Toward a Theory of Asset Subscription

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Abstract

This paper develops an understanding toward a theory of asset subscription. When a firm needs to raise cash from an asset that is too large or too risky for a single individual or financial institution so that an auction method is not applicable, the firm may use a subscription scheme. In this paper, we discuss a Nash subscription (NS) scheme and a sequential subscription (SS) scheme. We characterize the optimal strategy when the value of the asset is known. The comparison between a NS and a SS is provided. The difference between an auction scheme and a subscription scheme is discussed.

Key words: Nash subscription, sequential subscription, auction

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1 Introduction

When a firm needs to raise cash from an asset, it typically completes the task in two ways. It can hire an investment bank which may take up or arrange for a consortium of financial institutions to take up and distribute the asset. In this case, the price for the asset is set by the firm in consultation with the investment bank. Or an auction-type of schemes can be used to sell the asset. Between the two, the auction method, if applicable, often generates higher value to the firm due to enhanced competition among potential buyers. However, sometimes the asset that the firm would like to sell may be too large or too risky for a single individual or financial institution so that the auction method is not applicable[1]. Examples of such an asset could be an equity stake, a loan (performing or non-performing), a sovereign debt, a portfolio of stocks that a government owned after a market intervention (e.g. Hong Kong SAR). In this paper, we propose alternative schemes — subscription schemes — to sell an asset and develop an understanding toward a subscription theory.

As will become clear later, a subscription scheme is like a value competition where value equals quantity times price. The price and quantity are determined once all the subscribed values are finalized. This is in contrast to auction scheme in which the price is typically called or submitted for a given quantity. Also, in an auction scheme, there is typically a single winner wherein in a subscription scheme all participants share a portion of the asset. Hence, through a subscription scheme, the firm can sell an asset which

[1]See Che and Gale (1998) for a study on standard auctions with financially constrained bidders.

is too large for any single financial institution to a group of interested bidders. In a subscription scheme, by controlling the number of participants, the firm can select the extent of the discount to make the asset attractive to the participants.

An ultimate theory of asset subscription would require a study of optimal strategy when the value of the asset is uncertain and/or when the participants value the asset differently. This paper serves to bring out only the concept of asset subscription and will leave the more difficult and interesting problems of incorporating uncertainty as research topics for the future. Appendix I contains a preliminary setup for conducting such an analysis without attempting to solve for the optimal strategies. Another dimension we neglected is the fact that bidders may have different opportunity costs for their funds. Instead, we assume that each bidder has access to only an investment with a zero rate of return. So all bidders would like to make maximum profit from the asset subscription.

In section 2, we begin with a Nash subscription (NS) scheme with a known value in which the optimal strategy is relatively easy to characterize. In section 3, we characterize the optimal strategy in a sequential subscription (SS) scheme. The optimal strategy is highly non-linear. Hence it is useful to obtain some numerical examples. This is done in section 4. Section 5 contains discussions on the comparison between NS and SS and on the difference between an auction scheme and a subscription scheme. Section 6 concludes.

2 Nash Subscription Scheme

There is a financial asset with known value of *unity*, which needs to be sold to N investors. In an NS, investor i submits a sealed bid, b_i, that she is willing to subscribe for a portion (yet to be determined) of the asset, $i = 1, \dots, N$. The total value recovered by the seller is

$$\sum_{j=1}^{N} b_j.$$

In the end, investor i obtains

$$\frac{b_i}{\sum_{j=1}^{N} b_j}.$$

As mentioned in the introduction, we assume that each investor's alternative investment would give a zero rate of return. Hence investor i would maximize

$$\frac{b_i}{\sum_{j=1}^{N} b_j} - b_i,$$

taking the values subscribed by other investors $j \neq i$ as given (hence the name Nash subscription). The first order condition is

$$\frac{\sum_{j=1}^{N} b_j - b_i}{\left[\sum_{j=1}^{N} b_j\right]^2} - 1 = 0.$$

In equilibrium, $b_j = b$ for any j. Thus,

$$\frac{(N-1)b}{N^2 b^2} = 1$$

or

$$b = \frac{(N-1)}{N^2}.$$

Thus, the recovery rate is Nb, which equals $(N-1)/N$. The asset is thus sold at a discount of $1/N$. For example, for $N = 10$, it represents a 10% discount.

