MARKOVIAN QUADRATIC TERM STRUCTURE MODELS FOR RISK-FREE AND DEFAULTABLE RATES

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Abstract. In this paper, a class of regular quadratic Gaussian processes is defined to characterize quadratic term structure models (QTSMs) in a general Markovian setting. The primary motivation for this definition is to provide a more general model for the quadratic term structure of the forward curve, while maintaining the analytical tractability of the traditional QTSMs. It is demonstrated that the tractability of QTSMs does not necessarily rely on the Ornstein-Uhlenbeck state processes used in their traditional definition. Rather, the crucial element that provides analytical solutions for the prices of zero-coupon bonds and their options is a so-called quadratic Gaussian property as defined in this paper. In order to retain this property for a general Markov process, it is shown that, under the regularity conditions, no jumps are allowed in the infinitesimal generator of the process. It is further shown that the coefficient functions defined in the quadratic Gaussian property can be determined by multi-variate Riccati equations with a unique admissible parameter set. The implications of this result for modeling the term structure of risk-free rates and defaultable rates are discussed.

Key words and phrases. Quadratic term structure models, option pricing, defaultable rates, time-homogenous Markov processes.

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THE ORIGINAL PAPER IS NAMED "A GENERAL CHARACTERIZATION OF QUADRATIC TERM STRUCTURE MODELS". THIS IS A COMPLETE AND EXTENSION VERSION.

1. INTRODUCTION

Recently, quadratic term structure models (QTSMs) have been well studied in the contexts of both theoretical analysis and empirical testing (e.g., Ahn, Dittmar and Gallant (2002 [1]), Chen and Poor (2002 [2]), Leippold and Wu (2001 [8]) and Leippold and Wu (2002 [9])). It has been shown that QTSMs not only empirically outperform the affine term structure models (ATSMs) in that they are able to capture the nonlinearity of the relevant time series and are more flexible for model design, but they also exhibit a nice analytical tractability comparable to ATSMs, namely, the zero-coupon bond price has the exponential-quadratic form in the state variables and the prices of European type options can be calculated by Fourier analysis.

Consider a complete filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ and let X denote a d-dimensional underlying state process. The traditional QTSM (Ahn, Dittmar and Gallant, 2002 [1]) is defined in the framework of Itó's diffusion processes which specifies a state process X as a multi-variate Ornstein-Uhlenbeck process:

$$dX_t = [\mu + \Lambda X_t]dt + \Sigma dW_t \tag{1.1}$$

where $\mu \in \mathbb{R}^d$, $\Lambda, \Sigma \in \mathbb{R}^{d\times d}$ and W_t is a d-dimensional standard Brownian motion adapted to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then the short rate $r(X_t)$ is assumed to be a quadratic function of these state variables:

$$r(X_t) = R_0 + \langle R_1, X_t\rangle + \langle R_2 X_t, X_t\rangle. \tag{1.2}$$

In this paper, we will generalize this formulation to allow more general Markov state evolution than (1.1), while retaining the tractability of this model for pricing derivatives.

The classic quadratic term structure model of (1.1) and (1.2) yields a nice property for pricing zero-coupon bonds and their options shown as follows:

$$\begin{aligned} \mathbb{E}_t\left[e^{-\int_t^T r(X_s)ds} e^{\langle u, X_T\rangle + \langle V X_T, X_T\rangle}\right] &= \exp\left\{\tilde{A}(T-t,u,V) + \langle \tilde{B}(T-t,u,V), X_t\rangle\right. \\ &\qquad \left. + \langle \tilde{C}(T-t,u,V)X_t, X_t\rangle\right\}, \end{aligned} \tag{1.3}$$

for every $u \in \mathbb{C}^d$ and $V \in \mathbb{C}^{d\times d}$, where the functions $\tilde{A}(\tau,u,V)$, $\tilde{B}(\tau,u,V)$ and $\tilde{C}(\tau,u,V)$ can be determined from a series of multi-variate Riccati equations with initial conditions

$$\tilde{A}(0,u,V) = 0, \quad \tilde{B}(0,u,V) = u \quad \text{and} \quad \tilde{C}(0,t,V) = V.$$

This property turns out to be a crucial element in retaining the analytical tractability of QTSMs. This is because if this property holds, then on setting $u = 0$ and $V = 0$, (1.3) gives us price formulas for zero coupon bonds. Moreover, as shown in Leippold and Wu (2002 [9]), this property is the key to the Fourier analytic approach to pricing European options.

If we set R_0, R_1 and R_2 to 0 in (1.2), we can derive the following necessary condition for (1.3) to hold:

$$\begin{aligned} \mathbb{E}_t\left[e^{\langle u,X_T\rangle+\langle VX_T,X_T\rangle}\right] &= \exp\left\{A(T-t,u,V)+\langle B(T-t,u,V),X_t\rangle\right. \\ &\quad \left.+\langle C(T-t,u,V)X_t,X_t\rangle\right\}. \end{aligned} \tag{1.4}$$

As we shall see below, this is an essential property, which we define to be the "quadratic Gaussian" property of a state process X. In particular, as we will show in Section 5 (Proposition 5.1), if X is a time-homogenous Markov process, then (1.3) and (1.4) are equivalent given that the short rate r is a quadratic function of the state variables.

In order to characterize all time-homogeneous Markov processes such that (1.4) holds, we will define a class of regular quadratic Gaussian processes. As an application of this class of processes, a generalized quadratic term structure model (GQTSM) will be constructed and pricing problems under the GQTSM will also be discussed. It is worth mentioning that the GQTSMs proposed in this paper can be applied directly to modeling defaultable rates without specifying any auxiliary model for characterizing default risk.

The remainder of this paper is organized as follows. In Section 2, we give the basic notation used in this paper and some preliminary results from Markov semigroup theory. In Section 3, we define a class of regular quadratic Gaussian processes and provide some straightforward results that follow from this definition. We propose our main results in Section 4, which include an analytical expression for the infinitesimal generator of a regular quadratic Gaussian process and its unique characteristics. In Section 5, we give the definition of GQTSMs and deduce the option pricing formula under this family of models. The modeling of risk-free and defaultable rates by applying GQTSMs is discussed in Section 6. All mathematical proofs are included in the Appendix.

2. BASIC NOTATION AND PRELIMINARY RESULTS

In this section, we establish some preliminary notions that will be useful in the sequel. Some notation to be used throughout this paper is shown in Table 1.

Let us consider a time-homogenous Markov process X starting at $X_0 = x$ with state space D, and a positive contraction semigroup (P_t) on $B(D)$ with

$$P_0 f = f, \quad \text{for each} \quad f \in B(D). \tag{2.1}$$

Then according to Dynkin (1965, [6], Theorem 2.1), this semigroup corresponds to a transition function $p_t(x, \cdot)$, which satisfies, for $\forall(t,x) \in \mathbb{R}_+ \times D$,

$$P_t(f(x)) = \int_D f(\xi) p_t(x, d\xi).$$

By Kolmogorov's extension theorem, given the above contraction semigroup $(P_t)_{t\in\mathbb{R}_+}$, there exists a unique probability law $\mathbb{P}_x$ on the space $(\Omega, \mathcal{F}^\infty)$ such that X is a Markov process with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ that satisfies

$$\mathbb{E}_x[f(X_s)|\mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_{s-t})], \quad \mathbb{P}_x - \text{a.s.},$$

for any $s, t \in \mathbb{R}_+$, such that $s > t$ and for all $f \in B(D)$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

Remark 2.1. *It is not necessary to require the transition function $p_t(x, \cdot)$ to be conservative, since if $p_t(x, D) < 1$, we can expand D to D_Δ and let*

$$p_t(x, D_\Delta) = 1, \; p_t(\Delta, \{\Delta\}) = 1 \quad \textit{for} \quad \forall (t, x) \in \mathbb{R}_+ \times D. \tag{2.2}$$

This extension implies that once X goes to the state Δ, it will stay at Δ forever. This state is often called "coffin state". If the process X goes to the coffin state, it is said to be dead. In our case, we say X is dead if one of its component goes to infinity.

3. DEFINITION OF QUADRATIC GAUSSIAN PROCESSES

In this section, let us first define two sets that will be used frequently in this paper:

$$\begin{aligned} \mathcal{B} &:= \{(u, V) \in \mathbb{C}^d \times \mathbb{C}^{d \times d} : x \mapsto e^{\langle u, x\rangle + \langle Vx, x\rangle} \in B(D)\}, \\ \text{and} \quad \mathcal{E} &:= \{(\gamma, \delta, \Phi) : \gamma \in \mathbb{R}, \; \delta \in \mathbb{R}^d, \; \Phi \in \mathbb{R}^{d \times d}, \\ &\quad\;\; \gamma + \langle \delta, x\rangle + \langle \Phi x, x\rangle \geq 0, \text{ for each } x \in D.\}. \end{aligned}$$

Remark 3.1. *It is easy to see that*

i) The set

$$\{(u, V) : u \in \mathbb{C}^d \; , \; V \in Sem^d_{--} \oplus i\mathbb{R}^{d \times d}\} \subset \mathcal{B}.$$

ii) In particular, if $V \in \partial Sem^d_-$ and let $u = Re(u) + Im(u)i$, where $Re(u), Im(u) \in \mathbb{R}^d$, we have, for $\forall x \in D$,

$$\langle Vx, x\rangle = 0 \Rightarrow \langle Re(u), x\rangle = 0.$$

In particular, if $V = 0$, then $u \in i\mathbb{R}^d$.

iii) For any $(\gamma, \delta, \Phi) \in \mathcal{E}$, $\gamma \in \mathbb{R}_+$ and $\Phi \in Sem^d_+$. In particular, if $\Phi = 0$, then $\delta = 0$.

