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DUPLICATING CONTINGENT CLAIMS BY THE LAGRANGE METHOD

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Abstract. The problem of investing $y(0)$ dollars at time 0 to duplicate a contingent claim is formulated as a dynamic optimization problem and solved by the Lagrange method. As an example, the well-known formula of Black and Scholes on option pricing is derived. If the function defining $dy(t)$ is concave in $y(t)$, owing to costs of trading in incomplete markets, there is economy of scale in producing many claims simultaneously, thus explaining the profitability of institutions in providing such financial services.

1. INTRODUCTION

In a financial market with no arbitrage opportunities, the price of a contingent claim at time t which is represented by a random variable $B(T)$ at time $T > t$ is the value $y(t)$ of a portfolio which duplicates the claim with $y(T) = B(T)$. As in the literature, (e.g. Karatzas, 1989), this paper formulates the problem of finding the value $y(t)$ as an optimal control problem with $y(t)$ as the state variable and the portfolio π of stocks as a vector of control variables. It deals with the special problem when the dynamics of $y(t)$ are nonlinear because of market incompleteness as formulated by Peng and Yang (1999) and attempts to solve the problem by the Lagrange method of Chow (1997). It provides an alternative numerical method for solving the problem and an alternative proof of a proposition of Peng and Yang on the economy of scale in duplicating contingent claims in incomplete markets.

In section 2, a standard problem of duplicating a contingent claim is formulated as a dynamic optimization problem. The problem is solved by the Lagrange method. In section 3, an example is given in the pricing of a European call option. The problem is solved to obtain the well-known formula of Black and Scholes (1973) and Merton (1973). Section 4 shows that, under given assumptions, there is economy of scale in duplicating contingent claims simultaneously.

2. DUPLICATING A CONTINGENT CLAIM BY THE LAGRANGE METHOD

Using a fairly standard notation as adopted by Peng and Yang (1999), let $y(t)$ be the value of a portfolio which is used to duplicate a contingent claim $B(T)$, a random variable at time $T \geqslant t$. Assume that the market has one riskless asset

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and d risky assets. The price of the bond follows $dp_0(t) = p_0(t)r(t)dt$. The price of the ith stock follows $dp_i(t) = p_i(t)[b_i(t)dt + \Sigma_j \sigma_{ij}(t)dz_j(t)]$, where $z = (z_1, ..., z_d)'$ is a standard vector Brownian motion. Then, assuming no transaction costs, investing amounts $(\pi_1, ..., \pi_d) = \pi'$ in the d risky assets respectively and an amount $y - \Sigma_i \pi_i$ in the riskless asset will yield the following stochastic differential equation for the evolution of y:

$$
\begin{aligned}
dy(t) &= \sum_i (\pi_i/p_i)dp_i(t) + \left[\left(y - \sum_i \pi_i\right)/p_0\right]dp_0(t) \\
&= \left[r(t)y + \sum_{i=1}^{d} (b_i(t) - r(t))\pi_i\right]dt + \pi'\sigma\ dz(t)
\end{aligned} \tag{1}
$$

where σ is the matrix (σ_{ij}) and $dz = (dz_i, ..., dz_d)'$. To duplicate a contingent claim B at time T, we require an investment $y(0)$ at some initial time 0 such that $y(T) = B$. $y(0)$ is the price of the contingent claim at time 0. Hence to find the price of a contingent claim, we need to find $y(0)$ and the associated strategy $\pi(t)$, $0 \leqslant t \leqslant T$, such that $y(t)$ following (1) will reach $y(T) = B$, which is a terminal condition.

More generally, to duplicate a contingent claim $B(T)$, we find $y(0)$ and $\pi(t)$, $0 \leqslant t \leqslant T$, such that if $y(t)$ evolves according to the equation

$$
dy = f(t, y, \pi)dt + \pi'\sigma\ dz(t) \tag{2}
$$

we obtain $y(T) = B$. Peng and Yang (1999) assume that f is concave in y and π because of market incompleteness. Examples of market incompleteness are prohibition of short selling of assets i $(\pi_i(t) \geqslant 0)$, unavailability of assets i $(\pi_i(t) = 0)$ and prohibition of borrowing $y(t) - \Sigma_i \pi_i(t) \geqslant 0$. It is modeled in Peng and Yang (1999) by introducing a cost to the linear f in (1), thus reducing the increase in $y(t)$ and leading to a concave f in (2). The problem of duplicating a contingent claim can be formulated as an optimum control problem with y as the state variable and π as a vector control variable, the dynamics being given by equation (2). The objective is to maximize $-\frac{1}{2}E_0[y(T) - B(T)]^2$, where E_0 is expectation conditioned on information available at time 0. The maximum value, if achieved, is zero.