3 Sequential Subscription

Sequential subscriptions are much harder to analyze. In an SS, the investors are randomly ordered and the bids are submitted sequentially, hence the name: sequential subscription. The following analysis is done by backward induction.

Lemma 1 $B_N = \sqrt{B_{N-1}}$; $dB_N/dB_{N-1} = 1/(2B_N)$.

Proof. Let B_{N-1} be the sum of values subscribed by investors $i = 1, 2, ..., N-1$. Investor N chooses b_N to maximize

$$\frac{b_N}{B_{N-1} + b_N} - b_N,$$

taking B_{N-1} as given. Alternatively, we can think of investor N as choosing B_N to maximize

$$\left[\frac{B_N - B_{N-1}}{B_N}\right] - [B_N - B_{N-1}],$$

taking B_{N-1} as given. The first order condition is,

$$\frac{B_N - (B_N - B_{N-1})}{B_N^2} - 1 = 0,$$

which yields,

$$B_N = \sqrt{B_{N-1}}.$$

Thus,

$$\frac{dB_N}{dB_{N-1}} = \frac{1}{2B_N}$$

■

Obviously, $B_N = B_N(B_{N-1}) = ... = B_N(B_{N-1}(...(B_i)...)) \triangleq \Phi_i^{-1}(B_i)$ for any $i = 1, 2, ..N-1$. In other words, we have, $B_i = \Phi_i(B_N)$. And we can always write

$$\frac{dB_{j+1}}{dB_j}$$

as a function of B_N as in the lemma above for $j = 1, 2, ...N-1$, denoted by $\Gamma_j(B_N)$. For example, we know that

$$\begin{aligned} \Phi_{N-1}(B_N) &= B_N^2 \\ \Gamma_{N-1}(B_N) &= \frac{dB_N}{dB_{N-1}} = \frac{1}{2B_N}. \end{aligned}$$

But in general, what do we know about the functions $\Phi_j(B_N)$ and $\Gamma_j(B_N)$ for $j = 1, 2, ...N-2$?

Lemma 2 *Given $\Phi_j(B_N)$ and $\Gamma_j(B_N)$ for $j = N-1, ...i+1$. We can show that*

$$\Phi_i(B_N) = \Phi_{i+1}(B_N) - \frac{B_N(1-B_N)}{\prod_{j=i+1}^{N-1} \Gamma_j(B_N)}$$

Proof. The decision problem for investor $i+1$ can be written as follows

$$\max \frac{B_{i+1} - B_i}{B_N(B_{N-1}(...(B_{i+1})...))} - [B_{i+1} - B_i].$$

The first order condition is

$$\frac{B_N - [B_{i+1} - B_i] \prod_{j=i+1}^{N-1} \Gamma_j(B_N)}{B_N^2} - 1 = 0,$$

which yields

$$B_i = B_{i+1} - \frac{B_N - B_N^2}{\prod_{j=i+1}^{N-1} \Gamma_j(B_N)}$$

or, namely,

$$\Phi_i(B_N) = \Phi_{i+1}(B_N) - \frac{B_N(1 - B_N)}{\prod_{j=i+1}^{N-1} \Gamma_j(B_N)}$$

■

Lemma 3 *Given $\Gamma_j(B_N)$ for $j = N-1, ...i+1$, we can show that*

$$\Gamma_i(B_N) = \frac{\Gamma_{i+1}(B_N)}{1 + B_N(1 - B_N)\Gamma'_{i+1}(B_N)}.$$

Proof. Let $B_0 = 0$. Then, for any $i = 0, 1, ..., N-2$,

$$B_{i+1} - B_i = \frac{B_N - B_N^2}{\prod_{j=i+1}^{N-1} \Gamma_j(B_N)}.$$

Thus for $i = 0, 1, 2, ..., N-2$,

$$B_{i+2} - B_{i+1} = (B_{i+1} - B_i)\Gamma_{i+1}(B_N).$$

Take derivative with respect to B_i, we have,

$$\begin{aligned} \left[\tfrac{dB_{i+2}}{dB_{i+1}} - 1\right] \tfrac{dB_{i+1}}{dB_i} &= \left[\tfrac{dB_{i+1}}{dB_i} - 1\right] \Gamma_{i+1}(B_N) + (B_{i+1} - B_i)\Gamma'_{i+1}(B_N) \\ &\quad \times \Gamma_{N-1}(B_N)\Gamma_{N-2}(B_N)...\Gamma_{i+1}(B_N)\Gamma_i(B_N), \end{aligned}$$