For every $(u, V) \in \mathcal{B}$, we define the function $f_{u,V} : D \mapsto \mathbb{R}$ by

$$f_{u,V}(x) = e^{\langle u, x\rangle + \langle Vx, x\rangle}, \quad x \in D. \tag{3.1}$$

Therefore this definition indicates that $f_{u,V} \in B(D)$.

Now we can define a quadratic Gaussian process as follows:

Definition 3.1. *A Markov process $(X, (\mathbb{P}_x)_{x \in D}, (P_t)_{t \in \mathbb{R}_+})$ is said to be quadratic Gaussian if for every $(t, x, (u, V)) \in \mathbb{R}_+ \times D \times \mathcal{B}$, $P_t f_{u,V}(x)$ has an exponential quadratic form in x, i.e., there exist functions $A(t, u, V) \in \mathbb{C}$, $B(t, u, V) \in \mathbb{C}^d$ and $C(t, u, V) \in \mathbb{C}^{d \times d}$ such that*

$$P_t f_{u,V}(x) = \exp\left\{A(t, u, V) + \langle B(t, u, V), x\rangle + \langle C(t, u, V)x, x\rangle\right\}. \tag{3.2}$$

The corresponding semigroup $(P_t)_{t \in \mathbb{R}_+}$ is called a quadratic Gaussian semigroup. Without loss of generality, we can assume that $C(t, u, V)$ is a symmetric matrix for $\forall (t, u, V) \in \mathbb{R}_+ \times \mathcal{B}$.

Remark 3.2. *Since $(P_t)_{t\in\mathbb{R}_+}$ is a contraction semigroup, $P_t f_{u,V} \in B(D)$ and thus $(B(t,u,V), C(t,u,V)) \in \mathcal{B}$, for all $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$. And by (2.1), we can obtain the initial conditions for the functions $A(t,u,V)$, $B(t,u,V)$ and $C(t,u,V)$:*

$$A(0,u,V) = 0, \quad B(0,u,V) = u \quad \text{and} \quad C(0,u,V) = V. \tag{3.3}$$

It is easy to see that, given a time-homogeneous Markov process X, the definition of (3.2) is equivalent to the so-called "quadratic Gaussian" property of a stochastic process defined in (1.4). Therefore our definition of quadratic Gaussian processes entirely includes all possible time-homogeneous Markov processes such that (1.4) holds.

The uniqueness of Definition 3.1 can be easily seen. Since $i\mathbb{R}^d \times \{0_{d\times d}\} \in \mathcal{B}$, once we derive the functions $A(t,u,V)$, $B(t,u,V)$ and $C(t,u,V)$ for all $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$, then, for each $t \in \mathbb{R}_+$, the characteristic function of P_t is defined completely, and thus so is P_t, i.e., the law is unique under the definition of (3.2). The existence of a quadratic Gaussian process is given by the following lemma.

Lemma 3.1. *The multi-variate Ornstein-Uhlenbeck process defined in (1.1) is a quadratic Gaussian process.*

From Lemma 3.1, we know that the state process X defined in a traditional QTSM follows a quadratic Gaussian process. Now the remaining task is to derive the coefficient functions $A(t,u,V)$, $B(t,u,V)$ and $C(t,u,V)$. In order to specify sufficient regularity conditions to do so, we introduce the definition of regular quadratic Gaussian processes.

Definition 3.2. *A quadratic Gaussian process $(X, (\mathbb{P}_x)_{x\in D}, (P_t)_{t\in\mathbb{R}_+})$ is regular, if*

i) the functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ are continuous on $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$; and

ii)the weak infinitesimal generator

$$\tilde{\mathcal{A}} f_{u,V}(x) = \partial_t^+ P_t f_{u,V}(x)|_{t=0} = \lim_{t\downarrow 0} \frac{P_t f_{u,V}(x) - f_{u,V}(x)}{t}$$

exists for every $(x,(u,V)) \in D \times \mathcal{B}$.

If a Markov process $(X, (\mathbb{P}_x)_{x\in D}, (P_t)_{t\in\mathbb{R}_+})$ is quadratic Gaussian and regular, we can define

$$\begin{aligned} F(u,V) : &= \partial_t^+ A(t,u,V)|_{t=0}, \\ R(u,V) : &= \partial_t^+ B(t,u,V)|_{t=0}, \\ \text{and} \quad T(u,V) : &= \partial_t^+ C(t,u,V)|_{t=0} \end{aligned}$$

and thus we obtain

$$\tilde{\mathcal{A}} f_{u,V}(x) = \left(F(u,V) + \langle R(u,V), x\rangle + x'T(u,V)x\right) f_{u,V}(x), \tag{3.4}$$

for all $(x,(u,V)) \in D \times \mathcal{B}$.

Equation (3.4) indicates a relationship between the first derivatives of the functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ and the weak generator of X. This turns out to be a key property for obtaining $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$. The next lemma illustrates this relationship more clearly.

Lemma 3.2. *For a regular and quadratic Gaussian process, we have*

$$\partial_t^+ A(t,u,V) = F(B(t,u,V),C(t,u,V)), \tag{3.5}$$

$$\partial_t^+ B(t,u,V) = R(B(t,u,V),C(t,u,V)), \tag{3.6}$$

$$\partial_t^+ C(t,u,V) = T(B(t,u,V),C(t,u,V)), \tag{3.7}$$

for $\forall\,(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$*; and, moreover,*

$$F(u,V) = \tilde{\mathcal{A}} f_{u,V}(0), \tag{3.8}$$

$$R_i(u,V) = \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} - \frac{\tilde{\mathcal{A}} f_{u,V}(-e_i)}{f_{u,V}(-e_i)}\right], \tag{3.9}$$

$$T_{ii}(u,V) = \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} + \frac{\tilde{\mathcal{A}} f_{u,V}(-e_i)}{f_{u,V}(-e_i)}\right] - F(u,V), \tag{3.10}$$

$$\begin{aligned} \textit{and} \quad T_{ij}(u,V) &= \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i+e_j)}{f_{u,V}(e_i+e_j)} - \frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} - \frac{\tilde{\mathcal{A}} f_{u,V}(e_j)}{f_{u,V}(e_j)}\right] \\ &\quad + \frac{1}{2}F(u,V), \\ &\quad \textit{for} \quad i,j \in \{1,2,...,d\} \quad \textit{and} \quad i \neq j. \end{aligned} \tag{3.11}$$

Therefore, according to Lemma 3.2, if we can deduce the weak generator $\tilde{\mathcal{A}} f_{u,V}(\cdot)$, the derivative functions $F(u,V)$, $R(u,V)$ and $T(u,V)$ are determined. This gives us a method to solve for the coefficient functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$.

4. INFINITESIMAL GENERATORS OF REGULAR QUADRATIC GAUSSIAN PROCESSES

In this section, we obtain an analytical expression for the infinitesimal generator of a regular quadratic Gaussian process. We approach this task in several steps. First we focus on the function $f_{u,V}(x)$. Here we define a "cut-off" function $\chi = (\chi_1, \chi_2, ..., \chi_d)' : D \to [-1,1]^d$ by

$$\chi_k(x) = \begin{cases} x_k & \text{if} \quad |x_k| \leq 1, \\ sgn(x_k) & \text{if} \quad |x_k| > 1, \end{cases}$$

where $sgn(\cdot)$ denotes the algebraic sign of its argument. We also define a metric m on D as:

$$m(x,y) = \left[\sum_{k=1}^{d} \chi_k^2(x-y)\right]^{\frac{1}{2}}, \quad \forall\, x,y \in D.$$

It is easy to see that the metric m is bounded by $\sqrt{d}$.

We now give the following result as a first step toward deriving the infinitesimal generator of a quadratic Gaussian process.

Lemma 4.1. ***(Representation Results for Regular Processes)*** *Suppose X is a regular quadratic Gaussian process with weak infinitesimal generator $\tilde{\mathcal{A}}$. Then, for $\forall x \in D$, there exist elements*

$$\alpha(x) \in Sem_+^d, \quad \beta(x) \in \mathbb{R}^d \quad \textit{and} \quad \gamma(x) \in \mathbb{R}_+$$

and a positive measure $\nu(x, d\xi)$ *on* $D\backslash\{x\}$ *satisfying*

$$\int_{D\backslash\{x\}} m^2(x,\xi)\nu(x,d\xi) < \infty,$$

such that for all $(u, V) \in \mathcal{B}$,

$$\begin{aligned}\tilde{\mathcal{A}} f_{u,V}(x) &= tr\left(\alpha(x)\frac{\partial^2 f_{u,V}(x)}{\partial x \partial x'}\right) + \langle \beta(x), \nabla f_{u,V}(x)\rangle - \gamma(x) f_{u,V}(x) \\ &\quad + \int_{D\backslash\{x\}} \tilde{h}_{u,V}(x,\xi)\nu(x,d\xi),\end{aligned} \tag{4.1}$$

where

$$\tilde{h}_{u,V}(x,\xi) = f_{u,V}(\xi) - f_{u,V}(x) - \langle \nabla f_{u,V}(x), \chi(\xi - x)\rangle. \tag{4.2}$$

We now strengthen the definition of regularity and introduce the notion of an admissible parameter set.

