Option pricing with Levy Process

Eric Benhamou*

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Abstract

In this paper, we assume that log returns can be modelled by a Levy process. We give explicit formulae for option prices by means of the Fourier transform. We explain how to infer the characteristics of the Levy process from option prices. This enables us to generate an implicit volatility surface implied by market data. This model is of particular interest since it extends the seminal Black Scholes [1973] model consistently with volatility smile.

1 Introduction

It is now widely accepted that markets differ from the seminal Black Scholes [1973] model. The empirical literature has extensively reported on these abnormalities, especially on two of them, which indeed are closely linked. First, it is has been shown that unconditional return show excess kurtosis and skewness, inconsistent with normality assumptions (see Mandelbrot [1963] and Fama [1965] for the former ones, Kon [1984] and Jorion [1988] for more recent works, Bates [1996] for more references). Second, research has concentrated its attention on the implied volatility smile or skew (see Dumas et al. [1995] for a survey). Interestingly, the second fact is just another hint of

*Financial Markets Group, London School of Economics, and Centre de Mathématiques Appliquées, Ecole Polytechnique. Address: FMG, Room G303, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom, Tel 0171-955-7895, Fax 0171-242-1006. E-mail: benhamou_e@yahoo.com

the non-normality of returns. However, research has focussed at implied Black Scholes volatility since implied volatility has become a key concept for option pricing. Option prices are often quoted by their implied volatility. A more rigorous justi&cation is the less volatile character as well as the better predictability of volatility compared to prices.

Research has extensively offered methods to cope with the smile effect. Classically, these attempts can be divided into two different families: parametric and non parametric ones.

In the parametric methods, the equation of the evolution of the underlying process is speci&ed as a particular functional form. This description can consist either in a continuous diffusion process with a so called deterministic volatility (Rubinstein [1994], Dupire [1993] and Derman and Kani [1994]) or with a stochastic volatility process (Hull and White [1987], Wiggings [1987], Melino and Turnbull [1990], Stein and Stein [1991], Amin and Ng [1993] and Heston [1992]) or in a model with jumps (Aase [1993], Ahn and Thompson [1988], Amin [1993], Bates [1991] , Jarrow [1984], Merton [1976]).

Other works close in spirit are assuming constant elasticity of volatility distribution often called power-law(Cox Ross [1976]) or a mapping principle between normal and lognormal distributions (Hagan [1998], Pradier and Lewicki [1999]).

The second type of methods is the inference of the underlying distribution from market data. This has been called the expansion methods where one infers the different terms of the expansion and can reconstitute the distribution (Jarrow and Rud [1982], Bouchaud et al.[1998], Abken et al. [1996]).

The motivation of this paper is to present a semi-parametric method for modelling the smile effect. We assume that the underlying price process can be modelled as Levy process. However, we give no speci&c conditions on the underlying price process except some technical conditions. Since Levy processes include continuous time diffusion as well as jump process, this approach encompasses many of the previous method. It extends the Black Scholes model to any type of Levy process for the underlying.

The remainder of this paper is organized as follows. In section 2, we introduce some characteristic of Levy processes, its Laplace and Fourier transform. Section 3 explains how to compute option prices. Section 4 examines the volatility smile issue. We conclude brie! y giving further developments.