but

$$\begin{aligned} B_N - B_{N-1} &= (B_{N-1} - B_{N-2})\Gamma_{N-1}(B_N) \\ &= (B_{N-2} - B_{N-3})\Gamma_{N-2}(B_N)\Gamma_{N-1}(B_N) \\ &= ... \\ &= (B_{i+1} - B_i)\Gamma_{i+1}(B_N)...\Gamma_{N-2}(B_N)\Gamma_{N-1}(B_N), \end{aligned}$$

which can be substituted in the equation above to arrive at

$$[\Gamma_{i+1}(B_N) - 1]\,\Gamma_i(B_N) = [\Gamma_i(B_N) - 1]\,\Gamma_{i+1}(B_N) + B_N\,(1 - B_N)\,\Gamma'_{i+1}(B_N)\Gamma_i(B_N)$$

i.e.,

$$\Gamma_i(B_N) = \frac{\Gamma_{i+1}(B_N)}{1 + B_N\,(1 - B_N)\,\Gamma'_{i+1}(B_N)}.$$

■

By way of backward induction, we can find the functional forms to all $\Phi_j(B_N)$ and $\Gamma_j(B_N)$ for $j = N - 1, ...1$. The only task remaining is to find the condition that pins down B_N. We have the following proposition:

Proposition 4 *B_N can be obtained from the following "fundamental" equation:*

$$B_N(1 - B_N) = \Phi_1(B_N) \prod_{j=1}^{N-1} \Gamma_j(B_N).$$

Proof.

$$\begin{aligned} B_N - B_{N-1} &= (B_{N-1} - B_{N-2})\Gamma_{N-1}(B_N) \\ &= (B_{N-2} - B_{N-3})\Gamma_{N-2}(B_N)\Gamma_{N-1}(B_N) \\ &= ... \\ &= (B_1 - B_0)\Gamma_1(B_N)...\Gamma_{N-2}(B_N)\Gamma_{N-1}(B_N) \\ &= \Phi_1(B_N) \prod_{j=1}^{N-1} \Gamma_j(B_N). \end{aligned}$$

The last step uses $B_1 = \Phi_1(B_N)$ and $B_0 = 0$. ■

4 Concrete Examples

4.1 The case of two investors

In this case, all we need are $\Phi_1(B_2)$ and $\Gamma_1(B_2)$. They are given by

$$\begin{aligned}\Phi_1(B_2) &= B_2^2 \\ \Gamma_1(B_2) &= \frac{1}{2B_2}.\end{aligned}$$

The fundamental equation for B_2 is

$$B_2(1 - B_2) = B_2^2 \frac{1}{2B_2},$$

which results in

$$B_2 = \frac{1}{2}$$

and hence,

$$B_1 = B_2^2 = \frac{1}{4}.$$

Therefore,

$$b_1 = b_2 = \frac{1}{4}.$$

The bids are the same as in an NS scheme. This is a peculiar result because we would have expected the bids to be different under NS and SS, given the result in duopoly theory that the quantities produced by the duopolies are different under the Nash-Cournot game and the Stackelberg game (see Boyer and Moreaux, 1986 and Anderson and Engers, 1992).

4.2 The Case of Three Investors

Now, we need to find $\Phi_j(B_3)$ and $\Gamma_j(B_3)$ for $j = 1$ and 2. Let us do it backwards. First,

$$\begin{aligned}\Phi_2(B_3) &= B_3^2 \\ \Gamma_2(B_3) &= \frac{1}{2B_3}.\end{aligned}$$

Then,

$$\begin{aligned}\Phi_1(B_3) &= \Phi_2(B_3) - \frac{B_3(1-B_3)}{\Gamma_2(B_3)} \\ &= B_3^2 - 2B_3^2(1-B_3) \\ &= 2B_3^3 - B_3^2\end{aligned}$$

and

$$\begin{aligned}\Gamma_1(B_3) &= \frac{\Gamma_2(B_3)}{1 + B_3\,(1-B_3)\,\Gamma_2'(B_3)} \\ &= \frac{\frac{1}{2B_3}}{1 - B_3\,(1-B_3)\,\frac{1}{2B_3^2}} \\ &= \frac{1}{3B_3 - 1}.\end{aligned}$$

Now, use the fundamental equation

$$B_3(1-B_3) = \Phi_1(B_3)\Gamma_1(B_3)\Gamma_2(B_3)$$

or, namely,

$$B_3(1-B_3) = \left[2B_3^3 - B_3^2\right]\frac{1}{3B_3 - 1}\frac{1}{2B_3}$$

The only meaningful solution is $B_3 = \frac{1}{2} + \frac{1}{6}\sqrt{3} = .788\,68$ (other solutions to the equation will make some bs negative and are discarded). Note that this is greater than the value recovered under NS, which is 2/3.

