Modeling Credit Risk by Affine Processes

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Abstract

In this paper, the treasury rates and the credit migrations are jointly modeled by multi-dimensional affine processes. In order to capture the entire information, including credit migrations and default events, we construct non-conservative regular affine processes to model credit migrations and characterize the default by the death of the processes. In particular, two specific cases: purely jump affine models and affine diffusion models with potentials, are discussed. This affine approach not only produces the explicit formulas for the prices of corporate bonds and other credit derivatives, but also directly incorporates the credit rating information as a parameter into the pricing formulas. Moreover, our affine models allow to consider the joint credit migrations within an analytically tractable framework in order to capture the correlations of credit movements between firms. Finally, the empirical testing results of a simple affine model are presented to support the effectiveness of our models.

1 Introduction

The class of regular affine processes Y defined in the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ is a particular class of Markov processes which the following condition holds:

$$\mathbb{E}_y[e^{\langle u, Y_t \rangle}] = e^{\phi(t,u) + \langle \psi(t,u), y \rangle}, \quad t \in \mathbb{R}_+,\ u \in \mathbb{C}_-^m \times i\mathbb{R}^n, \tag{1.1}$$

where the coefficient functions ϕ and ψ can be determined from generalized Riccati equations (see Theorem 2.7 in Duffie, Filipović and Schachermayer, 2002 [4]). Since regular affine processes include continuous-state branching processes with immigration (CBI) and the Ornstein-Uhlenbeck (OU) type processes, therefore this class of processes has already been widely used in modeling the term structure of interest rates. For example, a dominant class of models, namely, affine term structure models (including jump diffusion models) specifies state processes as regular conservative affine processes and define the short rate as an affine function of state variables. Because of the affine property (1.1), the models become quite tractable, since they can produce nice pricing formulas for treasury bonds and even price European options by using transform analysis (see Duffie, Pan and Singleton 2000 [5]).

Besides these applications in risk-free rate modeling, affine processes can also be used in modeling credit risk. In Duffie, Filipović and Schachermayer (2002 [4]), the authors discussed that by adopting the doubly stochastic setup, the affine processes can also be applied to modeling default intensities. Therefore pricing defaultable bonds can be treated in the same manner as pricing treasury bonds. Furthermore, Filipović (2002 [7]) has constructed a new type of affine processes with one branch as simple point processes to model credit events. It turned out that the intensity based models can be embedded in this setup. Moreover, this "integrated" approach can be easily extended to consider multiple credit events in order to capture the correlations of default risk of firms.

However, all the above models only adopt conservative affine processes. In this paper, we will apply non-conservative affine processes to modeling credit risk. The non-conservativity allows us to characterize a credit event by the death of a process. Actually, in Chen and Poor (2002, [1]), the authors proposed to use a non-conservative regular quadratic Gaussian process to model the default intensity. But since there is no corresponding entity in the market for the default intensity, it is not tractable to model the intensity directly. Instead, here we will model credit migrations by positive regular affine processes. We have demonstrated that we can not only obtain explicit formulas for corporate bond prices and other credit derivatives, but also incorporate the credit rating information as a parameter into the pricing formula. Moreover, it is also straightforward to extend our affine approach to the higher dimension so that it allows to consider the joint credit migrations of several firms within an analytically tractable framework. It is worth mentioning that our model of credit migrations is different from the credit class model proposed by Jarrow, Lando and Turnbull (1997 [8]) in the way that the credit migration here is modeled as a real valued affine process, not a finite state Markov process proposed in [8]. This new approach admits a slight credit difference between firms in the same rating class, which seems closer to reality. On the other hand, in order to be consistent with the rating classes given by Moody's, the typical value region of the credit rating will be given for each investment grade class in Section 4.

The remainder of the paper is organized as follows. In Section 2, we provide a general affine framework of modeling credit risk. In particular, two specific cases: purely jump affine models and affine diffusion models with potentials, are discussed and corporate bond prices under three different recovery assumptions: zero recovery, recovery at maturity and recovery at default, are derived in both cases. In Section 3, we will discuss the measure change for non-conservative affine processes in order to calculate the default probability under the physical probability measure. Finally, we present the empirical testing results of a simple affine model in Section 4 to support the effectiveness of our new approach.

2 A General Affine Framework of Modeling Credit Risk

For the theory and notation of affine processes we refer to Duffie, Filipović and Schachermayer (2002 [4]). Consider a two dimensional non-conservative adapted regular affine process $Y = (Y^1, Y^2)$ in the state space $D := \mathbb{R}_+^2$ on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_y)$ satisfying the usual conditions, for all $y \in D$. Let D_Δ be the one-point compactication of D. Suppose that the state process Y governs the economy and $\mathbb{P}_y$ is the risk-neutral measure. We define Y^2 as a credit migration process. In order to model credit events, it is assumed that the default happens if and only if the affine process Y dies. Let τ be the default time, therefore we have

$$\tau = \inf \{t > 0 : Y_t = \Delta\},$$

which is a stopping time with respect to $(\mathcal{F}_t)$.

Now we define the short rate process r as follows.

$$r_t = Y_t^1, \quad \forall t < \tau.$$

Since it is not desirable if the economy breaks down at the default of a single firm, the "life" of the short rate process r should be extended after τ, and it is demonstrated that in most situations, r can be consistently extended. (see Lemma 2.2 and Lemma 2.3)

For the affine process Y in D, there exist some admissible parameters $(\alpha, b, \beta, c, \gamma, m, \mu)$, such that for all $f \in C_c^2(D)$, the infinitesimal generator of Y has the following generic form for all $y = (y_1, y_2) \in D$:

$$\begin{aligned} \mathcal{A}f(y) &= \alpha_1 y_1 \partial^2_{y_1} f(y) + \alpha_2 y_2 \partial^2_{y_2} f(y) + \langle b + \beta y, \nabla f(y) \rangle \\ &\quad -(c + \langle \gamma, y \rangle) f(y) + \int_{D \setminus \{0\}} (f(y+\xi) - f(y)) m(d\xi) \\ &\quad + \sum_{i=1}^{2} \int_{D \setminus \{0\}} (f(y+\xi) - f(y)) y_i \mu_i(d\xi). \end{aligned} \tag{2.1}$$

Under the above setup, the short rate process and credit migrations have been jointly modeled by affine processes. Moreover, as shown in Table 1, every parameter in (2.1) has its own implication to the model.

