h *(with* g *and* h *described in (H2)).*

Proof. Using condition (H2) we can write

$$\alpha_{i,j}^2(u, v) = (H(t_{i,j}) - H(v))^2(G(v) - G(u)).$$

As g and h are positive, G and H are increasing and all the factors in the previous formula are positive. If we inject that description in (4) and simplify some factors, we have as equation for κ

$$\sum_{j=0}^{n_2} c_{2,j} P(0, t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0, \theta_2) - H(t_{2,j})(\sqrt{G(\theta_1) - G(0)}X + \sqrt{G(\theta_2) - G(\theta_1)}\kappa)\right) = 0.$$

If we denote by Λ the term $\sqrt{G(\theta_1) - G(0)}X + \sqrt{G(\theta_2) - G(\theta_1)}\kappa$, which is independent of j, we have the result. □

In the case where the underlying swaps have the same cash-flows and cash-flow dates after the settlement of the second swap, the formula can be simplified further. The simplification consists in the analytical solution of the expected value of the first swap in case of exercise in θ_1. This is applicable in particular for cancelable swaps and bonds with embedded 2-Bermudan options.

Theorem 5. *Let* $\theta_1 \leq t_0 < t_1 < \cdots t_{k-1} < \theta_2 \leq t_k < \cdots < t_n$, $c_j > 0$ $(j = 1, \ldots, n)$, $c_0 < 0$ *and* $d_k < 0$. *We consider two receiver swaps which are represented by the cash-flows* $(c_0, c_1, \ldots, c_n)$ *on dates* $(t_0, \ldots, t_n)$ *and by the cash-flows* $(d_k, d_{k+1}, \ldots, d_n) = (d_k, c_{k+1}, \ldots, c_n)$ *on dates* $(t_k, \ldots, t_n)$.

In the HJM one-factor model, when the volatility term has the form (H2), the price of a 2-Bermudan receiver swaption with expiries θ_i *and underlying swap described above is given by*

$$\begin{aligned} V_0 &= \sum_{j=0}^{n} c_j P(0, t_j) N(\mu + \alpha_j(0, \theta_1)) \\ &\quad + \mathrm{E}\left(1\!\!1(X \geq \mu) \max\left(\sum_{j=0}^{n} c_j P(0, t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0, \theta_1) - \alpha_j(0, \theta_1)X\right),\right.\right. \\ &\qquad \left.\left.\sum_{j=k}^{n} d_j P(0, t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0, \theta_1) - \alpha_j(0, \theta_1)X\right) N(\kappa(X) + \alpha_j(\theta_1, \theta_2))\right)\right) \end{aligned}$$

where μ *is the smallest solution of*

$$\begin{aligned} &\sum_{j=0}^{n} c_j P(0, t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0, \theta_1) - \alpha_j(0, \theta_1)\mu\right) \\ &- \sum_{j=k}^{n} c_j P(0, t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0, \theta_1) - \alpha_j(0, \theta_1)\mu\right) N(\kappa(\mu) + \alpha_j(\theta_1, \theta_2)) = 0 \end{aligned}$$

and κ is the function defined by (4).

The price of the 2-Bermudan payer swaption is

$$\begin{aligned} V_0 &= -\sum_{j=0}^{n} c_j P(0,t_j) N\left(-\mu - \alpha_j(0,\theta_1)\right) \\ &\quad + \mathrm{E}\left(1\!\!1(X \le \mu) \max\left(-\sum_{j=0}^{n} c_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right), \right.\right. \\ &\quad \left.\left. -\sum_{j=k}^{n} d_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right) N\left(-\kappa(X) - \alpha_j(\theta_1,\theta_2)\right) \right)\right) \end{aligned}$$