Definition 4.1. *A quadratic Gaussian process* X *is said to be strongly regular if it is regular and the measure* $\nu(x, \cdot)$ *specified in Lemma 4.1 satisfies, for* $\forall x \in D$:

i)

$$\int_{D\backslash\{x\}} ||\chi(x-\xi)||\nu(x,d\xi) < \infty;$$

and

ii)

$$\int_{D\backslash\{x\}} ||\xi||^2\nu(x,d\xi) < \infty,$$

where $||\cdot||$ *denotes the Euclidean norm of a d-dimensional vector.*

Definition 4.2. *A parameter set* $(\alpha, \beta, b, \gamma, \delta, \Phi)$ *is said to be admissible if*

$$\begin{aligned} &\alpha \in Sem^d_+,\ \beta \in \mathbb{R}^d,\ b \in \mathbb{R}^{d\times d} \\ \text{and} \quad &(\gamma,\, \delta,\, \Phi) \in \mathcal{E}.\end{aligned}$$

From Lemma 3.2 and Lemma 4.1, we have the following mappings theorem for functions $F(u, V), R(u, V)$ and $T(u, V)$, which allows the coefficient functions $A(t, u, V)$, $B(t, u, V)$ and $C(t, u, V)$ to be determined given their initial conditions (3.3).

Proposition 4.1. ***(Mappings Theorem)***
For a strongly regular quadratic Gaussian process X, *there exists an admissible parameter set* $(\alpha, \beta, b, \gamma, \delta, \Phi)$ *such that for all* $(x, (u, V)) \in D \times \mathcal{B}$,

$$\begin{aligned}\tilde{\mathcal{A}} f_{u,V}(x) &= tr\left(\alpha\frac{\partial^2 f_{u,V}(x)}{\partial x \partial x'}\right) + \langle \beta + bx, \nabla f_{u,V}(x)\rangle \\ &\quad -(\gamma + \delta' x + x'\Phi x) f_{u,V}(x).\end{aligned} \tag{4.3}$$

Moreover, the functions $A(t,u,V)$, $B(t,u,V)$ and $C(t,u,V)$ satisfy the following Riccati equations:

$$\partial_t A(t,u,V) = F(B(t,u,V),C(t,u,V)), \quad A(0,u,V)=0, \tag{4.4}$$

$$\partial_t B(t,u,V) = R(B(t,u,V),C(t,u,V)), \quad B(0,u,V)=u, \tag{4.5}$$

$$\text{and} \quad \partial_t C(t,u,V) = T(B(t,u,V),C(t,u,V)), \quad C(0,u,V)=V, \tag{4.6}$$

with

$$F(u,V) = \langle \alpha u, u\rangle + 2tr(\alpha V) + \langle \beta, u\rangle - \gamma, \tag{4.7}$$

$$R(u,V) = 4V'\alpha u + b'u + 2V\beta - \delta, \tag{4.8}$$

$$T(u,V) = 4V'\alpha V + b'V + V'b - \Phi. \tag{4.9}$$

Remark 4.1. *Proposition 4.1 shows that the coefficient functions $A(t,u,V)$, $B(t,u,V)$ and $C(t,u,V)$ defined in the quadratic Gaussian property can be determined by a series of multi-variate Riccati equations (4.4)-(4.9). Moreover, since $T(u,V)$ does not depend on u, it follows that the function $C(t,u,V)$ can be rewritten as $C(t,V)$ and can be determined without knowledge of $A(t,u,V)$ and $B(t,u,V)$. This property gives us an efficient way of solving numerically for the coefficient functions.*

We now state the main result of this section, which gives us an analytic characterization of a strongly regular quadratic Gaussian process, namely, that it is a Feller process (Revuz and Yor 1994 [12], Definition III 2.5) and is associated with a unique admissible parameter set. As a straightforward consequence of the Feller property, we also derive the infinitesimal generator of a strongly regular quadratic Gaussian process.

Theorem 4.1. *Suppose X is a strongly regular quadratic Gaussian process; then it is a Feller process. Let $\mathcal{A}$ be its infinitesimal generator. Then there exists a unique admissible parameter set $(\alpha,\beta,b,\gamma,\delta,\Phi)$ such that, for all $f \in C_c^2(D)$,*

$$\mathcal{A}f(x) = tr\left(\alpha \frac{\partial^2 f(x)}{\partial x \partial x'}\right) + \langle \beta + bx, \nabla f(x)\rangle - (\gamma + \delta' x + x'\Phi x) f(x). \tag{4.10}$$

Conversely, given an admissible parameter set $(\alpha,\beta,b,\gamma,\delta,\Phi)$, there exists a regular quadratic Gaussian semigroup $(P_t)_{t\in\mathbb{R}_+}$ with the infinitesimal generator (4.10), and (3.2) holds for all $(u,V) \in \mathcal{B}$. The functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ are given by (4.4) through (4.9) with initial conditions (3.3).

Remark 4.2. *It follows from Theorem 4.1 that a strongly regular quadratic Gaussian process can be uniquely characterized by an admissible parameter set. Furthermore, in order for (3.2) to hold, there cannot exist a jump part in the infinitesimal generator of a strongly regular quadratic Gaussian process. Thus, for example, jump diffusion processes are not applicable in modeling state processes for which the "quadratic Gaussian" property holds.*

5. GENERALIZED QUADRATIC TERM STRUCTURE MODELS

In this section, we will define a class of generalized quadratic term structure models (GQTSMs) based on the construction of the previous sections, and discuss pricing problems under this class.

Definition 5.1. *A term structure model is a GQTSM with parameters* $(\alpha, \beta, b, \gamma, \delta, \Phi, R_0, R_1, R_2)$*, if*

i) $(\alpha, \beta, b, \gamma, \delta, \Phi)$ *is admissible and a d-dimensional state vector* X *follows a strongly regular quadratic Gaussian process under the risk-neutral measure* $\mathbb{P}_x$ *with this parameter set;*

ii) $(R_0, R_1, R_2) \in \mathcal{E}$ *and the short rate is a quadratic function of* X*:*

$$r(X_t) = R_0 + \langle R_1, X_t \rangle + \langle R_2 X_t, X_t \rangle, \quad (R_0, R_1, R_2) \in \mathcal{E};$$

and

(iii) for any fixed $x \in D$ *and* $t \in \mathbb{R}_+$*,*

$$\mathbb{E}_x \left[e^{\int_0^t r(X_s) ds} \right] < \infty.$$

Remark 5.1. *The condition* $(R_0, R_1, R_2) \in \mathcal{E}$ *is necessary for keeping the short rate* $r(X_t)$ *nonnegative, while the condition* (iii) *guarantees that the savings account is well defined.*

We now turn to the problems of bond pricing and option pricing under GQTSMs.

5.1. **Bond Pricing and Quadratic Pricing Class.**

First we define the family of the operators $(Q_t)_{t \in \mathbb{R}_+}$ as

$$Q_t f(x) := \mathbb{E}_x \left[e^{-\int_0^t r(X_s) ds} f(X_t) \right], \quad x \in D, \ t \in \mathbb{R}_+.$$

The following proposition proves that $(Q_t)_{t \in \mathbb{R}_+}$ is a pricing semigroup.

Proposition 5.1. ***(The Feynman-Kac Formula)***
Given a quadratic Gaussian term structure model with the parameters $(\alpha, \beta, b, \gamma, \delta, \Phi, R_0, R_1, R_2)$*, the family* $(Q_t)_{t \in \mathbb{R}_+}$ *forms a regular quadratic Gaussian semigroup with the parameter set* $(\alpha, \beta, b, \gamma + R_0, \delta + R_1, \Phi + R_2)$*.*

Remark 5.2. *Since* $(R_0, R_1, R_2) \in \mathcal{E}$*, it follows that* $(\gamma + R_0, \delta + R_1, \Phi + R_2) \in \mathcal{E}$ *and thus* $(\alpha, \beta, b, \gamma + R_0, \delta + R_1, \Phi + R_2)$ *is admissible.*

It is straightforward to derive the price $\pi(0, T)$ of a zero-coupon bond with maturity T at time 0 to be

$$\pi(0, T) = Q_T 1 = e^{A(T,0,0) + \langle B(T,0,0), x \rangle + x' C(T,0,0) x},$$

where $A(T, 0, 0)$, $B(T, 0, 0)$ and $C(T, 0, 0)$ can be solved from (4.4) to (4.9) with the initial value $(u, V) = (0, 0)$ and the parameter set $(\alpha, \beta, b, \gamma + R_0, \delta + R_1, \Phi + R_2)$.