Table 1: Implications of Parameters

Parameters	Implication
α_1	The diffusion of the short rate process Y^1
α_2	The diffusion of the credit migration process Y^2
$\frac{b_1}{\lvert\beta_{11}\rvert}$	The constant mean level of short rate Y^1
β_{11}	The mean reversion rate of short rate process Y^1
$\frac{b_2}{\lvert\beta_{22}\rvert}$	The constant mean level of the credit rating Y^2
β_{22}	The mean reversion rate of the credit migration process Y^2
$\frac{\beta_{12}}{\lvert\beta_{11}\rvert} Y^2$	The mean level of the short rate Y^1 impacted by the credit migration Y^2
$\frac{\beta_{21}}{\lvert\beta_{22}\rvert} Y^1$	The mean level of the credit rating Y^2 impacted by the short rate Y^1
$c + \langle \gamma, Y \rangle$	The default intensity
m	The constant jump measure of the joint process Y
μ_1	The jump measure of Y impacted by the short rate Y^1
μ_2	The jump measure of Y impacted by the credit rating Y^2

Remark 2.1 *It is straightforward to extend this simple approach to higher dimensional affine processes* $Y = (Y_1, ..., Y_m, Z_1, ..., Z_n)$ *with the state space* $D := \mathbb{R}^m_+ \times \mathbb{R}^n_+$. *One can construct* $(Y_1, ..., Y_m)$ *as an* m*-dimensional affine model for risk-free rates and use* $(Z_1, ..., Z_N)$ *modeling the joint credit migrations for* N *different firms in order to capture the correlations of credit movements between them.*

Since we model the default event as the death of state process Y, it follows from (2.1) that there are two possible causes: a sudden death of the process Y killed by the potential or an explosion of Y aroused by big jumps. Therefore, we now further differentiate these two cases by specifying the credit migration Y^2 either as a purely jump affine process or as a diffusion process with potential, and treat them separately.

2.1 Modeling Credit Migrations As Purely Jump Processes

In this part, it is assumed that $\alpha_2 = 0$, $c = 0$, $\gamma = 0$, $\beta_{12} = 0$ and the supports of m, μ_1 and μ_2 lie in

$$\mathcal{S} := \{(y_1, y_2) \in D\backslash\{0\} : y_1 \in \{0\} \text{ and } y_2 \in \mathbb{R}_+\},$$

which means that we can rewrite $m(d\xi)$, $\mu_1(d\xi)$ and $\mu_2(d\xi)$ as $\delta_0(\xi_1)\tilde{m}(d\xi_2)$, $\delta_0(\xi_1)\tilde{\mu_1}(d\xi_2)$ and $\delta_0(\xi_1)\tilde{\mu_2}(d\xi_2)$, respectively.

This setup gives us a purely jump affine process, and the jumps only occur in the credit migration Y^2. Moreover the short rate process Y^1 would not be influenced by Y^2. Therefore the death of Y is caused by the explosion of Y^2.

Remark 2.2 *According to Lemma 9.2 in Duffie, Filipović and Schachermayer (2002 [4]), in order to retain the non-conservativity of a purely jump affine process. One of the jump measures m, μ_1 and μ_2 should necessarily satisfy the following condition:*

$$\int_{\mathbb{R}_+\backslash\{0\}} (\xi \wedge \xi^2)\mu(d\xi) = \infty. \tag{2.2}$$

Therefore for all $f(y) \in C_c^2(D)$, we must have

$$\begin{aligned}\mathcal{A}f(y) \quad = \quad & \alpha_1 y_1 \partial_{y_1}^2 f(y) + \alpha_2 y_2 \partial_{y_2}^2 f(y) + (b_1 + \beta_{11} y_1)\partial_{y_1} f(y) \\ & + (b_2 + \beta_{21} y_1 + \beta_{22} y_2)\partial_{y_2} f(y), \\ & + \int_{D\backslash\{0\}} (f(y+\xi) - f(y)) m(d\xi) \\ & + \sum_{i=1}^{2} \int_{D\backslash\{0\}} (f(y+\xi) - f(y)) y_i \mu_i(d\xi).\end{aligned} \tag{2.3}$$

Lemma 2.1 *If the infinitesimal generator of an affine process $Y = (Y^1, Y^2)$ is defined by (2.3), then the process Y_t^1 before time τ can be driven by the following diffusion process with the initial value y_1.*

$$dY_t^1 = (b_1 - \beta_{11} Y_t^1)dt + \sqrt{2\alpha_1 Y_t^1} dW_t, \quad t < \tau, \tag{2.4}$$

where W is a Brownian motion independent of Y^2.

Proof. Since for $\forall s_1, s_2 \in \mathbb{R}_+$, we have $f(y) = e^{-s_1 y_1 - s_2 y_2}$ satisfies (2.3).

$$\begin{aligned} \mathcal{A}f(y) &= \alpha_1 y_1 \partial_{y_1} f(y) + \alpha_2 y_2 s_2^2 f(y) + (b_1 + \beta_{11} y_1)\partial_{y_1} f(y) \\ &\quad -(b_2 + \beta_{21} y_1 + \beta_{22} y_2) s_2 f(y), \\ &\quad + \int_{\mathbb{R}_+ \setminus \{0\}} e^{-s_1 y_1 - s_2 y_2} (e^{-s_2 \xi_2} - 1) \tilde{m}(d\xi_2) \\ &\quad + \sum_{i=1}^{2} \int_{\mathbb{R}_+ \setminus \{0\}} e^{-s_1 y_1 - s_2 y_2} (e^{-s_2 \xi_2} - 1) y_i \tilde{\mu}_i(d\xi). \end{aligned} \tag{2.5}$$

Since

$$\lim_{s_2 \downarrow 0} f(y) = e^{-s_1 y_1},$$

and by the closeness of the infinitesimal generator (see Lemma 1.3, Dynkin 1965 [6]), we have

$$\mathcal{A}e^{-s_1 y_1} = \alpha_1 y_1 \partial_{y_1}^2 e^{-s_1 y_1} + (b_1 + \beta_{11} y_1)\partial_{y_1} e^{-s_1 y_1}, \quad \forall s_1 \in \mathbb{R}_+, \tag{2.6}$$

which completes the proof.

However, once the credit migration process Y^2 explodes, the process Y will die, therefore the remaining task is to extend the "life" of the short rate process r. By Lemma 2.1, we know that the dynamics of Y^1 is identical to the short rate process of Cox, Ingersoll and Ross (CIR) Model (1985 [2]) before the default time τ. Hence the solution of extending this process can be given by Lemma 2.2.

Lemma 2.2 *Suppose* $r_t = Y_t^1$, *if* $t < \tau$, *and* Y^1 *is defined by (2.4), then we have*

1) $\lim_{t \uparrow \tau} Y_t^1$ *exists and is finite almost surely;*

2) if we extend r *as follows:*

$$r_t = \tilde{r}_{t-\tau} \quad \textit{if} \quad t \geq \tau, \tag{2.7}$$

where $\tilde{r}$ *is a strong solution of the stochastic differential equation*

$$d\tilde{r}_t = (b_1 - \beta_{11}\tilde{r}_t)dt + \sqrt{2\alpha_1 \tilde{r}_t} d\tilde{W}_t, \quad \tilde{r}_0 = \lim_{t \to \tau^-} Y_t^1, \tag{2.8}$$

and $\tilde{W}$ *is a Brownian motion independent of* Y, *then* r *is still a one-dimensional* *affine process on R_+ with the infinitesimal generator*

$$\mathcal{A}g(y_1) = \alpha_1 y_1 \frac{d^2 g(y_1)}{dy_1^2} + (b_1 + \beta_{11} y_1)\frac{dg(y_1)}{dy_1}, \quad \forall g \in C_c^2(\mathbb{R}_+). \tag{2.9}$$

Proof. Since by Lemma 2.1, we know that Y^1 is a continuous process, therefore for $\forall \omega \in \Omega$, we have the $Y^1(\omega)$ is a continuous function on a compact support $[0, \tau(\omega)]$. Therefore Y^1 is bounded and $\lim_{t\uparrow\tau} Y^1(\omega)$ exists. Therefore we have proved the first argument. Since our extension guarantees that the Y^1 is a continuous solution of the SDE (2.4). Therefore it is easy to see (2.9). This completes the proof of Lemma 2.2.