2 Levy process and properties

2.1 Modelling assumptions

We consider a continuous time trading economy with in&nite horizon. The uncertainty in this economy is classically modeled by a complete probability space $(\Omega, F, \mathbb{Q})$. The information evolves according to the augmented &ltration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$ generated by a standard Brownian motion $(W_t)_{t\in\mathbb{R}^+}$. We assume that there exits a Levy process $(X_t)_{t\in\mathbb{R}^+}$. We de&ne a Levy process as a stochastic process, adapted to the Brownian &ltration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$, which satis&es the following property: $(X_t)_{t\in\mathbb{R}^+}$ is with independent stationary increments and is centered in its origin. $X_0 = 0$ almost surely. This Levy process is not assumed to be a stable Levy process, with stable Levy law. This process does not necessary satisfy a scaling law. We assume as well that the Laplace transform of the Levy process is bounded. There exits $\tau > 0$, $\lambda_u > 0$ so that for every $\lambda_d \in]-\infty, \lambda_u]$, $t \in [0, \tau]$, the Laplace transform $\lambda \longmapsto \mathbb{E}\left[e^{\lambda X_t}\right]$ is bounded by two strictly positive constant over $[\lambda_d, \lambda_u]$. There exist $B_d > 0$, $B_u > 0$ so that $\forall \lambda \in [\lambda_d, \lambda_u]$, $\forall t \in [0, \tau]$

$$B_d \leq \mathbb{E}\left[e^{\lambda X_t}\right] \leq B_u \tag{1}$$

We assume that the underlying $(S_t)_{t\in\mathbb{R}^+}$ can be modelled as a continuous time process, written as a function of a geometric Brownian motion times the exponential of a Levy process with no Brownian part. This leads to the following decomposition

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T + X_t} \tag{2}$$

It is worth noticing that the condition on the Laplace transform of the Levy process for negative values of λ implies that the event that the underlying equals zero, is of nul measure: $P\left(S_t = 0\right) = 0$ for every $t \in \mathbb{R}^+$. It is worth noticing as well that we take the de&nition of an "extended" Laplace transform since we allow for both positive and negative values for λ as opposed to the traditional Laplace only de&ned for positive values of λ. We can then introduce two characterizations of the Levy process. The &rst one is based on the Laplace transform, whereas the second on the Fourier transform

2.2 First Characterization of the Levy process

2.2.1 Exponent for the Laplace transform

Proposition 1 *There exists a function* $\phi :]-\infty, \lambda_u] \to \mathbb{R}$ *de&ned as for every* $t \in \mathbb{R}^+$, $\lambda \in]-\infty, \lambda_u]$

$$E\left[e^{\lambda X_t}\right] = e^{t\phi(\lambda)}$$

This function is called the Levy-Laplace exponent.

Proof: Let take $\lambda \in]-\infty, \lambda_u]$. We can &rst show that for every $t \in \mathbb{R}^+$, $\mathbb{E}\left[e^{\lambda X_t}\right]$ exits and is &nite $\left[e^{\lambda X_t}\right] < +\infty$. For every $n \in N^*$, $t \in \mathbb{R}^+$,

$$\begin{aligned} \mathbb{E}\left[e^{\lambda X_{nt}}\right] &= \mathbb{E}\left[e^{\lambda\left(X_{nt}-X_{(n-1)t}\right)} e^{\lambda X_{t(n-1)}}\right] \\ &= \mathbb{E}\left[e^{\lambda\left(X_{nt}-X_{(n-1)t}\right)}\right] \mathbb{E}\left[e^{\lambda X_{t(n-1)}}\right] \\ &= \mathbb{E}\left[e^{\lambda X_t}\right] \mathbb{E}\left[e^{\lambda X_{t(n-1)}}\right] \end{aligned}$$

where we have used in the last two equation &rst the independence between increments and second the stationarity of increments. This leads to

$$\mathbb{E}\left[e^{\lambda X_t}\right] = \mathbb{E}\left[e^{\lambda X_{t/n}}\right]^n$$

For n so that $t/n < \tau$, this proves that the above quantity exists and is bounded. The independence and stationarity of increments leads as well that for every $u \in \mathbb{R}^+$and $v \in \mathbb{R}^+$,

$$\mathbb{E}\left[e^{\lambda X_u}\right] \mathbb{E}\left[e^{\lambda X_v}\right] = \mathbb{E}\left[e^{\lambda X_{u+v}}\right]$$