By the Markov property we also have

$$\begin{aligned}\pi(t,T) &= \mathbb{E}_x[e^{-\int_t^T r(X_s)ds}1|\mathcal{F}_t] \\ &= \mathbb{E}_x[e^{-\int_0^{T-t}(R_0+R_1'X_s+X_s'R_2X_s)ds}\circ\theta_t 1|X_t] \\ &= \mathbb{E}_{X_t}[e^{-\int_0^{T-t}(R_0+R_1'X_s+X_s'R_2X_s)ds}] \\ &= e^{A(T-t,0,0)+\langle B(T-t,0,0),X(t)\rangle+X_t'C(T-t,0,0)X_t}, \qquad (5.1)\end{aligned}$$

where $\theta_t(\omega)(s)=\omega(t+s)$, which is a shift operator: $\Omega\mapsto\Omega$.

Definition 5.2. [1] *A model is said to belong to the Quadratic Pricing Class (QPC) if the prices of zero-coupon bonds, $\pi(t,T)$, are exponential-quadratic functions of a Markov process X; i.e.,*

$$\pi(t,T)=e^{A(T-t)+\langle B(T-t),X_t\rangle+X_t'C(T-t)X_t}. \qquad (5.2)$$

Under Definition 5.2, we have the following proposition.

Proposition 5.2. *A model belongs to the QPC if it is a GQTSM.*

5.2. **Option Pricing.** For a European zero-coupon bond option, the price can be easily obtained by taking advantage of the quadratic Gaussian pricing semigroup. Let the payoff $h(X_t)=(K-\pi(t,T))^+$, where $K(K>0)$ is the strike price. We have

$$\begin{aligned}Q_th(x) &= \mathbb{E}_x\left[e^{-\int_0^t r(X_s)ds}h(X_t)\right] \qquad (5.3)\\ &= \mathbb{E}_x\left[e^{-\int_0^t r(X_s)ds}\pi(t,T)1_{\{\pi(t,T)\le K\}}\right] \\ &\quad -K\mathbb{E}_x\left[e^{-\int_0^t r(X_s)ds}1_{\{\pi(t,T)\le K\}}\right]. \qquad (5.4)\end{aligned}$$

Here we define

$$G_{u_1,V_1;u_2,V_2}(y,x,t)=\mathbb{E}_x[e^{-\int_0^t r(X_s)ds}e^{u_1'X_t+X_t'V_1X_t}1_{\{u_2'X_t+X_t'V_2X_t\le y\}}], \qquad (5.5)$$

where u_1,u_2 are two d-dimensional vectors and two V_1,V_2 are $d\times d$ matrices, such that $(u_1,V_1)\in\mathcal{B}\cap\left(\mathbb{R}^d\times\mathbb{R}^{d\times d}\right)$. Therefore, given (5.2), (5.4) and (5.5), we have

$$Q_th(x)=KG_{0,0,;B,C}(K',x,t)-e^AG_{B,C;B,C}(K',x,t),$$

where

$$K'=\ln(K)-A,$$

and we rewrite $A(T-t)$, $B(T-t)$ and $C(T-t)$ as A, B, C for convenience. Thus, we see that finding an analytical solution for the function G is sufficient for solving European option pricing problems.

Remark 5.3. *Since the price of a zero-coupon bond is a positive real value bounded by 1 for any initial state x, we can reasonably assume that $(u_1,V_1)\in\mathcal{B}\cap\left(\mathbb{R}^d\times\mathbb{R}^{d\times d}\right)$.*

It is seen from (5.5) that $G_{u_1,V_1;u_2,V_2}(y,x,t)$ is the cumulative distribution function of $u_2'X_t+X_t'V_2X_t$ under the measure $e^{-\int_0^t r(X_s)ds}e^{u_1'X_t+X_t'V_1X_t}\mathbb{P}_x$. Therefore

[1]Here we follow the same definition as that given in Leippold and Wu (2002, [9])

we can determine the function G by calculating its characteristic function. On letting $\mathcal{G}$ denote this characteristic function, we have

$$\begin{aligned}
\mathcal{G}_{u_1,V_1;u_2,V_2}(z,x,t) &= \int_{\mathbb{R}} e^{izy} dG_{u_1,V_1;u_2,V_2}(y,x,t) \\
&= \int_{\mathbb{R}} e^{izy} d\mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} 1_{\{u_2' X_t + X_t' V_2 X_t \le y\}}] \\
&= \int_{\mathbb{R}} e^{izy} \mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} \delta(y - u_2' X_t + X_t' V_2 X_t)] \\
&= \mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{(u_1+izu_2)' X_t + X_t'(V_1+izV_2)X_t}] \\
&= Q_t f_{u_1+izu_2,V_1+izV_2}(x). \qquad (5.6)
\end{aligned}$$

Because $(u_1+izu_2, V_1+izV_2) \in \mathcal{B}$, by Proposition 5.1, we can calculate the Fourier transform of (5.5), and thus by applying the inverse Fourier transform, we can easily obtain the prices of European options on zero coupon bonds. This technique was originally proposed by Heston (1993, [7]), generalized by Duffie, Pan and Singleton (2000 [4]) to the affine jump-diffusion model, and by Leippold and Wu (2002, [9]) to the traditional quadratic term structure model.

6. MODELING DEFAULTABLE RATES BY GQTSMS

In this section, we demonstrate that generalized quadratic terms structure models can be applied to modeling defaultable rates without specifying any auxiliary model for characterizing default risk, which is often used in the literatures (see e.g., Madam and Unal (1996 [11]), Duffie and Singleton (1999 [5])). In order to achieve this, we use the death of the underlying state process to indicate the default. Because the quadratic Gaussian state processes defined in GQTSMs are possibly non-conservative, we can characterize defaultable rates by non-conservative GQTSMs given the definition below.

Definition 6.1. *A GQTSM is said to be non-conservative (resp., conservative) if the underlying quadratic Gaussian state process is non-conservative (resp., conservative).*

6.1. Conservative GQTSMs.

Since the conservativity of a state process guarantees that it will never die, this feature captures the nature of non-defaultness. Therefore risk-free rates can be modeled by conservative GQTSMs. From Dynkin (1965 [6], Lemma 2.3), it follows that a quadratic Gaussian process X with parameter set $(\alpha, \beta, b, \gamma, \delta, \Phi)$ is conservative if and only if $\gamma = 0$, $\delta = 0$ and $\Phi = 0$.

Under this conservative condition, the infinitesimal generator of the state process can be written as

$$\mathcal{A}f(x) = tr\left(\alpha \frac{\partial^2 f(x)}{\partial x \partial x'}\right) + \langle \beta + bx, \nabla f(x) \rangle, \quad \text{for} \quad \forall\ f \in C_c^2(D).$$

Therefore the state process X is Gaussian and can be modeled by the following diffusion process:

$$dX_t = (\beta + bX_t)dt + \sigma dW_t, \quad \text{where} \quad \sigma\sigma' = 2\alpha. \qquad (6.1)$$

Comparing (6.1) and (1.1), we concluded that the traditional QTSMs are the conservative GQTSMs, which can characterize only risk-free rates.

6.2. **Non-Conservativity and Default Risks.** In this part, we will show how the non-conservativity of a quadratic Gaussian state process captures the default risk. In order to illustrate this question, we will give a formula for the default probability and prove that it will be strictly positive under the non-conservative condition. For simplicity, we only discuss an example in a one-factor case. One can easily extend it to multivariate cases by applying numerical integration.

6.2.1. *One-factor Non-Conservative Case.* Suppose we have a one-factor non-conservative GQTSM with parameters $(\alpha, \beta, b, \gamma, \delta, \Phi, R_0, R_1, R_2)$; i.e., we suppose that γ, δ and Φ are not all zero. Therefore the state process X can possibly die sometime, which indicates a default. According to Remark 2.1, the event that default happens before time T is equivalent to the event that $\{X_T^2 = \infty\}$, and therefore the probability $P_d(T)$ that default happens before time T under the risk neutral measure can be given by

$$\begin{aligned} P_d(T) &= \mathbb{E}\left[1_{\{X_T^2=\infty\}}\right] \\ &= 1 - \mathbb{E}\left[1_{\{X_T^2<\infty\}}\right] \\ &= 1 - \mathbb{E}\left[\lim_{\kappa\to\infty} e^{-\frac{1}{\kappa}X_T^2}\right] \\ &= 1 - \lim_{\kappa\to\infty} \mathbb{E}\left[e^{-\frac{1}{\kappa}X_T^2}\right]. \end{aligned}$$

Since X is a quadratic Gaussian process, it follows from (3.2) that

$$\begin{aligned} P_d(T) &= 1 - \lim_{\kappa\to\infty} \exp\left\{A(T,0,\frac{1}{\kappa}) + \langle B(T,0,\frac{1}{\kappa}), x\rangle + \langle C(T,0,\frac{1}{\kappa}x, x\rangle\right\} \\ &= 1 - \exp\left\{A(T,0,0) + \langle B(T,0,0), x\rangle + \langle C(T,0,0)x, x\rangle\right\}. \end{aligned}$$

