

Credit Derivatives in an Affine Framework

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Abstract

We develop a general and efficient method for valuating credit derivatives based on multiple entities in an affine framework. This includes interdependence of market and credit risk, joint credit migration and counterparty default risk of multiple firms.

As an application we provide closed form expressions for the joint distribution of default times, default correlations, and credit default spreads in the presence of counterparty default risk.

1 Introduction

The rapid growth of the credit derivatives market (see Credit Risk 2000 [18]) generates an upsurge for valuation models of various credit derivatives, including credit default swaps (CDSs). It seems that, so far, no proposed method has been able to provide an analytically tractable model which incorporates an appropriate dependence structure between market and credit risk, credit migration and default risk of multiple firms. These aspects are inevitable for the accurate pricing of credit derivatives and an efficient model calibration.

In this paper we present a general method to value default-sensitive securities based on multiple entities in an affine framework. We model risk-free rates and the credit states of multiple firms jointly as an affine state process. Due to a simple mathematical trick, which allows to replace indicator variables by exponential-affine functions of the state process, we obtain closed form expressions for the conditional expectations of a variety of joint credit events. This allows us to derive closed form expressions for joint distributions of default times, default correlations, and CDS spreads in the presence of counterparty default risk. Using an affine approximation technique, which goes back to Singleton and Umantsev [21], one can also obtain analytically tractable expressions for swaption prices.

The state of a firm is expressed by a tuple consisting of a credit index and credit indicator. The credit index, as mentioned in [13], is regarded as the

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firm’s credit score, which can be related to its asset value or its credit rating. It is assumed that the higher the credit index value, the worse a firm’s financial situation and zero-value of the corresponding credit index implies the perfect financial health of a firm. The indicator variable is defined to follow a simple point process starting at zero with a constant jump size one. The first jump of this process indicates the default of the corresponding firm (this method is originally proposed by Lando (1998 [16])). To model risk-free rates, for simplicity, here we only employ an one-factor affine model and define the factor as the short rate. It is straightforward to include an (affine) multi-factor interest rate model. Additional, e.g. industry specific, factors can easily be built in, as long as they comply with the affine structure. Although, for simplicity, we consider affine diffusion and simple point processes when it comes to computations, the following carries over to more general affine jump-diffusion state processes. We sketch alternative affine regimes including jumps, which contribute to more weight in the tail distribution of the credit index process.

It is worth mentioning that, instead of modeling default by setting a barrier for the credit index as in structural models (see e.g. [7, 1]), we add an extra indicator in order to overcome the difficulty of calculating the default probability. That is, we are in a fully reduced form (intensity based) framework. However, instead of conditioning on different filtrations to value default events, such as e.g. the “total hazard construction” in [22], we use an analytic approach, involving Laplace transforms and ODEs. As a result, we provide a general and efficient intensity based valuation method for credit derivatives in an affine framework.

A similar method is proposed for a hybrid of a structural and reduced form model in Chen and Filipović (2003 [5]). The present setup can in fact be considered as a limit case of the barrier model considered in [5]: the credit index process “jumps to infinity” at default, thereby hitting the “barrier at infinity”. However, since in [5] the lifetime of the state process itself is limited (the process explodes), it is difficult to use that approach for a multiple firm situation. The present approach keeps a structural flavour, in that we still identify one of the firm’s state variables as credit index. We also point out that we allow for simultaneous default of several firms, which is often ignored by other models.

As for the recovery issue of a credit derivative, we adopt the convention of recovery at default and assume that the recovery rate is a random variable depending on both risk-free rates and the credit index of the default firm, which is more reasonable than assuming recovery at maturity as in [15] or that the recovery rate is stochastically independent of default probability and risk-free rates as in [12].

The literature on credit risk modeling is huge and fastly growing. We do not intend to provide a comprehensive reference list. Instead, we refer to the recent books by Duffie and Singleton (2003 [11]) and Schönbucher (2003 [19]) for an overview. Here, we mention in particular Jarrow and Yildirim (2002 [14]), who introduce correlation between market and credit risk by using an intensity based model where risk-free rates and default intensities depend on some common macroeconomic factors. Meanwhile, motivated by the catenated downfalls of firms during the financial crises in East Asia, Jarrow and Yu (2001

[15]), and further developed by Yu (2003 [22]), propose to consider the credit risk induced by the interdependence structure between firms by generalizing the intensity based models to allow a firm exposed to some firm-specific default risk, as well as to common risk factors. However, due to the complexity of the analysis, they confine their discussion to the situation where the default intensity follows a simple point process and only price the “idealized” default swaps with the simplified assumption that the recovery payment is made at the maturity of the CDS. Schönbucher and Schubert (2001 [20]) use copulas to introduce default dependency in intensity based models.

Different from the intensity based approach, Zhou (2001 [23]) generalizes the structural models originally pioneered by Merton (1974 [17]) to characterize default correlations by establishing the asset value correlation between two firms. Within this particular first-passage-time model, Zhou derives the joint default probability. However, it is hard to further obtain the prices of defaultable bonds and other credit derivatives unless it is assumed that the risk-free rates and default arrivals are independent. Similarly, Hull and White (2001 [13]) characterize credit risk by importing a credit index for each company and model the default by the event that the credit index hits a certain barrier. This credit index concept allows to consider all credit information of a firm including its asset value and its credit rating. In order to avoid the burdensome calculation of the hitting probability (default probability), they assume that generally a credit index process can be transformed to a Wiener process. This is restrictive since it requires that the credit risk is independent of the risk-free security market.

The remainder of the paper is organized as follows. Section 2 introduces the basic three-firm model, based on affine diffusion and simple point processes. We then discuss and illustrate the joint distribution of default times, the density function and default correlation, and present alternative dynamics including jumps. In Section 3 we derive closed form expressions for the valuation of a CDS in the presence of counterparty risk. In Section 4 we sketch how to price a swaption by affine approximation. Brief concluding remarks are given in Section 5.

2 The Basic Three-Firm Model

In this section we describe the basic model incorporating three firms and a one-factor short rate model. The extension to an m -factor interest rate and n -firm model along the following lines is straightforward, see also Section 2.3 below.

