Pricing Credit Default Swaps Under Default Correlations and Counterparty Risk*

Li Chen Damir Filipović

Princeton University

Abstract

In this paper, we develop a generalized affine model to characterize correlated credit risk of multi-firms. When valuing credit derivatives, this new approach allows to incorporate correlative market and credit risk, interdependent default risk structure and counterparty risk into consideration. We have demonstrated our affine model not only combines the existing structural models and intensity based models, but also produces explicit formulas for the prices of credit default swaps and other credit derivatives.

1 Introduction

Recently the exponential growth of the credit derivative market (see "Credit Risk" 2000 [2]) generates an upsurge in the fair valuation of various credit derivatives including credit default swaps (CDSs). All the existing methods, however, seem unable to provide an analytically tractable model to entirely incorporate the concerns of

- the correlative market and credit risk;
- the joint credit migrations of multi-firms;
- the interdependence of the default risk structure,

which are essentially important accounted for precisely pricing the credit derivatives. Actually, Jarrow and Yildirim (2002 [7]) start to model the correlation between market and credit risk by the intensity based models in that this correlation arises because of the mutual-dependence of risk-free rates and default

*This version: Mar. 25 2003 (First Draft)

intensity on some common macroeconomic factors. Meanwhile, motivated by the catenated downfalls of firms during the financial crises in East Asia, Jarrow and Yu (2001 [8]) propose to consider the credit risk induced by the interdependence structure between firms by generalizing the intensity based models to allow a firm exposed to some firm-specific default risk, as well as to common risk factors. However, due to the complexity of the analysis, they confine their discussion to the situation that the default intensity follows a simple point process and only price the "idealized" default swaps with the simplified assumption that the recovery payment is made at the maturity of the CDS. Different from the intensity based approach, Hull and White (2001 [6]) characterize credit risk by importing a "credit index" for each company and model the default by the event that the credit index hits a certain barrier. This approach generalizes the structural model originally proposed by Merton (1974 [10]) to consider all credit information of a firm including its asset value and its credit rating. In order to avoid the burdensome calculation of the hitting probability (default probability), they assume that generally a credit index process can be transformed to a Wiener process, which is in doubt since it implies that the credit risk is independent of risk-free security market.

Distinctive from all these methods, we will model the risk-free rates, credit indices and default events altogether by a multi-dimensional affine process. In this way, not only the dynamics of a credit index can be substantially extended from a Wiener process to any affine process (including affine jump-diffusion processes), but also this generalized affine model provides us an analytical framework to consider all the essential concerns mentioned before because of the analytical tractability and the rich structure of affine Markov processes. Moreover, we have demonstrated that this model can produce explicit formulas for the prices of default swaps and other credit derivatives.

In order to build an affine credit risk model for N different firms of interest, first we assign two positive variables to each firm: one for modeling its credit index, the other for the indicator of its default. The credit index of a firm, as mentioned in [6], is regarded as its credit score, which can be related to its asset value or its credit rating (see [1] for an affine rating-based model). It is assumed that the higher the credit index value, the worse a firm's financial situation and zero-value of the corresponding credit index implies the perfect financial health of a firm. The indicator variable is defined to follow a simple point process starting at 0 with a constant jump size one. The first jump of this process indicates the default of the corresponding firm[1]. To model the risk-free

[1] This method is originally proposed by Lando (1998 [9])

rates, for simplicity, here we only employ an one-factor affine model and define the factor as the short rate. It is straightforward to extend the model to multi-factor cases. Hence we can construct a $2N+1$-dimensional affine process jointly modeling the dynamics of all these factors. By the specification analysis of this affine process, we illustrate how to incorporate market-credit risk correlation, joint credit migrations and firm-specific default risk altogether into the model.

It is worth mentioning that, instead of modeling default by setting a barrier for the credit index as in structural models, we add an extra indicator in order to overcome the difficulty of calculating the default probability. Accordingly, the jump intensity of this extra variable should depend on the corresponding credit index[2]. On the other hand, if we only model the default indicator process of each firm without considering credit indices, our model degenerates into the integrated affine models proposed by Filipović (2002, [4]). Since there is no corresponding market entity for the default intensity, this class of models is not quite tractable for implementation. However, Filipović has demonstrated that the traditional doubly stochastic setup for modeling credit risk can be embedded in these integrated affine models. Therefore, by adding credit indices, our new approach combines virtues of both structural models and intensity based models. Furthermore, our affine approach allows to model a limit case of the default interdependence between firms, i.e., the possibility of simultaneous defaults of several firms is under consideration, which is ignored by all the previous models.

As to the recovery issue of a credit derivative, we adopt the convention of "recovery at default" and assume that the recovery rate is a random variable depending on both risk-free rates and the credit index of the default firm, which is much more reasonable than assuming "recovery at maturity" as in [8] or the recovery rate is stochastically independent of default probability and risk-free rates as in [5].

The remainder of the paper is organized as follows. First we will briefly introduce some preliminary theorem of positive regular affine processes in Section 2. Then in Section 3, we construct our generalized affine model and interpret how to capture all the default correlations. In Section 4, we derive the prices of default swaps with and without considering counterparty risk. Brief concluding remarks are given in Section 5.

[2] A different approach can be found in Chen and Filipović (2003 [1]) who apply a non-conservative affine process to modeling credit migration and characterize default events by the death of the processes.

2 Fundamental Theory of Positive Affine Processes

In order to make this paper self-contained, first we introduce some fundamental theory of positive conservative affine processes[3]. Consider a time-homogeneous Markov process $X = (X^1, ..., X^m)$ with state space $D := \mathbb{R}_+^m$ starting at x and its transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ acting on bD,

$$P_t f(x) = \int_D f(\xi) p_t(x, d\xi), \quad \forall\ f \in bD.$$

Definition 2.1 *Under some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$, a Markov process X and its transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ is called affine if, for every $(t, u) \in \mathbb{R}_+ \times \mathbb{C}_-^m$, there exist $\phi(t, u) \in \mathbb{C}$ and $\psi(t, u) \in \mathbb{C}^m$ such that*

$$P_t f_u(x) = e^{\phi(t,u)+\langle\psi(t,u),x\rangle}, \quad \forall\ x \in D, \tag{2.1}$$

where $f_u(x) := e^{\langle u,x\rangle}$.

For the further analysis, we also need to provide some regularity conditions.

Definition 2.2 *The affine process $(X, \mathbb{P}_x, (P_t))$ is said to be regular if it is stochastic continuous and the right hand side derivative $\partial_t^+ P_t f_u(x)|_{t=0}$ exists for each $(x, u) \in D \times \mathbb{C}_-^m$, and is continuous at $u = 0$.*

2.1 Representation Theorem and Feynman-Kac Formula

Now we start to discuss some basic properties of positive conservative regular affine (PCRA) processes.