The last step follows from the Lipschitz continuity of $A(T,u,\cdot)$, $B(T,u,\cdot)$ and $C(T,u,\cdot)$ at 0. $A(T,0,0)$, $B(T,0,0)$ and $C(T,0,0)$ can be determined from (4.4)-(4.9) with initial conditions:

$$A(T,0,0) = 0, \quad B(T,0,0) = 0, \quad \text{and} \quad C(T,0,0) = 0.$$

In particular, in the one dimensional case, they have the following analytic expressions. (see Levendorskiĭ (2002 [10]))

$$\begin{aligned} C(T,0,0) &= \Phi C_1 C_2 \frac{1-e^{\omega T}}{C_2 - C_1 e^{\omega T}}, \\ B(T,0,0) &= \frac{2}{(C_2-C_1)(C_2-C_1e^{\omega T})} \\ &\quad \times \left[\frac{C_2(C_1\beta\Phi - \frac{\delta}{2})}{\omega_1}(e^{\omega_1 T}-1) - \frac{C_1(C_2\beta\Phi - \frac{\delta}{2})}{\omega_1 - \omega}(e^{\omega_1 T} - e^{\omega T})\right]. \end{aligned}$$

$$\text{and} \quad A(T,0,0) = \int_0^T \left[2\alpha C(s,0,0) + \alpha B^2(s,0,0) + \beta B(s,0,0) - \gamma\right] ds,$$

where $C_1 \leq 0 \leq C_2$ are roots of the quadratic equation $4\alpha\Phi C^2 + 2bC - 1 = 0$, $\omega = 4\alpha\Phi(C_1 - C_2) \leq 0$ and $\omega_1 = 4\alpha\Phi C_1 + b$. Therefore since γ, δ and Φ are not all zero as assumed before, we have

i) if $\Phi > 0$, it follows that $C(T, 0, 0) < 0$, and thus we have $P_d(T) > 0$ generally except for two specific values of x that are the roots of the quadratic equation $C(T, 0, 0)x^2 + B(T, 0, 0)x + A(T, 0, 0) = 0$;

ii) if $\Phi = 0$, by Remark 3.1, then $\delta = 0$ and $\gamma > 0$. Therefore the functions C, B are always zero and the function $A(T, 0, 0) = -\gamma T < 0$, for $T > 0$. In this case, $P_d(T)$ is still strictly positive for $T > 0$.

On the other hand, if the state process X is conservative which indicates that $\gamma = \delta = \Phi = 0$, then we have $A(T, 0, 0) = B(T, 0, 0) = C(T, 0, 0) = 0$, for every $T \in \mathbb{R}_+$, which means that no default can happen (i.e., $P_d(T) \equiv 0$).

The above discussion shows that non-conservativity of the state process introduces default risk into quadratic term strucuture models. Therefore, the class of GQTSMs provides a unifying framework for modeling risk-free and defaultable rates.

7. Conclusion

The main contributions of this paper can be summarized briefly as follows. Firstly, we have developed a class of GQTSMs, which extends the traditional QTSMs to a general Markovian setting, while retaining the analytical tractability of the traditional QTSMs. Secondly, we have demonstrated that the pricing kernel for GQTSMs is a quadratic Gaussian semigroup so that the zero coupon bond pricing and option pricing formulas are still easy to derive. Finally, we have shown that our GQTSMs can directly model defaultable rates as well as risk-free rates. This new feature of GQTSMs provides a new approach to pricing interest rate derivatives subject to credit risk.

A. APPENDIX

A.1. **Proof of Lemma 3.1.** We can rewrite (1.1) as

$$d(\Gamma(t)X_t) = \mu dt + \Sigma dW_t,$$

where

$$\frac{d\Gamma(t)}{dt} = -\Lambda\Gamma(t), \Gamma(0) = I.$$

Therefore for $\forall t \in \mathbb{R}_+$, we have

$$X_t = \Gamma^{-1}(t)x + \Gamma^{-1}(t)\int_0^t \Gamma(r)\mu dr + \Gamma^{-1}(t)\int_0^t \Gamma(r)\Sigma dW_r. \tag{A.1}$$

From (A.1), it follows that X_t is Gaussian with mean

$$M(t) = \Gamma^{-1}(t)x + \Gamma^{-1}(t)\int_0^t \Gamma(r)\mu dr,$$

and variance

$$\sigma(t) = \Gamma^{-1}(t)\int_0^t \Gamma(r)\Sigma\sigma'\Gamma(r)'dr\Gamma^{-1}(t)'.$$

Thus we obtain

$$P_t f_{u,V}(x) = \exp\left\{A(t,u,V) + \langle B(t,u,V), x\rangle + x' C(s,t,u,V)x\right\},$$

where

$$\begin{aligned}
A(t,u,V) &= -\log(|\sigma(t)||\sigma(t)-2V|) + \frac{1}{2}(\sigma^{-1}(t)b(t)+u)'(\sigma^{-1}(t)-2V)^{-1} \\
&\quad (\sigma^{-1}(t)b(t)+u) - \frac{1}{2}b(t)'\sigma^{-1}(t)b(t), \\
B(t,u,V) &= \Gamma^{-1}(t)'\sigma^{-1}(t)[(\sigma^{-1}(t)-2V)^{-1}(\sigma^{-1}(t)b(t)+u) - b(t)], \\
\text{and} \quad C(t,u,V) &= -\frac{1}{2}[\Gamma^{-1}(t)'\sigma^{-1}(t)(I-(\sigma^{-1}(t)-2V)^{-1}\sigma^{-1}(t))\Gamma^{-1}(t)] \\
\text{with} \quad b(t) &= \Gamma^{-1}(t)\int_0^t \Gamma(r)\mu dr.
\end{aligned}$$

This completes the proof.

A.2. **Proof of Lemma 3.2.** By the Chapman-Kolmogorov equation we can derive the following relationships, for $\forall(s,t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times D$:

$$A(t+s,u,V) = A(t,u,V) + A(s,B(t,u,V),C(t,u,V)), \tag{A.2}$$

$$B(t+s,u,V) = B(s,B(t,u,V),C(t,u,V)), \tag{A.3}$$

$$\text{and} \quad C(t+s,u,V) = C(s,B(t,u,V),C(t,u,V)). \tag{A.4}$$

Given (A.2)-(A.4) and (3.3), we have

$$\begin{aligned}
A(t+s,u,V) - A(t,u,V) &= A(s,B(t,u,V),C(t,u,V)) - A(0,B(t,u,V),C(t,u,V)), \\
B(t+s,u,V) - B(t,u,V) &= B(s,B(t,u,V),C(t,u,V)) - B(0,B(t,u,V),C(t,u,V)), \\
\text{and} \quad C(t+s,u,V) - C(t,u,V) &= C(s,B(t,u,V),C(t,u,V)) - C(0,B(t,u,V),C(t,u,V)).
\end{aligned}$$

Now, on letting $s \to 0^+$, (3.5) through (3.7) follow. By using (3.4), we can easily derive (3.8) through (3.11). This completes the proof.

A.3. **Proof of Lemma 4.1.** A proof of a more general result can be found in Duffie, Filipović and Schachermayer (2002 [3], Section 4).

A.4. **Proof of Proposition 4.1.** In order to prove Proposition 4.1, first we give a lemma concerning the uniqueness of representations.

Lemma A.1. *Given (A,β,γ,ν), where $A \in \mathbb{R}^{d\times d}$, $\beta \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$ and ν is a measure on $\mathbb{R}^d$, the representation*

$$\begin{aligned}
\hat{\mu}(u) &= \langle u, Au\rangle + \langle \beta, u\rangle + \gamma \\
&\quad + \int_{D\setminus\{0\}} \left[e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\right]\nu(d\xi), \quad u \in i\mathbb{R}^d
\end{aligned} \tag{A.5}$$

is unique.

Proof. See Sato (1999 [15], Theorem 8.1).

The following corollary is a direct extension of this lemma.

Corollary A.1. *Given functions (A,β,γ), where $A : D \mapsto \mathbb{R}^{d\times d}$, $\beta : D \mapsto \mathbb{R}^d$ and $\gamma : D \mapsto \mathbb{R}$, the representation*

$$\begin{aligned}\bar{\mu}(x,u,V) &= (u+2Vx)'A(x)(u+2Vx) + 2tr(A(x)V) + \langle \beta(x), u+2Vx\rangle \\ &\quad +\gamma(x), \quad \textit{for each} \quad (x,(u,V)) \in D\times\mathcal{B}\end{aligned}$$

is unique.

We now prove Proposition 4.1.