This optimal control problem can be solved by the Lagrange method of Chow (1997). Operationally the method consists of forming the Lagrangean

$$
\begin{aligned}
L = &-\tfrac{1}{2}E_0[y(T) - B(T)]^2 \\
&- \int_0^T E_t\lambda(t + dt)[y(t + dt) - y(t) - f(t, y, \pi)dt - \pi'\sigma\ dz(t)]
\end{aligned} \tag{3}
$$

where λ is the Lagrange multiplier. The first-order conditions for optimum are obtained by setting the partial derivatives of L with respect to π and y equal to zero. Mathematically this derivation is not rigorous because the integral in (3) is not properly defined, but the first-order conditions are rigorously derived by

Situ Rong (1999, equation (16)). Furthermore, the key result (8) below can be obtained by defining a Lagrange multiplier process as given by (5) without relying on the derivation of the first-order conditions.

$$\begin{aligned}\frac{\partial L}{\partial \pi(t)} &= E_t\left\{ \lambda(t+dt)\left[\frac{\partial f}{\partial \pi}\, dt + \sigma\ dz \right] \right\} \\ &= E_t\,[\lambda(t) + d\lambda]\left[\frac{\partial f}{\partial \pi}\, dt + \sigma\ dz \right] \\ &= E_t\left[\lambda(t)\,\frac{df}{\partial \pi}\, dt + \lambda\sigma\ dz + d\lambda\,\frac{\partial f}{\partial \pi}\, dt + d\lambda \cdot \sigma\ dz \right] = 0.\end{aligned}$$

By applying Itô's lemma to $\lambda(t, y)$ (with Σ denoting $\sigma\sigma'$)

$$d\lambda = \left\{ \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial y} f + \frac{1}{2}\frac{\partial^2 \lambda}{\partial y^2} \cdot \pi' \Sigma \pi \right\} dt + \frac{\partial \lambda}{\partial y}\, \pi' \sigma\ dz$$

we obtain the first condition for determining the optimum π:

$$\lambda\,\frac{\partial f}{\partial \pi} + \frac{\partial \lambda}{\partial y}\, \Sigma \cdot \pi = 0. \tag{4}$$

The second condition is obtained by

$$\begin{aligned}\frac{\partial L}{\partial y(t)} &= E_t\left\{ \lambda(t+dt)\left[1 + \frac{\partial f}{\partial y}\, dt \right] - \lambda(t) \right\} \\ &= E_t\left[(\lambda + d\lambda)\left(1 + \frac{\partial f}{\partial y}\, dt \right) - \lambda \right] \\ &= E_t\left[\lambda + \lambda\,\frac{\partial f}{\partial y}\, dt + d\lambda\left(1 + \frac{\partial f}{\partial y}\, dt \right) - \lambda \right] = 0\end{aligned}$$

implying $-\lambda(\partial f/\partial y)\ dt = E(d\lambda)$. Hence, using (4):

$$d\lambda = -\lambda\,\frac{\partial f}{\partial y}\, dt + \frac{\partial \lambda}{\partial y}\, \pi' \sigma\ dz = -\lambda\left[\frac{\partial f}{\partial y}\, dt + \left(\frac{\partial f}{\partial \pi} \right)' \sigma'^{-1}\ dz \right] \tag{5}$$

which is a stochastic differential equation for λ.

To find the price $y(t)$ at time t, we apply Itô's lemma to the function $y\lambda$:

$$d(y\lambda) = \left[\lambda f - y\lambda \frac{\partial f}{\partial y} + \frac{1}{2} \operatorname{tr} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi'\Sigma\pi & -\lambda\pi' \dfrac{\partial f}{\partial \pi} \\ -\lambda\pi' \dfrac{\partial f}{\partial \pi} & \lambda^2 \left(\dfrac{\partial f}{\partial \pi} \right)' \Sigma^{-1} \dfrac{\partial f}{\partial \pi} \end{bmatrix} \right) \right] dt$$

$$+ \lambda \left[\pi'\sigma - y \left(\frac{\partial f}{\partial \pi} \right)' \sigma'^{-1} \right] dz. \tag{6}$$

Using (6) we evaluate

$$y(T)\lambda(T) = y(t)\lambda(t) + \int_t^T \lambda(s) \left[f(s, y, \pi) - y(s) \frac{\partial f}{\partial y} - \pi'(s, y) \frac{\partial f}{\partial \pi} \right] ds$$

$$+ \int_t^T \lambda(s) \left[\pi'(s, y)\sigma - y(s) \left(\frac{\partial f}{\partial \pi} \right)' \sigma'^{-1} \right] dz(s). \tag{7}$$

Taking conditional expectation E_t on both sides of (7), letting $\lambda(t) = 1$ and noting the conditional expectation of the second integral of (7) to be zero, we obtain

$$y(t) = E_t[B(T)\lambda(T)] - E_t \int_t^T \lambda(s) \left[f(s, y, \pi) - y(s) \frac{\partial f}{\partial y} - \pi'(s, y) \frac{\partial f}{\partial \pi} \right] ds \tag{8}$$

which is a formula for evaluating the price $y(t)$ of a contingent claim $B(T)$.