In this case,

$$
\begin{aligned}
B_2 &= \left[\frac{3+\sqrt{3}}{6}\right]^2 \\
&= .622
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= 2B_3^3 - B_3^2 \\
&= 2\left[\frac{3+\sqrt{3}}{6}\right]^3 - \left[\frac{3+\sqrt{3}}{6}\right]^2 \\
&= .359.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
b_1 &= .359 \\
b_2 &= .263 \\
b_3 &= .167.
\end{aligned}
$$

We will put the analysis for the cases of 4, 5 and 6 investors in Appendix II. Here, let us construct a table, summarizing our findings on SS for a known value of unity.

Table 1: Sequential Subscription

Investors	b	B	profit	return	share
1	0.250	0.250	0.250	1.000	0.500
2	0.250	0.500	0.250	1.000	0.500

Investors	b	B	profit	return	share
1	0.359	0.359	0.096	0.268	0.455
2	0.263	0.622	0.070	0.268	0.333
3	0.167	0.789	0.045	0.268	0.211

Investors	b	B	profit	return	share
1	0.412	0.412	0.042	0.101	0.454
2	0.261	0.674	0.026	0.101	0.287
3	0.151	0.825	0.015	0.101	0.167
4	0.083	0.908	0.008	0.101	0.092

Investors	b	B	profit	return	share
1	0.442	0.442	0.019	0.043	0.461
2	0.258	0.701	0.011	0.043	0.269
3	0.142	0.843	0.006	0.043	0.149
4	0.076	0.919	0.003	0.043	0.079
5	0.040	0.959	0.002	0.043	0.041

Investors	b	B	profit	return	share
1	0.461	0.461	0.009	0.020	0.470
2	0.256	0.717	0.005	0.020	0.261
3	0.137	0.854	0.003	0.020	0.140
4	0.072	0.925	0.001	0.020	0.073
5	0.037	0.962	0.001	0.020	0.038
6	0.019	0.981	0.000	0.020	0.019

Here are some conclusions we can draw from the table:

- The discount required to sell the asset decreases rapidly as the number of investors increases. When $N = 4$, the discount is already smaller than 10 percent, the discount achieved in NS when $N = 10$.
- All investors obtain the same rate of return, which is decreasing as N increases.
- There is a first mover advantage: the investor who bids first obtains the highest share and profit.
- For all investors, the profit declines monotonically as N increases.
- The share of the first investor is *not* monotonically decreasing as N increases. It decreases when N goes from 2 to 3 and from 3 to 4, but starts to increase as N goes from 4 to 5. This is because the share of the first investor is b_1/B_1; both the numerator and denominator are increasing functions of N.
- We conjecture that when N approaches infinity, the first investor's subscription and share approach 1/2, the second investor's approach 1/4, the third investor's approach 1/8, etc. We call this "the bisection conjecture."[2]

[2] In Stackelberg oligopoly games, Boyer and Moreaux (1986) and Anderson and Engers (1992) show that a linear demand curve induces a "law of bisection." In a setting with incomplete information studied by Zhang and Zhang (1999), they proved a variant of this

5 Discussions

5.1 NS and SS

So far, we have characterized the optimal strategies in a Nash subscription and a sequential subscription. The optimal strategies in a sequential subscription are given in a recursive fashion. We show that a small number of investors are needed in an SS than in an NS to obtain a high recovery rate for the seller of the asset. We also show that the first mover has substantial advantage in capturing a larger profit. The limitation of the above analysis is the assumption that the value of the asset is known and that the only alternative investment has a zero rate of return. The extension to the case with uncertainty in the value of the asset is required for any potential real-world applications.[3] Some of the results may be overturned. For example, the first

law. In their notation,

$$\frac{dq_{n+1}}{dq_n} = -\frac{1}{2}.$$

Our conjecture can be given an intuitive "proof". First, $B_N \to 1$ as $N \to \infty$. Second, $\Gamma'_{i+1}(B_N)|_{B_N=1}$ is bounded. Hence Lemma 3 says that as $N \to \infty$,

$$\begin{aligned} \Gamma_1(B_N) &= \Gamma_2(B_N) = ... = \Gamma_{N-1}(B_N) \\ &= \frac{1}{2B_N} = \frac{1}{2}. \end{aligned}$$