Therefore Lemma 2.2 gives us a way of the extension for the short rate process, which is essentially important for pricing risk-free and some defaultable securities. Now we will discuss the pricing issue under this purely jump affine model. Throughout the following part, it is assumed that the current time is 0 and no default happens before.

2.1.1 Pricing Risk-Free Securities

Since now the short rate process r is extended to $\mathbb{R}_+$, which is a positive affine process, by Lemma 2.2 and the affine property, the price of a zero-coupon treasury bond maturing at T is given by

$$B^t(T, y) = \mathbb{E}_y\left[e^{-\int_0^T r_s ds}\right] = e^{\phi^t(T) + \psi^t(T) y_1}, \quad \forall\, T > 0, \tag{2.10}$$

where

$$\begin{aligned} \psi^t(T) &= -\frac{2(e^{\rho T} - 1)}{(\rho - \beta_{11})(e^{\rho T} - 1) + 2\rho}, \\ \phi^t(T) &= \frac{b_1}{\alpha_1} \log\left(\frac{2\rho e^{\frac{1}{2}(\rho - \beta_{11})T}}{(\rho - \beta_{11})(e^{\rho T} - 1) + 2\rho}\right), \\ \text{with} \quad \rho &= \sqrt{\beta_{11}^2 + 4\alpha_1}, \end{aligned}$$

and the corresponding yield is given by

$$y^t(T, y) = -\frac{1}{T} \log B^t(T, y) = -\frac{1}{T}(\phi^t(T) + \psi^t(T) y_1).$$

Remark 2.3 *By using Markov property, it is straightforward to derive the* *prices for treasury bonds at time* $t(t < T)$*.*

$$\begin{aligned} B^t(t,T,y) &= \mathbb{E}_y\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right] \\ &= \mathbb{E}_y\left[e^{-\int_0^{T-t} \theta_t \circ r_s ds}|\mathcal{F}_t\right] \\ &= \mathbb{E}_{Y_t}\left[e^{-\int_0^{T-t} r_s ds}\right] \\ &= e^{\phi^t(T-t)+\psi^t(T-t)Y_t^1}, \end{aligned}$$

where θ_t *is a shift operator, such that*

$$\theta_t \circ \omega_s = \omega_{t+s}, \quad \forall t, s \in \mathbb{R}_+.$$

This is also true for pricing defaultable bonds.

2.1.2 Pricing Default Bonds With Zero Recovery and Recovery at Maturity

Under the assumption of zero recovery at default, the payoff H_0 of a zero-coupon corporate bond with maturity T has the form

$$H_0 = 1_{\{\tau > T\}} = 1_{\{Y_T \neq \Delta\}} = 1_{\{Y_T^2 < \infty\}}.$$

Therefore the price of this bond can be given by

$$\begin{aligned} B_0^c(T,y) &= \mathbb{E}_y\left[e^{-\int_0^T r_s ds} 1_{\{Y_T^2 < \infty\}}\right] \\ &= \mathbb{E}_y\left[e^{-\int_0^T Y_s^1 ds} \lim_{\kappa \to \infty} e^{\frac{1}{\kappa} Y_T^2}\right] \\ &= \lim_{\kappa \to \infty} \mathbb{E}_y\left[e^{-\int_0^T Y_s^1 ds} e^{-\frac{1}{\kappa} Y_T^2}\right] \\ &= \lim_{\kappa \to \infty} e^{\phi(T,0,-\frac{1}{\kappa})+\psi_1(T,0,-\frac{1}{\kappa})y_1+\psi_2(T,0,-\frac{1}{\kappa})y_2}, \end{aligned}$$

where the function ϕ, ψ_1 and ψ_2 can be determined from the following generalized Riccati equations which can be easily solved by numerical integration.

$$
\begin{aligned}
\frac{d\psi_2\left(t,0,-\frac{1}{\kappa}\right)}{dt} &= \beta_{22}\psi_2\left(t,0,-\frac{1}{\kappa}\right)+\int_{\mathbb{R}_+}(e^{\psi_2\xi}-1)\mu_2(d\xi) \qquad (2.11)\\
\frac{d\psi_1\left(t,0,-\frac{1}{\kappa}\right)}{dt} &= \alpha_1\psi_1^2\left(t,0,-\frac{1}{\kappa}\right)+\beta_{11}\psi_1\left(t,0,-\frac{1}{\kappa}\right)+\beta_{21}\psi_2\left(t,0,-\frac{1}{\kappa}\right)\\
&\quad -1-\int_{\mathbb{R}_+}(e^{\psi_2\xi}-1)\mu_1(d\xi), \qquad (2.12)\\
\frac{d\phi\left(t,0,-\frac{1}{\kappa}\right)}{dt} &= b_1\psi_1\left(t,0,-\frac{1}{\kappa}\right)+b_2\psi_2\left(t,0,-\frac{1}{\kappa}\right)-\int_{\mathbb{R}_+}(e^{\psi_2\xi}-1)m(d\xi) \qquad (2.13)
\end{aligned}
$$

with the initial conditions

$$\phi\left(0,0,-\frac{1}{\kappa}\right)=0,\quad \psi_1\left(0,0,-\frac{1}{\kappa}\right)=0\quad\text{and}\quad \psi_2\left(0,0,-\frac{1}{\kappa}\right)=-\frac{1}{\kappa}. \qquad (2.14)$$

On setting

$$
\begin{aligned}
\phi^c(t) &= \lim_{\kappa\to\infty}\phi\left(t,0,-\frac{1}{\kappa}\right),\\
\psi_1^c(t) &= \lim_{\kappa\to\infty}\psi_1\left(t,0,-\frac{1}{\kappa}\right),\\
\text{and}\quad \psi_2^c(t) &= \lim_{\kappa\to\infty}\psi_2\left(t,0,-\frac{1}{\kappa}\right),
\end{aligned}
$$

we can rewrite $B_0^c(T,y)$ as

$$B_0^c(T,y)=e^{\phi^c(T)+\psi_1^c(T)y_1+\psi_2^c(T)y_2}. \qquad (2.15)$$

Therefore the semiannual coupon bond price with coupon rate c and the coupon payment date $\vec{T}=(T_1,T_2,...,T_m)$ can be derived as

$$P_0^c(c,\vec{T},y)=\sum_{i=1}^{m}\frac{c}{2}B_0^c(T_i,y)+B_0^c(T_m,y). \qquad (2.16)$$

Remark 2.4 *For our affine model, we have demonstrated that both the short rate and the credit rating enter the corporate bond pricing formula as parameters. Moreover, since the coefficient function* $\psi_2^c(t)$ *is negative for each* $t\in\mathbb{R}_+$*, we can see that the lower the credit rating* y_2*, the higher the bond price, which* *means the better the financial status of the company. This is consistent with our assumption that the default is equivalent to the explosion of Y^2, when the corresponding $Y^2 = \infty$.*

By Lemma 9.2 in Duffie, Filipović and Schachermayer (2002 [4]), because of the non-conservativity of the process Y, $\psi_2(t, 0, u)$ is not locally Lipschitz continuous in $u = 0$. Consequently, we have

$$\psi_2^c(T) < 0, \quad \psi_1^c(T) < \psi^t(T), \text{ and } \phi^c(T) < \phi^t(T), \quad \forall\, T > 0.$$

Therefore even if the initial credit rating y_2 is 0, which indicates the perfection of a firm's financial health, the yield spread remains positive. This fixed spread is given by

$$y^f(T, y_1) = -\frac{1}{T}(\phi^c(T) - \phi^t(T) + (\psi_1^c(T) - \psi^t(T))y_1) > 0.$$

This result also coincides with the empirical evidence observed by Duffee (1999 [3]). It not only implies that there still exists a potential default possibility even for very safe firms, but also reflects liquidity effects and state taxes imposed on the corporate bonds.