Definition 2.3 *The parameters $(\alpha, b, \beta, c, \gamma, m_0, \mu)$ is said to be admissible if*

- $\alpha = (\alpha_1, ..., \alpha_m)$, *with* $\alpha_i \in \mathbb{R}_+$, $1 \le i \le m$;
- $b = (b_1, ..., b_m)$, *with* $b_i \in \mathbb{R}_+$, $1 \le i \le m$;
- $\beta = (\beta_{ij})_{1\le i,j\le m} \in \mathbb{R}^m$, *with* $\beta_{i,j} \in \mathbb{R}_+$, *if* $i \ne j$;
- m_0 *is a Borel measure on* $D\backslash\{0\}$ *satisfying*

$$\int_{D\backslash\{0\}} \left(\sum_{i=1}^m \chi(\xi_i)\right) m_0(d\xi) < \infty;$$

[3]For the complete theory of affine processes we refer to Duffie, Filipović and Schachermayer (2002 [3]).

- *$\mu = (\mu_1, ..., \mu_m)$, where μ_i is a Borel measure on $D\backslash\{0\}$ satisfying*

$$\int_{D\backslash\{0\}} \left(\sum_{j=1, j\neq i}^{m} \chi(\xi_j) + \chi^2(\xi_i) \right) \mu_i(d\xi) < \infty, \quad \forall\ i = 1, 2, ..., m.$$

Lemma 2.1 *Suppose $(X, \mathbb{P}_x)$ is a positive regular affine process and let $\mathcal{A}$ be its infinitesimal generator, then there exist some admissible parameters $(\alpha, b, \beta, c, \gamma, m_0, \mu)$ such that, for $\forall\ f \in C_c^2(D)$,*

$$\begin{aligned} \mathcal{A}f(x) &= \sum_{i=1}^{m} \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=1}^{m} (b_i + \sum_{j=1}^{m} \beta_{i,j} x_j) \partial_{x_i} f(x) - c - \langle \gamma, x \rangle f(x) \\ &+ \int_{D\backslash\{0\}} (f(x+\xi) - f(x)) m_0(d\xi) \\ &+ \sum_{i=1}^{m} \int_{D\backslash\{0\}} (f(x+\xi) - f(x) - \partial_{x_i} f(x) \chi(\xi_i)) x_i \mu_i(d\xi). \qquad (2.2) \end{aligned}$$

Moreover, (2.1) holds for all $(t, u) \in \mathbb{R}_+ \times \mathbb{C}_-^m$, where ϕ and ψ solve the generalized Riccati equations,

$$\begin{aligned} \phi(t, u) &= \int_0^t F(\psi(s, u)) ds, \qquad (2.3) \\ \partial\psi(t, u) &= R(\psi(t, u)), \quad \psi(0, u) = u \qquad (2.4) \end{aligned}$$

with

$$\begin{aligned} F(u) &= \sum_{i=1}^{m} b_i u_i - c + \int_{D\backslash\{0\}} (e^{\langle u, x \rangle} - 1) m_0(d\xi), \qquad (2.5) \\ R_i(u) &= \alpha_i u_i^2 + \sum_{j=1}^{n} \beta_{j,i} u_j - \gamma_i + \int_{D\backslash\{0\}} (e^{\langle u, x \rangle} - 1 - \langle u_i, \chi(\xi_i) \rangle) \mu_i(d\xi), \\ &\quad \forall\ i = 1, 2, ..., m. \qquad (2.6) \end{aligned}$$

If X is conservative, then in addition we have $c = 0$, $\gamma = 0$ and

$$\int_{D\backslash\{0\}} (||\xi|| \wedge ||\xi||^2) \mu_i(d\xi) < \infty, \quad \forall\ i = 1, 2, ..., m. \qquad (2.7)$$

Proof. A general proof can be found in [3] (Theorem 2.7 and Lemma 9.2).

However for many applications in finance, especially for pricing a contingent t-claim with payoff $f(X_t)$, we usually need to consider a discount factor.

Therefore, for every $f \in bD$, we define

$$Q_t f(x) := \mathbb{E}_x \left[e^{-\int_0^t \rho(X_s)ds} f(X_t) \right], \quad \forall\, t \in \mathbb{R}_+, \tag{2.8}$$

where $\rho(x)$ denotes the short rate. In affine term structure models, it is defined to be an affine function of the state variables X:

$$\rho(X) = \rho_0 + \langle \rho_1, X \rangle.$$

The following lemma gives us a version of Feynman-Kac Formula for PCRA processes.

Lemma 2.2 *Suppose X is a PCRA process with parameters $(\alpha, b, \beta, 0, 0, m, \mu)$, the family $(Q_t)_{t\in\mathbb{R}_+}$ defined in (2.8) forms a regular affine semigroup with infinitesimal generator*

$$\mathcal{B}f = \mathcal{A}f - \rho f, \quad \forall\, f \in C_c^2(D).$$

The corresponding admissible parameters are $(\alpha, b, \beta, \rho_0, \rho_1, m, \mu)$.

Proof. See Proposition 11.1 in [3].

Lemma 2.1 and Lemma 2.2 provide us a powerful tool for pricing any contingent claim with the exponential affine form of state variables. This affine property turns out to be essentially useful when pricing various credit derivatives. On the other hand, in order to build models based on affine processes, it is necessary to briefly discuss the specification for a given PCRA process.

2.2 Specification Analysis of Affine Processes

Consider an positive regular affine process X with infinitesimal generator given by (2.2), we discuss the specifications of this affine process in four aspects.

- Drifts: for each $i = 1, 2, ..., m$, the drift of the branch process X^i is given by $b_i + \sum_{j=1}^m \beta_{i,j} X_t^j$ at each time $t > 0$.
- Diffusions: for each $i = 1, 2, ..., m$, the diffusion of the branch process X^i at time t is given by $\alpha_i X_t^i$.
- Jumps: the jump of X comes from the jump measures m_0 and $\mu = (\mu_1, \mu_2, ..., \mu_m)$ with the intensity given by

$$m_0(d\xi) + \sum_{i=1}^m X_t^i \mu_i(d\xi)$$

at each time $t > 0$.

- Potential: generally the affine process can be killed with the killing $c + \langle \gamma, X \rangle$, however, for a PCRA process, we have $c = 0$ and $\gamma = 0$.

