Projecting the Forward Rate Flow on a Finite Dimensional Manifold

Erhan Bayraktar * Li Chen † H. Vincent Poor ‡

Princeton University

Abstract

Given an Heath-Jarrow-Morton (HJM) interest rate model $\mathcal{M}$ and a parametrized family of finite dimensional forward rate curves $\mathcal{G}$, this paper provides us a way to project this infinite dimensional HJM forward rate curve r_t to the finite dimensional manifold $\mathcal{G}$. This projection characterizes banks' behavior of calibrating forward curves by applying a certain family of curves (e.g., Nelson-Seigel family). Moreover, we derive the Stratonovich dynamics of the projected finite dimensional forward curve. This leads an implicit algorithm for parametric estimation of the original HJM model. We have demonstrated the feasibility of this method by applying generalized method of moments and methods of simulated moments.

1 Introduction

We consider our problem in a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual condition and $\mathbb{P}$ is the equivalent martingale measure. We consider an HJM model $\mathcal{M}$ (1992, [7]). With Musiela parameterization, the dynamics of the forward rate curve under $\mathbb{P}$ can be given by the following infinite dimensional stochastic differential equation (SDE).

$$dr_t(x) = \tilde{\mu}(r_t, x)dt + \sigma(r_t, x)dW_t, \tag{1}$$

where W is a m-dimensional standard $\mathbb{P}$-Brownian motion. From the arbitrage-free condition, it follows that

$$\tilde{\mu}(r_t, x) := \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, u)du.$$

The main problems of interest are

*Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, ebayrak@ee.princeton.edu

†Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, lichen@princeton.edu

‡Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, poor@princeton.edu

- Under what condition does the infinite dimensional forward rate curve $r_t(\cdot)$ defined in (1) admits a finite-dimensional realization?
- Under what condition is this forward curve consistent with a given family of finite-dimensional curves?

Actually these two problems are highly correlated. Björk *et.al* [2], [1] give the necessary and sufficient conditions for the consistency problem, as well as the existence of a finite-dimensional realization, in terms of the volatility structure, under the assumption of arbitrary smoothness of the volatility function. Their results are extended by Filipović and Teichmann [5]. Moreover, in [4], Filipović shows that there does not exist a nontrivial forward rate model consistent with Nelson-Siegel family which has widely been used in calibrating forward curves. Technically speaking, it means that starting from Nelson-Siegel (NS) manifold, any forward rate flow will leave this manifold immediately.

However it is of interest to find the projection of an infinite dimensional forward rate curve onto a finite dimensional manifold and obtain the corresponding dynamics of coordinates. This particularly characterizes banks' behavior, since most banks construct forward rate curves from cross sectional market forward rate data by interpolation using Nelson-Siegel curves. Since Nelson-Siegel family of curves is a 4-dimensional manifold, therefore deriving the dynamics of the coordinates obtained by projecting the forward rate curve can be interpreted as the construction behavior of banks.

2 Projection

We will use a weighted Sobolev space as the space of forward rate curves. For any $\gamma > 0$ we define

$$\mathcal{H}_\gamma := \left\{ r : \mathbb{R}_+ \to \mathbb{R} : \|\mathrm{r}\|_\gamma^2 := \int_0^\infty \left(\mathrm{r}^2(\mathrm{x}) + \left(\frac{\mathrm{dr}}{\mathrm{dx}}(\mathrm{x}) \right)^2 \right) \mathrm{e}^{-\gamma \mathrm{x}} \mathrm{dx} < \infty \right\} \qquad (2)$$

With the inner product

$$\langle r, q \rangle := \int_0^\infty r(x) q(x) e^{-\gamma x} dx + \int_0^\infty \left(\frac{dr}{dx}(x) \right) \left(\frac{dq}{dx}(x) \right) e^{-\gamma x} dx, \qquad (3)$$

$\mathcal{H}_\gamma$ becomes a Hilbert space. The exponential weighting in the definition of H_γ is to make constant curves a subset of this space.