By simply substituting (4.2) into (4.1), we have

$$\begin{aligned}\frac{\mathcal{A}f_{u,V}(x)}{f_{u,V}(x)} &= (u+2Vx)'\alpha(x)(u+2Vx) + 2tr(\alpha(x)V) + \langle\beta(x), u+2Vx\rangle + \gamma(x) \\ &\quad + \int_{D\setminus\{x\}} \left(e^{\langle u,\xi-x\rangle + \xi'V\xi - x'Vx} - 1 - \langle u+2Vx, \chi(\xi-x)\rangle\right)\nu(x,d\xi). \end{aligned} \tag{A.6}$$

By applying (3.8) to (A.6), we obtain

$$\begin{aligned}F(u,V) &= \langle\alpha u,u\rangle + 2tr(\alpha V) + \langle\beta,u\rangle + \gamma \\ &\quad + \int_{D\setminus\{0\}}\left(e^{\langle u,\xi\rangle} - 1 - \langle u,\chi(\xi)\rangle\right)m(d\xi),\end{aligned}$$

where

$$\alpha = \alpha(0),\ \beta=\beta(0),\ \gamma=\gamma(0) \text{ and } m(d\xi) = \nu(0,d\xi).$$

In the same way, by applying (3.9), (3.10) to (A.6), we derive that

$$\begin{aligned}R_i(u,V) &= \langle\hat{\alpha}_i u,u\rangle + 4\langle\bar{\alpha}_i u + \hat{\alpha}_i V^i, V^i\rangle \\ &+ 2tr(\hat{\alpha}_i V) + \langle b_i,u\rangle + 2\langle\bar{\beta}_i, V^i\rangle + \delta_i \\ &+ \frac{1}{2}\int_{D\setminus\{0\}}\left(e^{\langle u+2V^i,\xi\rangle + \xi'V\xi} - 1 - \langle u+2V^i,\chi(\xi)\right)\nu_i(d\xi) \\ &- \frac{1}{2}\int_{D\setminus\{0\}}\left(e^{\langle u-2V^i,\xi\rangle + \xi'V\xi} - 1 - \langle u-2V^i,\chi(\xi)\right)\nu_{-i}(d\xi),\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}T_{ii}(u,V) &= \langle(\bar{\alpha}_i-\alpha)u,u\rangle + 4\langle\hat{\alpha}_i u \\ &+ \bar{\alpha}_i V^i, V^i\rangle + 2tr((\bar{\alpha}_i-\alpha)V) + \langle(\bar{\beta}_i-\beta),u\rangle + 2\langle b_i, V^i\rangle \\ &+ \Phi_{ii} + \frac{1}{2}\int_{D\setminus\{0\}}\left(e^{\langle u+2V^i,\xi\rangle + \xi'V\xi} - 1 - \langle u+2V^i,\chi(\xi)\right)\nu_i(d\xi) \\ &+ \frac{1}{2}\int_{D\setminus\{0\}}\left(e^{\langle u-2V^i,\xi\rangle + \xi'V\xi} - 1 - \langle u-2V^i,\chi(\xi)\right)\nu_{-i}(d\xi) \\ &+ \int_{D\setminus\{0\}}\left(e^{\langle u,\xi\rangle} - 1 - \langle u,\chi(\xi)\rangle\right)m(d\xi), \quad \text{for each} \quad 1\le i\le d,\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}\hat{\alpha}_i &= \frac{1}{2}(\alpha(e_i) - \alpha(-e_i)), \quad \bar{\alpha}_i = \frac{1}{2}(\alpha(e_i) + \alpha(-e_i)),\\ b_i &= \frac{1}{2}(\beta(e_i) - \beta(-e_i)), \quad \bar{\beta}_i = \frac{1}{2}(\beta(e_i) + \beta(-e_i)),\\ \delta_i &= \frac{1}{2}(\gamma(e_i) - \gamma(-e_i)), \quad \Phi_{ii} = \frac{1}{2}(\gamma(e_i) + \gamma(-e_i)) - \gamma,\\ \nu_i(\cdot) &= \nu(e_i, \cdot), \quad \text{and} \quad \nu_{-i}(\cdot) = \nu(-e_i, \cdot).\end{aligned}$$

Now by (3.4), we have, for each $s \in \mathbb{R}$,

$$\frac{\tilde{\mathcal{A}} f_{u,V}(se_i)}{f_{u,V}(se_i)} = F(u, V) + sR_i(u, V) + s^2 T_{ii}(u, V). \tag{A.9}$$

This equation holds for all $(i, (u, V)) \in \{1, 2, ..., d\} \times \mathcal{B}$.

We approach the proof in two steps. First we let $V = 0$. Then according to Lemma A.1 and given (A.7) - (A.8), we have

$$\alpha(se_i) = \alpha + s\hat{\alpha}_i + s^2(\bar{\alpha}_i - \alpha), \tag{A.10}$$

$$\beta(se_i) = \beta + sb_i + s^2(\bar{\beta}_i - \beta), \tag{A.11}$$

$$\gamma(se_i) = \gamma + s\delta_i + s^2\Phi_{ii}, \tag{A.12}$$

$$\text{and} \quad \nu(se_i, \cdot) = \left(\frac{1}{2}[\nu_i(\cdot) + \nu_{-i}(\cdot)] - m(\cdot)]\right) s^2$$

$$\frac{1}{2}(\nu_i(\cdot) - \nu_{-i}(\cdot))s + m(\cdot), \quad \text{for } \forall s \in \mathbb{R}. \tag{A.13}$$

Since $\nu(se_i, \cdot)$ is a nonnegative measure for each $s \in \mathbb{R}$, we can deduce the following constraints among the measures $m(\cdot)$, $\nu_i(\cdot)$ and $\nu_{-i}(\cdot)$:

i)

$$\nu_i(\cdot) + \nu_{-i}(\cdot)] - 2m(\cdot) \geq 0; \quad \text{and} \tag{A.14}$$

ii) if

$$\nu_i(\cdot) + \nu_{-i}(\cdot) - 2m(\cdot) = 0, \tag{A.15}$$

then

$$\nu_i(\cdot) = \nu_{-i}(\cdot) = m(\cdot). \tag{A.16}$$

Secondly, we let $u = 0$. By applying (A.10) and (A.11), we can rewrite (A.9) as

$$
\begin{aligned}
& 4(s^2-1)\langle(\alpha(se_i)-\alpha)V^i, V^i\rangle + 2(s^3-s)\langle(\bar{\beta}_i-\beta), V^i\rangle \\
+ & \int_{D\backslash\{0\}} \left[e^{\xi' V\xi + 2s\langle V^i,\xi\rangle} - 1 - 2s\langle V^i, \chi(\xi)\rangle\right] \nu(se_i, d\xi) \\
- & \int_{D\backslash\{0\}} \left[e^{\xi' V\xi} - 1\right] m(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s}{2}\left[e^{\xi' V\xi + 2\langle V^i,\xi\rangle} - 1 - 2\langle V^i, \chi(\xi)\rangle\right] \nu_i(d\xi) \\
+ & \int_{D\backslash\{0\}} \frac{s}{2}\left[e^{\xi' V\xi - 2\langle V^i,\xi\rangle} - 1 + 2\langle V^i, \chi(\xi)\rangle\right] \nu_{-i}(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s^2}{2}\left[e^{\xi' V\xi + 2\langle V^i,\xi\rangle} - 1 - 2\langle V^i, \chi(\xi)\rangle\right] \nu_i(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s^2}{2}\left[e^{\xi' V\xi - 2\langle V^i,\xi\rangle} - 1 + 2\langle V^i, \chi(\xi)\rangle\right] \nu_{-i}(d\xi) \\
+ & \int_{D\backslash\{0\}} s^2\left[e^{\xi' V\xi} - 1\right] m(d\xi) = 0, \quad \text{for } \forall s \in \mathbb{R}.
\end{aligned}
\tag{A.17}
$$

Given any d-dimensional vector ϕ on the unit hyper-sphere and $r \in \mathbb{R}_+$, let $V(r,\phi)$ be a symmetric matrix whose jth column and jth row are equal to $ir\phi$ and $ir\phi'$, respectively, and all the other entries are zero. Since (A.17) is true for all $V \in Sem^d_-$, in particular we can apply $V(r,\phi)$ to (A.17). Note that

$$
\begin{aligned}
e^{\xi' V\xi + 2s\langle V^j,\xi\rangle} - 1 - 2s\langle V^j, \chi(\xi)\rangle \quad & \leq \quad (2||V|| + 2s^2||V^j||^2)||\xi||^2 + ||V||^2(2|s|||\xi||^3 + ||\xi||^4) \\
& \leq \quad 4r||\xi||^2 + 2r^2(s^2||\xi||^2 + 4|s|||\xi||^3 + 2||\xi||^4) \text{ for } ||\xi|| \leq d.
\end{aligned}
$$

Moreover, the left-hand side of the expression is bounded when $||\xi|| > d$. Therefore by Lemma 4.1 and the dominated convergence theorem, after dividing both sides by r^2 and letting $r \to +\infty$, it follows that

$$4(s^2-1)\langle(\alpha(se_i)-\alpha)\phi, \phi\rangle = 0.$$

Because this is true for each ϕ on the unit hyper-sphere and $s \in \mathbb{R}$, we have

$$\alpha(se_i) = \alpha, \quad \text{for each} \quad (i,s) \in \{1,2,...,d\} \times \mathbb{R}.$$

Therefore the first item in the left-side of (A.17) vanishes.