By the rich structure of PCRA processes, we sketch how to embed a traditional intensity based credit risk model with single party into this affine framework. Consider an m-factor affine model (process) with the state vector X, an intensity based model typically defines the short rate $\rho(x) = \rho_0 + \langle \rho_1, x \rangle$ and captures the default event by the first jump of a compound Poisson process with the conditional jump intensity $h(x) = h_0 + \langle H, x \rangle$ (see [9]). It is demonstrated by Filipović [4] that this setup can be In order to replicate this doubly stochastic setup, we construct an extended $m+1$-dimensional affine process $X' = (X, Z)$ and set

$$\alpha_{m+1} = b_{m+1} = \beta_{m+1,m+1} = \beta_{m+1,i} = 0, \quad m_0(d\xi) = h_0 \delta_1(\xi_{m+1})(\prod_{j=1}^{m} \delta_0(\xi_j)),$$

and $$\mu_i(d\xi) = H_i \delta_1(\xi_{m+1})(\prod_{j=1}^{m} \delta_0(\xi_j)), \quad \mu_{m+1} = 0, \quad \forall\ i = 1, 2, ..., m,$$

where $\delta_x(\cdot)$ denotes the Dirac measure sitting at x. Therefore it is demonstrable that for $\forall\ f(x') = g(z) \in C_c^2(\mathbb{R}_+)$, we have

$$\begin{aligned} \mathcal{A}f(x') &= \int_{D\setminus\{0\}} (f(x' + \xi) - f(x'))m_0(d\xi) \\ &\quad + \sum_{i=1}^{m+1} \int_{D\setminus\{0\}} (f(x' + \xi) - f(x') - \partial_{x'_i} f(x)\chi(\xi'_i))x'_i \mu_i(d\xi) \\ &= (h_0 + \langle H, x \rangle)(g(z+1) - g(z)), \end{aligned}$$

which implies that the jump intensity of the branch process Z is equal to $h_0 + \langle H, X_t \rangle$ at each time $t > 0$ and the jump measure is the Dirac measure sitting at 1. Further it is proved by Filipović that these two model approaches are exactly same up to the first default time.

2.3 Diffusion Processes and Stable Jumps

The randomness of a PCRA process X comes from its diffusion and jumps. Since X is conservative, therefore the jump measures $\mu = (\mu_1, ..., \mu_m)$ are stable which satisfy (2.7). In particular, for each $i = 1, 2, ..., m$, on setting

$$\mu_i(d\xi) = \alpha_i' \frac{\theta(\theta+1)}{\Gamma(1-\theta)} \frac{1}{\xi_i^{2+\theta}} (\prod_{j\neq i} \delta_0(\xi_j)), \quad \theta \in (0,1), \tag{2.9}$$

it follows from (2.6) that

$$\begin{aligned} R_i(u) &= \alpha_i u_i^2 + \sum_{j=1}^{n} \beta_{j,i} u_j - \gamma_i + \int_{D\setminus\{0\}} (e^{\langle u,x\rangle} - 1 - \langle u_i, \chi(\xi_i)\rangle)\mu_i(d\xi) \\ &= \alpha_i u_i^2 + \sum_{j\neq i} \beta_{j,i} u_j + \tilde{\beta}_{i,i} u_i + \alpha_i'(-u_i)^{1+\theta}, \end{aligned} \tag{2.10}$$

where $\Gamma(\cdot)$ denotes the Gamma function and

$$\begin{aligned} \tilde{\beta}_{i,i} &= \beta_{i,i} + \alpha_i' \int_1^\infty (\xi_i - 1) \frac{\theta(\theta+1)}{\Gamma(1-\theta)} \frac{1}{\xi_i^{2+\theta}} \\ &= \beta_{i,i} + \frac{\alpha_i'}{\Gamma(1-\theta)}. \end{aligned}$$

From (2.10), we can see that the diffusion can be regarded as a limit case ($\theta \to 1^-$) of the stable jumps. Therefore in the following discussion, instead of speaking of diffusion and jumps separately, we will unite these two cases by extending the region of θ to $(0, 1]$ and call this type of processes θ-stable affine processes.

Remark 2.1 *In this way, the jump defined in (2.9) can be treated as a "pseudo-diffusion" with the coefficients $\alpha' = (\alpha_1', ..., \alpha_m')$. By defining θ-stable affine processes, we do not differentiate the parameters α and α'.*

3 Construction of Generalized Affine Models

Considering N different firms of interest, we construct a $2N+1$ dimensional PCRA process $X = (r, Y^1, ..., Y^N, Z^1, ..., Z^N)$ and $(\mathbb{P}_x, (P_t))$[4]), where r represents the short rate process. For each $i = 1, 2, ..., N$, Y^i denotes the credit index process of firm i and Z^i, the default indicator of firm i, follows a simple point process with constant jump size one. Therefore by letting τ^i denote the default time of firm i, it follows that

$$\tau^i = \inf\left\{t > 0 : Z_t^i > 0\right\}, \quad 1 \leq i \leq N. \tag{3.1}$$

[4]Here $\mathbb{P}_x$ denotes the risk neutral measure and $X_0 = x = (r_0, y_1, ..., y_N, z_1, ..., z_N)$.

According to Lemma 2.1 and the previous discussion, there exist some admissible parameters $(\alpha, b, \beta, m_0, m_r, \mu, \nu)$ characterizing X, where

- $\alpha = (\alpha_0, \alpha_1, ..., \alpha_N)$;
- $b = (b_0, b_1, ..., b_N)$;
- $\beta = (\beta_{ij})_{0 \le i,j \le N}$;
- m_0, m_r, $\mu = (\mu_1, ..., \mu_N)$, $\nu = (\nu_1, ..., \nu_N)$ are all Borel measures with the support on $\{\mathbb{R}_+^{N+1} \times \{0,1\}^N\}$.

Therefore, for each function $f \in C_c^2(D)$, the infinitesimal generator $\mathcal{A}$ of X has the following generic form:

$$\begin{aligned}\mathcal{A}f(x) \quad = \quad & \alpha_0 r_0 \partial_r^2 f(x) + \sum_{i=1}^N \alpha_i y_i \partial_{y_i}^2 f(x) + (b_0 + \beta_{0,0} r_0 + \sum_{j=1}^N \beta_{0,j} y_j) \partial_r f(x) \\ & + \sum_{i=1}^N (b_i + \beta_{i,0} r + \sum_{j=1}^N \beta_{i,j} y_j) \partial_{y_i} f(x) + \int_{D \setminus \{0\}} (f(x+\xi) - f(x)) m_0(d\xi) \\ & + \int_{D \setminus \{0\}} (f(x+\xi) - f(x) - \partial_r f(x) \chi(\xi_0)) r_0 m_r(d\xi) \\ & + \sum_{i=1}^N \int_{D \setminus \{0\}} (f(x+\xi) - f(x) - \partial_{y_i} f(x) \chi(\xi_i)) y_i \mu_i(d\xi) \\ & + \sum_{i=1}^N \int_{D \setminus \{0\}} (f(x+\xi) - f(x) - \partial_{z_i} f(x) \chi(\xi_{i+N})) z_i \nu_i(d\xi), \qquad (3.2)\end{aligned}$$

where $\xi = (\xi_0, \xi_1, ..., \xi_{2N}) \in D$[5].