For background and theory of affine processes we refer to [8]. With e_i we denote the i -th standard basis vector in \mathbb{R}^7 , $i = 0, 1, \dots, 6$. Moreover, we shall frequently use the multi-index notation

$$\mathbf{p} = (p_4, p_5, p_6), \mathbf{q} = (q_4, q_5, q_6) \in \mathbf{I} := \{0, 1\}^3.$$

We now consider the affine jump-diffusion process $X = (X^0, \dots, X^6)$ in \mathbb{R}_+^7

with generator

$$\begin{aligned} \mathcal{A}f(x) = & \sum_{i=0}^3 \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=0}^3 (b_i + \langle \beta_i, x \rangle) \partial_{x_i} f(x) \\ & + \sum_{\mathbf{p} \in \mathbf{I}} (f(x + p_4 e_4 + p_5 e_5 + p_6 e_6) - f(x)) (\ell_{\mathbf{p}} + \langle \lambda_{\mathbf{p}}, x \rangle), \end{aligned} \quad (1)$$

where

$$\alpha_i, b_i \geq 0, \quad \beta_i \in \mathbb{R}^7 \text{ with } \beta_{ij} \geq 0 \quad \forall j \neq i, \quad \ell_{\mathbf{p}} \geq 0, \quad \lambda_{\mathbf{p}} \in \mathbb{R}_+^7.$$

We let X be realized on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions (e.g. the “canonical” space of càdlàg paths in \mathbb{R}_+^7). Depending on the context, \mathbb{P} stands either for the real-world or risk-neutral measure. In the latter case, prices are computed as \mathbb{P} -(conditional) expectations. Equivalent measure changes which preserve the respective affine structures exist and are feasible. For a discussion we refer to [3] and [4].

X^0 denotes the short rate process. The pair (X^i, X^{3+i}) represents the credit state of firm i , $i = 1, 2, 3$. We let $X_0^{3+i} = 0$ for $i = 1, 2, 3$. Then the first jump time

$$\tau_i := \inf\{t \mid X_t^{3+i} > 0\}$$

of X^{3+i} models the default time of firm i . We see from (1) that the firms can default simultaneously in all possible combinations, since we sum up over all jumps in the directions of $p_4 e_4 + p_5 e_5 + p_6 e_6$, for $\mathbf{p} \in \mathbf{I}$ (to exclude one of these combinations, simply put the corresponding intensity coefficients, $\ell_{\mathbf{p}}$ and $\lambda_{\mathbf{p}}$, equal to zero). X^i plays the role of a credit index for firm i . The larger X^i the more likely is a default of firm i (this effect can be achieved by an appropriate choice of the model parameters $\lambda_{\mathbf{p},i}$).

The generator (1) implies a rich interdependence structure between the components X^i :

- The interest rates, X^0 , influence all credit risk related variables, X^1, \dots, X^6 , by β_{i0} (mean-reversion level of X^i) and the respective $\lambda_{\mathbf{p},0}$ (jump intensity of X^{3+i}).
- The credit index of firm i , X^i , $i = 1, 2, 3$, drives the intensities for (joint) defaults of firms 1, 2 and 3 by the respective $\lambda_{\mathbf{p},i}$. This type of correlation has already been used by e.g. [9].

X^i also influences the mean reversion level for X^j by β_{ji} , $j = 0, \dots, 3$ (however, typically we let the short rates evolve autonomously, that is, we set $\beta_{0i} = 0$).

- The counter process for firm i , X^{3+i} , $i = 1, 2, 3$, influences the intensities for (joint) defaults of firms 1, 2 and 3 by the respective $\lambda_{\mathbf{p},3+i}$. Note that this introduces “infectious defaults” or a “loop dependent default risk structure” as proposed in [6] and [15], respectively: the default of either

firm increases the default intensity of the other firm. See also Example 2 below.

X^{3+i} also influences the mean reversion level for X^j by $\beta_{j,3+i}$, $j = 0, \dots, 3$, another form of default contagion.

Fix $\delta \geq 0$. The basic affine property to be found in [8] reads

$$\mathbb{E} \left[e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_t \rangle} \mid \mathcal{F}_t \right] = e^{\phi(T-t, v; \delta) + \langle \psi(T-t, v; \delta), X_t \rangle} \quad (2)$$

for all $v \in \mathbb{R}_-^7$, where the \mathbb{R}_- -valued functions $\phi = \phi(t, v; \delta)$ and $\psi_i = \psi_i(t, v; \delta)$ solve the following generalized Riccati equations (GREs)

$$\begin{aligned} \partial_t \phi &= \sum_{k=0}^3 b_k \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \ell_{\mathbf{p}} \left(e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1 \right), \\ \phi(0, v; \delta) &= 0, \\ \partial_t \psi_i &= \alpha_i \psi_i^2 + \sum_{k=0}^3 \beta_{ki} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p}, i} \left(e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1 \right) - \delta \mathbf{1}_{\{i=0\}} \quad (3) \\ \psi_i(0, v; \delta) &= v_i, \\ \partial_t \psi_j &= \sum_{k=0}^3 \beta_{kj} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p}, j} \left(e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1 \right), \\ \psi_j(0, v; \delta) &= v_j \end{aligned}$$

for $i = 0, 1, 2, 3$ and $j = 4, 5, 6$.

2.1 Basic Trick

The following basic trick allows to express a variety of possible joint credit events as (limits) of exponential functions of X : let $i, j \in \{1, 2, 3\}$, then

$$\begin{aligned} \mathbf{1}_{\{t < \tau_i\}} &= \lim_{k \rightarrow \infty} e^{-k X_t^{3+i}}, \\ \mathbf{1}_{\{s < \tau_i \leq t\}} &= \mathbf{1}_{\{s < \tau_i\}} - \mathbf{1}_{\{t < \tau_i\}} = \lim_{k \rightarrow \infty} \left(e^{-k X_s^{3+i}} - e^{-k X_t^{3+i}} \right), \quad s < t, \\ \mathbf{1}_{\{s < \tau_i, t < \tau_j\}} &= \mathbf{1}_{\{s < \tau_i\}} \mathbf{1}_{\{t < \tau_j\}} = \lim_{k \rightarrow \infty} e^{-k(X_s^{3+i} + X_t^{3+j})}, \\ &\text{etc.} \end{aligned} \quad (4)$$

This asks for the following general result.