Proof. By the implicit function theorem applied to equation (4), κ is continuous (as a function of X). Let $Q_j = c_j P(0,t_j)\exp(-\frac{1}{2}\alpha_j^2(0,\theta_1))$. Note that $Q_0 < 0$ and $Q_j > 0$ ($j = 1,\ldots,n$). The difference between the value of the first swap and the swaption can be written as

$$\exp(-\alpha_0 X)\left(Q_0 + \sum_{j=1}^{k-1} Q_j \exp((\alpha_0 - \alpha_j)X) + \sum_{j=k}^{n} Q_j (1 - \frac{d_j}{c_j} N(\kappa(X) + \alpha_j)) \exp((\alpha_0 - \alpha_j)X) \right).$$

As $\alpha_0 - \alpha_j < 0$, (see [5] for a proof of it) all the coefficient in the exponentials are negative. Moreover as $d_j = c_j$ for $j > k$, $d_k < 0$ and $0 < N < 1$, all the factors of the exponential are positive. The only negative term is Q_0.

Using all those elements, the term within the parenthesis tends to $+\infty$ as X tends to $-\infty$ and converges to $Q_0 < 0$ when X tends to $+\infty$.

By continuity at least one point exists for which the difference is 0. Also as it converges to $+\infty$ in $-\infty$, the set of zeros is bounded from below. This proves that the set of solutions, which is closed, has a finite minimum. □

Remark: It is not clear if it is possible to have an equation for μ with several solutions. If the solution is unique, the price formula can be simplified further by removing the first term of the max.

6. Numerical implementation

Expected values are usually computed through a numerical integration. The expected value we have to compute is the one of a random variable written as the function of a standard normal random variable. It means that we know quite well the weight of the distribution underlying our expected value. We can use points for the numerical integration with equal *weight* with respect to the underlying normal distribution. By using equally weighted points we concentrate the computation where they have a greater importance and so increase the precision for a given number of points in the numerical integration.

All the models used in the section have been implemented using the same language (Matlab[3]) and the computation time was measured at the same moment running all of them in a loop, without operator intervention.

6.1. **European swaption speed.** In this section we briefly study the pricing speed of several Hull-White model implementation for European swaptions. It may seem strange to study the European swaption computation speed in a paper on Bermudan swaptions. But a 2-Bermudan option is equal to an expected value involving European swaption or composed European options. So when you compute the external expectation you have the choice of computing the internal one independently. In this section we show that the explicit method on which this paper relies is more efficient when a good precision is required.

[3]Matlab code available from the author.

Even if we show this it does not mean that it necessarily has to be used for Bermudan options. Efficient use of the second step computations can lead to large time saving. The recombining property of the Hull-white tree is one of them. Most of the points of the second expiry date are used to compute several points at the first date.

But this section will show that there is no hope to use a method for the first expiry independently (without efficient use of previous computation) of the method used for the second expiry without using the explicit approach for the second step.

For this we use a numerical integration technique with *equi-probable* points (not equi-distant). The Hull-White trinomial tree implementation is a standard one (as described in Brigo and Mercurio [1] with long term discount factors recovered form the one-step one as described in Hull [9]). The explicit solution is the one described in [5].

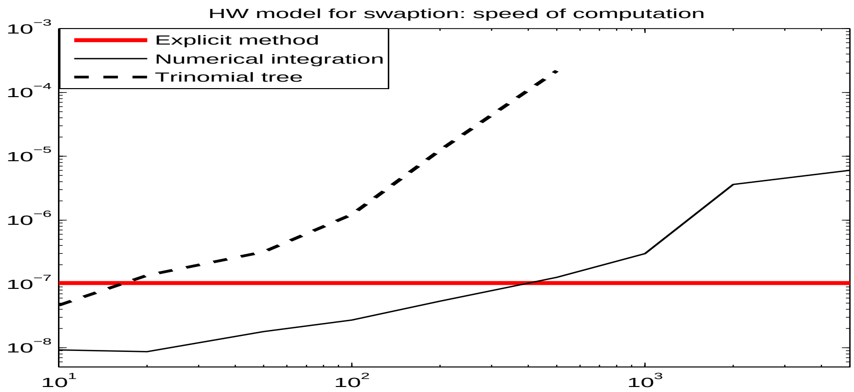


FIGURE 1. Computation time for European swaptions with the semi-explicit method, the numerical integral method and Hull-White tree

Figure 1 represents the computation time for $n = 10, 20, 50, 100, 200$ and 500 steps for the Hull-White tree and numerical integration (for the numerical integration we also added $n = 1000, 2000$ and 5000) and the constant time for the explicit solution. By step we mean the Hull-White tree equivalent. In a Hull-White trinomial tree, for n steps there are (approximatively) $2n + 1$ final points. To have a correct comparison we also use $2n + 1$ points in the numerical integration procedure.