Then Lemma 2 and Proposition 4 can be used to derive

$$b_1 = 1/2,\ b_2 = 1/4, ...$$

[3] In Appendix I, we show that even in a 2-investor SS game, the first investor's optimal bidding strategy is not trivial.

investor may not feel as secure as the last investor, because the first does not observe the private valuation of the last investor when making her decision on the subscription.

When the value of the asset is known to all the investors, the investors prefer to be in a Nash subscription scheme, which results in a deeper discount. In auction theory, the result is stronger. It is known that in a more general environment with uncertainty and private information, English auction (sequential bidding) generates higher expected revenue (recovery rate) than other standard auctions, such as first-price sealed-bid (simultaneous bidding), since the former allows more information revelation. However, in sequential subscription, the information pooling may be contaminated by strategic underbids made by earlier investors to influence later investors who use Bayesian updating. In any case, the sequential subscription scheme, although fair ex ante, is unfair ex post in the sense that profits for the investors could be different. This unfairness may be moderated if there are multiple assets and investors are reordered after the sale of each asset.

Allowing for more realistic alternative investment opportunities will certainly complicate the characterization of optimal bidding strategies.

The following is a list of possible use of NS and SS:

- In the United States, the treasury bills are sold by "auction." This auction is different from a traditional one in the sense that the bidders bid both quantity and prices (the yields). Would an SS scheme be easier to conduct and more transparent?

- Given the widespread use of Internet, it also seems possible that an

IPO can be done by a sequential subscription scheme.

- The model of SS may have empirical relevance in a study of market shares and rates of return for oligopolies. If an empirical pattern is found which is compatible with the theory here or an extended version, the outliers may be identified as firms who have committed strategic mistakes. This analysis of NS and SS suggests that when a country opens its market in a particular industry (telecom, for example) to N foreign investors, a sequential admission (SS) gives the country a better deal than a simultaneous admission (NS).

- The SS scheme can also be used to run an experiment, testing whether the second player can keep her promise in a two-player, one-shot game when (1) the two players never either know or see each other (if the game is played by phone or Internet), or (2) they do not know each other before the game but have a face-to-face discussion of what they will do. The first player is the insider planted by the conductor of the experiment.

5.2 Auction or Subscription

Auctions are typically used to sell items or projects to bidders who have private valuations for these items. The main concerns are that an item goes to the bidder who values it the most and a project is undertaken by the bidder who is the most efficient.

In the banking systems of many developing countries, there are problems of bad loans. In some cases, asset management companies are set up to dispose of these bad loans. Because of the large risk involved, no single financial institution would be willing to take the assets, and hence an auction method is difficult to apply. When a subscription method is used, either a Nash subscription or a sequential subscription, better risk sharing can be achieved. Each institutional investor can participate in subscriptions for a diversified set of bad loans.

In the literature of oligopoly, there were theories of quantity competition (Nash-Cournot) and price competition (Bertrant). As we can see from the above, a subscription scheme is like a value competition where value equals quantity times price. The price and quantity are determined once all the subscribed values are finalized. This is in contrast to auction scheme in which the price is typically called or submitted for a given quantity.

Any auction mechanism is vulnerable to collusion. So is any subscription scheme, whether NS or SS. In practice, one way to mitigate collusive behavior is to use the following procedure that attaches a sealed-bid auction feature to a sequential subscription scheme. First, information on the financial asset is released to potential institutional investors. Suppose that after a period of inquiry, consultation and research, 20 investors showed interest to divide up the asset. The 20 investors are invited to a room and they are randomly grouped into 5 teams, each with 4 investors. The teams are then separated into 5 closed rooms. For each team, a sequential subscription game is played with a representative of the seller present. The winning team is the one that pays the most for the asset, which is then distributed to the winning members according to their subscription shares. This makes collusion more difficult.

Of course, the investors have to be given enough incentive to participate in such a game in the first place.