Remark 2.5 *It is easy to see that the credit spread is a linear function of Y. Therefore, by the properties of affine processes, it is straightforward to deduce the pricing formulas for European credit spread options. (See Duffie, Filipović and Schachermayer (2002 [4], Section 13.2)*

Now we assume that the defaultable bond admits a recovery with the recovery rate δ paid at maturity T, the payoff H_δ at T becomes

$$H_\delta = 1_{\{\tau > T\}} + \delta 1_{\{\tau \le T\}} = (1 - \delta)1_{\{Y_T^2 < \infty\}} + \delta.$$

Therefore the price can be derived as follows.

$$\begin{aligned}
B_\delta^c(T, y) &= \mathbb{E}_y\left[e^{-\int_0^T r_s ds} H_\delta\right] \\
&= \mathbb{E}_y\left[e^{-\int_0^T r_s ds}[(1-\delta)1_{\{Y_T^2 < \infty\}} + \delta]\right] \\
&= (1-\delta)\mathbb{E}_y\left[e^{-\int_0^T Y_s^1 ds} 1_{\{Y_T^2 < \infty\}}\right] + \delta \mathbb{E}_y\left[e^{-\int_0^T r_s ds}\right] \\
&= (1-\delta)B_0^c(T, y) + \delta B^t(T, y).
\end{aligned}$$

2.1.3 Recovery at Default

However, as mentioned by J.P. Morgan Research Report (2001, [9]), a reasonable assumption for recovery is recovery at default. Let δ' denote the recovery rate. Conditioned on the default time $\tau = t_0 (t_0 < T)$, the present value of the payoff of recovery can be obtained by

$$\pi(\delta', y, t_0) = \delta' \mathbb{E}_y \left[e^{-\int_0^{t_0} r_s ds} \right] = \delta' B^t(t_0, y).$$

Therefore the price of a corporate bond with recovery rate δ' is given by

$$B^c_{rd}(\delta', T, y) = \int_0^T \pi(\delta', y, t) \varphi(y, t) dt + B^c_0(T, y), \tag{2.17}$$

where $\varphi(y, t)$ denotes the density function of the default time τ under the risk neutral measure $\mathbb{P}_y$, and it can be derived as follows. Since

$$\begin{aligned} \mathbb{P}_y[\tau \le t] &= \mathbb{E}_y \left[1_{\{Y_t = \Delta\}} \right] \\ &= 1 - \mathbb{E}_y \left[1_{\{Y_t^2 < \infty\}} \right] \\ &= 1 - \mathbb{E}_y \left[\lim_{\kappa \to \infty} e^{-\frac{1}{\kappa} Y_t^2} \right] \\ &= 1 - \lim_{\kappa \to \infty} \mathbb{E}_y \left[e^{-\frac{1}{\kappa} Y_t^2} \right], \end{aligned}$$

by the affine property of Y, it follows that

$$\mathbb{P}_y[\tau \le t] = 1 - \lim_{\kappa \to \infty} \exp \left\{ \phi' \left(t, 0, -\frac{1}{\kappa} \right) + \psi'_1 \left(t, 0, -\frac{1}{\kappa} \right) y_1 + \psi'_2 \left(t, 0, -\frac{1}{\kappa} \right) y_2 \right\},$$

where ϕ', ψ'_1 and ψ'_2 can be determined from the following generalize Riccati equations:

$$\begin{aligned} \frac{d\psi'_2 \left(t, 0, -\frac{1}{\kappa} \right)}{dt} &= \beta_{22} \psi'_2 \left(t, 0, -\frac{1}{\kappa} \right) + \int_{\mathbb{R}_+} (e^{\psi'_2 \xi} - 1) \mu_2(d\xi) \\ \frac{d\psi'_1 \left(t, 0, -\frac{1}{\kappa} \right)}{dt} &= \alpha_1 \psi'_1 \left(t, 0, -\frac{1}{\kappa} \right)^2 + \beta_{11} \psi'_1 \left(t, 0, -\frac{1}{\kappa} \right) + \beta_{21} \psi'_2 \left(t, 0, -\frac{1}{\kappa} \right) \\ &\quad - \int_{\mathbb{R}_+} (e^{\psi'_2 \xi} - 1) \mu_1(d\xi), \\ \frac{d\phi' \left(t, 0, -\frac{1}{\kappa} \right)}{dt} &= b_1 \psi'_1 \left(t, 0, -\frac{1}{\kappa} \right) + b_2 \psi'_2 \left(t, 0, -\frac{1}{\kappa} \right) - \int_{\mathbb{R}_+} (e^{\psi'_2 \xi} - 1) m(d\xi) \end{aligned}$$

with the initial conditions

$$\phi'\left(0,0,-\frac{1}{\kappa}\right)=0, \quad \psi_1'\left(0,0,-\frac{1}{\kappa}\right)=0 \quad \text{and} \quad \psi_2'\left(0,0,-\frac{1}{\kappa}\right)=-\frac{1}{\kappa}.$$

On setting

$$\begin{aligned}\phi^0(t) &= \lim_{\kappa\to\infty}\phi'\left(t,0,-\frac{1}{\kappa}\right),\\ \psi_1^0(t) &= \lim_{\kappa\to\infty}\psi_1'\left(t,0,-\frac{1}{\kappa}\right),\\ \text{and} \quad \psi_2^0(t) &= \lim_{\kappa\to\infty}\psi_2'\left(t,0,-\frac{1}{\kappa}\right),\end{aligned}$$

we can rewrite $\mathbb{P}_y[\tau \le t]$ as

$$\mathbb{P}_y[\tau \le t] = 1 - e^{\phi^0(t)+\psi_1^0(t)y_1+\psi_2^0(t)y_2},$$

and therefore the density $\varphi(y,t) = -\partial_t e^{\phi^0(t)+\psi_1^0(t)y_1+\psi_2^0(t)y_2}$. By numerical integration, it is easy to calculate (2.17).