Using the same strategy, after dividing both sides by r and letting $r \to +\infty$, by the dominated convergence theorem, we obtain

$$\bar{\beta}_i - \beta = \int_{D\backslash\{0\}} \chi(\xi)\left[\frac{1}{2}(\nu_i(d\xi) + \nu_{-i}(d\xi) - 2m(d\xi))\right]. \tag{A.18}$$

Now (A.17) remains as

$$
\begin{aligned}
0 &= (1-s^2)\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-1\right] m(d\xi) \\
&+ \frac{1}{2}(s^2+s)\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-e^{2\langle V^i,\xi\rangle}\right]\nu_i(d\xi) \\
&+ \frac{1}{2}(s^2-s)\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-e^{-2\langle V^i,\xi\rangle}\right]\nu_{-i}(d\xi), \qquad \text{(A.19)}\\
&\text{for each} \quad (i,s,(u,V))\in\{1,2,...,d\}\times\mathbb{R}\times\mathcal{B}.
\end{aligned}
$$

Let $\mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i})$ be equal to the right-hand side of the above equation, and therefore for each $(i,s,(u,V))\in\{1,2,...,d\}\times\mathbb{R}\times\mathcal{B}$, $\mathcal{G}_i=0$, which will yield the following equations:

$$
\begin{aligned}
\text{(A.20)} \qquad & \nabla_u\mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) = 0, \\
\text{(A.21)} \qquad & \nabla_{V^i}\mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) = 0, \\
\text{(A.22)} \qquad \text{and} \quad & \nabla_u\nabla_{V^i}\mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) = 0, \\
& \text{for each} \quad (i,s,(u,V)) \in \{1,2,...,d\}\times\mathbb{R}\times\mathcal{B}.
\end{aligned}
$$

On setting $V^i=0$, (A.21) can be rewritten as

$$
(s^3-s)\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\xi[\nu_i(d\xi)+\nu_{-i}(d\xi)-2m(d\xi)]=0,
$$
$$
\text{for each} \quad (u,V)\in\mathcal{B} \quad \text{with} \quad V^i=0.
$$

Then, on setting $u=0$ and $V=0$, by the definition of strong regularity, (A.22) yields

$$
(s^3-s)\int_{D\setminus\{0\}} \xi\xi'[\nu_i(d\xi)+\nu_{-i}(d\xi)-2m(d\xi)]=0,
$$
$$
\text{for each} \quad s\in\mathbb{R}.
$$

By (A.14), the above equation just tells us that the measure $\nu_i(\cdot)+\nu_{-i}(\cdot)-2m(\cdot)$ is zero. By (A.15) and (A.16), we have

$$
\nu_i(\cdot)=\nu_{-i}(\cdot)=m(\cdot), \quad \text{for each} \quad i\in\{1,2,...,d\}. \qquad \text{(A.23)}
$$

Then we can rewrite (A.19) as

$$
\begin{aligned}
&\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}e^{2s\langle V^i,\xi\rangle}m(d\xi) \\
&= \frac{1}{2}s^2\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left(e^{\langle V^i,\xi\rangle}-e^{-\langle V^i,\xi\rangle}\right)^2 m(d\xi) \\
&+ \frac{1}{2}s\int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left(e^{2\langle V^i,\xi\rangle}-e^{-2\langle V^i,\xi\rangle}\right) m(d\xi) \\
&+ \int_{D\setminus\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}m(d\xi). \qquad \text{(A.24)}
\end{aligned}
$$

Now let $u=0$ and all the entries of V be 0, except for V_{ii} which is set to be -1. After differentiating both sides of (A.24) twice with respect to s and setting $s=0$, we have

$$\int_{D\backslash\{0\}} e^{-\xi_i^2} 4\xi_i^2 m(d\xi) = \int_{D\backslash\{0\}} e^{-\xi_i^2} \left(e^{-\xi_i} - e^{\xi_i}\right)^2 m(d\xi). \tag{A.25}$$

Since the following inequality holds for all $\xi \neq 0$

$$4\xi_i^2 < \left(e^{-\xi} - e^{\xi}\right)^2, \tag{A.26}$$

and (A.25), (A.26) are true for all $i \in \{1, 2, ..., d\}$, it thus follows that $m(\cdot)$ is a zero measure. From (A.18), we have

$$\bar{\beta}_i = \beta, \quad \text{for each} \quad i \in \{1, 2, ..., d\}. \tag{A.27}$$

Thus we have proved (4.7) and (4.8) by simply denoting $b = (b_1, b_2, ..., b_d)$. We can rewrite (A.6) as

$$\begin{aligned} \frac{\tilde{\mathcal{A}} f_{u,V}(x)}{f_{u,V}(x)} &= \langle \alpha(x)(u + 2Vx), u + 2Vx\rangle + 2tr(\alpha(x)V) \\ &\quad + \langle \beta(x), u + 2Vx\rangle + \gamma(x). \end{aligned} \tag{A.28}$$

As to (4.9), we use the same arguments on the equation

$$\frac{\tilde{\mathcal{A}} f_{u,V}(se_i + te_j)}{f_{u,V}(se_i + te_j)} = \frac{\tilde{\mathcal{A}} f_{u,V}(se_i)}{f_{u,V}(se_i)} + \frac{\tilde{\mathcal{A}} f_{u,V}(te_j)}{f_{u,V}(te_j)} + 2stT_{ij}(u, V) - F(u, V),$$
$$\text{for all} \quad (s, t) \in \mathbb{R}^2.$$

Then we can obtain

$$\begin{aligned} \alpha(se_i + te_j) &= \alpha, \\ \beta(se_i + te_j) &= \beta + sb_i + tb_j, \\ \gamma(se_j + te_j) &= \gamma + s\delta_i + t\delta_j + s^2\Phi_{ii} + t^2\Phi_{jj} + 2st\Phi_{ij}, \\ \text{and} \quad \nu(se_i + te_j, \cdot) &= 0, \\ \text{for each} \quad & i, j \in \{1, 2, ..., d\}. \end{aligned}$$

This completes the proof of (4.9) by denoting $\Phi = [\Phi_{ij}]$.

Now by (3.4) and (4.7) through (4.9), it follows that

$$\begin{aligned} \frac{\tilde{\mathcal{A}} f_{u,V}(x)}{f_{u,V}(x)} &= F(u, V) + \langle R(u, V), x\rangle + x'T(u, V)x \\ &= (u + 2Vx)'\alpha(u + 2Vx) + 2tr(\alpha V) \\ &\quad + \langle \beta + bx, u + 2Vx\rangle + \gamma + \delta' x + x'\Phi x. \end{aligned} \tag{A.29}$$

Thus comparing (A.29) with (A.28), by Corollary A.1, we have proved that

$$\alpha(x) = \alpha, \quad \beta(x) = \beta + bx \quad \text{and} \quad \gamma(x) = \gamma + \delta' x + x'\Phi x.$$

Since $\gamma(x)$ is nonnegative, for each $x \in D$, it follows that $(\gamma, \delta, \Phi) \in \mathcal{E}$. This completes the proof of (4.3).

By the properties of Riccati equations, we know that there exist unique continuous solutions $A(\cdot, u, V)$, $B(\cdot, u, V)$ and $C(\cdot, u, V)$ for all initial values $(u, V) \in \mathcal{B}$ and all admissible parameter sets. Furthermore, since $F(u, V)$, $R(u, V)$ and $T(u, V)$ are also continuous functions, it follows that $\partial_t A(t, u, V)$, $\partial_t B(t, u, V)$ and $\partial_t C(t, u, V)$

are continuous. Then by Lemma 3.1 and (3.3), we derive (4.4), (4.5) and (4.6). This completes the proof of Proposition 4.1.

A.5. **Proof of Theorem 4.1.** Let $\mathcal{T}$ denote the Fréchet space of rapidly decreasing C^∞-functions on D defined by Rudin (1991 [14], Definition 7.1), and let $\mathcal{L}(\mathcal{T})$ denote its complex linear hull. From Rudin (1991 [14], Theorem 7.15), it follows that $C_c^\infty(D)$ is dense in $\mathcal{T}$ and the Fourier transform is a linear, continuous, one to one mapping on $\mathcal{T}$. Therefore there exists a subset $\mathcal{T}_0$ of $\mathcal{T}$ such that its complex linear hull $\mathcal{L}(\mathcal{T}_0)$ is also dense in $\mathcal{L}(\mathcal{T})$, and for every $g \in \mathcal{T}_0$

$$\begin{aligned} g(x) &= \int_D e^{i\langle q,x\rangle}\tilde{g}(q)dq \\ &= \int_D f_{iq,0}(x)\tilde{g}(q)dq, \end{aligned}$$

for some $\tilde{g} \in C_c^\infty(D)$.

For $t \in \mathbb{R}_+$, we have

$$\begin{aligned} P_t g(x) &= \int_D P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \int_D e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x}\tilde{g}(q)dq. \end{aligned}$$

Because $B(\cdot, iq, 0)$ and $C(\cdot, iq, 0)$ are continuous functions on $\mathbb{R}_+$, it follows that $P_t g \in C^\infty(D)$, for $g \in \mathcal{L}(\mathcal{T})$. Moreover, it is easy to show that

$$\mathcal{A}^\sharp P_t f_{u,V}(x) = \partial_t P_t f_{u,V}(x), \quad \text{for} \quad \forall\ (u, V) \in \mathcal{B},$$

where $\mathcal{A}^\sharp f(x)$ is equal to the right-side of (4.10) for every $f \in C_c^2(D)$. Then we have

$$\begin{aligned} \partial_t P_t g(x) &= \int_D \partial_t P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \int_D \mathcal{A}^\sharp P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \mathcal{A}^\sharp P_t g(x), \text{ for each } (t,x) \in \mathbb{R}_+ \times D. \end{aligned} \tag{A.30}$$

In particular,

$$\lim_{t\downarrow 0} P_t g(x) = g(x),$$

for each $x \in D$ and $g \in \mathcal{L}(\mathcal{T}_0)$.