Proposition 2.1. *For $t \leq T$, $v \in \mathbb{R}_-^7$, $\delta \geq 0$ and $\mathbf{p} \in \mathbf{I}$ we have*

$$\begin{aligned} \mathbb{E} \left[e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k \rightarrow \infty} e^{-k(p_4 X_T^4 + p_5 X_T^5 + p_6 X_T^6)} \mid \mathcal{F}_t \right] \\ = e^{\Phi(T-t, v; \delta; \mathbf{p}) + \sum_{i \in \{0, \dots, 3\} \cup J_0(\mathbf{p})} \Psi_i(T-t, v; \delta; \mathbf{p}) X_t^i} \prod_{j \in J_1(\mathbf{p})} \mathbf{1}_{\{X_t^j = 0\}}, \quad (5) \end{aligned}$$

where $J_0(\mathbf{p}) := \{4 \leq j \leq 6 \mid p_j = 0\}$, $J_1(\mathbf{p}) := \{4 \leq j \leq 6 \mid p_j = 1\}$ and the \mathbb{R}_- -valued functions

$$\Phi = \Phi(t, v; \delta; \mathbf{p}) \quad \text{and} \quad \Psi_i = \Psi_i(t, v; \delta; \mathbf{p})$$

satisfy

$$\begin{aligned} \partial_t \Phi &= \sum_{k=0}^3 b_k \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \ell_{\mathbf{q}} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \ell_{\mathbf{q}}, \\ \Phi(0, v; \delta; \mathbf{p}) &= 0, \\ \partial_t \Psi_i &= \alpha_i \Psi_i^2 + \sum_{k=0}^3 \beta_{ki} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q}, i} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) \\ &\quad - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q}, i} - \delta 1_{\{i=0\}}, \\ \Psi_i(0, v; \delta; \mathbf{p}) &= v_i, \\ \partial_t \Psi_j &= \sum_{k=0}^3 \beta_{kj} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q}, j} (e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q}, j}, \\ \Psi_j(0, v; \delta; \mathbf{p}) &= v_j \end{aligned} \tag{6}$$

for $i = 0, 1, 2, 3$ and $j \in J_0(\mathbf{p})$, where $\mathbf{I}_0(\mathbf{p}) := \{\mathbf{q} \in \mathbf{I} \mid q_j = 0 \forall j \in J_1(\mathbf{p})\}$ and $\mathbf{I}_1(\mathbf{p}) := \mathbf{I} \setminus \mathbf{I}_0(\mathbf{p}) = \{\mathbf{q} \in \mathbf{I} \mid q_j = 1 \text{ for some } j \in J_1(\mathbf{p})\}$.

Proof. By dominated convergence, the left hand side of (5) equals

$$\begin{aligned} &\lim_{k \rightarrow \infty} \mathbb{E} \left[e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} e^{-k(p_4 X_T^4 + p_5 X_T^5 + p_6 X_T^6)} \mid \mathcal{F}_t \right] \\ &= \lim_{k \rightarrow \infty} e^{\phi(T-t, v-k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta) + \langle \psi(T-t, v-k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta), X_t \rangle}. \end{aligned}$$

Since $\partial_t \psi_j \leq 0$ for all $j = 4, 5, 6$, we have that $\psi_j(t, v-k(p_4 e_4 + p_5 e_5 + p_6 e_6); \delta) \rightarrow -\infty$ uniformly on compacts for $k \rightarrow \infty$ if $j \in J_1(\mathbf{p})$. We conclude that the right hand side of the GREs (3) converges uniformly on compacts to the right hand side of (6), which proves the proposition. \square

Remark 2.2. Notice that $\Phi(t, v; \delta; \mathbf{p})$ and $\Psi_i(t, v; \delta; \mathbf{p})$ in Proposition 2.1 do in fact not depend on v_j for $j \in J_1(\mathbf{p})$.

Example 1 Let $t \leq T$. The \mathcal{F}_t -conditional Laplace transform of X_T with respect to the T -forward measure conditional on $\{T < \tau_1 \wedge \tau_2\}$ is

$$\frac{\mathbb{E} \left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} 1_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}{\mathbb{E} \left[e^{-\int_t^T X_s^0 ds} 1_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}, \quad v \in \mathbb{R}_-^7,$$

where

$$\begin{aligned}
& \mathbb{E} \left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \mathbf{1}_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k \rightarrow \infty} e^{-k(X_T^4 + X_T^5)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(T-t, v; 1; 1, 1, 0) + \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, v; 1; 1, 1, 0) X_t^i} \mathbf{1}_{\{X_t^4 = X_t^5 = 0\}}.
\end{aligned}$$

2.2 Joint Distribution of Default Times

With the aid of (4) and Proposition 2.1 we now discuss the dependence structure of the default times τ_1 and τ_2 .

Fix $s \geq 0$. For the \mathcal{F}_s -conditional joint distribution of (τ_1, τ_2) we have

$$\begin{aligned}
F(t, T) &= \mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{F}_s] \\
&= 1 - \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mid \mathcal{F}_s] - \mathbb{E}[\mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s] + \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s], \tag{7}
\end{aligned}$$

for $t, T \geq s$. The terms involved are

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mid \mathcal{F}_s] &= \mathbb{E} \left[\lim_{k \rightarrow \infty} e^{-kX_t^4} \mid \mathcal{F}_s \right] \\
&= e^{\Phi(t-s, 0; 0; 1, 0, 0) + \sum_{i \in \{0, \dots, 3, 5, 6\}} \Psi_i(t-s, 0; 0; 1, 0, 0) X_s^i} \mathbf{1}_{\{X_s^4 = 0\}}, \\
\mathbb{E}[\mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s] &= \mathbb{E} \left[\lim_{k \rightarrow \infty} e^{-kX_T^5} \mid \mathcal{F}_s \right] \\
&= e^{\Phi(T-s, 0; 0; 0, 1, 0) + \sum_{i \in \{0, \dots, 3, 4, 6\}} \Psi_i(T-s, 0; 0; 0, 1, 0) X_s^i} \mathbf{1}_{\{X_s^5 = 0\}},
\end{aligned}$$

and, for $t \leq T$,

$$\begin{aligned}
& \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s] \\
&= \mathbb{E} \left[\lim_{k \rightarrow \infty} e^{-kX_t^4} \mathbb{E} \left[\lim_{l \rightarrow \infty} e^{-lX_T^5} \mid \mathcal{F}_t \right] \mid \mathcal{F}_s \right] \\
&= e^{\Phi(T-t, 0; 0; 0, 1, 0)} \mathbb{E} \left[\lim_{k \rightarrow \infty} e^{-k(X_t^4 + X_t^5)} e^{\sum_{i \in \{0, \dots, 3, 4, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) X_t^i} \mid \mathcal{F}_s \right] \\
&= e^{\Phi(T-t, 0; 0; 0, 1, 0) + \Phi(t-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) e_i; 0; 1, 1, 0)} \\
&\quad \times e^{\sum_{j \in \{0, \dots, 3, 6\}} \Psi_j(t-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, 0; 0; 0, 1, 0) e_i; 0; 1, 1, 0) X_s^j} \mathbf{1}_{\{X_s^4 = X_s^5 = 0\}}, \tag{8}
\end{aligned}$$

and similarly for $t \geq T$,

$$\begin{aligned}
& \mathbb{E}[\mathbf{1}_{\{t < \tau_1\}} \mathbf{1}_{\{T < \tau_2\}} \mid \mathcal{F}_s] \\
&= e^{\Phi(t-T, 0; 0; 1, 0, 0) + \Phi(T-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(t-T, 0; 0; 1, 0, 0) e_i; 0; 1, 1, 0)} \\
&\quad \times e^{\sum_{j \in \{0, \dots, 3, 6\}} \Psi_j(T-s, \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(t-T, 0; 0; 1, 0, 0) e_i; 0; 1, 1, 0) X_s^j} \mathbf{1}_{\{X_s^4 = X_s^5 = 0\}}. \tag{9}
\end{aligned}$$

Remark 2.3. For simplicity, we will set $s = 0$ in what follows and use the convention $X_0^j = 0$ for $j = 4, 5, 6$. All results carry over after a slight modification to the general case $s \geq 0$.