2.2 Modeling Credit Migrations As Diffusion Processes with Potential

Here it is assumed that $\beta_{12}=0$, the jump measures m, μ_1 and μ_2 are all zero. Therefore, for all $f \in C_c^2(D)$, we have the infinitesimal generator

$$\begin{aligned}\mathcal{A}f(y) &= \alpha_1 y_1 \partial_{y_1}^2 f(y) + \alpha_2 y_2 \partial_{y_2}^2 f(y) + (b_1 + \beta_{11}y_1)\partial_{y_1} f(y)\\ &\quad +(b_2+\beta_{21}y_1+\beta_{22}y_2)\partial_{y_2} f(y) - (c+\langle\gamma,y\rangle)f(y),\\ &= \alpha_1 y_1 \partial_{y_1}^2 \tilde{f}(y) + \alpha_2 y_2 \partial_{y_2}^2 \tilde{f}(y) + (b_1 + \beta_{11}y_1)\partial_{y_1} \tilde{f}(y)\\ &\quad +(b_2+\beta_{21}y_1+\beta_{22}y_2)\partial_{y_2} \tilde{f}(y)\\ &\quad +\int_{D_\Delta\setminus\{0\}} (c+\langle\gamma,y\rangle)(\tilde{f}(y+\xi)-\tilde{f}(y))\delta_\Delta(d\xi), \end{aligned} \tag{2.18}$$

where $\delta_\Delta(\cdot)$ is the Dirac measure sitting at Δ and $\tilde{f}$ is defined as

$$\tilde{f}(y) = \begin{cases} f(y) & \text{if} \quad y \in D, \\ 0 & \text{if} \quad y = \Delta, \end{cases}$$

which is in $C_c^2(D_\Delta)$. If we extend jump measures m, μ_1 and μ_2 to the space D_Δ by letting

$$\frac{m(\cdot)}{c} = \frac{\mu_1(\cdot)}{\gamma_1} = \frac{\mu_2(\cdot)}{\gamma_2} = \delta_\Delta(\cdot),$$

then the affine process defined by (2.3) can be regarded jump processes with diffusions. Therefore it is easy to extend Lemma 2.1 and Lemma 2.2.

Lemma 2.3 *If the infinitesimal generator of an affine process $Y = (Y^1, Y^2)$ is defined by (2.18), then the process Y^1 before time τ can be driven by the square root diffusion process (2.4) with the initial value y_1. Moreover, if we extend the short rate r by (2.7) and (2.8). Then r is still a one-dimensional positive affine process with the infinitesimal generator given by (2.9).*

2.2.1 Pricing Defaultable Bonds

Since by Lemma 2.3, it follows that the risk-free bond price formula is identical to (2.10). Hence we only focus on pricing defaultable bonds under this affine setup. By using the same price notation and assumptions as in the previous section, we have the price of a zero-coupon corporate bond with zero recovery as

$$\begin{aligned}
B_0^c(T, y) &= \mathbb{E}_y\left[e^{-\int_0^T r_s ds} 1_{\{Y_T^2 < \infty\}}\right] \\
&= \mathbb{E}_y\left[e^{-\int_0^T Y_s^1 ds} \lim_{\kappa\to\infty} e^{\frac{1}{\kappa} Y_T^2}\right] \\
&= \lim_{\kappa\to\infty} \mathbb{E}_y\left[e^{-\int_0^T Y_s^1 ds} e^{-\frac{1}{\kappa} Y_T^2}\right] \\
&= \lim_{\kappa\to\infty} e^{\phi(T,0,-\frac{1}{\kappa}) + \psi_1(T,0,-\frac{1}{\kappa}) y_1 + \psi_2(T,0,-\frac{1}{\kappa}) y_2}, \\
&= e^{\phi(T,0,0) + \psi_1(T,0,0) y_1 + \psi_2(T,0,0) y_2}, \qquad (2.19)
\end{aligned}$$

where the function ϕ, ψ_1 and ψ_2 can be determined from the following Riccati equations which can be easily solved numerically.

$$\begin{aligned}
\frac{d\psi_2\left(t, 0, -\frac{1}{\kappa}\right)}{dt} &= \alpha_2 \psi_2^2\left(t, 0, -\frac{1}{\kappa}\right) + \beta_{22}\psi_2\left(t, 0, -\frac{1}{\kappa}\right) - \gamma_2 \\
\frac{d\psi_1\left(t, 0, -\frac{1}{\kappa}\right)}{dt} &= \alpha_1 \psi_1^2\left(t, 0, -\frac{1}{\kappa}\right) + \beta_{11}\psi_1\left(t, 0, -\frac{1}{\kappa}\right) + \beta_{21}\psi_2\left(t, 0, -\frac{1}{\kappa}\right) \\
&\quad -1 - \gamma_1, \\
\frac{d\phi\left(t, 0, -\frac{1}{\kappa}\right)}{dt} &= b_1\psi_1\left(t, 0, -\frac{1}{\kappa}\right) + b_2\psi_2\left(t, 0, -\frac{1}{\kappa}\right) - c
\end{aligned}$$

with the initial conditions

$$\phi\left(0,0,-\frac{1}{\kappa}\right)=0, \quad \psi_1\left(0,0,-\frac{1}{\kappa}\right)=0 \quad \text{and} \quad \psi_2\left(0,0,-\frac{1}{\kappa}\right)=-\frac{1}{\kappa}.$$

Remark 2.6 *The last step of deducing in (2.19) is based on the Lipschitz continuity of $\psi_2(t,0,u)$ at $u=0$.*

Therefore the price of defaultable bonds with the assumption of recovery at maturity can be given by

$$\begin{aligned} B^c_\delta(T,y) &= (1-\delta)B^c_0(T,y)+\delta B^t(T,y), \\ &= (1-\delta)e^{\phi(T,0,0)+\psi_1(T,0,0)y_1+\psi_2(T,0,0)y_2}+\delta e^{\phi^t(T)+\psi^t(T)y_1}. \end{aligned}$$

Similarly, we can derive the pricing formula for recovery at default as in (2.17), where the density function $\varphi(y,t)$ of the default time τ can be derived as follows.

$$\varphi(y,t)=-\partial_t e^{\phi^0(t)+\psi^0_1(t)y_1+\psi^0_2(t)y_2},$$

where $\phi^0(t)$, $\psi^0_1(t)$ and $\psi^0_2(t)$ can be determined by the following Reccati equations:

$$\begin{aligned} \frac{d\psi^0_2(t)}{dt} &= \alpha_2\psi^0_2(t)^2+\beta_{22}\psi^0_2(t)-\gamma_2 \\ \frac{d\psi^0_1(t)}{dt} &= \alpha_1\psi^0_1(t)^2+\beta_{11}\psi^0_1(t)+\beta_{21}\psi^0_2(t)-\gamma_1, \\ \frac{d\phi^0(t)}{dt} &= b_1\psi^0_1(t)+b_2\psi^0_2(t)-c. \end{aligned}$$

3 Empirical Testing of Affine Models

3.1 A Simple Purely Jump Credit Risk Model

Here we apply a purely jump credit risk model for empirical testing. For simplicity, it is further assumed that

$$\beta_{21}=0, \quad b_2=0 \tag{3.1}$$

$$\frac{m(d\xi)}{\lambda_0}=\frac{\mu_1(d\xi)}{\lambda_1}=\frac{\mu_2(d\xi)}{\lambda_2}=\frac{\varsigma}{\Gamma(1-\varsigma)}\frac{1}{\xi^{1+\varsigma}}d\xi, \tag{3.2}$$

where ς is a real number between $(0,1)$ and $\Gamma(\cdot)$ denotes the Gamma function.