As it can be seen from the graph, the tree approach is a lot slower when a lot of points are used (50 or above). The numerical integration is faster up-to 200 points, but slower for more points. For the option used, one need more than 200 points to have a price that is within 0.1% of the correct one.

At this stage there is no clear evidence of the speed superiority of the explicit solution to the numerical integration when high precision is not required. But as the next section evidences, an intelligent use of intermediary computations, as described in Theorem 4 will prove that method largely superior.

6.2. **Tree and numerical integration speed.** We now come back to our main subject, 2-Bermudan swaptions, and compare the speed of different implementations. On one hand we use the same classical Hull-White tree implementation and on the other hand for the numerical integration we use three different implementations: the *brute* implementation using directly Theorem 3, the *fast* implementation described in Theorem 4 and the *semi-analytical* implementation of Theorem 5. The brute implementation recomputes the full price (and in particular the κ) of the European swaption at each point of the numerical integration. The two others compute only once the Λ of Theorem 4 and use it to deduce the κ for each point.

The examples are all on a 1y x 5y and 1.5y x 4.5y receiver swaption. The strike is close to the at-the-money rate of the first option. The yield curve used is the one of 28 October 2004[4]. We measure the speed for $n = 10, 20, 50, 100, 200$ and 500 steps. For the semi-explicit and numerical integration we do it also for $n = 1000, 2000$ and 5000. As in the previous section for n steps there are $2n + 1$ points at first date in the tree and we use the same number of points in the numerical integration. The tree is developed up to the second expiry date while the numerical integration stops at the first date. The number of steps is for each of the two periods. So what we call a n step computation means $2n$ total steps and $4n + 1$[5] final points in the tree and $2n + 1$ points in the numerical integration. For the semi-explicit approach only the points corresponding to the non-explicit part are computed. So for an option with a 50% probability of exercise at the first date, only one half of the points are computed.

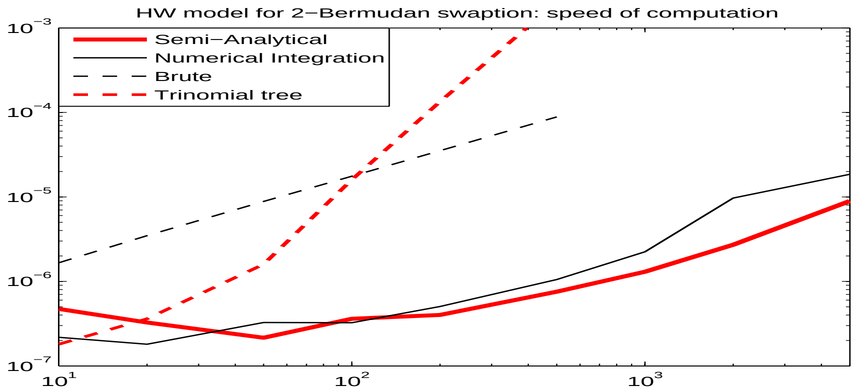


FIGURE 2. Computation time for 2-Bermudan swaptions with brute, fast and semi-analytic methods, and Hull-White trinomial tree

The results are graphically represented in Figure 2. The graph is on a log-log scale. So lines represent *exponent* laws. A regression of the log-points with the log-speed gives a slope (exponent) of 2.1 for the tree and 1.0 for the brute force. This is what was expected from the number of computations. For the other methods the numbers are 0.6 and 0.5. There the situation is more complex as there is a large part of the computation (Λ and μ) that is independent of the number of points before starting the point computations.