This indicates that the function $f_\lambda : t \mapsto \mathbb{E}\left[e^{\lambda X_t}\right]$ satis&es

$$f_\lambda(u+v) = f_\lambda(u) . f_\lambda(v) \tag{3}$$

This function is as well continuous in zero. First, it is easy to see that this function should satisfy $f_\lambda(0) = 0$ or 1. Because of assumption (1), the only possible case is $f_\lambda(0) = 1$. Second, if the function were not continuous in zero, it would imply that there would exist a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of real numbers strictly decreasing to zero so that $\underset{\varepsilon\downarrow 0}{lim}\mathbb{E}\left[e^{-\lambda X_\varepsilon}\right]$ is not equal to 1. This would mean that there exist $\eta > 0$ so that

$$\left|\mathbb{E}\left[e^{\lambda X_\varepsilon}\right] - 1\right| > \eta$$

Denoting by $(q_n)_{n\in\mathbb{N}}$ a sequence of rational numbers so that $\varepsilon_n q_n \in \left[\frac{\tau}{2}, \tau\right]$, we have $q_n \underset{n\to+\infty}{\rightarrow} +\infty$ leading to

$$\mathbb{E}\left[e^{\lambda X_{q_n \varepsilon_n}}\right] = \left(\mathbb{E}\left[e^{\lambda X_{\varepsilon_n}}\right]\right)^{q_n}$$

We have then two cases: either $\mathbb{E}\left[e^{\lambda X_{\varepsilon_n}}\right] \geq 1+\eta$ for an in&nity of terms and the value $+\infty$ is an accumulation point of the sequence $(X_{\varepsilon_n})_{n\in\mathbb{N}}$ or it is the case of $\mathbb{E}\left[e^{\lambda X_{\varepsilon_n}}\right] \leq 1-\eta$ to be satis&ed by an in&nity of terms and the value 0 is an accumulation point of the sequence $(X_{\varepsilon_n})_{n\in\mathbb{N}}$. Both of cases contradict the original assumption (1). We have proved that the function $f_\lambda : t \mapsto \mathbb{E}\left[e^{\lambda X_t}\right]$ is an automorphism from $(R^+, +)$ to $(R^+, *)$ which is continuous in zero. It can therefore be written as an exponential function

$$f_\lambda(t) = e^{t\phi(\lambda)}$$

The preceding arguments assume only $\lambda \in]-\infty, \lambda_u]$. As a conclusion, we have built a function $\phi : t \mapsto \phi(\lambda)$.□

Remark 1 *It is straightforward to get the different momentum of the underlying process*

$$\mathbb{E}\left[S_t^\lambda\right] = S_0^\lambda e^{t\left(\lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2} + \phi(\lambda)\right)} \tag{4}$$

2.2.2 Particular Cases

Let us de&ne $(W_t)_{t\in\mathbb{R}^+}$ as a Brownian motion, $(N_t)_{t\in\mathbb{R}^+}$ a Poisson process of intensity θ and a sequence of independent variable independent identically distributed $(U_j)_{j\in\mathbb{N}^*}$ with value in $]-1,+\infty[$ and $U_0 = 1$ almost surely. The common law of the variables $(U_j)_{j\in\mathbb{N}^*}$ is denoted by $\mathcal{L}$, associated with a variable U. For $u \in]-\infty, 1]$, we assume that $\mathbb{E}\left[(1+U)^u\right]$ is &nite. Let us assume that the respective &ltrations spanned by $(W_t)_{t\in\mathbb{R}^+}$, $(N_t)_{t\in\mathbb{R}^+}$ and $(U_j)_{j\in\mathbb{N}^*}$ are independent. Let us denote by $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ the &ltration spanned by the stochastic variables $(W_s)_{s\leq t}$, $(N_s)_{s\leq t}$ and $(U_j)_{j\leq N_t}$. In this framework, $(W_t)_{t\in\mathbb{R}^+}$ is still a Brownian motion adapted to the &ltration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ respectively $(N_t)_{t\in\mathbb{R}^+}$ a Poisson process of intensity θ. The jump time of the Poisson process denoted by $(\tau_j)_{j\in\mathbb{N}^*}$ are still stopping time for the &ltration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$. We assume that the underlying security can be modelled as a risky asset with some stochastic jumps of stochastic intensity $(U_j)_{j\in\mathbb{N}^*}$ which occur according to the Poisson process $(N_t)_{t\in\mathbb{R}^+}$. Between two jumps, the risky asset can be modelled by a standard geometric Brownian motion, as in the Black Scholes model [1973] with a determinist drift μ and a volatility σ. This leads to the following formula for the underlying security