Remark 3.1 *By Remark 2.2, it is easy to see that the jump measures defined* *above satisfy the necessary condition (2.2) for the non-conservativity. Moreover, a straightforward calculation shows that*

$$\int_{\mathbb{R}_+\setminus\{0\}} (e^{v\xi}-1)\frac{\varsigma}{\Gamma(1-\varsigma)}\frac{1}{\xi^{1+\varsigma}}d\xi = -(-v)^{\varsigma}, \quad v\in\mathbb{R}_-.$$

Therefore, by (2.11), we know that for each $t>0$*,* $\psi(t,0,u)$ *is not Lipschitz continuous at* $u=0$*. Hence*

$$\mathbb{P}_y[Y_t^2<\infty]<1, \quad \forall t>0, y\in D$$

and this purely jump process Y *is not conservative, which means the explosion can occur in finite time.*

The above purely jump setup gives us a simple but reasonable class of models for credit migrations. The assumptions that $b_2=0$ and $\beta_{21}=0$ implies the mean level of credit rating of a firm can only stay stable at 0, the state of prefect financial health. Otherwise, it always has a tendency to move to this perfection state.

The zero coupon corporate bond price $B_0^c(T)$ with zero-recovery and maturity T is given by (2.15). Although there are no explicit formulas for the solutions $\phi^c(T)$ and $\psi_1^c(T)$ of (2.11)-(2.13). However, since $\psi_2^c(T)$ satisfies a Bernoulli equation, we can derive its explicit expression as follows, .

$$\psi_2^c(T) = -\left[-\frac{\lambda_2}{\beta_{22}}\left(1-e^{(1-\varsigma)\beta_{22}T}\right)\right]^{\frac{1}{1-\varsigma}}, \quad \forall T>0. \tag{3.3}$$

Therefore we can easily calibrate the parameters by the market quotes of bond prices.

3.2 Data Description

Both treasury and corporate bond data are downloaded from Bondpage.com. This dataset includes a snapshot of 50 observations of treasury note and bond prices and more than 800 month-end quoted prices of corporate bonds issued by the investment-grade firms. All the bonds we used are non-callable and have at least one year remaining to maturity. All the bonds share the same settlement date. A summary of the data is shown in Table 2.

Table 2: Summary of Statistics for Bond Data

	Maturity (years)						
Quality	0-2	2-4	4-6	6-8	8-10	10-20	Total
Treasury	3	12	8	6	4	17	50
Aaa	26	32	44	31	32	35	200
Aa	25	32	34	46	23	40	200
A	22	44	38	32	37	27	200
Baa	23	37	59	32	26	23	200

The dataset includes 650 bond prices of industrial firms and 150 for financial firms. The complete data are available from the author.

3.3 Estimation Strategy

Since our data is just a snapshot of the market at a certain time, the traditional filtering technique (e.g., extended Kalman Filters) is not applicable. Instead, we will estimate the parameters in our model by using nonlinear optimization. It would be more efficient to estimate the default-free interest rate and credit default processes jointly instead of separately, since, as derived in the previous section, a firm's bond price depends on both the risk-free rate and its credit rating. However, since the risk-free rate is independent of credit migration processes, and it does impact on the evolutions of credit ratings, therefore it is better for us to approach the estimations by two steps. First step is to calibrate our default-free parameters by using the prices of treasury notes and bonds, and based on the risk-free estimation results, then we estimate the credit migration processes by using corporate bond prices.

3.3.1 Estimation of Default-free Parameters

We apply the month-end prices of 50 Treasury notes and bonds estimating the default-free parameters $(b_1, \beta_{11}, \alpha_1, r)$. Since the data we use here are are all from coupon bonds, one way to estimate the parameters is to apply a simple bootstrapping method to derive a series of corresponding zero-coupon bond prices or yields, then the default-free parameters can be calibrated by applying nonlinear optimizations.

Suppose that vector $p = (p_1, p_2, ..., p_N)^T$ represents our market observations of coupon bond prices and $C = (c_{i,j})_{N\times M}$ represents the cash flows of coupon payments, where $c_{i,j}$ denotes the jth coupon payoff of the ith bond, N is the number of treasury notes and bonds (here $N = 50$) and M is the number of different dates of coupon payments. Let $D^* = (B^t(T_1), B^t(T_2), ..., B^t(T_M))^T$ denotes the corresponding prices of zero-coupon bonds, it follows that we can obtain D^* by using ordinary least square (OLS).

$$\begin{aligned} D^* &= \min_{D\in\mathbb{R}^M_+} \left\{ ||p - C\cdot D||^2 \right\}, \\ \text{or equivalently,} \quad (C^TC)D^* &= C^TP. \end{aligned}$$

However, in our case, we have $N \ll M$, which makes the OLS algorithm infeasible. Therefore, instead, we directly apply nonlinear optimization algorithm on coupon bond prices:

$$\theta^* = \min_{\theta\in\mathbb{R}^4} \left\{ \sum_{i=1}^{N} (p_i - P^t(\theta, \vec{T_i}, c_i))^2 \right\},$$

where

$$P^t(\theta, \vec{T_i}, c_i) = \sum_{j=1}^{m_i} \frac{c_i}{2} B(0, T_{i,j}) + B(0, T_{i,m_i}).$$

$\theta_0 = (b_1, \beta_{11}, \alpha_1, r)$, $\vec{T_i} = (T_{i,1}, T_{i,2}, ..., T_{i,m_i})$, c_i denote the vector of coupon payment dates and the coupon rate of ith bonds, respectively. Here it is assumed that the coupon is paid semiannually.

In order to test the robustness of our nonlinear optimization method, we have done thirty independent experiments by choosing different initial values of the parameters. The statistic results of the estimation are summarized in Table 3. As Table 3 indicates, one-factor CIR model does a relatively poor job in capturing a short term treasury yield curve, the mean error of predicting yields with maturities less than four years is around 2.6 basis point. This limitation of the model will consequently influence the performance of our model when pricing short term corporate bonds. However, Figure 1 shows that it does

Table 3: Nonlinear Optimization Estimation of Default-free Parameters

Parameters	Mean	Median	Std. ($\times 10^{-6}$)
b_1	0.011705	0.011714	41.014
β_{11}	-0.15459	-0.15461	90.167
α_1	0.0002962	0.0002967	55.817
r	0.0104467	0.010374	476. 082

Bond Maturity	Mean Error (basis points)	Median Error (basis points)	$\sqrt{\text{Mean square error}}$ ($\times 10^{-6}$)
0-2 years	2.538	2.582	54.9
2-4 years	2.705	2.844	22.9
4-6 years	1.446	1.389	10.0
6-8 years	0.987	0.849	46.3
8-10 years	0.771	0.692	51.54
10-20 years	0.297	0.279	23.55

capture long-term treasury yields pretty well in view of lower mean errors of prediction. Therefore it implies that our simplest affine model could be applied to characterize the long-term credit risk.