Since $V = 0$, for any fixed $t \in \mathbb{R}_+$, we know that $C(t, iq, 0) \in Sem_-^d$ by Remark 3.2. Then if $C(t, iq, 0) \in Sem_{--}^d$,

$$\lim_{|x|\to\infty} e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x} = 0, \text{ for each } q \in \mathbb{R}^d.$$

Also, because $\tilde{g} \in C_c^\infty(D)$, by the dominated convergence theorem, we have

$$P_t g(x) = \int_D e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x}\tilde{g}(q)dq \in C_0(D).$$

If $C(t, iq, 0) \in \partial Sem_-^d$, then by (4.5) and (4.8), it follows that

$$B(t, iq, 0) = \Gamma(t)iq + \eta(t),$$

where $\eta(t) \in \mathbb{R}^d$ and $\Gamma(t) \in \mathbb{R}^{d\times d}$ satisfy

$$\begin{aligned} \frac{d\Gamma(t)}{dt} &= (4C(t,iq,0)\alpha + b')\Gamma(t), \Gamma(0) = I, \\ \text{and} \quad \eta(t) &= \Gamma^{-1}(t)\int_0^t \Gamma(s)(2C(s,iq,0)\beta - \delta)ds. \end{aligned}$$

Let $\Theta = \{x \in D : x'C(t,iq,0)x = 0\}$. From Remark 3.1, we know that, for every $x \in \Theta$,

$$\langle \eta(t), x\rangle = 0.$$

Thus, by the Riemann-Lebesgue lemma, we have that

$$\lim_{x\in\Theta\&|x|\to\infty}\int_D e^{A(t,iq,0)+\langle i\Gamma(t)q,x\rangle}\tilde{g}(q)dq = 0.$$

It is also clear that

$$\lim_{x\in D\setminus\Theta\&|x|\to\infty} e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle+x'C(t,iq,0)x} = 0, \quad \text{for } \forall\ q\in D.$$

Therefore we have shown that

$$P_t g \in C_0(D) \text{ for each } g \in \mathcal{L}(\mathcal{T}_0). \tag{A.31}$$

From Duffie, Filipović and Schachermayer (2002 [3], Lemma 8.4), we can extend (A.5) and (A.31) to all the functions $g \in C_0(D)$. Thus by Revuz and Yor (1994 [12], Proposition III 2.4), we have proved that X is a Feller process.

Because X is a Feller process, by Sato(1999 [15], Lemma 31.7), in order to finish the proof, it is enough to prove that (A.30) is true for all $g \in C_c^2(D)$. Since (A.30) holds for $g \in \mathcal{L}(\mathcal{T}_0)$, and by the closeness of $\mathcal{A}$, we have $\mathcal{L}(\mathcal{T}) \subset \mathcal{D}(\mathcal{A})$ and (A.30) holds for $g \in \mathcal{L}(\mathcal{T})$. By Duffie, Filipović and Schachermayer (2002 [3], Lemma 8.4), it is easy to see that (4.10) holds for all $g \in C_c^2(D)$. This completes the proof of the first part of the theorem.

It is easy to prove the converse part of the theorem. Given an admissible parameter set $(\alpha,\beta,b,\gamma,\delta,\Phi)$, there exist unique continuous solutions for $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ for (4.4) to (4.6), which means that we have well defined (3.2). Because the law is unique under Definition 3.1, the proof of Theorem 4.1 is complete.

A.6. **Proof of Proposition 5.1.** By the Markov property it is straightforward to prove that Q_t satisfy the Chapman-Kolmogorov equation such that, for any $f \in B(D)$,

$$Q_{t+s}f(x) = Q_tQ_sf(x).$$

and since $Q_0 f = f$ and $Q_t 1 \in [0,1]$, we proved that (Q_t) is a positive contraction semigroup on $B(D)$. Now since X is a Feller process and thus admits a cadlag modification, then by the definition of Q_t, we know that $(t,x) \mapsto Q_tf(x)$ is measurable with respect to $\mathbb{R}_+ \times D$. Therefore as shown in Rogers and Williams (1994 [13], III.2), for every $\lambda > 0$, the resolvent of (Q_t),

$$R_\lambda^Q g(x) = \int_{\mathbb{R}_+} e^{-\lambda t}Q_tg(x)dt,$$

is a linear operator mapping: $B(D) \mapsto B(D)$. The following deductions proceed the same as an analogous argument in Rogers and Williams (1994 [13]):

$$\begin{aligned} & R_\lambda g(x) - R_\lambda^Q g(x) \\ = \ & \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_t) \left(1 - e^{-\int_0^t r(X_s)ds}\right) dt \right] \\ = \ & \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_t) \left(\int_0^t r(X_u) e^{-\int_0^u r(X_s)ds} du \right) dt \right] \\ = \ & \mathbb{E}_x \left[\int_{\mathbb{R}_+} r(X_u) e^{-\int_0^u r(X_s)ds} \left(\int_{\mathbb{R}_+} e^{-\lambda(t+u)} P_{t+u} g(X_{t+u}) dt \right) du \right] \\ = \ & \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda u} r(X_u) e^{-\int_0^u r(X_s)ds} \left(\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_u) dt \right) du \right] \\ = \ & \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda u} e^{-\int_0^u r(X_s)ds} r(X_u) R_\lambda g \right] \\ = \ & R_\lambda^Q (r(x) R_\lambda g). \end{aligned} \tag{A.32}$$

By Revuz and Yor (1994 [12], Proposition VII 1.4), let $f \in C_c^2(D) \subset \mathcal{D}(\mathcal{A})$, there exists a unique $g \in C_0(D)$ such that $R_\lambda g = f$, therefore $g = (\lambda I - \mathcal{A})f$. Then by (A.32), we derive that

$$f = R_\lambda^Q (g(x) + r(x) f(x)). \tag{A.33}$$

Since $g + rf \in C_0(D)$, $f \in \mathcal{D}(\mathcal{A}_Q)$ and thus $C_c^2(D) \subset \mathcal{D}(\mathcal{A}_Q)$. Now by (A.33), we have

$$(\lambda I - \mathcal{A}_Q) f = g + rf = (\lambda I - \mathcal{A}) f + rf,$$

where $\mathcal{A}_Q$ denote the infinitesimal generator of the semigroup (Q_t). Hence

$$\mathcal{A}_Q f(x) = \mathcal{A} f(x) - r(x) f(x), \quad f \in C_c^2(D).$$

By using Theorem 4.1, we finish the proof.

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Table 1. Summary of Notation

Notations	Implications
X	A time-homogeneous Markov process
D	The state space: $D := \mathbb{R}^d$
D_Δ	One point compactification of D; i.e., $D_\Delta := D \cup \{\Delta\}$
$(\mathcal{F}_t)_{t\in\mathbb{R}_+}$	The natural filtration generated by X; i.e., $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$, for each $t \in \mathbb{R}_+$
$\mathcal{F}_\infty$	The σ-algebra $\bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$
$C(D)$	The Banach space of continuous functions on D
$B(D)$	The Banach space of bounded complex-valued Borel-measurable functions on D
$C_b(D)$	The Banach space $C(D) \bigcap B(D)$
$C_0(D)$	The Banach space consisting of elements of $C(D)$ that vanishes at infinity
$C_c(D)$	The Banach space consisting of elements of $C(D)$ having a compact support
$C^k(D)$	The space of k-times differentiable functions f on the interior of D such that all partial derivatives of f up to order k are continuous
$C^\infty(D)$	The space $\bigcap_{k\in\mathbb{N}} C^k(D)$
∇f	The gradient of the function f on D
e_i	The ith standard orthonormal basis vector on $\mathbb{R}^d$
$i\mathbb{R}^d$	The set $\{i\eta : \eta \in \mathbb{R}^d\}$
$\langle \alpha, \beta \rangle$	The inner product on $\mathbb{C}^d$
Sem^d_{--}, Sem^d_{-}	The collections of $d \times d$ negative and semi-negative definite matrices, respectively
Sem^d_{++}, Sem^d_{+}	The collections of $d \times d$ positive and semi-positive definite matrices, respectively
∂Sem^d_+, ∂Sem^d_-	The sets given by $Sem^d_+ \backslash Sem^d_{++}$ and $Sem^d_- \backslash Sem^d_{--}$, respectively
$\mathcal{A}$	The infinitesimal generator of X
$\mathcal{D}(\mathcal{A})$	The space that contains all the functions f such that $\mathcal{A}f$ exists
$A \oplus B$	The set $\{x + y : x \in A, y \in B\}$.

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