$$S_t = e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t}\prod_{j=1}^{N_t}(1+U_j) \tag{5}$$

with the convention that $\prod_{j=1}^{0}(1+U_j)=1$. This can be seen as a particular example of our example since in this case the Levy process is

$$X_t = \ln\left(\prod_{j=1}^{N_t}(1+U_j)\right)$$

We have the following proposition so as to calculate the Levy exponent $\phi :]-\infty, \lambda_u] \rightarrow \mathbb{R}$.

Proposition 2

$$\mathbb{E}\left[S_t^\lambda\right] = e^{t\left(\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2}+\theta\left(\mathbb{E}(1+U)^\lambda-1\right)\right)}$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[S_t^\lambda\right] &= \mathbb{E}\left(e^{\lambda\left(\mu-\frac{\sigma^2}{2}\right)t+\lambda\sigma W_t}\prod_{j=1}^{N_t}(1+U_j)^\lambda\right)\\
&= e^{\lambda\left(\mu-\frac{\sigma^2}{2}\right)t+\frac{(\lambda\sigma)^2 t}{2}}\,\mathbb{E}\left(\prod_{j=1}^{N_t}(1+U_j)^\lambda\right)
\end{aligned}$$

The last equation holds because of the independence of the stochastic variables $(W_s)_{s\leq t}$, $(N_s)_{s\leq t}$ and $(U_j)_{j\leq N_t}$. We have then

$$\begin{aligned}
\mathbb{E}\left(\prod_{j=1}^{N_t}(1+U_j)^\lambda\right) &= \sum_{n=0}^{+\infty}\mathbb{E}\left(\prod_{j=1}^{N_t=n}(1+U_j)^\lambda \,\middle|\, N_t=n\right)\mathbb{P}\left(N_t=n\right)\\
&= \sum_{n=0}^{+\infty\theta}\left(\mathbb{E}\left((1+U)^\lambda\right)\right)^n e^{-\theta t}\frac{(\theta t)^n}{n!}\\
&= \exp\left(\theta t\left(\mathbb{E}\left((1+U)^\lambda\right)-1\right)\right)
\end{aligned}$$

□

The proposition 2 leads to the following Levy exponent function:

$$\phi(\lambda) = \theta\left(\mathbb{E}\left((1+U)^\lambda\right)-1\right)$$

2.3 Second Characterization: Characteristic function

Another characteristic of the Levy process is its characteristic function which can be analyzed as its Fourier Transform. This has the advantage to be very ! exible since the Fourier transform $\mathbb{E}\left[e^{i\lambda X}\right]$ is always de&ned and no condition is required on the momentum. It is also numerically very efficient by means of Fast Fourier transform algorithms.

2.3.1 Fourier transform and characteristic function

Let us remind some preliminary results. We assume that X_t is a Levy process. We have the following proposition:

Proposition 3 *There exists a function $\psi :]-\infty,+\infty[\to \mathbb{R}$ so that for every $t \in \mathbb{R}^+$, for every $\lambda \in \mathbb{R}$*

$$\mathbb{E}\left[e^{i\lambda X_t}\right] = e^{t\psi(\lambda)}$$

with $\psi(\lambda) = i\mu\lambda - \frac{\sigma^2}{2}\lambda^2 - \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}} \Pi(dx)\right)$ with $\mu \in \mathbb{R}$ and Π, called the Levy measure, is a positive measure on $\mathbb{R}$ so that $\int_{\mathbb{R}} Min\left(1, x^2\right) \Pi(dx) < \infty$. The term $\frac{\sigma^2}{2}\lambda^2$ is called the Brownian part of the process. The parameter μ is the drift of the Levy process.