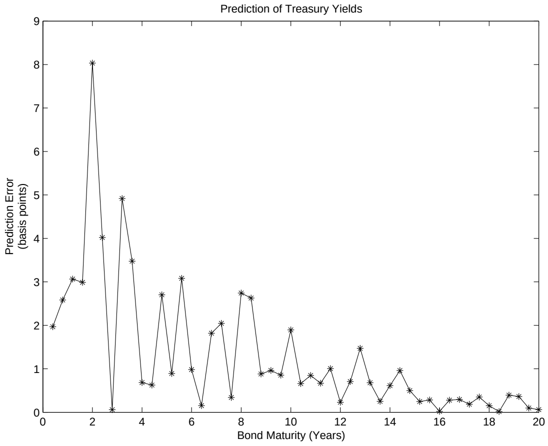

Figure 1: The Prediction of Treasury Yield Curve

3.4 Estimation of Parameters in Credit Migration Processes

Since we already derived the bond price formulas for corporate bonds, in principle, we are able to calibrate the parameters $(\beta_{22}, \lambda_0, \lambda_1, \lambda_2, \varsigma)$ and the credit ratings of firms by applying the nonlinear optimization strategy as mentioned before. However, two problems come up. First, although our dataset includes 800 non-callable corporate bond prices, no individual firm has more than 10 observations. Therefore the credit rating estimation for each individual firm is subject to substantial uncertainty. Duffee (1999 [3]) encountered the similar problems when estimating the default intensity of each firm. Here instead of assigning different credit rating to each firm, we assume that the credit ratings of firms in the same investment grade class given by Moody's are same. We will discuss the difference of credit ratings between the firms in the same class later. The second problem we face is the difficulty of estimating ς which determines the jump measure in the infinitesimal generator. Since the parameter ς turns out to be dominant over the other parameters, which means changing the values ς will result in significant value changes of other parameters, but the differences between measurement errors are rather small. It implies that estimating the parameter ς by minimizing the mean square error is infeasible. Therefore, instead of taking it as a parameter to be estimated, we set ς at several fixed values when estimating other parameters and compare the results among different ς. It follows that the credit migration parameter set is $\theta_d = (\beta_{22}, \lambda_0, \lambda_1, \lambda_2, X_{Aaa}, X_{Aa}, X_A, X_{Baa})$, where $(X_{Aaa}, X_{Aa}, X_A, X_{Baa})$ denotes the credit ratings of different investment grade classes.

3.4.1 Zero Recovery and Recovery at Maturity

Table 4 summarizes the parameter estimates of credit migration processes under the assumption of recovery at maturity. The table reports the mean and standard error (in parentheses) of thirty independent experiments for every parameter.

Table 4: Estimates of Parameters in Credit Migration Processes

ς	β_{22}	λ_0	λ_1	λ_2	Meas. Error
0.3	-9.3697 (0.137)	6.233E-08 (1.011E-09)	2.2512 (0.0320)	0.1385 (0.0022)	0.0539(0.00310)
0.5	-7.9615 (0.148)	5.036E-05 (8.391E-07)	6.5716 (0.125)	0.4612(0.0087)	0.0511(0.00271)
0.6	-8.4053 (0.146)	0.0042 (0.00012)	9.7114 (0.1357)	0.9678(0.0198)	0.0490(0.002506)
0.65	-7.9352 (0.256)	0.062356 (0.0045)	9.7668 (0.133)	1.3283(0.273)	0.0440(0.00043)
0.7	-7.7485(0.134)	0.27896 (0.0122)	10.0000(0.098)	1.5916 (0.0232)	0.0472 (0.00061)
0.75	-1.6093 (0.092)	0.1634(0.0092)	4.8122(0.065)	0.6017(0.0101)	0.0117 (0.00081)
0.8	-2.5556(0.0410)	0.2394(0.0046)	6.6401 (0.065)	1.1209 (0.0173)	0.0144(0.00084)

As indicated by Table 4, when $\varsigma = 0.75$, the mean square error is smallest. Therefore, in the following discussion, we fix ς at 0.75. From Table 4, we can obtain the following results:

- The credit rating migration process is mean reverting (reversion rate $\beta_{22} \approx -1.6$ under the risk-neutral measure.
- The short rate does impact on the jumps of credit ratings, which is supported by the positive estimate $\lambda_1 \approx 4.8$. Therefore this empirical result contradicts the hypothesis of independence between short rate and default risk as assumed by many literatures.

Now we examine the performance of our model in predicting the yield curves. Table 6 shows that there exists a severe prediction error on short-term corporate bond yields (average around 100 basis points) comparing with the long-term predicting errors (average around 9 basis points). This short term distortion is caused by the innate defects of our short rate model. As mentioned before, as estimating the default-free process, we already notice that the one-factor CIR model can not capture short term yields. However, in order to calibrate the defaultable parameters, the trueness of our estimates of risk-free rate parameters is assumed, therefore the existing short-term distortion has been inherited and enlarged when we estimate the defaultable processes. Another reason of this distortion is because of the illiquidity of short term corporate bonds. Figure 3.4.1 shows the prediction error distributions of corporate bonds.

Table 5: The Distributions of Prediction Errors

Bond Maturity	Mean Error (basis points)	Median Error (basis points)	$\sqrt{\text{Mean square error}}$ (basis points)
0-2 years	108.85	107.41	71.49
2-4 years	52.36	62.27	30.24
4-6 years	6.72	7.02	2.98
6-8 years	6.35	5.98	3.31
8-10 years	9.04	10.41	8.63
10-20 years	9.99	7.82	4.19

Meanwhile we obtain the estimates of the credit ratings $(X_{Aaa}, X_{Aa}, X_A, X_{Baa})$ as shown in Table 6. Different rating classes show different credit values as expected. Aaa class has the lowest (best) average credit value (around 2.9), while Baa class has the highest (worst) average credit value (around 8.0). The value of average credit rating increases as the corresponding investment grade of class goes down, as we expected. However, when we inversely calculate the credit rating X of each individual firm by applying our estimated parameters and Equation (2.16), we find that there exists a quite big credit rating downward shift from the short-term bonds to long-term bonds as shown in Figure 3.4.1. The longer the time to the maturity, the lower the credit rating. As a result, these exists an overlapping of credit value range between adjacent rating classes. One reason of this shift comes from the short-term distortion as mentioned before. Another reason is coming from the steep slope of coefficients function ψ_t when t is between (0,4), as shown in Figure 3.4.1. Finally Figure 3.4.1 shows the spread distributions of each class.

Therefore we have repeated the optimization only focusing on the long-term bonds (from 4 years to 20 years), then the distinction of credit ratings among each investment grade classes becomes significant, as indicated by Figure 3.4.1. The estimates of other parameters are shown in Table 7.

Finally when assuming recovery at maturity, we obtain similar results. Here we only shown the estimated recovery rates in each investment grade classes