Under the above affine setup, the short rate and credit risk of N firms are jointly modeled by a multi-dimensional affine process. In the following discussion, we demonstrate that the affine process defined in (3.2) provides us a unifying analytical framework for modeling the correlative market and credit risk, joint credit migrations and interdependent default risk.

3.1 Construction of Correlative Market and Credit Risk

For the model defined in (3.2) and each i, $1 \le i \le N$, there are two ways to establish the correlations between the short rate r and credit risk of firm i. To model the impact of r on credit index Y^i, we may set $\beta_{i,0}$ non-zero, through

[5]Here ξ_0 corresponds to the short rate, ξ_i corresponds to the term Y^i and ξ_{i+N} corresponds to the term Z^i, for each $i = 1, 2, ..., N$.

which the drift of Y^i can be affected by r. In particular, since the positivity of this affine process requires $\beta_{i,0}$ to be positive, we notice the increase of the short rate will result in the increase of the drift of the credit index. As shown in [1], the corresponding corporate bond price will accordingly drop, which can be interpreted by the depreciation of risk-free bonds. This effect will also lead to the increase of the jump intensity of Z^i because it has a positive correlation with Y^i. A more direct way to model this impact is to construct the jump measure m_r by including the term: $\lambda_{r,i}\delta_1(\xi_{i+N})\left(\prod_{j\neq i}\delta_0(\xi_j)\right)$. Then, in (3.2), we have

$$\int_{D\setminus\{0\}}(f(x+\xi)-f(x)-\partial_r f(x)\xi_0)r_0 m_r(d\xi)=(f(x+e_{i+N})-f(x))\lambda_{r,i}r_0+....$$

Thus we can conclude that $\lambda_{r,i}r$ is the default intensity of firm i impacted by the short rate r.

Since usually the risk-free security market will not be affected by the default of a certain company, it is assumed that

$$\beta_{0,i}=0,\quad \forall\ i=1,2,...,N,$$

and the supports of measures $\mu_i(d\xi)$ and $\nu_i(d\xi)$ are on the set

$$\left\{\xi\in D:\xi_0=0\ \&\ (\xi_{N+1},...,\xi_{2N})\in\{0,1\}^N\right\},\quad \forall\ i=1,2,...,N.$$

Therefore, the correlations between market and credit risk are well established.

3.2 The Interactions Between Credit Indices and Default Indicators

In order to avoid the numerical calculation of the hitting probability (default probability) of a general credit index process, we add another variable to indicate the default. Accordingly, the credit index, serving as a benchmark for a firm's financial health, should dominate the jump intensity of the indicator process. This interdependence structure between the credit index and the jump intensity of the indicator process can be achieved by appropriately constructing the jump measure μ_i. For each $i=1,2,...,N$, on setting

$$\mu_i(d\xi)=\lambda_{y_i,z_i}\left(\prod_{j\neq i+N}\delta_0(\xi_j)\right)\delta_1(\xi_{i+N})+...,$$

it follows that

$$\int_{D\setminus\{0\}} (f(x+\xi)-f(x)-\partial_{y_i} f(x)\xi_i) y_i \mu_i(d\xi) = (f(x+e_{i+N})-f(x))\lambda_{y_i,z_i} y_i + ... \tag{3.3}$$

The first term in RHS of (3.3) implies that the default intensity of firm i (mainly) contributed by its credit index is equal to $\lambda_{y_i.z_i} Y^i$. This is consistent with our assumption that the higher value the credit index Y^i, the worse the financial situation, which just corresponds to the higher default intensity of firm i.

3.3 Characterizing Default Correlations and Firm-Specific Default Risk

Previously we discussed the constructions of correlations between the market and credit risk and the interaction between the credit index and the corresponding default indicator process. Now we show our model is capable of capturing the default correlations and the firm-specific default risk defined in [8]. For each $i, j \in \{1, 2, ..., N\}$, $i \neq j$, we will construct the correlation of credit risk between firm i and firm j, especially the impact of firm i on the credit migration of firm j, in three different ways.

One way to create this correlation is to set $\beta_{i,j}$ strictly positive. The interdependence of credit index processes arises because of their interference through the drift terms of Y^i and Y^j. This leads to the correlated default risk between firms. The increase of credit index Y^i can lead to a faster systematic increasing tendency of credit index Y^j.

The second way is to bridge connections between the credit index Y^i and indicator process Z^j by constructing the measure μ_i to include the term

$$\lambda_{y_i,z_j} \delta_1(\xi_{j+N}) \prod_{k\neq j+N} \delta_0(\xi_k).$$

Then, in (3.2) we can rewrite

$$\int_{D\setminus\{0\}} (f(x+\xi)-f(x)-\partial_{y_i} f(x)\xi_i) y_i \mu_i(d\xi) = (f(x+e_{j+N})-f(x))\lambda_{y_i,z_j} y_i + ...,$$

which just implies that the default intensity of firm j contributed by firm i is $\lambda_{y_i,z_j} Y^i$.

The third way is more direct to characterize the firm-specific risk of firm j

upon the default of firm i. On setting

$$\nu_i(d\xi) = \lambda_{z_i,z_j}\delta_1(\xi_{j+N})\left(\prod_{k\neq j+N}\delta_0(\xi_k)\right) + ...,$$

it follows that

$$\int_{D\setminus\{0\}}(f(x+\xi)-f(x)-\partial_{z_i}f(x)\xi_{i+N})z_i\nu_i(d\xi) = (f(x+e_{j+N})-f(x))\lambda_{z_i,z_j}z_i+...,$$

which means that after the default of firm i, the default intensity of firm j increases by λ_{z_i,z_j}. All these three ways can be used to model the default correlations, however, for simplicity, we only apply the latter two methods in our model. Therefore, we can assume without loss of generality that $\beta_{i,j} = 0$, for each $1 \leq i, j \leq N$ and $i \neq j$, throughout the following discussion.

Further, in order to characterize the significant interdependence between two firms i and j, or the "loop dependent default risk structure" proposed in [8], which implies that the default of either firm will cause the default of the other firm, we can even construct the intensity for this co-default by appropriately setting m_0, m_r, μ_i and μ_j as

$$\begin{aligned}
m_0(d\xi) &= \lambda_{0,i,j}\delta_1(\xi_{i+N})\delta_1(\xi_{j+N})\prod_{k\neq i,j}\delta_0(\xi_k),\\
m_r(d\xi) &= \lambda_{r,i,j}\delta_1(\xi_{i+N})\delta_1(\xi_{j+N})\prod_{k\neq i,j}\delta_0(\xi_k),\\
\mu_i(d\xi) &= \lambda_{y_i,z_i,z_j}\delta_1(\xi_{i+N})\delta_1(\xi_{j+N})\prod_{k\neq i,j}\delta_0(\xi_k),\\
\mu_j(d\xi) &= \lambda_{y_j,z_i,z_j}\delta_1(\xi_{i+N})\delta_1(\xi_{j+N})\prod_{k\neq i,j}\delta_0(\xi_k), \quad \forall\, i,j = 1,2,..,N\ , i\neq j.
\end{aligned}$$

Therefore in RHS of (3.2), we have the term

$$(f(x+e_{i+N}+e_{j+N})-f(x))(\lambda_{0,i,j}+\lambda_{r,i,j}r_0+\lambda_{y_i,z_i,z_j}y_i+\lambda_{y_j,z_i,z_j}y_j),$$

which indicates that the default intensity of both firm i and firm j is $\lambda_{0,i,j} + \lambda_{r,i,j}r + \lambda_{y_i,z_i,z_j}Y^i + \lambda_{y_j,z_i,z_j}Y^j$.