2.2.1 Joint Density

Notice that the joint distribution function (7) is twice continuously differentiable in (t, T) for $t \neq T$, but not on the diagonal $\Delta := \{(t, t) \mid t \geq 0\}$ in general. Below we illustrate the three cases where i) a jointly continuous density function f with

$$F(t, T) = \int_0^t \int_0^T f(u, v) dudv, \quad \forall (t, T) \in \mathbb{R}_+^2, \quad (10)$$

exists (Example 4), ii) f is only piecewise continuous (Example 2), iii) the entire mass of the distribution is concentrated on Δ and hence a density does not exist (Example 3).

Example 2 Let $\ell := \ell_{(1,0,0)} > 0$ and $\lambda := \lambda_{(0,1,0),4} > 0$ and all the other parameters be zero. Then the generator (1) is of the form

$$\mathcal{A}f(x) = (f(x + e_4) - f(x)) \ell + (f(x + e_5) - f(x)) \lambda x_4.$$

That is, firm 1 defaults with a constant intensity ℓ and the default intensity of firm 2 is zero first, jumps to λ at the default time of firm 1 (“infectious default”) and increases by the amount of λ at any further jump time of X^4 . Accordingly, we have

$$\begin{aligned} \partial_t \Phi(t, v; 0; 1, 0, 0) &= \partial_t \Phi(t, v; 0; 1, 1, 0) = -\ell \\ \partial_t \Phi(t, v; 0; 0, 1, 0) &= \ell (e^{v_4 - \lambda t} - 1) \\ \partial_t \Psi_4(t, v; 0; 1, 0, 0) &= \lambda (e^{v_5} - 1) \\ \partial_t \Psi_4(t, v; 0; 0, 1, 0) &= \partial_t \Psi_4(t, v; 0; 1, 1, 0) = -\lambda \end{aligned}$$

and $\partial_t \Psi_i(t, v; 0; \mathbf{p}) \equiv 0$ for all $i \neq 4$. So that

$$\begin{aligned} \Phi(t, v; 0; 1, 0, 0) &= \Phi(t, v; 0; 1, 1, 0) = -\ell t \\ \Phi(t, v; 0; 0, 1, 0) &= \ell \left(\frac{e^{v_4}}{\lambda} (1 - e^{-\lambda t}) - t \right) \\ \Psi_4(t, v; 0; 1, 0, 0) &= v_4 + \lambda (e^{v_5} - 1) t \\ \Psi_4(t, v; 0; 0, 1, 0) &= \Psi_4(t, v; 0; 1, 1, 0) = v_4 - \lambda t \end{aligned}$$

and $\Psi_i(t, v; 0; \mathbf{p}) \equiv v_i$ for all $i \neq 4$. In view of (8) and (9) we obtain for $G(t, T) = \mathbb{E}[1_{\{t < \tau_1\}} 1_{\{T < \tau_2\}}]$,

$$G(t, T) = \begin{cases} e^{\frac{\ell}{\lambda} (1 - e^{-\lambda(T-t)}) - \ell T}, & t \leq T, \\ e^{-\ell t}, & t \geq T. \end{cases}$$

It is easy to see that $\partial_t G(t, T)$ and $\partial_T G(t, T)$ are jointly continuous in (t, T) and absolutely continuous in T and t , respectively. Hence (10) holds. But f is not continuous at Δ since

$$\partial_t \partial_T G(t, T) = \begin{cases} (\ell \lambda e^{-\lambda(T-t)} + \ell e^{-\lambda(T-t)} (1 - e^{-\lambda(T-t)})) G(t, T), & t < T, \\ 0, & t > T, \end{cases}$$

and we see that $\frac{\partial_t \partial_T G(t^-, t)}{G(t, t)} = \ell \lambda \neq 0$.

Example 3 It is rather obvious that the distribution (7) is singular if defaults of different firms can occur simultaneously. For illustration consider the generator

$$\mathcal{A}f(x) = f(x + e_4 + e_5) - f(x),$$

that is, we set $\ell_{(1,1,0)} = 1$ and all other parameters are zero. A straightforward calculation shows that

$$F(t, T) = 1 - e^{-t \wedge T}.$$

This distribution carries the entire mass on the diagonal Δ , and therefore has no density.

Example 4 As we have seen above, the joint distribution function (7) for infectious defaults (Example 2) and simultaneous defaults (Example 3) does not admit a (continuous) density. We now consider an example where τ_1 and τ_2 are conditionally independent given the information $\mathcal{G} = \sigma(X_t^0, \dots, X_t^3 \mid t \geq 0)$ generated by X^0, \dots, X^3 . We let the generator (1) be of the form

$$\begin{aligned} \mathcal{A}f(x) &= \alpha_0 x_0 \partial_{x_0}^2 f(x) + (b_0 + \beta_{00} x_0) \partial_{x_0} f(x) \\ &+ \sum_{i=1}^2 \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=1}^2 (b_i + \beta_{i0} x_0 + \beta_{ii} x_i) \partial_{x_i} f(x) \\ &+ (f(x + e_4) - f(x)) (\lambda_{(1,0,0),0} x_0 + \lambda_{(1,0,0),1} x_1 + \lambda_{(1,0,0),2} x_2) \\ &+ (f(x + e_5) - f(x)) (\lambda_{(0,1,0),0} x_0 + \lambda_{(0,1,0),1} x_1 + \lambda_{(0,1,0),2} x_2), \end{aligned} \quad (11)$$

with the symmetric structure

$$\begin{aligned} \alpha_1 &= \alpha_2, \quad b_1 = b_2, \quad \beta_{10} = \beta_{20}, \quad \beta_{11} = \beta_{22}, \\ \lambda_{(1,0,0),0} &= \lambda_{(0,1,0),0}, \quad \lambda_{(1,0,0),1} = \lambda_{(0,1,0),2}, \quad \lambda_{(1,0,0),2} = \lambda_{(0,1,0),1}. \end{aligned} \quad (12)$$

Since here we have

$$\mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{G}] = \mathbb{P}[\tau_1 \leq t \mid \mathcal{G}] \cdot \mathbb{P}[\tau_2 \leq T \mid \mathcal{G}]$$

and both of the \mathcal{G} -conditional distribution functions on the right hand side have a \mathcal{G} -measurable continuous density, it is rather obvious that $F(t, T) = \mathbb{E}[\mathbb{P}[\tau_1 \leq t, \tau_2 \leq T \mid \mathcal{G}]]$ admits a continuous density.