It can be seen that using 5000 points with one of the efficient numerical integration approaches take still less time than for 100 points in the tree or brute approaches. Without discussing the convergence, it is clear that the proposed approaches are significantly more efficient than the tree of brute approaches.

Also the semi-explicit approach is faster for large number of points than the full numerical integration. Even if an extra equation is solved to find μ, the number of points is reduced in proportion of the probability of exercise at the first date.

6.3. **Convergence of the results.** We compare the convergence of the results for several implementations. We still use the tree implementation but now for the numerical integration we use two versions of Theorem 4, one with *equi-distant* and one with *equi-probable* points, and the semi-explicit implementation of Theorem 5.

The results for $n = 10, 20, 50, 100, 200$ and 500 steps for the tree and also $n = 1000, 2000$ and 5000 for the integration approaches are given in Figure 3.

[4]There is nothing special with that date, except it is my sister birth date!

[5]Actually as the second step is shorter (6m) the distance between points is also smaller and more than $4n + 1$ final points are used.

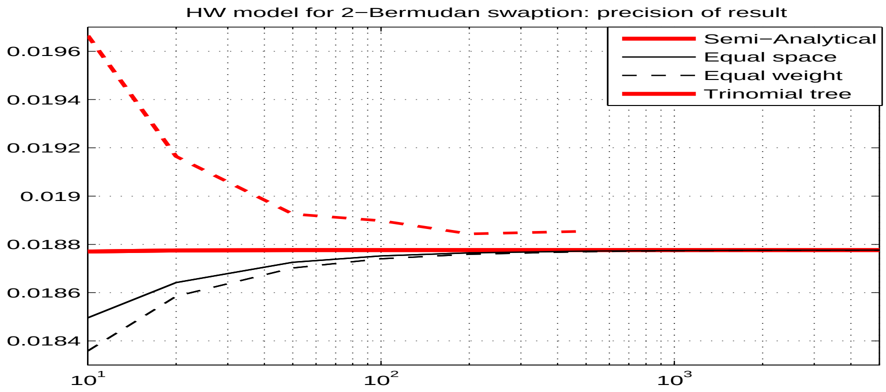


FIGURE 3. Computation time for 2-Bermudan swaptions with brute, fast and semi-analytic methods, and Hull-White trinomial tree

The graph clearly indicates that the semi-analytical is the best approach in terms of convergence. This can be explained by the way a Bermudan swaption behaves. The most valuable part is the first option. This part is valued explicitly and so is as precise as the double precision of the computer. The rest, which is small, is computed in the integral and converge to its true value. The second best is the implementation with equally probable points. This is also not surprising as the computations are concentrated on the more relevant points. Finally come the equally spaced point and the trinomial tree approaches. The tree has also equally spaced points at each step, it is therefore not surprising that they perform in a similar way.

6.4. **Delta and gamma.** The results on price convergence can be extended even more successfully to delta and in an unrivalled way to Gamma.

The tree approaches are notoriously unstable for *greeks* computations and gamma numbers are dominated by numerical noises (see [6] for some computation examples in the case of European swaptions).

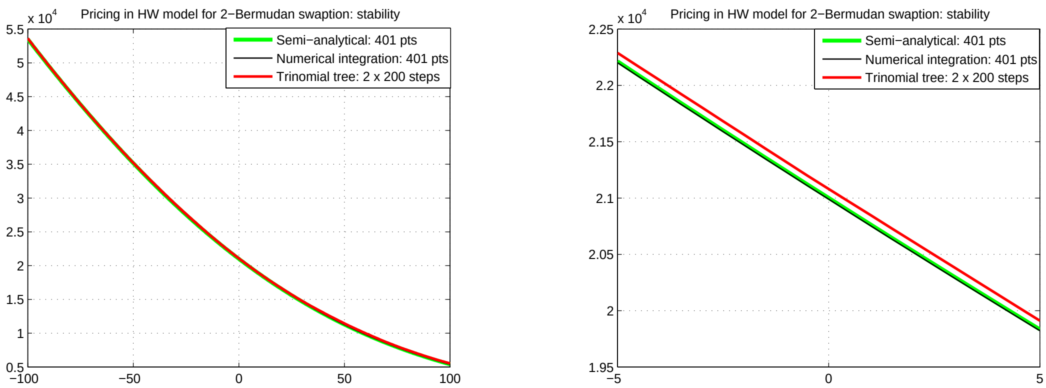
We take the 2-Bermudan swaption of the previous section and compute its yield curve delta and gamma. By this we mean that we try to assess the first and second order price change coming from a parallel movement of one basis point of all market rates that compose the curve.