Note that equation (8) remains valid if the volatility matrix $\sigma(t)$ is stochastic, following an Itô process for example. A stochastic $\sigma(t)$ would affect the properties of the processes (2) and (5) for y and λ, but equations (4) and (5) remain valid. The reason is that the Lagrangean (3) remains unchanged when $\sigma(t)$ is stochastic. Although $\sigma(t)$ can be considered a new matrix state variable, no new Lagrange multipliers need to be introduced as long as these state variables are not affected by the control variables (Chow, 1997, p. 33). Thus a stochastic $\sigma(t)$ affects the evaluation of conditional expectations of functions of λ and y in (8) but not equation (8) itself.

For model (1) the last term of (8) is zero. Let μ be a vector with $\mu_i = b_i - r$ and solve (5) for $\lambda(T)$ in (8) to yield

$$y(0) = E_0[\lambda(T)B(T)] = E_0 \left[\exp\left\{ -rT - \frac{1}{2} \mu'(\sigma\sigma')^{-1}\mu T - \mu'\sigma'^{-1} z(T) \right\} B(T) \right]$$

$$= E_0[e^{-rT} M(T) B(T)] \tag{9}$$

where $M(t) = e^{rt}\lambda(t)$ defines a risk neural measure Q related to the original measure P by $dQ = M(t)dP$. (9) gives a well-known result on duplicating contingent claims, as in Harrison and Pliska (1981, pp. 246–249).

In the general case, Peng (1999) has provided a linear approximation algorithm to solve equation (2) with terminal condition $y(T) = B$ based on equations (5) and (8). The algorithm solves (y^{i+1}, π^{i+1}) in the $(i+1)$st iteration, given (y^i, π^i). It amounts to linearizing f in equation (2) about (y^i, π^i) to solve for dy^{i+1} in his equation (4). It is shown to coverage under stated assumptions. Peng's equation (4), together with an equation for $d\lambda^{i+1}$ given by our equation (5), will yield the same $y^{i+1}(t)$ as given by equation (8) above, as can be shown by applying Itô's lemma to $d(y^{i+1}\lambda^{i+1})$, integrating and taking expectation in the derivation of (8).

3. PRICING A EUROPEAN CALL OPTION

As an example, let the portfolio consist of π dollars of a particular stock and $y - \pi$ dollars of the bond. The price of the stock is assumed to follow $dp = p(\alpha\ dt + \sigma\ dz)$. Bond price follows $dp_0 = p_0 r\ dt$. A European call option is represented by the random variable $B(T) = \max(P(T) - c, 0)$, where c is the strike price. To duplicate a European call option by investing y dollars, with π dollars in the stock and $y - \pi$ dollars in the bond, the value of the portfolio will evolve according to

$$dy = [ry + (\alpha - r)\pi]dt + \pi\sigma\ dz. \tag{10}$$

We form the Lagrangean

$$\begin{aligned} L = {} & -\frac{1}{2}\int_0^T E_t[y(T) - B(T)]^2\ dt \\ & - \int_0^T E_t\lambda(t+dt)\{y(t+dt) - y(t) - [ry + (\alpha - r)\pi]dt - \pi\sigma\ dz\} \end{aligned}$$

and obtain the first-order conditions

$$(\alpha - r)\lambda + \frac{\partial\lambda}{\partial y}\sigma^2\pi = 0 \tag{11}$$

$$d\lambda = -r\lambda\ dt + \frac{\partial\lambda}{\partial y}\pi\sigma\ dz = -r\lambda\ dt - (\alpha - r)\lambda\sigma^{-1}\ dz \tag{12}$$

which are a special case of (4) and (5). Since the function f is linear and homogenous in y and π in this example, the integral on the right-hand side of (8) is zero. We obtain the well-known Black–Scholes formula for option pricing with $M(T)$ defined by (9):

$$y(t) = E_t[\lambda(T)B(T)] = e^{-r(T-t)}E_t[M(T)B(T)]. \tag{13}$$

As pointed out in the paragraph below equation (8), if $\sigma(t)$ is stochastic, the first half of equation (13) remains valid but the properties of (9) and (12), and thus the evaluation of $E_t[\lambda(T)B(T)]$, will be affected.