We write short $\Phi(v; \mathbf{p}) = \Phi(t, v; 0; \mathbf{p})$ and $\Psi_i(v, \mathbf{p}) = \Psi_i(t, v; 0; \mathbf{p})$. The relevant ODEs (6) are

$$\begin{aligned} \partial_t \Phi(0; \mathbf{p}) &= b_0 \Psi_0(0; \mathbf{p}) + b_1 (\Psi_1(0; \mathbf{p}) + \Psi_2(0; \mathbf{p})), \\ \partial_t \Psi_0(0; \mathbf{p}) &= \alpha_0 \Psi_0^2(0; \mathbf{p}) + \beta_{00} \Psi_0(0; \mathbf{p}) + \beta_{10} (\Psi_1(0; \mathbf{p}) + \Psi_2(0; \mathbf{p})) - \lambda_{\mathbf{p},0}, \\ \partial_t \Psi_i(0; \mathbf{p}) &= \alpha_1 \Psi_i^2(0; \mathbf{p}) + \beta_{11} \Psi_i(0; \mathbf{p}) - \lambda_{\mathbf{p},i}, \quad i = 1, 2, \end{aligned}$$

for $\mathbf{p} = (1, 0, 0)$, $(0, 1, 0)$, and

$$\begin{aligned} \partial_t \Phi(v; 1, 1, 0) &= b_0 \Psi_0(v; 1, 1, 0) + 2b_1 \Psi_1(v; 1, 1, 0), \\ \partial_t \Psi_0(v; 1, 1, 0) &= \alpha_0 \Psi_0^2(v; 1, 1, 0) + \beta_{00} \Psi_0(v; 1, 1, 0) + 2\beta_{10} \Psi_1(v; 1, 1, 0) - 2\lambda_{\mathbf{p},0}, \\ \partial_t \Psi_1(v; 1, 1, 0) &= \alpha_1 \Psi_1^2(v; 1, 1, 0) + \beta_{11} \Psi_1(v; 1, 1, 0) - \lambda_{(1,0,0),1} - \lambda_{(0,1,0),1}, \end{aligned}$$

with $\Psi_2(v; 1, 1, 0) = \Psi_1(v; 1, 1, 0)$, by symmetry.

Note that Ψ_1 and Ψ_2 above solve autonomous Riccati equations. The following solution formula is well know:

Lemma 2.4. *The function*

$$G = G(t, r_0) = -\frac{2C(e^{\rho t} - 1) - (\rho(e^{\rho t} + 1) + B(e^{\rho t} - 1))r_0}{\rho(e^{\rho t} + 1) - B(e^{\rho t} - 1) - 2A(e^{\rho t} - 1)r_0} \quad (13)$$

with $\rho := \sqrt{B^2 + 4AC}$ is the unique solution of the Riccati equations

$$\partial_t G = AG^2 + BG - C, \quad G(0, r_0) = r_0,$$

for $A, C \geq 0$, $B \in \mathbb{R}$ and $r_0 \leq 0$.

With formula (13) at hand it is possible—but cumbersome (we used Mathematica for the formal calculations)—to show that (7) is smooth enough to allow for a continuous density. Figures 1–3 show this density function for

$$\begin{aligned} \alpha_0 &= 1.736 \times 10^{-5}, & \alpha_1 &= 3.2648, \\ b_0 &= 0.01167, & b_1 &= 1.6328 \times 10^{-5}, \\ \beta_{00} &= -0.15492, & \beta_{10} &= 0.23006, & \beta_{11} &= -1.472, \\ \lambda_{(1,0,0),0} &= 0.26365, & \lambda_{(1,0,0),1} &= 0.10613, \\ X_0^0 &= 0.0105, & X_0^1 &= X_0^2 = 0.07, & X_0^3 &= 0, \end{aligned} \quad (14)$$

and different values for $\lambda_{(1,0,0),2}$, the impact of the second firm's credit rating, X_t^2 , on the default intensity of firm 1, and vice versa. The larger $\lambda_{(1,0,0),2}$, the stronger the dependence of the default times. The above parameters were obtained by the model calibration in [5]. $X_0^1 = 0.07$ corresponds to Moody's rating class Aaa.

2.2.2 Default Correlation

Although the joint distribution function (7) contains all the information about the dependence of the default times τ_1 and τ_2 , it is of particular interest to look at the correlation of the events $\{\tau_1 \leq T\}$ and $\{\tau_2 \leq T\}$,

$$\frac{Cov_{12}(T)}{\sqrt{Cov_{11}(T)Cov_{22}(T)}}$$

with

$$\begin{aligned} Cov_{ij}(T) &:= \mathbb{E}[1_{\{\tau_i \leq T\}}1_{\{\tau_j \leq T\}}] - \mathbb{E}[1_{\{\tau_i \leq T\}}]\mathbb{E}[1_{\{\tau_j \leq T\}}] \\ &= \begin{cases} \mathbb{E}[1_{\{\tau_i \leq T\}}] - (\mathbb{E}[1_{\{\tau_i \leq T\}}])^2, & i = j, \\ F(T, T) - \mathbb{E}[1_{\{\tau_i \leq T\}}]\mathbb{E}[1_{\{\tau_j \leq T\}}], & i \neq j, \end{cases} \end{aligned}$$

for varying $T \geq 0$. The terms involved are

$$\mathbb{E}[1_{\{\tau_i \leq T\}}] = 1 - \mathbb{E} \left[\lim_{k \rightarrow \infty} e^{-kX_T^{3+i}} \right] = 1 - e^{\Phi(T,0;0;\mathbf{p}(i)) + \sum_{j=0}^3 \Psi_j(T,0;0;\mathbf{p}(i))X_0^j},$$

where $\mathbf{p}(1) := (1, 0, 0)$ and $\mathbf{p}(2) := (0, 1, 0)$.

Figure 4 shows the default correlations for the model (11), (12), (14) with different values for $\lambda_{(1,0,0),2}$.

2.3 Alternative Dynamics

As already mentioned at the beginning of Section 2, it is straightforward to extend the preceding $(1+6)$ -dimensional factor process, $X = (X^0, \dots, X^7)$ (1-factor interest rates and 3 firms) to the $(m+2n)$ -dimensional analog,

$$X = (X^0, \dots, X^m, X^{m+1}, \dots, X^{m+n}, X^{m+n+1}, \dots, X^{m+2n})$$

(m -factor interest rates and n firms) with (X^{m+i}, X^{m+n+i}) describing the credit state of firm $i = 1, \dots, n$. Additional, e.g. industry specific, factors can easily be built in.