We compute those numbers for the initial yield curve but also for the other yield curves resulting from shifts by half a basis point increment up to 100 basis points away from the current curve. The similar experience for European swaptions [6] indicates that there is very little hope to obtain a correct result through the tree approach. We show the results for 2-Bermudan swaptions in Figures 4, 5, and 6. Those results are obtained with 200 steps (with the meaning of step described above)[6].

The price seems acceptable for all the methods if one doesn't look from too close. When one goes to the first order sensitivity, the delta, the results are bad for the tree method but there is still some hope to improve it by increasing the number of steps.

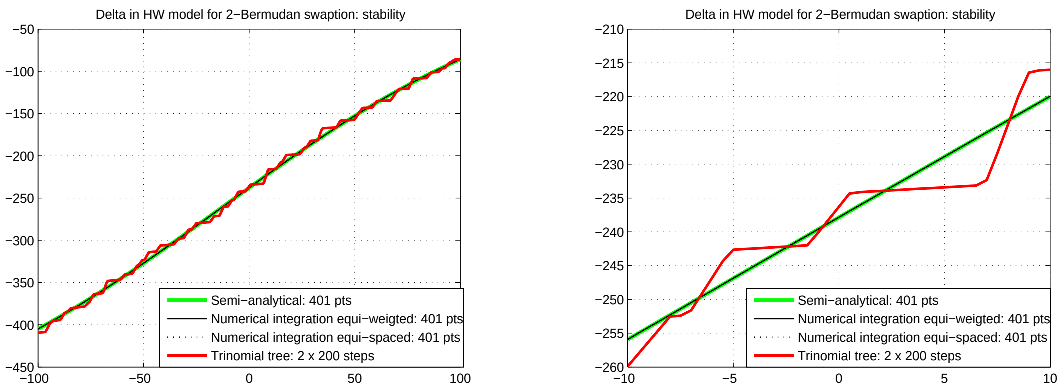
In the case of the gamma, the tree numbers are meaningless. The cure of the problem would require more points in the tree than is possible for the memory of the computer. The two numerical integration approaches give acceptable results. They exhibit small oscillations, but still acceptable ones.

[6]It took around four hours on my computer to run the (non-optimized) code to compute 3 x 401 curves and prices with the 4 implementations using 200 steps precisions. As can be inferred from Figure 2, most of the time was devoted to the tree computations.



(a) Price on the full range (b) Enlarged picture

Figure 4. Price of 2-Bermudan swaption with numerical integration, semi-analytic methods, and Hull-White trinomial tree



(a) Delta on the full range (b) Enlarged picture

Figure 5. Delta of 2-Bermudan swaption with numerical integration, semi-analytic methods, and Hull-White trinomial tree

One of the tree problem is that even if a lot of points are used, the extreme ones are almost useless. In the example we study the *value* (as defined by the Q function in [1]) of the 200 first points and 200 last points at the second expiry date have an average value of $4 * 10^{-95}$. This is to be compared with an average of 0.0095 for the 101 central points. The central points of the tree are the only ones that bring value.

With 200 steps (401 points), the numerical integration gamma still lacks of precision. But thanks to the speed efficiency we can increase the number of points without problems. Figure 7 gives the gamma profile for 401, 801 and 1601 points.

At initial scale only one line is visible. This is why we increase the scale with the number of points. The increase number of points gives results that all practical purposes are smooth and precise enough. This is done with a computation time below the one for 50 steps in the trinomial tree approach!!!