4 Pricing Credit Default Swaps

To illustrate some applications of our new model, we start to consider the valuation of a plain vanilla credit default swap (CDS) with \$1 notional principal. A CDS is a contract that the seller provides the buyer insurance against the risk of default of the third firm called reference entity. As return, the buyer makes periodic payments to the seller. The payment dates are denoted by a vector $\vec{T} = (T_1, ..., T_n)$. At default of the reference entity, the seller settles this contract by paying the buyer in cash to par the value of bonds held by the buyer. It is assumed that the settlement is made at the next immediate payment date after the default and without loss of generality, suppose the current time is $T_0 = 0$ and no default observed before.

4.1 No Counterparty Default Risk

First we assume the buyer and seller are default-free and let τ denote the default time of the reference entity. Hence we only need to model the short rate and the credit risk of reference entity by applying a 3-dimensional affine process $X = (R, Y^1, Z^1)$. For convenience of the analysis, we consider X a θ-stable affine process with coefficients $\alpha = (\alpha_0, \alpha_1)$, where $\theta \in (0, 1]$. It is further assumed that

$$\frac{m_0(d\xi)}{\lambda_{0,1}} = \frac{m_r(d\xi)}{\lambda_{r,1}} = \frac{\mu_1(d\xi)}{\lambda_{y_1,z_1}} = \left(\prod_{i=0}^{2} \delta_0(\xi_i)\right) \delta_1(\xi_3),$$

and $\nu_1(\cdot) = 0$.

Under the above setup, the present value of the buyer payments is given by

$$\begin{aligned} B(x, \vec{T}) &= \sum_{k=1}^{n} c_0 \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} 1_{\{T_k < \tau\}} \right] \\ &= \sum_{k=1}^{n} c_0 \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} 1_{\{Z_{T_k} = 0\}} \right] \\ &= \sum_{k=1}^{n} c_0 \lim_{\kappa \to \infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} e^{-\kappa Z_{T_k}} \right], \end{aligned}$$

where c_0 denotes the buyer payment each time. By the affine property (2.1), Lemma 2.1 and Lemma 2.2, we can derive, for each $k = 1, 2, ..., n$,

$$\lim_{\kappa \to \infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} e^{-\kappa Z_{T_k}} \right] = e^{\phi^b(T_k) + \psi_r^b(T_k) r + \psi_1^b(T_k) y_1},$$

where the coefficient functions ϕ^b, ψ_r^b and ψ_1^b can be determined from the following Reccati equations:

$$\frac{d\psi_1^b(t)}{dt} = \alpha_1(-\psi_1^b(t))^{1+\theta} + \beta_{1,1}\psi_1^b(t) - \lambda_{y_1,z_1}, \tag{4.1}$$

$$\frac{d\psi_r^b(t)}{dt} = \alpha_0(-\psi_r^b(t))^{1+\theta} + \beta_{0,0}\psi_r^b(t) + \beta_{1,0}\psi_1^b(t) - 1 - \lambda_{r,1}, \tag{4.2}$$

$$\frac{d\phi^b(t)}{dt} = b_0\psi_r^b(t) + b_1\psi_1^b(t) - \lambda_{0,1}, \tag{4.3}$$

with initial conditions

$$\phi^b(0) = \psi_r^b(0) = \psi_1^b(0) = 0.$$

Equations (4.1) through (4.3) can be easily solved numerically. Moreover, when $\theta = 2$ (the diffusion case), ψ_1^b has an analytical solution:

$$\begin{aligned} \psi_1^b(t) &= -2\lambda_{y_1,z_1}\frac{e^{\omega t}-1}{(\beta_{1,1}+\gamma)(e^{\omega t}-1)+2\gamma}, \\ \text{with} \quad \gamma &= \sqrt{\beta_{1,1}^2 + 4\alpha_1\lambda_{y_1,z_1}}, \\ \omega &= \frac{\gamma}{\lambda_{y_1,z_1}\alpha_1}. \end{aligned}$$

Therefore we obtain

$$B(x,\vec{T}) = \sum_{k=1}^{n} c_0 e^{\phi^b(T_k)+\psi_r^b(T_k)r+\psi_1^b(T_k)y_1}. \tag{4.4}$$

Now we can move on to the valuation of the payoff the seller of a CDS. Since we assume that the settlement is made at the next immediate payment date after default, we can derive the following formula for the seller's payoff $S(x,\vec{T})$.

$$\begin{aligned} S(x,\vec{T}) &= \mathbb{E}_x\left[\sum_{k=1}^{n} e^{-\int_0^{T_k} r_s ds}(1-G(X_{T_k}))1_{\{T_{k-1}<\tau\le T_k\}}\right] \\ &= \sum_{k=1}^{n}\mathbb{E}_x\left[e^{-\int_0^{T_k} r_s ds}(1-G(X_{T_k}))1_{\{Z_{T_{k-1}}=0, Z_{T_k}>0\}}\right] \\ &= \sum_{k=1}^{n}\lim_{\kappa\to\infty}\mathbb{E}_x\left[e^{-\int_0^{T_k} r_s ds}(1-G(X_{T_k}))\left(e^{-\kappa Z_{T_{k-1}}} - e^{-\kappa Z_{T_k}}\right)\right], \end{aligned}$$

where $G(\cdot)$ denotes the recovery rate for the bonds issued by reference entity at the default. On setting it as an exponential affine function of X

$$G(X) = e^{g_0 + g_r r + g_1 Y^1}, \forall\ g_0, g_r, g_1 \in \mathbb{R}_-,$$

we can obtain the analytical expression for $S(x, \vec{T})$. First let us denote

$$\begin{aligned} S(x,\vec{T}) &= \sum_{k=1}^{n} \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} (1 - G(X_{T_k})) \left(e^{-\kappa Z_{T_{k-1}}} - e^{-\kappa Z_{T_k}} \right) \right] \\ &= \sum_{k=1}^{n} C_k(x) - D_k(x) - E_k(x) + F_k(x), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} C_k(x) &= \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} e^{-\kappa Z_{T_{k-1}}} \right], \\ D_k(x) &= \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} e^{-\kappa Z_{T_k}} \right], \\ E_k(x) &= \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} G(X_{T_k}) e^{-\kappa Z_{T_{k-1}}} \right], \\ \text{and} \quad F_k(x) &= \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} G(X_{T_k}) e^{-\kappa Z_{T_k}} \right], \quad \forall\ 1 \le k \le n. \end{aligned}$$