We now briefly sketch a possible change of the characteristics of the factor process X , replacing or extending the continuous diffusion parts by jumps. For the mathematical justification of what follows we refer, again, to [8]. The basic observation is that, for $\theta \in (1, 2)$,

$$\mu_\theta(d\xi) := \frac{\theta(\theta-1)}{\Gamma(2-\theta)} \frac{d\xi}{\xi^{1+\theta}}$$

satisfies $\int_{\mathbb{R}_{++}} (\xi \wedge \xi^2) \mu_\theta(d\xi) < \infty$ and

$$\int_{\mathbb{R}_{++}} (e^{v\xi} - 1 - v\xi) \mu_\theta(d\xi) = (-v)^\theta, \quad v \in \mathbb{R}_-. \quad (15)$$

The diffusion part

$$\alpha_i x_i \partial_{x_i}^2 f(x)$$

in (1) can now selectively for $i \in \{0, \dots, 3\}$ be replaced (or extended) by

$$\alpha_i x_i \int_{\mathbb{R}_{++}} (f(x + \xi e_i) - f(x) - \partial_{x_i} f(x) \xi) \mu_{\theta_i}(d\xi) \quad (16)$$

for $\theta_i \in (1, 2)$. The equation for ψ_i in (3) accordingly changes to

$$\partial_i \psi_i = \alpha_i (-\psi_i)^{\theta_i} + \sum_{k=0}^3 \beta_{ki} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p},i} (e^{p_4 \psi_4 + p_5 \psi_5 + p_6 \psi_6} - 1) - \delta 1_{\{i=0\}}. \quad (17)$$

Replacing the diffusion part of X^i by jumps (16) leads to a heavier tail distribution of X_t^i in general. Indeed, since the right hand side of (17) is monotonic increasing in θ_i for $|\psi_i|$ large enough, a comparison argument for ODEs

(see e.g. [2]) yields that $|\psi_i(t, v)| = -\psi_i(t, v)$ is monotonic decreasing in θ_i for $|v|$ large enough. That is, the smaller $\theta_i \in (1, 2)$, the smaller $\mathbb{E}[e^{-sX_t^i}] = e^{\phi(t, -se_i) + \langle \psi(t, -se_i), X_0 \rangle}$, for $s > 0$ large enough, indicating that there is more weight in the tail of X_t^i .

Finally, note that the limit case $\theta \rightarrow 2$ corresponds to the diffusion setup (1).

3 Valuing Credit Default Swaps

We now consider the valuation of a plain vanilla credit default swap (CDS) with notional principal \$1. The *seller* (firm 3) of a CDS contract provides the *buyer* (firm 2) insurance against the risk of default of a third party called the *reference entity* (firm 1). In return, the buyer makes periodic payments to the seller. We denote by T_0 the start date of the CDS and the payment dates by T_1, \dots, T_n . We assume that $T_k - T_{k-1} \equiv \Delta$ for all $k = 1, \dots, n$. We consider a Bermudan setup. That is, cashflows take place at dates T_k only, given the events that happened in the preceding periods $(T_{j-1}, T_j]$, $j = 1, \dots, k$.

At time T_k :

- if no default has occurred yet ($T_k < \tau_1 \wedge \tau_2 \wedge \tau_3$) then the buyer pays to the seller a fixed rate c ;
- if the reference entity has defaulted in period $(T_{k-1}, T_k]$ ($T_{k-1} < \tau_1 \leq T_k$) and the seller has not defaulted yet ($T_k < \tau_3$) and the buyer has not defaulted by T_{k-1} ($T_{k-1} < \tau_2$) then the seller pays $1 - G(X_{T_k})$ and the contract terminates, where

$$G(x) = e^{r + \langle \rho, x \rangle} \leq 1$$

denotes the recovery rate for the bond issued by the reference entity, for some $r \in \mathbb{R}_-$ and $\rho \in \mathbb{R}_-^7$;

- in all other cases there is no payment and the contract terminates.

The value at time $t \leq T_0$ of the buyer's payments accordingly is cB_t , where

$$\begin{aligned} B_t &= \mathbb{E} \left[\sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} \Delta \mathbf{1}_{\{T_k < \tau_1 \wedge \tau_2 \wedge \tau_3\}} \mid \mathcal{F}_t \right] \\ &= \Delta \sum_{k=1}^n \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} \lim_{l \rightarrow \infty} e^{-l(X_{T_k}^4 + X_{T_k}^5 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\ &= \Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0; 1; 1, 1, 1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0; 1; 1, 1, 1) X_t^i} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}}. \end{aligned} \tag{18}$$

The value at time $t \leq T_0$ of the seller's payment is

$$\begin{aligned}
S_t &= \mathbb{E} \left[\sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) \mathbf{1}_{\{T_{k-1} < \tau_1 \leq T_k\}} \mathbf{1}_{\{T_{k-1} < \tau_2\}} \mathbf{1}_{\{T_k < \tau_3\}} \mid \mathcal{F}_t \right] \\
&= \sum_{k=1}^n \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) \right. \\
&\quad \left. \times \lim_{l, m \rightarrow \infty} \left(e^{-lX_{T_{k-1}}^4} - e^{-mX_{T_k}^4} \right) e^{-lX_{T_{k-1}}^5 - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= \sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k},
\end{aligned}$$

where, taking into account Remark 2.2,

$$\begin{aligned}
S_t^{1k} &= \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} \lim_{l, m \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5) - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} \lim_{m \rightarrow \infty} e^{-mX_{T_k}^6} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 0; 0, 1)} \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i=0}^5 \Psi_i(\Delta, 0; 1; 0; 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 0; 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 0; 0, 1) e_i; 1; 1, 1, 1) X_t^j} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{19}$$

$$\begin{aligned}
S_t^{2k} &= \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} e^{r + \langle \rho, X_{T_k} \rangle} \lim_{l, m \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5) - mX_{T_k}^6} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k} \rangle} \lim_{m \rightarrow \infty} e^{-mX_{T_k}^6} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 0; 0, 1)} \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i=0}^5 \Psi_i(\Delta, \rho; 1; 0; 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 0; 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 0; 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 0; 0, 1) e_i; 1; 1, 1, 1) X_t^j} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{20}$$