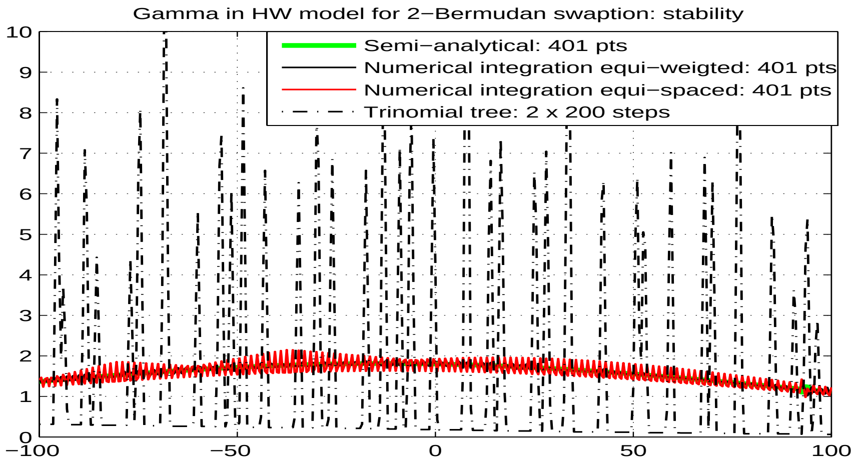
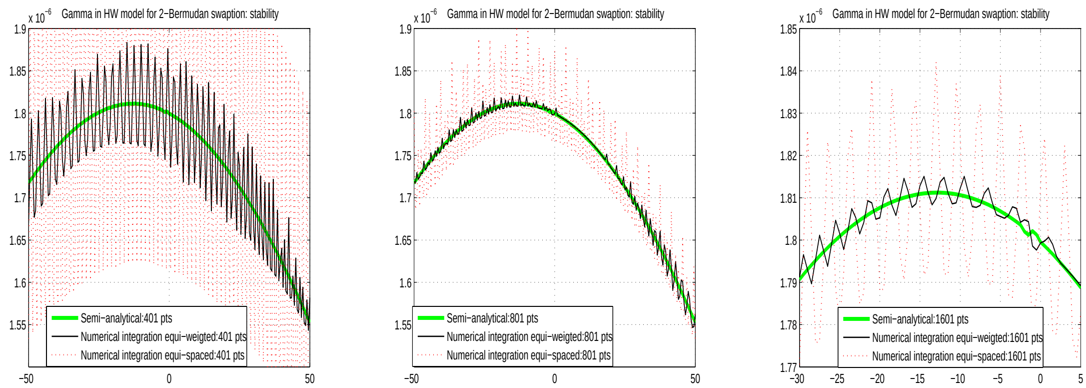


FIGURE 6. Gamma of 2-Bermudan swaption with numerical integration, semi-analytic methods, and Hull-White trinomial tree



(a) With 401 points, (b) with 801 points,... (c) and with 1601 points.

FIGURE 7. Gamma of 2-Bermudan swaption with numerical integration and semi-analytic methods.

7. Extension to general Bermudan swaption

This approach will not work directly in practice for n-Bermudan swaption $(n \geq 3)$ as $n-1$ integrations would be required for a total of points of the order of p^{n-1} where a Hull-White tree has a number of final points in pn (total of the order of $(pn)^2$).

Nevertheless some extra analytical manipulation and selection of the points for the integration can bring the number of computation for an integration-like formula to pn. This will be developed in a forthcoming article in preparation [4].

8. Conclusion

Both in terms of speed and convergence the semi-explicit approach proposed in Theorem 5 performs better than more simple methods described in this paper. The improvements, especially in terms of speed, are even more impressive (several order of magnitude) with respect to a standard Hull-White trinomial tree. The scope of the improvement is currently limited to 2-Bermudan swaptions but part of the method can be extended efficiently to more general swaptions.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements. The author wishes to thank the referee and his colleagues for valuable comments on previous versions of the paper.

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