Now by affine property, all $C_k(x)$, $D_k(x)$, $E_k(x)$ and $F_k(x)$ are easy to calculate. First we have

$$C_k(x) = e^{\phi^c(T_k) + \psi_r^c(T_k) r + \psi_1^c(T_k) y_1}, \tag{4.6}$$

$$E_k(x) = e^{\phi^e(T_k) + \psi_r^e(T_k) r + \psi_1^e(T_k) y_1}, \tag{4.7}$$

where the coefficient functions $(\phi^c, \psi_r^c, \psi_1^c)$ and $(\phi^e, \psi_r^e, \psi_1^e)$ can be both determined from the same Reccati equations:

$$\begin{aligned} \frac{d\psi_1(t)}{dt} &= \alpha_1(-\psi_1(t))^{1+\theta} + \beta_{1,1}\psi_1(t) - \lambda_{y_1,z_1} 1_{\{t \ge T_k - T_{k-1}\}}, \\ \frac{d\psi_r(t)}{dt} &= \alpha_0(-\psi_r(t))^{1+\theta} + \beta_{0,0}\psi_r(t) + \beta_{1,0}\psi_1(t) - 1 - \lambda_{r,1} 1_{\{t \ge T_k - T_{k-1}\}}, \\ \frac{d\phi(t)}{dt} &= b_0\psi_r(t) + b_1\psi_1(t) - \lambda_{0,1} 1_{\{t \ge T_k - T_{k-1}\}}, \end{aligned}$$

with different initial conditions

$$\begin{aligned} &\phi^c(0) = \psi_r^c(0) = \psi_1^c(0) = 0, \\ \text{and} \qquad &\phi^e(0) = g_0, \quad \psi_r^e(0) = g_r, \quad \psi_1^e(0) = g_1. \end{aligned}$$

The next step is to derive $D_k(x)$ and $F_k(x)$ as follows.

$$D_k(x) = e^{\phi^d(T_k)+\psi_r^d(T_k)r+\psi_1^d(T_k)y_1}, \tag{4.8}$$
$$F_k(x) = e^{\phi^f(T_k)+\psi_r^f(T_k)r+\psi_1^f(T_k)y_1}, \tag{4.9}$$

where both $(\phi^d, \psi_r^d, \psi_1^d)$ and $(\phi^f, \psi_r^f, \psi_1^f)$ follow the same Riccati equations:

$$\begin{aligned}
\frac{d\psi_1(t)}{dt} &= \alpha_1(-\psi_1(t))^{1+\theta} + \beta_{1,1}\psi_1(t) - \lambda_{y_1,z_1}, \\
\frac{d\psi_r(t)}{dt} &= \alpha_0(-\psi_r(t))^{1+\theta} + \beta_{0,0}\psi_r(t) + \beta_{1,0}\psi_1(t) - 1 - \lambda_{r,1}, \\
\frac{d\phi(t)}{dt} &= b_0\psi_r(t) + b_1\psi_1(t) - \lambda_{0,1},
\end{aligned}$$

but with different initial conditions

$$\begin{aligned}
&\phi^d(0) = \psi_r^d(0) = \psi_1^d(0) = 0, \\
\text{and} \quad &\phi^f(0) = g_0, \quad \psi_r^f(0) = g_r, \quad \psi_1^f(0) = g_2.
\end{aligned}$$

Therefore we have the following proposition for the CDS spread c_0 with no counterparty default risk assumptions.

Proposition 4.1 *Under the assumption of no counterparty default risk, the spread of a CDS without considering counterparty default risk is given by Mc_0, where M denotes the number of payment in one year and*

$$\begin{aligned}
c_0 &= \frac{\sum_{k=1}^n e^{\phi^c(T_k)+\psi_r^c(T_k)r+\psi_1^c(T_k)y_1} - e^{\phi^d(T_k)+\psi_r^d(T_k)r+\psi_1^d(T_k)y_1}}{\sum_{k=1}^n e^{\phi^b(T_k)+\psi_r^b(T_k)r+\psi_1^b(T_k)y_1}} \\
&\quad - \frac{e^{\phi^e(T_k)+\psi_r^e(T_k)r+\psi_1^e(T_k)y_1} - e^{\phi^f(T_k)+\psi_r^f(T_k)r+\psi_1^f(T_k)y_1}}{\sum_{k=1}^n e^{\phi^b(T_k)+\psi_r^b(T_k)r+\psi_1^b(T_k)y_1}}.
\end{aligned} \tag{4.10}$$

Proof. By letting $S(x, \vec{T}) = B(x, \vec{T})$. By (4.4)-(4.9), it is straightforward to obtain (4.10).

4.2 Pricing CDS with Counterparty Default Risk

In this part, we start to take the buyer and seller's default possibility into account and make the following assumptions:

- Upon the credit event of either of three firms, settlement is made at the next immediate payment date T^*.

- At T^*, if the reference entity goes to default only, this CDS contract is settled same as previously discussed (no counterparty risk case).
- At T^*, if the buyer goes to default only, it stops paying to seller and the CDS contract terminates immediately with no obligation of the seller.
- At T^*, if the seller goes to default only, the buyer stops paying and and the CDS contract terminates immediately with no obligation of the seller.
- At T^*, if the buyer and the reference entity go to default only, the seller should make a recovery payment to the buyer with a recovery rate G.
- At T^*, if the seller and the reference entity go to default only, the buyer stops paying and the CDS contract terminates immediately.
- At T^*, if three parties all go to default, the CDS contract terminates immediately.