6 Conclusion

There are many choices as to how to sell an item: through organized markets, over the counter, auction schemes, rationing schemes, auction-rationing schemes, lottery schemes, etc. In this paper, subscription schemes are proposed as an alternative and the optimal strategies are studied in a simple certainty framework. Ultimately, the subscription schemes will have to be studied under uncertainty and the results compared with traditional auction schemes (see a survey by Klemperer, 1999 on auctions) and auction-rationing schemes (see Parlour and Rajan, 2001). The simple model shows that Nash subscription and sequential subscription schemes allow a firm to sell an asset at a discount of its choice by picking the number of subscribers. We anticipate that when this is extended to the case of uncertainty, the sequential subscription scheme would allow information revelation and risk sharing among investors and hence will be a useful scheme.

Appendix I

This appendix sets up a simple subscription game under uncertainty. Suppose there is a loan with face value of unity with a recovery value, μ, uniformly distributed in interval (0,1). Given, μ, investors receive imperfect signals s, uniformly distributed in $[\mu - \mu(1-\mu),\ \mu + \mu(1-\mu)] \subset (0,1)$, which has a mean equal to μ and a variance equal to $2[\mu(1-\mu)]^3/3$. The intuition here is that if μ is close to 0 or 1, there is a smaller noise in the signal. When μ is in the middle, the signals tend to be noisier.

In an N-investor NS game, each investor observes only her own signal. The best she can do is to treat her signal as the possible realization of μ. Hence, each bids

$$b_i = \frac{N-1}{N^2} s_i,$$

where s_i is investor i's signal. And hence the expected sales value equals

$$B^{NS} = \frac{N-1}{N^2} \sum_{i=1}^{N} s_i.$$

In an N-investor SS game, the situation is much more complicated. Each investor should infer what signals the bidders in front of her received. Let us think about the simplest case of $N = 2$. We need to work backward.

Given b_1 and the inferred $\tilde{s}_1 = f(b_1)$, the best estimate of μ by investor 2 is

$$\hat{\mu}_2 = \frac{\tilde{s}_1 + s_2}{2}.$$

She would then maximize

$$\frac{b_2}{b_1 + b_2} \left[\frac{\tilde{s}_1 + s_2}{2} \right] - b_2.$$

The first order condition implies

$$\begin{aligned} b_2 &= \sqrt{\left[\frac{f(b_1)+s_2}{2}\right] b_1} - b_1 \\ &= b_2(b_1, s_2) \end{aligned}$$

with

$$\frac{db_2}{db_1} = \frac{\left[\frac{f(b_1)+s_2}{2}\right] + \frac{f'(b_1)}{2} b_1}{2\sqrt{\left[\frac{f(b_1)+s_2}{2}\right] b_1}} - 1.$$

Thus the first investor's problem becomes

$$\max\ E \frac{b_1}{b_1 + b_2(b_1, s_2)} \left[\frac{s_1+s_2}{2}\right] - b_1,$$

where the expectation is taken with respect to s_2, which is considered by investor 1 to be uniformly distributed in $(s_1^2, 2s_1 - s_1^2)$— namely, centered in s_1.

The first order condition for investor 1 is

$$E \frac{[b_1 + b_2(b_1, s_2)] - b_1 \left[1 + \frac{db_2}{db_1}\right]}{[b_1 + b_2(b_1, s_2)]^2} \left[\frac{s_1+s_2}{2}\right] = 1.$$

This yields,

$$E \frac{\left[\left[\frac{f(b_1)+s_2}{2}\right] - \frac{f'(b_1)}{2} b_1\right]}{2\sqrt{\left[\frac{f(b_1)+s_2}{2}\right] b_1}} = 1. \qquad (1)$$

Note that in the above notation, s_2 is considered by investor 1 to be uniformly distributed in $(f(b_1)^2,\ 2f(b_1) - f(b_1)^2)$. Therefore, we need to find a function $f(b_1)$ such that

$$\frac{1}{2f(b_1)(1-f(b_1))} \int_{f(b_1)^2}^{2f(b_1)-f(b_1)^2} \frac{\left[\frac{f(b_1)+s_2}{2}\right] - \frac{f'(b_1)}{2} b_1}{2\sqrt{\left[\frac{f(b_1)+s_2}{2}\right] b_1}} ds_2 = 1 \text{ for any } b_1.$$

This results in a differential equation in f with $\lim_{b_1 \to 0} f(b_1) = 0$. Once we obtain f, the bidding function, $b_1 = f^{-1}(s_1)$, is easily available. It is straightforward to check that the naive strategy mimicking the certainty case, $b_1 = s_1/4$, namely $f(b_1) = 4b_1$ is not a solution (Jensen's inequality is at work).