$$\begin{aligned}
S_t^{3k} &= \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} \lim_{l, m \rightarrow \infty} e^{-lX_{T_{k-1}}^5 - m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\
&= \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} \lim_{m \rightarrow \infty} e^{-m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-lX_{T_{k-1}}^5} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 1, 0, 1)} \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i \in \{0, \dots, 3, 5\}} \Psi_i(\Delta, 0; 1; 1, 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(\Delta, 0; 1; 1, 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 1, 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, 0; 1; 1, 0, 1) e_i; 1; 1, 1, 1) X_t^j} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}},
\end{aligned} \tag{21}$$

$$\begin{aligned}
S_t^{4k} &= \mathbb{E} \left[e^{-\int_t^{T_k} X_s^0 ds} e^{r + \langle \rho, X_{T_k} \rangle} \lim_{l, m \rightarrow \infty} e^{-lX_{T_{k-1}}^5 - m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} \mathbb{E} \left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k} \rangle} \lim_{m \rightarrow \infty} e^{-m(X_{T_k}^4 + X_{T_k}^6)} \mid \mathcal{F}_{T_{k-1}} \right] \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-lX_{T_{k-1}}^5} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 1, 0, 1)} \mathbb{E} \left[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i \in \{0, \dots, 3, 5\}} \Psi_i(\Delta, \rho; 1; 1, 0, 1) X_{T_{k-1}}^i} \right. \\
&\quad \left. \times \lim_{l \rightarrow \infty} e^{-l(X_{T_{k-1}}^4 + X_{T_{k-1}}^5 + X_{T_{k-1}}^6)} \mid \mathcal{F}_t \right] \\
&= e^{r + \Phi(\Delta, \rho; 1; 1, 0, 1) + \Phi(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 1, 0, 1) e_i; 1; 1, 1, 1)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j(T_{k-1} - t, \sum_{i=0}^3 \Psi_i(\Delta, \rho; 1; 1, 0, 1) e_i; 1; 1, 1, 1) X_t^j} \mathbf{1}_{\{X_t^4 = X_t^5 = X_t^6 = 0\}}.
\end{aligned} \tag{22}$$

The forward CDS spread $C(t)$ at time $t \leq T_0$ is the fixed rate at which we have $C(t)B_t = S_t$. From the above, we obtain

Lemma 3.1. *The forward CDS spread is given by*

$$C(t) = \frac{S_t}{B_t} = \frac{\sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}}{\Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0; 1; 1, 1, 1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0; 1; 1, 1, 1) X_t^i}}, \tag{23}$$

where the terms B_t and S_t^{ik} are given by (18) and (19)–(22), respectively.

Figure 5 shows the CDS spread $C(T_0)$ for different CDS lengths, $T_n - T_0$, for the case of single-party risk (only the reference entity can default, that is, $X^2 = X^3 = 0$) with different rating classes: $X_{T_0}^1 = 0.07$ (Moody's Aaa), 0.13102 (Aa), 0.465 (A), 0.80907 (Baa) and $X_{T_0}^4 = 0$ (no default by T_0). The remaining parameters are according to (14).

4 Swaption Pricing by Affine Approximation

In this section we sketch a method for approximating swaption prices as proposed by Singleton and Umantsev [21]. Consider a call option on the above CDS (a swaption) with strike rate K and expiry date T_0 . Its payoff at T_0 is

$$(S_{T_0} - KB_{T_0})^+ = S_{T_0}1_{\{C(T_0) > K\}} - KB_{T_0}1_{\{C(T_0) > K\}}.$$

Since S_{T_0} and B_{T_0} are sums of exponential-affine functions in X_{T_0} , there is hope for a good analytic approximation of the swaption price at time $t < T_0$,

$$P_{\text{swpt}}(t) = \mathbb{E} \left[e^{-\int_t^{T_0} X_s^0 ds} S_{T_0} 1_{\{C(T_0) > K\}} \mid \mathcal{F}_t \right] - K \mathbb{E} \left[e^{-\int_t^{T_0} X_s^0 ds} B_{T_0} 1_{\{C(T_0) > K\}} \mid \mathcal{F}_t \right],$$

if there is an accurate affine approximation of $\log C(T_0) = \log S_{T_0} - \log B_{T_0}$ as a function of X_{T_0} , say

$$\log C(T_0) \approx P + \langle Q, X_{T_0} \rangle \quad (24)$$

for some $P \in \mathbb{R}$ and $Q \in \mathbb{R}^7$. The computation of $P_{\text{swpt}}(t)$ then boils down to the Fourier-inversion methods for conditional distributions with Laplace transforms of the form

$$\mathbb{E} \left[e^{-\int_t^{T_0} X_s^0 ds} e^{\langle R+v, X_{T_0} \rangle} \mid \mathcal{F}_t \right] = e^{\phi(T_0-t, R+v; 1) + \langle \psi(T_0-t, R+v; 1), X_t \rangle},$$

see (2), for $v \in \mathbb{R}_-^7 - R$, for some $R \in \mathbb{R}^7$, as discussed in [10].

To illustrate the effectiveness of this approximation, we show in Figure 6 the level sets for different levels K on the cross-sectional surface $(x_0, x_1) \mapsto C(T_0, X_{T_0} = (x_0, x_1, 0, \dots, 0))$, with the same parameter values as at the end of Section 3. It is obvious that a linear approximation (24) will yield accurate swaption prices.

5 Conclusion

This paper provides the basic and efficient techniques for valuating credit derivatives in an affine framework.

The state of a firm is characterised by its credit index and default indicator process. The joint evolution of risk-free rates and multiple firm's state processes can incorporate complex dependence structures. Due to a simple mathematical trick, which allows to replace indicator variables by exponential-affine functions of the state process, we obtain closed form expressions for the conditional expectations of a variety of joint credit events.

We demonstrate the efficiency of this approach by explicitly calculating the joint distribution and density (provided it exists) of default times, default correlations, and CDS spreads in the presence of counter-party default risk. Also we sketch the pricing of swaptions by using an affine approximation technique, as proposed by Singleton and Umantsev [21].

Our empirical results, for simplicity, are based on affine diffusion and simple point processes. An extension towards more general affine jump-diffusion processes, including multi-factor interest rate models and additional industry specific factors, is straightforward and an alternative affine regime is sketched in this paper. It remains future research to compare their empirical performances.

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Figure 1: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0$.