Therefore we need to construct a 7-dimensional θ-stable affine process

$$X = (r, Y^1, Y^2, Y^3, Z^1, Z^2, Z^3)$$

to model the short rate and three parties' credit risk, and denote the default time as τ^1, τ^2 and τ^3, where the superscripts "1", "2" and "3" denote the reference entity, the buyer and the seller, respectively. In order to model counterparty risk, we build the interdependent default risk structure of these three parties as follows. Consider the reference entity as a primary firm (whose default risk is not affected by the other two firms) and the buyer and seller as two secondary firms (whose default intensity depends on the status of the primary firm). Therefore, as discussed in the previous section, we can set

$$\begin{aligned}
m(d\xi) &= \sum_{i=1}^{3} \lambda_{0,i}\delta_1(\xi_{i+3}) \prod_{j=0,j\neq i}^{6} \delta_0(\xi_j), \\
m_r(d\xi) &= \sum_{i=1}^{3} \lambda_{r,i}\delta_1(\xi_{i+3}) \prod_{j=0,j\neq i}^{6} \delta_0(\xi_j), \\
\mu_1(d\xi) &= \sum_{i=1}^{3} \lambda_{y_1,z_i}\delta_1(\xi_{i+3}) \prod_{j=0,j\neq i}^{6} \delta_0(\xi_j) \\
\nu_1(d\xi) &= \sum_{i=2}^{3} \lambda_{z_1,z_i}\delta_1(\xi_{i+3}) \prod_{j=0,j\neq i}^{6} \delta_0(\xi_j) \\
\mu_i(d\xi) &= \lambda_{y_i,z_i}\delta_1(\xi_{i+3}) \prod_{j=0,j\neq i}^{6} \delta_0(\xi_j), \\
\text{and} \quad \nu_i(d\xi) &= 0, \quad \forall\ i = 2, 3.
\end{aligned}$$

By using the same notation as the previous part, we have

$$\begin{aligned}
B(x, \vec{T}) &= \sum_{k=1}^{n} c_r \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} 1_{\{T_k < \tau^1\}} 1_{\{T_k < \tau^2\}} 1_{\{T_k < \tau^3\}} \right] \\
&= \lim_{\kappa \to \infty} \sum_{k=1}^{n} c_r \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds} e^{-\kappa (Z^1_{T_k} + Z^2_{T_k} + Z^3_{T_k})} \right] \\
&= \sum_{k=1}^{n} c_r e^{\phi^b(T_k) + \psi_r^b(T_k) r + \psi_1^b(T_k) y_1 + \psi_2^b(T_k) y_2 + \psi_3^b(T_k) y_3}, \qquad (4.11)
\end{aligned}$$

where c_r denotes the amount of payment made at each date and the coefficient functions ϕ^b, ψ_r^b, ψ_1^b, ψ_2^b and ψ_3^b can be determined from the following Riccati equations shown in Appendix.

In the same way, by assuming the recovery rate $G(x) = e^{g_0 + g_r r + \langle \vec{g}, \vec{y} \rangle}$, where $\vec{y} = (y_1, y_2, y_3)$ and $\vec{g} = (g_1, g_2, g_3) \in \mathbb{R}^3_-$, we can formulate the present value of the seller's payment as

$$
\begin{aligned}
S(x,\vec{T}) &= \sum_{k=1}^{n} \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds}(1-G(X_{T_k}))1_{\{T_{k-1}<\tau^1\le T_k\}}1_{\{T_{k-1}<\tau^2\}}1_{\{T_k<\tau^3\}} \right] \\
&= \sum_{k=1}^{n} \lim_{\kappa\to\infty} \mathbb{E}_x \left[e^{-\int_0^{T_k} r_s ds}(1-G(X_{T_k}))(e^{-\kappa Z^1_{T_{k-1}}} - e^{-\kappa Z^1_{T_k}})e^{-\kappa Z^2_{T_{k-1}}} e^{-\kappa Z^3_{T_k}} \right] \\
&= \sum_{k=1}^{n} \Big(e^{\phi^c(T_k)+\psi^c_r(T_k)r+\langle\vec{\psi}^c(T_k),\vec{y}\rangle} - e^{\phi^d(T_k)+\psi^d_r(T_k)r+\langle\vec{\psi}^d(T_k),\vec{y}\rangle} \\
&\qquad e^{\phi^e(T_k)+\psi^e_r(T_k)r+\langle\vec{\psi}^e(T_k),\vec{y}\rangle} - e^{\phi^f(T_k)+\psi^f_r(T_k)r+\langle\vec{\psi}^f(T_k),\vec{y}\rangle} \Big), \qquad (4.12)
\end{aligned}
$$

where $\vec{\psi}^{\cdot} = (\psi^{\cdot}_1, \psi^{\cdot}_2, \psi^{\cdot}_3)$ and the coefficient functions $(\phi^c, \psi^c_r, \vec{\psi}^c)$ and $(\phi^e, \psi^e_r, \vec{\psi}^e)$ can be both determined from the same Reccati equations (A.1)-(A.6) and both $(\phi^d, \psi^d_r, \vec{\psi}^d)$ and $(\phi^f, \psi^f_r, \vec{\psi}^f)$ follow the same Reccati equations (A.7)-(A.12) shown in Appendix. Therefore we derive the CDS spread under the counterparty risk.

Proposition 4.2 *Under the consideration of counterparty risk, the CDS spread is given by Mc_r, where*

$$
\begin{aligned}
c_r &= \frac{\sum_{k=1}^{n} e^{\phi^c(T_k)+\psi^c_r(T_k)r+\langle\vec{\psi}^c(T_k),\vec{y}\rangle} - e^{\phi^d(T_k)+\psi^d_r(T_k)r+\langle\vec{\psi}^d(T_k),\vec{y}\rangle}}{\sum_{k=1}^{n} e^{\phi^b(T_k)+\psi^b_r(T_k)r+\langle\vec{\psi}^b(T_k),\vec{y}\rangle}} \\
&\quad - \frac{e^{\phi^e(T_k)+\psi^e_r(T_k)r+\langle\vec{\psi}^e(T_k),\vec{y}\rangle} - e^{\phi^f(T_k)+\psi^f_r(T_k)r+\langle\vec{\psi}^f(T_k),\vec{y}\rangle}}{\sum_{k=1}^{n} e^{\phi^b(T_k)+\psi^b_r(T_k)r+\langle\vec{\psi}^b(T_k),\vec{y}\rangle}}.
\end{aligned}
$$

5 Conclusion

This paper develops a new credit risk model by applying affine processes. Our generalized affine model combines the virtues of structural models and intensity based models together by jointly modeling credit indices and credit indicators of firms. Moreover, we have shown in this paper how to construct the model to include the correlative market and default correlations. Finally, as a direct application of this model, we value default swaps by considering all these factors and derive an explicit formula the the default swap spread.

For simplicity, here we only apply an θ-stable affine process (including the case of affine diffusion processes) when pricing default swaps. It can be straightforwardly extended to any other type of affine processes. Further, the model can also be generalized to employ a multi-factor affine model for risk-free rates in order to fit the risk-free yield curve better.

The implications of this model can be empirically investigated. The parameters appearing in the model may be implicitly estimated by the corporate bond prices and then be applied to price default swaps. We can also implement this scheme reversely because of the expanding default swap market.