Figure 2: The Prediction of Corporate Bond Yield Curve

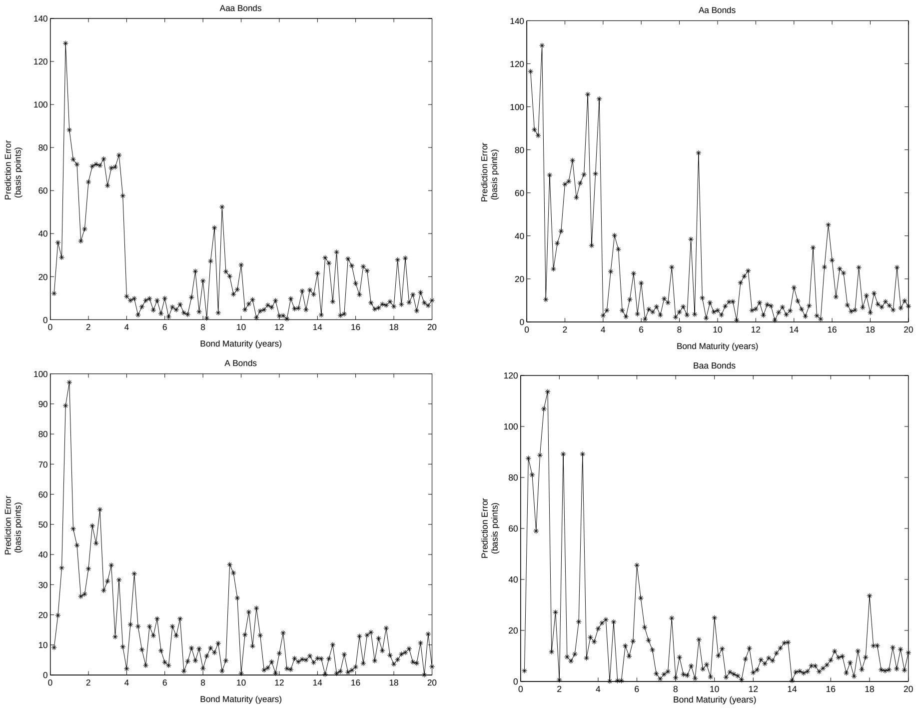

Table 6: Estimates of Parameters of Credit Migrations

Rating Classes	Mean	Median	Std. Error
X_{Aaa}	2.9355	2.9454	0.3239
X_{Aa}	3.8901	3.9024	0.6577
X_A	5.3406	5.4033	0.4942
X_{Baa}	8.0593	8.1222	0.5877

Table 7: Estimates of Defaultable Parameters for Long Term Bonds

ς	β_{22}	λ_0	λ_1	λ_2	Meas. Error
0.75	-1.1873(0.083)	0.1428 (0.0074)	1.9387 (0.0012)	0.4758 (0.0091)	0.0032 (0.000095)

Rating Classes	Mean	Median	Std. Error
X_{Aaa}	4.0122	4.0812	0.0232
X_{Aa}	4.5901	4.4024	0.0717
X_A	6.0096	5.9403	0.0924
X_{Baa}	8.3569	8.3329	0.0432

in Table 8. It is worth mentioning that the average recovery rates of bonds in Class A and Baa are much smaller than those in Class Aaa and Aa.

Table 8: Estimates of Recovery Rates with Recovery at Maturity

Rating Classes	Mean	Maximum	Minimum	Std. Error
Aaa	0.2682	0.7854	0.1239	0.0834
Aa	0.2995	0.9024	0.0577	0.1213
A	0.0925	0.3033	0.00942	0.0721
Baa	0.0001416	0.01222	0.00005877	0.001232

Figure 3: The Credit Rating Distributions of Corporate Bonds

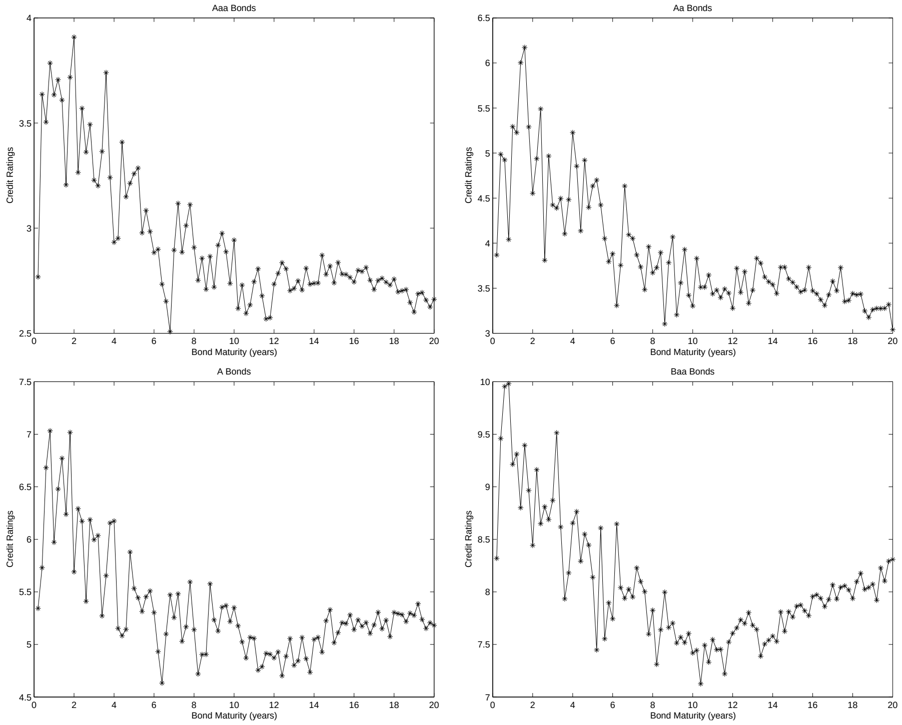

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Figure 4: The Coefficient Function $-\psi_2^c(t)$

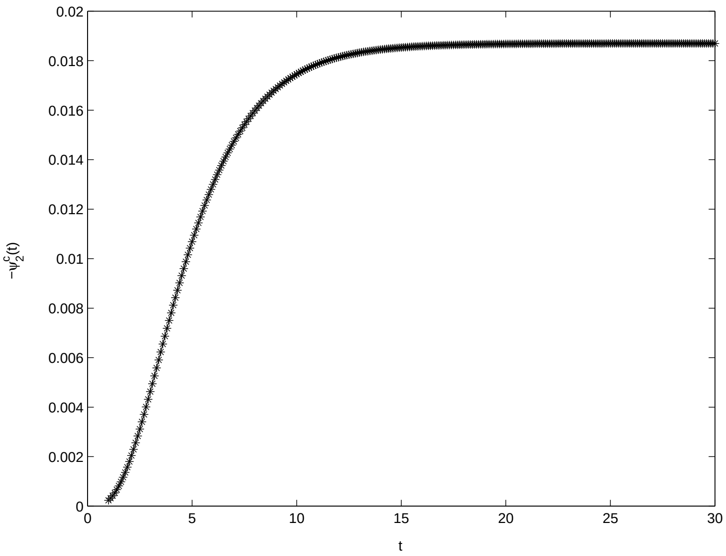

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Figure 5: The Spreads of Corporate Bonds

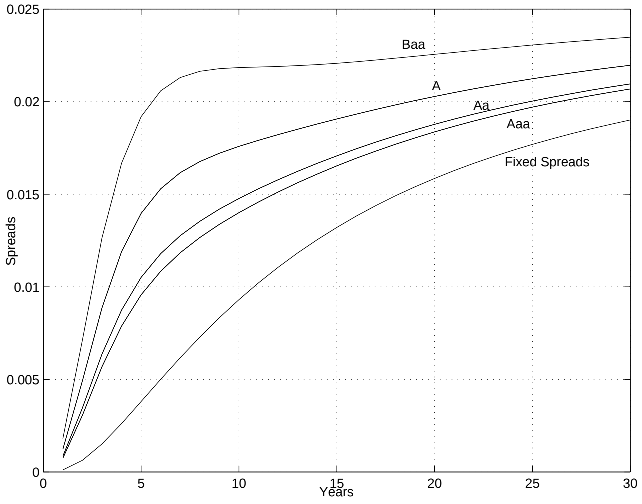

Figure 6: Comparison of Credit Ratings Among Different Classes