Let $G := \{G(\vec{z}, x), \vec{z} = (z^1, ..., z^n) \subset \mathbb{Z} \in \mathbb{R}^n\}$ be a family of forward rate curves. $\mathbb{Z}$ is called the *parameter space*. Consider a curve β in this manifold of the form $\beta : h \to G(z(h), \cdot)$, where $h \to z(h)$ is a curve in $\mathbb{R}^n$. Then, according to the chain rule, we compute the following Frechét derivative:

$$D\beta(0) = DG(z(h), \cdot)|_{h=0} = \sum_{k=1}^n \frac{\partial G(z, .)}{\partial z^k} \dot{z}^k(0), \qquad (4)$$

where the Frechret derivative is defined by

$$\lim_{|h|\to 0} \frac{\|\beta(h) - \beta(0) - hD\beta(0)\|}{|h|} = 0. \tag{5}$$

The tangent space for a point in this manifold is then obtained by considering all the curves passing through this given point and considering the tangent vectors. However it is clear that the tangent vector space is given as

$$\mathcal{S} = span\{\frac{\partial G(z,\cdot)}{\partial z^1}, ..., \frac{\partial G(z,\cdot)}{\partial z^n}\}. \tag{6}$$

Let's denote these spanning vectors by $w_1, ..., w_n$.

Let us now consider the orthogonal projection between any linear space V containing $\mathcal{S}$ (finite dimensional) and the tangent vector space $\mathcal{S}$ itself. Since the basis of $\mathcal{S}$ is not necessarily orthogonal, we have the following projection formula $\Pi_z : V \mapsto span\{w_1, ..., w_n\}$,

$$v \to \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \bar{\lambda}_{ij} \langle v, w_j \rangle \right] w_i, \tag{7}$$

where $(\bar{\lambda}_{ij}) := \Lambda^{-1}$ with $\Lambda := (\langle w_i, w_j \rangle)$.

Now let us consider the forward rate dynamics in Musiela parametrization which we will project this flow on the G-manifold.

$$dr_t(x) = \left(\frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, x)^T du \right) dt + \sigma(r_t, x) dW_t. \tag{8}$$

Using infinite dimensional Itô formula ([3]) (8)can be written in Stratonovich form as

$$dr_t(x) = \left(\frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, x)^T du - \frac{1}{2}[\sigma_r'(r_t)\sigma(r_t)](x) \right) dt + \sigma(r_t, x) \circ dW_t, \tag{9}$$

where

$$\sigma_r'(r_t)\sigma(r_t) = \sum_{i=1}^{m} \sigma_{ir}'(r_t)\sigma_i(r_t), \tag{10}$$

where m is the dimension of W. σ_r' denotes the Frechrét derivative of the vector field σ with respect to the infinite dimensional variable r. Let us denote

$$\mu(r_t) := \frac{\partial}{\partial x} r_t + \sigma(r_t) \int_0^x \sigma(r_t)^T du - \frac{1}{2}\sigma_r'(r_t)\sigma(r_t). \tag{11}$$

The projected flow on the manifold G of the forward rate is then given by

$$dG(\vec{z}_t, \cdot) = \Pi_{\vec{z}_t} \left[\mu(G(\vec{z}_t, \cdot)) \right] dt + \Pi_{\vec{z}_t} \left[\sigma(G(\vec{z}_t, \cdot)) \right] \circ dW_t. \tag{12}$$

3 The Dynamics of Finite Dimensional Manifolds

In order to derive the diffusion process for $\vec{z}$, first we give the following lemma.

Lemma 3.1 *For any continuous semimartingales X_t and Y_t such that $[X]_T < \infty$, and $[Y]_T < \infty$ is finite for each $T > 0$. Then for $\forall\ T > 0$, we have*

$$\int_0^T (X_t Y_t) \circ dW_t = \int_0^T X_t \circ (Y_t \circ dW_t), \quad a.s.. \tag{13}$$

Proof. From the definitions of Stratonovich's integral and variation, it is sufficient to prove that

$$\lim_{||\Psi|| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) = 0, \quad \text{a.s.}, \tag{14}$$

where $\Psi = \{t_0, t_1, ..., t_n\}$ is a partition of $[0, T]$ and $||\Psi||$ is the mesh of this partition, namely, $||\Lambda|| = \max_{1 \le i \le n}\{|t_i - t_{i-1}|\}$. Since for any Λ, we have

$$\begin{aligned} \left| \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \right| &\le \max_{1 \le i \le n} |W_{t_i} - W_{t_{i-1}}| \times \\ & \sqrt{\sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2} \sqrt{\sum_{i=1}^{n} \left(Y_{t_i} - Y_{t_{i-1}}\right)^2}, \end{aligned}$$

by the uniform continuity of the Brownian motion W on the compact support $[0, T]$ and the finiteness of $[X]_T$, and $[Y]_T$, it follows that (14) is true. Therefore we proved Lemma 3.1.