Appendix II

$N = 4$. We need to find $\Phi_j(B_3)$ and $\Gamma_j(B_3)$ for $j = 1$, 2 and 3. Again, do it backward. First,

$$\begin{aligned}\Phi_3(B_4) &= B_4^2 \\ \Gamma_3(B_4) &= \frac{1}{2B_4}.\end{aligned}$$

Then,

$$\Phi_2(B_4) = 2B_4^3 - B_4^2$$

and

$$\Gamma_2(B_4) = \frac{1}{3B_4 - 1}.$$

Use the updating rule,

$$\begin{aligned}\Phi_1(B_4) &= \Phi_2(B_4) - \frac{B_4(1 - B_4)}{\prod_{j=2}^{3} \Gamma_j(B_4)} \\ &= 2B_4^3 - B_4^2 - \frac{B_4(1 - B_4)}{\left(\frac{1}{3B_4 - 1}\right)\frac{1}{2B_4}} \\ &= 6B_4^4 - 6B_4^3 + B_4^2\end{aligned}$$

and

$$\begin{aligned}\Gamma_1(B_4) &= \frac{\Gamma_2(B_4)}{1 + B_4\left(1 - B_4\right)\Gamma_2'(B_4)} \\ &= \frac{\frac{1}{3B_4 - 1}}{1 + B_4\left(1 - B_4\right)\left(-\frac{3}{(3B_4 - 1)^2}\right)} \\ &= \frac{3B_4 - 1}{12B_4^2 - 9B_4 + 1}.\end{aligned}$$

Now, to calculate B_4, we use the fundamental equation

$$B_4(1-B_4) = \left(6B_4^4 - 6B_4^3 + B_4^2\right)\left(\frac{3B_4-1}{12B_4^2-9B_4+1}\right)\left(\frac{1}{3B_4-1}\right)\frac{1}{2B_4}.$$

The only sensible answer is $B_4 = \frac{1}{2} + \frac{1}{6}\sqrt{6} = .908\,25$. Therefore

$$\begin{aligned}
b_1 &= .412\,46 \\
b_2 &= .26108 \\
b_3 &= .15138 \\
b_4 &= .08288.
\end{aligned}$$

Now look at the case of $N = 5$. First, we have

$$\begin{aligned}
\Phi_4(B_5) &= B_5^2 \\
\Gamma_4(B_5) &= \frac{1}{2B_5}.
\end{aligned}$$

Then we have

$$\Phi_3(B_5) = 2B_5^3 - B_5^2$$

and

$$\Gamma_3(B_5) = \frac{1}{3B_5-1}.$$

Using the updating rule, we have

$$\Phi_2(B_5) = 6B_5^4 - 6B_5^3 + B_5^2$$

and

$$\Gamma_2(B_5) = \frac{3B_5-1}{12B_5^2-9B_5+1}.$$

The last items to update are

$$\begin{aligned}
\Phi_1(B_5) &= \Phi_2(B_5) - \frac{B_5(1-B_5)}{\prod_{j=2}^{4}\Gamma_j(B_5)} \\
&= 6B_5^4 - 6B_5^3 + B_5^2 - \frac{B_5(1-B_5)}{\left(\frac{3B_5-1}{12B_5^2-9B_5+1}\right)\left(\frac{1}{3B_5-1}\right)\frac{1}{2B_5}} \\
&= 24B_5^5 - 36B_5^4 + 14B_5^3 - B_5^2
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_1(B_5) &= \frac{\Gamma_2(B_5)}{1 + B_5\,(1-B_5)\,\Gamma_2'(B_5)} \\
&= \frac{\frac{3B_5-1}{12B_5^2-9B_5+1}}{1 + B_5\,(1-B_5)\left(-6\frac{6B_5^2-4B_5+1}{\left(12B_5^2-9B_5+1\right)^2}\right)} \\
&= \frac{12B_5^2 - 9B_5 + 1}{60B_5^3 - 72B_5^2 + 21B_5 - 1}.
\end{aligned}$$