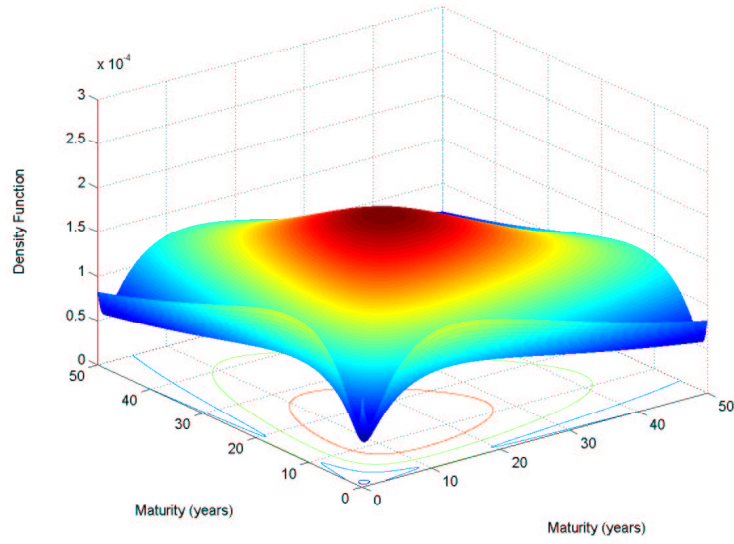


Figure 2: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0.01$.

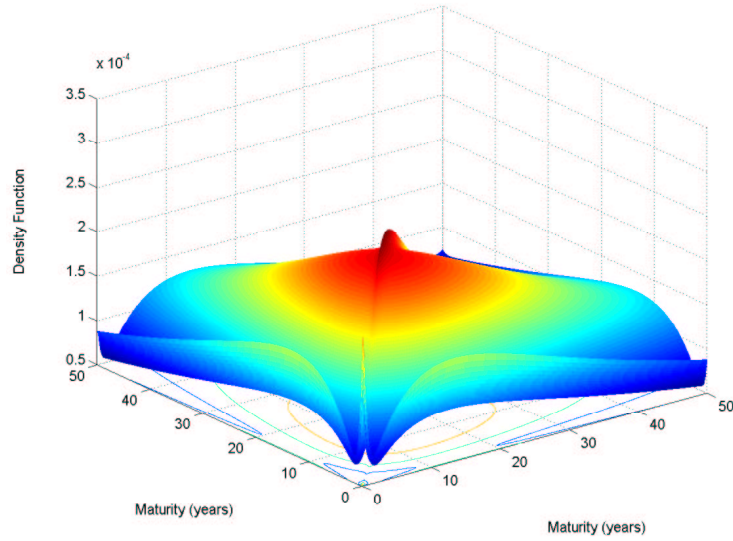


Figure 3: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0.01$ (zoomed).

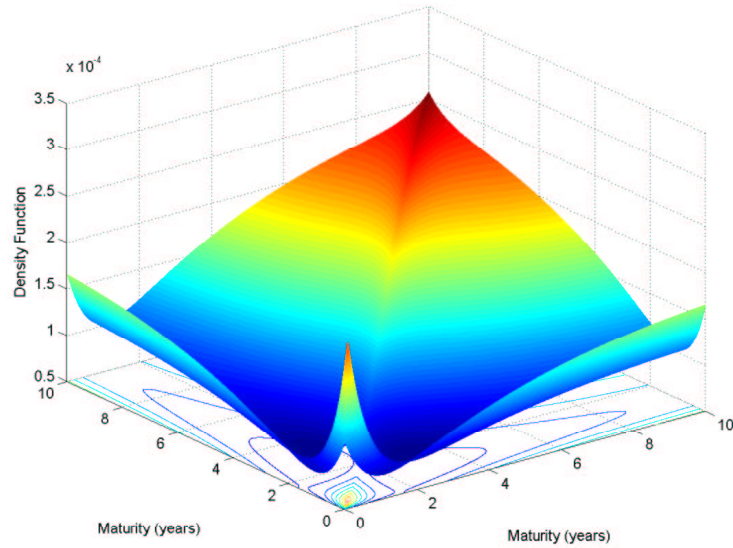


Figure 4: Default Correlations Between Two 'Aaa' Rated Firms for Different $\lambda_{(1,0,0),2}$.

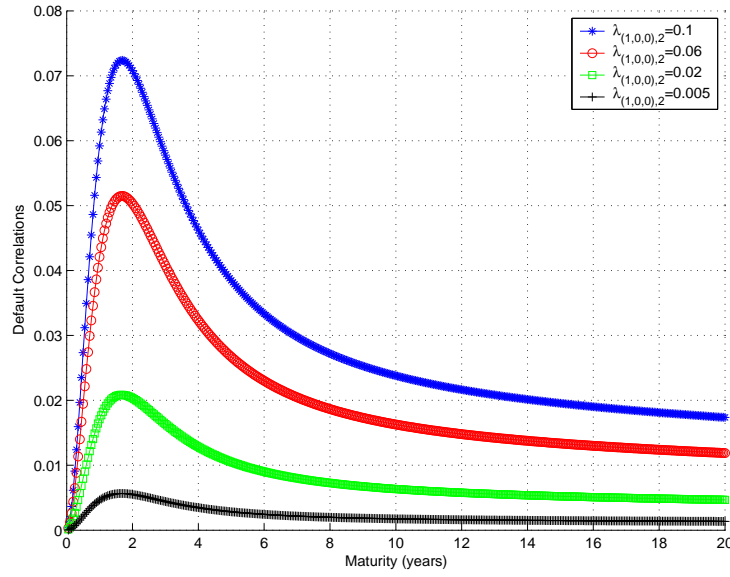


Figure 5: CDS spreads with Single-Party Risk (Reference Entity)

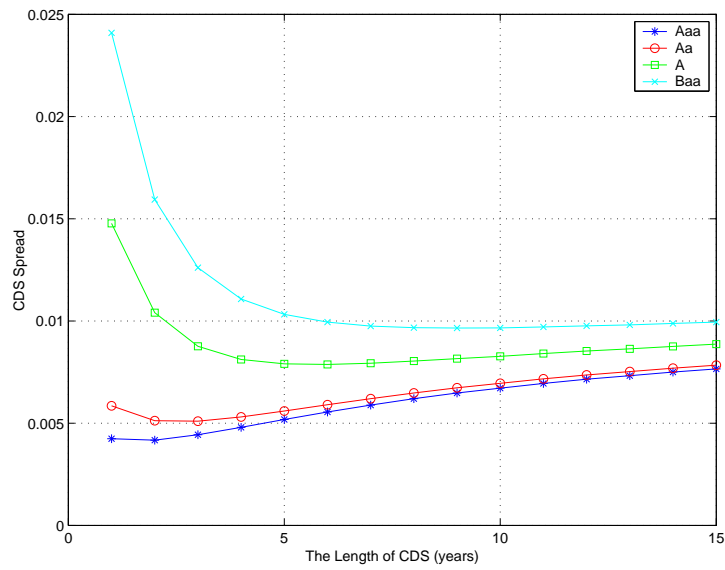


Figure 6: Exercising Boundary of a Default Swaption with Maturity 5 Years.