By applying Stratonovic chain rule, we can derive that

$$dG(\vec{z}_t, \cdot) = \sum_{i=1}^{n} \frac{\partial G(\vec{z}_t, \cdot)}{\partial z^i} \circ dz_t^i. \tag{15}$$

On the other hand, by (7) and (12), we also have

$$\begin{aligned} dG(\vec{z}_t, \cdot) &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \bar{\lambda}_{ij} \langle \mu(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z^j} \rangle \right] \frac{\partial G(\vec{z}_t, \cdot)}{\partial z^i} dt \\ &+ \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \bar{\lambda}_{ij} \langle \sigma(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z^j} \rangle \right] \frac{\partial G(\vec{z}_t, \cdot)}{\partial z^i} \circ dW_t. \end{aligned} \tag{16}$$

By comparing (15) and (16) and using Lemma 3.1, we conclude that the finite dimensional vector $\vec{z}$ follows a diffusion process:

$$\begin{aligned} dz_t^i &= \left[\sum_{j=1}^n \bar{\lambda}_{ij} \langle \mu(G(\vec{z}_t,\cdot)), \frac{\partial G(\vec{z}_t,\cdot)}{\partial z^j} \rangle\right] dt \\ &+ \left[\sum_{j=1}^n \bar{\lambda}_{ij} \langle \sigma(G(\vec{z}_t,\cdot)), \frac{\partial G(\vec{z}_t,\cdot)}{\partial z^j} \rangle\right] \circ dW_t, \quad \forall\, i \in \{1,...,n\}. \quad (17) \end{aligned}$$

3.1 Nelson Seigel Family

For Nelson Seigel family, the dimension of the vector $\vec{z}$ is 4 and the corresponding $G(\vec{z},\cdot)$ can be written as

$$G(\vec{z},x) = z_1 + (z_2 + z_3 x)e^{-z_4 x}, \quad \forall\, x > 0.$$

For $z \neq 0$ the Frechét derivative with respect to z is

$$\partial_{\vec{z}} G(\vec{z},x) = (1 \quad e^{-z_4 x} \quad xe^{-z_4 x} \quad -z_4(z_2 + z_3 x)e^{-z_4 x})^T, \tag{18}$$

and with respect to x is

$$\partial_x G(\vec{z},x) = [z_3 - z_4(z_2 + z_3 x)]e^{-z_4 x}.$$

Note that for the image of these maps to be in $\mathbb{H}_\gamma$ $z_4 > -\gamma/2$. Therefore we choose our parameter space $\mathcal{Z} = \{z \in \mathrm{IR}^4 : \mathrm{z}_4 \neq 0, \mathrm{z}_4 > -\gamma/2\}$.

Now we choose $\sigma(r_t, x) = \sigma$ for the HJM model, then by (17), we can obtain the dynamics of $\vec{z}$ which is a projection of this forward rate curve as follows.

Since $\sigma(r_t, x)$ is a constant, therefore by (11), we can derive that

$$\mu(G(\vec{z},x)) = [z_3 - z_4(z_2 + z_3 x)]e^{-z_4 x} + \sigma^2 x.$$

Using the inner product defined in (3), it is straightforward to obtain the dynamics of the projected multi-factor $\vec{z}$. Because it is tedious, we include it in Appendix.

4 Parametric Estimation of the Original HJM Models

In this section, we discuss parametric estimations of the original HJM model $\mathcal{M}$ by using its projection on some finite dimension manifold $\mathcal{G}$. Generally speaking, estimating parameters of an infinite dimensional SDE is usually hard to implement. However, since we have already derived the SDE (17) governing the dynamics of its projected finite dimensional vector $\vec{z}$, this provides us an much easier way to empirically investigate of the HJM original model.

Suppose θ is the parameter set in the HJM model and thus by projecting this forward curve into a finite dimensional manifold G, we have the following general form for the diffusion process of $\vec{z}$.

$$d\vec{z}_t = A(\vec{z}_t, \theta) + B(\vec{z}_t, \theta) \circ dW_t, \tag{19}$$

where W is a m-dimensional standard Brownian motion.

Here we apply a generalized method of moments (GMM) proposed by Hansen (1982 [6]) to estimating θ. For banks, periodic calibrating the initial forward curve produces a time series $\{\vec{z}_{t_k}\}_{1\leq k\leq N}$. Assume that $\Delta = t_{k+1} - t_k$, for each $k = 1, ..., N$. By discretizing (19), we obtain a discrete-time model:

$$\vec{z}_{t_{k+1}} - \vec{z}_{t_k} = A(\vec{z}_{t_k}, \theta)\Delta + \frac{1}{2}\left[B(\vec{z}_{t_{k+1}}, \theta) + B(\vec{z}_{t_k}, \theta)\right]\epsilon_k, \quad \forall\ k = 1, 2, ..., N, \tag{20}$$

where ϵ_k is a m-dimensional Gaussian random vector with mean 0 and covariance matrix ΔI, i.e., $\{\epsilon_k\}_{1\leq k\leq N}$ are mutually independent.