Finally, to solve for B_5, use the fundamental equation

$$B_5(1-B_5) = \Phi_1(B_5)\prod_{j=1}^{4}\Gamma_j(B_5)$$

or, namely,

$$\begin{aligned}
B_5(1-B_5) &= \left(24B_5^5 - 36B_5^4 + 14B_5^3 - B_5^2\right)\left(\frac{12B_5^2-9B_5+1}{60B_5^3-72B_5^2+21B_5-1}\right) \\
&\quad\times\left(\frac{3B_5-1}{12B_5^2-9B_5+1}\right)\left(\frac{1}{3B_5-1}\right)\frac{1}{2B_5}.
\end{aligned}$$

The only sensible solution is $B_5 = \frac{1}{2} + \frac{1}{60}\sqrt{\left(450 + 30\sqrt{105}\right)} = .958\,68$. Therefore,

$$b_1 = .44235$$

$$
\begin{aligned}
b_2 &= .25828 \\
b_3 &= .14249 \\
b_4 &= .07595 \\
b_5 &= .03961.
\end{aligned}
$$

$N = 6$. We know that

$$
\begin{aligned}
\Phi_1(B_6) &= \Phi_2(B_6) - \frac{B_6(1-B_6)}{\prod_{j=2}^{5}\Gamma_j(B_6)} \\
&= 24B_6^5 - 36B_6^4 + 14B_6^3 - B_6^2 \\
&\quad - \frac{B_6(1-B_6)}{\left(\frac{12B_6^2-9B_6+1}{60B_6^3-72B_6^2+21B_6-1}\right)\left(\frac{3B_6-1}{12B_6^2-9B_6+1}\right)\left(\frac{1}{3B_6-1}\right)\frac{1}{2B_6}} \\
&= 120B_6^6 - 240B_6^5 + 150B_6^4 - 30B_6^3 + B_6^2
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_1(B_6) &= \frac{\Gamma_2(B_6)}{1 + B_6\,(1-B_6)\,\Gamma_2'(B_6)} \\
&= \frac{\frac{12B_6^2-9B_6+1}{60B_6^3-72B_6^2+21B_6-1}}{1 + B_6\,(1-B_6)\left(-12\frac{60B_6^4-90B_6^3+48B_6^2-10B_6+1}{\left(60B_6^3-72B_6^2+21B_6-1\right)^2}\right)} \\
&= \frac{60B_6^3 - 72B_6^2 + 21B_6 - 1}{360B_6^4 - 600B_6^3 + 300B_6^2 - 45B_6 + 1}.
\end{aligned}
$$

Finally, to solve for B_6, use the fundamental equation

$$
B_6(1-B_6) = \Phi_1(B_6)\prod_{j=1}^{5}\Gamma_j(B_6)
$$

or, namely,

$$
B_6(1-B_6) = \left(120B_6^6 - 240B_6^5 + 150B_6^4 - 30B_6^3 + B_6^2\right)
$$

$$
\begin{aligned}
&\times\left(\frac{60B_6^3-72B_6^2+21B_6-1}{360B_6^4-600B_6^3+300B_6^2-45B_6+1}\right)\\
&\times\left(\frac{12B_6^2-9B_6+1}{60B_6^3-72B_6^2+21B_6-1}\right)\\
&\times\left(\frac{3B_6-1}{12B_6^2-9B_6+1}\right)\left(\frac{1}{3B_6-1}\right)\frac{1}{2B_6}.
\end{aligned}
$$

The only sensible solution is $B_6 = \frac{1}{2} + \frac{1}{30}\sqrt{\left(150 + 15\sqrt{15}\right)} = .980\,85$. Therefore,

$$B_5 = B_6^2 = .962\,07$$

$$B_4 = 2B_6^3 - B_6^2 = .92522$$

$$B_3 = 6B_6^4 - 6B_6^3 + B_6^2 = .853\,64$$

$$B_2 = 24B_6^5 - 36B_6^4 + 14B_6^3 - B_6^2 = .716\,68$$

$$B_1 = 120B_6^6 - 240B_6^5 + 150B_6^4 - 30B_6^3 + B_6^2 = .46068$$

Thus,

$$
\begin{aligned}
b_1 &= .46068\\
b_2 &= .25600\\
b_3 &= .13696\\
b_4 &= .07158\\
b_5 &= .03685\\
b_6 &= .01878.
\end{aligned}
$$

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