4. ECONOMY OF SCALE IN DUPLICATING CONTINGENT CLAIMS

This section shows that if the function $f(t, y, \pi)$ in (2) is increasing and concave in y and π and is homogeneous of degree one in y and π (i.e. $f(t, ny, n\pi) = nf(t, y, \pi)$ for $n > 0$), then the cost of duplicating two contingent claims B_1 and B_2 together is lower than the sum of the costs of duplicating them separately. Hence there is profit to be made by financial institutions in producing a large number of these claims.

We first show that if $f(t, y, \pi)$ is increasing and concave in y and π, the cost $y(0)$ required under optimum portfolio policy $\pi(t), 0 \leqslant t \leqslant T$, to duplicate a contingent claim $B(T)$ at time $T > t$ is a convex function $F(B)$ of $B(T)$; i.e. for $0 \leqslant \alpha \leqslant 1$:

$$F(\alpha B_1 + (1-\alpha)B_2) \leqslant \alpha F(B_1) + (1-\alpha)F(B_2). \tag{14}$$

Consider $F(\alpha B_1 + (1-\alpha)B_2) = g(\alpha)$ as a function of α. The convexity of $F(B)$ with respect to B is equivalent to the convexity of g with respect to α for any B_1 and B_2. Thus we need to show only that the second derivative of g with respect to α, denoted by $g''(\alpha)$, is positive. Observe that $g''(\alpha)$ is the second derivative of $y(0)$ with respect to α, where $(y(t), \pi(t))$ is the solution of the optimal control problem with dynamics given by equation (2) and the end-point condition $y(T) = \alpha B_1 + (1-\alpha)B_2$. The optimum path $(y(t, \alpha), \pi(t, \alpha))$ is treated as a function of α.

Denote the first derivative of $(y(t, \alpha), \pi(t, \alpha))$ with respect to α by $(y'(t), \pi'(t))$, and the second derivative by $(y''(t), \pi''(t))$. Differentiating both sides of equation (2) twice with respect to α, one obtains (with subscripts of f denoting partial derivatives)

$$dy''(t) = [a(t, \alpha)y'' + (b(t, \alpha))'\pi'' + A(t, \alpha)]dt + (\pi'')'\sigma\, dz$$

$$a(t, \alpha) = f_y(y(t, \alpha), \pi(t, \alpha)); \qquad b(t, \alpha) = f_\pi(y(t, \alpha), \pi(t, \alpha)) \tag{15}$$

and

$$A(t, \alpha) = f_{yy} \cdot (y')^2 + 2y'(f_{y\pi'})\pi' + (\pi')'(f_{\pi\pi'})\pi'. \tag{16}$$

Since f is concave in y and π, $A(t, \alpha)$ is nonpositive. Note that $y''(T) = 0$ as $y(T) = \alpha B_1 + (1-\alpha)B_2$.

To the stochastic differential equation (15), we introduce the dual equation analogous to (5):

$$d\lambda(t) = -\lambda[a(t, \alpha)dt + (b(t, \alpha))'\sigma'^{-1}\, dz] \tag{17}$$

with initial condition $\lambda(0)=1$. As in equations (6)–(8) we apply Itô's lemma to $y''(t,\alpha)\lambda(t)$ and take expectations to obtain

$$y''(0)\lambda(0) = E_0[y''(T)\lambda(T)] - E_0 \int_0^T \lambda(t)A(t,\alpha)dt$$

$$= -E_0 \int_0^T \lambda(t)A(t,\alpha)dt. \tag{18}$$

Since A is nonpositive and $\lambda(0)=1$, it follows that $y''(0)$ is nonnegative, and (14) is proved.

In addition, if $f(t,y,\pi)$ is homogeneous of degree one in y and π, duplicating $nB(T)$ will require only n times the cost $y(t)$ for duplicating one $B(T)$. This can be seen by solving the constrained dynamic optimization problem as formulated in equation (3) by changing the variables y and π to $y^*=ny$ and $\pi^*=n\pi$, respectively; if $y(0)$ and $\pi(t)$ solve the problem of duplicating $B(T)$, $y^*(0)=ny(0)$ and $\pi^*(t)=n\pi(t)$ will solve the problem of duplicating $nB(T)$. We thus have

$$F(nB)=nF(B).$$

Letting $\alpha=1/2$, $B_1=2B_1^*$ and $B_2=2B_2^*$, we use equation (14) and the above equation to obtain

$$F(B_1^*+B_2^*)\leqslant F(B_1^*)+F(B_2^*).$$

The cost of duplicating two contingent claims together is thus equal to or lower than the cost of duplicating them separately. Hence there is economy of scale, and possible profit, in duplicating many contingent claims simultaneously, as pointed out by Peng and Yang (1999).

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