Now we construct the moment functions $h_k(\theta)$ as follows:

$$h_k(\theta) = \vec{z}_{t_{k+1}} - \vec{z}_{t_k} - A(\vec{z}_{t_k}, \theta)\Delta, \tag{21}$$

and denote the sample average by $f_N(\theta) := \frac{1}{N}\sum_{k=1}^{N} h_k(\theta)$, then by simply letting

$$\hat{\theta}_N = \min_{\theta}\{\langle f_N(\theta), f_N(\theta)\rangle\}, \tag{22}$$

we obtain the least square estimator of θ. Under fairly general conditions (see [6]), the estimator $\hat{\theta}_N$ offers a consistent estimator of θ_0.

Remark 4.1 *If the dimension of θ is high, it is straightforward to strengthen this algorithm by adding more moment functions.*

As argued by Hansen in [6], the estimator (22) is generally not efficient as far as its convergence rate is concerned. The least squares estimator of (22) can be improved by taking a weighted least squares estimator. Suppose evaluated at the true value $\{h_k(\theta_0)\}$ is strictly stationary, then define

$$\Gamma_\nu := E\left\{h_k(\theta_0)h_{k-\nu}(\theta_0)'\right\}. \tag{23}$$

Assuming these quantities are absolutely summable, we define

$$S := \sum_{\nu=-\infty}^{\infty} \Gamma_\nu. \tag{24}$$

The optimal GMM estimator is given by

$$\hat{\theta}^*_N = \min_{\theta}\{f_N(\theta)S^{-1}f^T_N(\theta)\}, \tag{25}$$

(25) needs an initial estimate of S, which calls for an initial estimate θ. The initial estimate of θ is given by (22), which can be used to calculate an estimate for S as follows([8]):

$$S_N = \hat{\Gamma}_{0,N} + \sum_{\nu=1}^{q} (1 - (\nu/(q+1))) \left(\hat{\Gamma}_{\nu,N} + \hat{\Gamma}_{\nu,N} \right) \tag{26}$$

where

$$\hat{\Gamma}_{\nu,N} = \frac{1}{N} \sum_{n=\nu+1}^{N} h_n(\hat{\theta}) h_{n-\nu}(\hat{\theta})' \tag{27}$$

where $\hat{\theta}$ is the estimator obtained using (22). This estimate of S is then used to compute an estimator of θ using 25. This recursion can be carried until the estimator becomes stable.

5 Appendix

5.1 The Results of Projected Nelson Siegal Dynamics

First, we have

$$\begin{aligned}
\langle \mu(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z_1} \rangle &= \frac{\gamma z_3 - z_2 z_4 (\gamma + z_4)}{(\gamma + z_4)^2} + \frac{\sigma^2}{\gamma^2}, \\
\langle \mu(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z_2} \rangle &= \frac{\sigma^2}{(\gamma + z_4)^2} + \frac{z_3}{\gamma + 2z_4} + \frac{2 z_3 z_4^2}{\gamma + 2z_4} \\
&\quad + z_4 \left(-\frac{\sigma^2}{\gamma + z_4} - \frac{z_3}{(\gamma + 2z_4)^2} - \frac{z_2}{\gamma + 2z_4} \right) - \frac{z_4^3 (z_3 + z_2(\gamma + 2z_4))}{(\gamma + 2z_4)^2}, \\
\langle \mu(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z_3} \rangle &= \frac{z_3}{(\gamma + 2z_4)^2} + \frac{\sigma^2 (2 + (\gamma + z_4)^2)}{(\gamma + z_4)^3} \\
&\quad - \frac{z_4^3 (2z_3 + z_2(\gamma + 2z_4))}{(\gamma + 2z_4)^3} + \frac{z_4^2 (3z_3 + z_2(\gamma + 2z_4))}{(\gamma + 2z_4)^2} \\
&\quad + z_4 \left(-\frac{\sigma^2}{(\gamma + z_4)^2} - \frac{z_2}{(\gamma + 2z_4)^2} + z_3 \left(-\frac{2}{(\gamma + 2z_4)^3} - \frac{2}{\gamma + 2z_4} \right) \right), \\
\langle \mu(G(\vec{z}_t, \cdot)), \frac{\partial G(\vec{z}_t, \cdot)}{\partial z_4} \rangle &= z_4 \left[\frac{z_3^2 (-\gamma + 2\gamma^2 z_4 + 5\gamma z_4^2 + 4z_4^3)}{(\gamma + 2z_4)^3} \right. \\
&\quad - \frac{z_3 (\sigma^2 (\gamma + 2z_4)^2 (2 + \gamma^2 + \gamma z_4) + z_2 (\gamma + z_4)^3 (\gamma + 3\gamma z_4^2 + 4z_4^3))}{(\gamma + z_4)^3 (\gamma + 2z_4)^2} \\
&\quad \left. + z_2 \left(-\frac{\sigma^2}{(\gamma + z_4)^2} + \frac{z_2 z_4^3}{\gamma + 2z_4} + z_4 \left(\frac{\sigma^2}{\gamma + z_4} + \frac{z_2}{\gamma + 2z_4} \right) \right) \right],
\end{aligned}$$