A Appendix

A.1 Derivations of Several Results

The coefficient functions ϕ^b, ψ_r^b, ψ_1^b, ψ_2^b and ψ_3^b in (4.11) satisfying

$$
\begin{aligned}
\frac{d\psi_i^b(t)}{dt} &= \alpha_i(-\psi_1^b(t))^{1+\theta} + \beta_{i,i}\psi_1^b(t) - \lambda_{y_i,z_i} - \sum_{j=2}^{3}\lambda_{y_i,z_j}1_{\{i=1\}}, \quad \forall\ i = 1,2,3, \\
\frac{d\psi_r^b(t)}{dt} &= \alpha_0(-\psi_r^b(t))^{1+\theta} + \beta_{0,0}\psi_r^b(t) + \sum_{i=1}^{3}\beta_{i,0}\psi_i^0(t) - \sum_{i=1}^{3}\lambda_{r,i} - 1, \\
\frac{d\phi^b(t)}{dt} &= b_0\psi_r^b(t) + \sum_{i=1}^{3} b_i\psi_i^0(t) - \sum_{i=1}^{3}\lambda_{0,i},
\end{aligned}
$$

with initial conditions

$$\phi^b(0) = \psi_r^b(0) = \psi_i^b(0) = 0, \quad i = 1,2,3.$$

The coefficient functions $(\phi^c, \psi_r^c, \vec{\psi}^c)$ and $(\phi^e, \psi_r^e, \vec{\psi}^e)$ in (4.12) can be both determined from the same Reccati equations

$$\frac{d\psi_3(t)}{dt} = \alpha_3(-\psi_3(t))^{1+\theta} + \beta_{3,3}\psi_3(t) - \lambda_{y_3,z_3}, \tag{A.1}$$

$$\frac{d\psi_2(t)}{dt} = \alpha_2(-\psi_2(t))^{1+\theta} + \beta_{2,2}\psi_2(t) - \lambda_{y_2,z_2}1_{\{t\geq T_k - T_{k-1}\}}, \tag{A.2}$$

$$\frac{d\psi_1(t)}{dt} = \alpha_1(-\psi_1(t))^{1+\theta} + \beta_{1,1}\psi_1(t) - \lambda_{y_1,z_3} - (\lambda_{y_1,z_1} + \lambda_{y_1,z_2})1_{\{t\geq T_k - T_{k-1}\}}, \tag{A.3}$$

$$\frac{d\psi_r(t)}{dt} = \alpha_0(-\psi_r(t))^{1+\theta} + \beta_{0,0}\psi_r(t) + \sum_{i=1}^{3}\beta_{i,0}\psi_i(t) - \lambda_{r,3} - (\lambda_{r,1} + \lambda_{r,2})1_{\{t\geq T_k - T_{k-1}\}} - 1, \tag{A.4}$$

$$\frac{d\phi(t)}{dt} = b_0\psi_r(t) + \sum_{i=1}^{3} b_i\psi_i(t) - \lambda_{0,3} - (\lambda_{0,1} + \lambda_{0,2})1_{\{t\geq T_k - T_{k-1}\}}, \tag{A.5}$$

with different initial conditions

$$\begin{aligned} & \phi^c(0) = \psi_r^c(0) = 0, \quad \vec{\psi}^c(0) = 0, \\ \text{and} \qquad & \phi^e(0) = g_0, \quad \psi_r^e(0) = g_r, \quad \vec{\psi}^e(0) = \vec{g}. \end{aligned} \tag{A.6}$$

The coefficient functions $(\phi^d, \psi_r^d, \vec{\psi}^d)$ and $(\phi^f, \psi_r^f, \vec{\psi}^f)$ in (4.12) solve the same generalized Riccati equations.

$$\frac{d\psi_3(t)}{dt} = \alpha_3(-\psi_3(t))^{1+\theta} + \beta_{3,3}\psi_3(t) - \lambda_{y_3,z_3}, \tag{A.7}$$

$$\frac{d\psi_2(t)}{dt} = \alpha_2(-\psi_2(t))^{1+\theta} + \beta_{2,2}\psi_2(t) - \lambda_{y_2,z_2}1_{\{t \geq T_k - T_{k-1}\}}, \tag{A.8}$$

$$\frac{d\psi_1(t)}{dt} = \alpha_1(-\psi_1(t))^{1+\theta} + \beta_{1,1}\psi_1(t) - \lambda_{y_1,z_1} - \lambda_{y_1,z_3} - \lambda_{y_1,z_2}1_{\{t \geq T_k - T_{k-1}\}}, \tag{A.9}$$

$$\frac{d\psi_r(t)}{dt} = \alpha_0(-\psi_r(t))^{1+\theta} + \beta_{0,0}\psi_r(t) + \sum_{i=1}^{3}\beta_{i,0}\psi_i(t) - \lambda_{r,1} - \lambda_{r,3} - \lambda_{r,2}1_{\{t \geq T_k - T_{k-1}\}} - 1, \tag{A.10}$$

$$\frac{d\phi(t)}{dt} = b_0\psi_r(t) + \sum_{i=1}^{3} b_i\psi_i(t) - \lambda_{0,1} - \lambda_{0,3} - \lambda_{0,2}1_{\{t \geq T_k - T_{k-1}\}}, \tag{A.11}$$

but with different initial conditions

$$\begin{aligned} & \phi^d(0) = \psi_r^d(0) = 0, \quad \vec{\psi}^d(0) = 0, \\ \text{and} \qquad & \phi^f(0) = g_0, \quad \psi_r^f(0) = g_r, \quad \vec{\psi}^f(0) = \vec{g}. \end{aligned} \tag{A.12}$$

References

[1] Chen, L. and D. Filipović (2003): Modeling Credit Risk by Affine Processes. Working Paper, Princeton University.

[2] Credit Risk: A *Risk* Special Report. *Risk*, March 2000.

[3] Duffie, D., D. Filipović, and W. Schachermayer (2002): Affine Processes and Applications in Finance. Forthcoming, *The Annals of Applied Probability.*

[4] Filipović, D. (2002): Defaultable Securities in the Affine Framework. Working Paper, Princeton University.

[5] Hull, J. and A. White (2000): Valuing Credit Default Swaps I: No Counterparty Default Risk. *The Journal of Derivatives*, 1, 29-40.

[6] Hull, J. and A. White (2001): Valuing Credit Default Swaps II: Modeling Default Correlations. *The Journal of Derivatives* 8(3), 12-22.

[7] Jarrow, R. A. and Y. Yildirim (2002): Valuing Default Swaps Under Market and Credit Risk Correlation. *The Journal of Fixed Income* 11(4), 7-19.

[8] Jarrow, R. A. and F. Yu (2001): Counterparty Risk and the Pricing of Defaultable Securities. *The Journal of Finance* 56(5), 1765-1799.

[9] Lando, D. (1998): On Cox processes and credit risky securities. *The Review of Derivatives Research* 2(2), 99-120.

[10] Merton, R. (1974): On the Pricing of Corporate Debt: the Risk Structure of Interest Rates. *Journal of Finance* 29, 449-470.