and

$$\begin{aligned}
\langle\sigma(G(\vec{z}_t,\cdot)),\frac{\partial G(\vec{z}_t,\cdot)}{\partial z_1}\rangle &= \frac{\sigma}{\gamma},\\
\langle\sigma(G(\vec{z}_t,\cdot)),\frac{\partial G(\vec{z}_t,\cdot)}{\partial z_2}\rangle &= \frac{\sigma}{\gamma+z_4},\\
\langle\sigma(G(\vec{z}_t,\cdot)),\frac{\partial G(\vec{z}_t,\cdot)}{\partial z_3}\rangle &= \frac{\sigma}{(\gamma+z_4)^2},\\
\langle\sigma(G(\vec{z}_t,\cdot)),\frac{\partial G(\vec{z}_t,\cdot)}{\partial z_4}\rangle &= -\frac{\sigma(z_3+z_2(\gamma+z_4))}{(\gamma+z_4)^2}.
\end{aligned}$$

We also can derive the symmetric matrix W.

$$\begin{aligned}
W_{11} &= \frac{1}{\gamma},\\
W_{12} &= \frac{1}{\gamma+z_4},\\
W_{13} &= \frac{1}{(\gamma+z_4)^2},\\
W_{14} &= -z_4\frac{z_3+z_2(\gamma+z_4)}{(\gamma+z_4)^2},\\
W_{22} &= \frac{1+z_4^2}{\gamma+2z_4},\\
W_{23} &= \frac{-1+\gamma z_4+z_4^2}{(\gamma+2z_4)^2},\\
W_{24} &= -z_4\frac{z_3+z_2(\gamma+2z_4)}{(\gamma+2z_4)^2}-\frac{z_4^2(-z_3(\gamma+z_4)+z_2z_4(\gamma+2z_4))}{(\gamma+2z_4)^2},\\
W_{33} &= \frac{2+\gamma^2+2\gamma z_4+2z_4^2}{(\gamma+2z_4)^3},\\
W_{34} &= \frac{z_4(-z_3(2+\gamma^2+2\gamma z_4+2z_4^2)+z_2(-\gamma+(-2+\gamma^2)z_4+3\gamma z_4^2+2z_4^3))}{(\gamma+2z_4)^3},\\
W_{44} &= \frac{1}{(\gamma+2z_4)^3}(z_4^2(z_2^2(\gamma+2z_4)^2(1+z_4^2)+z_3^2(2+\gamma^2+2\gamma z_4+2z_4^2)\\
&\quad -2z_2z_3(-\gamma+(-2+\gamma^2)z_4+3\gamma z_4^2+2z_4^3))).
\end{aligned}$$

References

[1] Björk, T. and B. J. Christensen (1999): Interest Dynamics and Consistent Forward Rate Curves, *Mathematical Finance*, 9, 4, 323-348.

[2] Björk, T. and L. Svensson (2001): On the existence of Finite Dimensional Nonlinear Realizations of Interest Rate Models, *Mathematical Finance*, 205-243.

[3] Da Prato, G. and J. Zabczyk (1992): *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

[4] Filipoviĉ, D. (1999) : A Note on the Nelson-Siegel Family, *Mathematical Finance*, 9, 349-359.

[5] Filipoviĉ, D. and J. Teichman (2002): Existence of Invariant Manifolds for Stochastic Equations in Infinite Dimension, *forthcoming in Journal of Functional Analysis.*

[6] Hansen, L. P. (1982): Large Sample Properties of Generalized Method of Moments Estimators. *Econometrica* 50(4), 1029-1054.

[7] Heath, D., R. Jarrow and A. Morton (1992): Bond Pricing and the Term Structure of interest rates, *Econometrica 60*, 1, 77-106.

[8] Newey, W. K. and K. D. West(1985): A Simple Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix , *Econometrica*, 55, 703-708.