Proof: See Bertoin [1997].□

The function $\psi :]-\infty,+\infty[\to \mathbb{R}$ is called the Levy-Khintchine exponent. To estimate the Brownian part, we use the following proposition:

Proposition 4 *There exists a limit to the following ratio:*

$$\lim_{|\lambda|\to\infty} \frac{\psi(\lambda)}{\lambda^2} = -\frac{\sigma^2}{2}$$

Proof: We &rst notice that

$$\lim_{|\lambda|\to\infty} \frac{1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}}{\lambda^2} = 0$$

It can be shown (see Bertoin [1997]) for every $x \in \mathbb{R}$

$$|1 - \cos x| \leq 2Min\left(1, x^2\right)$$

as well as for every $x \in \mathbb{R}$, for every $\lambda \in \mathbb{R}$

$$\left|-\sin(\lambda x) + \lambda x 1_{\{|x|<1\}}\right| \leq Min\left(1 + |\lambda|, (\lambda x)^2\right)$$

Combining the different results, we get that for $|\lambda| \geq 2$

$$\frac{\left|1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\right|}{\lambda^2} \leq 3Min\left(1, x^2\right)$$

We can conclude by dominated convergence that

$$\lim_{|\lambda|\to\infty} \frac{1}{\lambda^2} \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}} \Pi(dx)\right) = 0$$

□

2.3.2 Particular case

In the case of the process given in the section 2.2.2 by equation (5), we can calculate explicitly the characteristic function. This is summarized by the following proposition

Proposition 5 *For every $t \in \mathbb{R}$, and every $\lambda \in \mathbb{R}$*

$$\mathbb{E}\left[e^{i\lambda X_t}\right] = e^{t\left(i\lambda\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\sigma^2\lambda^2}{2} + \theta\left(\mathbb{E}(1+U)^{i\lambda} - 1\right)\right)}$$

Proof: same as in proposition 2.□

3 Option price

The interest of this extension of the Black Scholes model is its tractability. We can &nd explicit formula for the price of vanilla option as shown in the following subsection. To get a price, we assume that the discounted asset is a martingale under the natural probability measure $\mathbb{Q}$ of our probability space $(\Omega, F, \mathbb{Q})$. We impose this restriction so as to be have a unique martingale-measure used for pricing purposes. It can be shown that there exists an in&nity of equivalent martingale measures under which $\left(e^{-rT} S_T\right)_{T\in\mathbb{R}}$ is a martingale. The correct price is obtained as the expected discounted payoff under this measure. The price of the call option of strike K, maturity T, with an initial underlying level of S_0 calculated as the expectation of the discounted payoff $e^{-rT}(S_T - K)^+$ under one of this equivalent martingale measure is dense in the interval of $\left[\left(S_0 - e^{-rT}K\right)^+, S_0\right]$ (see Erberlein and Jacod [1997]).

3.1 Laplace transform and option price

The &rst restriction on the process is that $\left(e^{-rT}S_T\right)$ is a martingale under $\mathbb{P}$. This leads to the following constraints:

Proposition 6

$$\mu = r - \phi(1) \tag{6}$$

Proof: $\left(e^{-rt}S_t\right)_{t\in\mathbb{R}^+}$ is a martingale. This implies that

$$\mathbb{E}\left[e^{-rt}S_t\right] = S_0$$

or using the proposition 2 this leads to

$$e^{t\left(-r+\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}+\phi(1)\right)} = 1$$

or the condition (6).□

3.1.1 General formula

Let us denote by P^{Call} respectively P^{Put} the price of a call option, respectively the price of a put option, and by $P^{Call}_{BS}(S_0, K, T, r, \sigma)$ (respectively $P^{Put}_{BS}(S_0, K, T, r, \sigma)$) the price of a Black Scholes call (respectively put) with an initial underlying level of S_0 a strike of K, a maturity of T, a risk free rate of r and a volatility of σ. Similarly to the formula given by Hull and White [1987] for stochastic volatility, the price of the option is given by the following proposition

Proposition 7 *The price of a vanilla option is given by*

$$P^i = \mathbb{E}\left[P^i_{BS}\left(S_0e^{-T\phi(1)}e^{X_T}, K, T, r, \sigma\right)\right] \tag{7}$$

where i stands for either call or put.

Proof: Since a put option on S_T with strike can be seen as a call option on $(-S_T)$ with a strike $-K$ $(K - S_T)^+ = (-S_T - (-K))^+$, we only examine the case of the call option. The price of the option is calculated as the expectation of the discounted payoff

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] \\
= \;&\mathbb{E}\left[e^{-rT}\left(S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_t}e^{X_T} - K\right)^+\right] \\
= \;&\mathbb{E}\left[e^{-rT}\left(\left(S_0e^{-T\phi(1)}\right)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_t}e^{X_T} - K\right)^+\right]
\end{aligned}$$

where we have used the no-arbitrage condition on the drift term (6). Using conditional expectation, we get

$$\begin{aligned}&\mathbb{E}\left[e^{-rT}\left(S_T-K\right)^+\right]\\ =\;&\mathbb{E}\left[\mathbb{E}\left[e^{-rT}\left(\left(S_0e^{-T\phi(1)}e^{X_T}\right)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_t}-K\right)^+\middle| X_T\right]\right]\end{aligned}$$

Where the conditional expectation can be interpreted in the Black Scholes model as a closed formula, leading to the &nal result.□

The same methodology can be applied to binary and range option with payoff equal to

$$f(x)=1_{\{e^x>K\}}$$

for a binary option and

$$f(x)=\left(e^x-K_1\right)1_{\{K_1\leq e^x<K_2\}}$$

Denoting by P_{Bs}^{Bin} respectively P_{BS}^{Range} the Black Scholes price of a binary respectively a range option, we have the following proposition

Proposition 8 *The price of a binary respectively a range option is given by*

$$P^i=\mathbb{E}\left[P_{BS}^i\left(S_0e^{-T\phi(1)}e^{X_T},K,T,r,\sigma\right)\right] \tag{8}$$

where i stands for either Bin or Range.

Proof: same as above.□

3.1.2 Particular case

In the case of the process given in the section 2.2.2 by equation (5) this leads to the following results

Proposition 9 *The no arbitrage condition (6) leads to*

$$\mu=r-\theta\mathbb{E}(U) \tag{9}$$

and the price of a vanilla option is given by

$$P^i=\mathbb{E}\left[P_{BS}^i\left(S_0e^{-T\theta\mathbb{E}(U)}\prod_{j=1}^{N_T}\left(1+U_j\right),K,T,r,\sigma\right)\right] \tag{10}$$

Let us now assume that the intensity process of the jumps modelled by the variable $(1+U)$ follows a lognormal distribution with mean m and volatility v^2. We then get an explicit formula for the vanilla option price, as stated by the following proposition, similar to the Merton s formula [1976]:

Proposition 10

$$P^i = e^{-\theta(1+c)T} \sum_{i=1}^{+\infty} \frac{(\theta(1+k)T)^n}{n!} P^i_{BS}(S_0, K, T, r_n, \sigma_n) \tag{11}$$

with

$$\begin{aligned} c &= \mathbb{E}[U] = e^{m+v^2/2} - 1 \\ r_n &= r - \theta c + \frac{n\left(m + v^2/2\right)}{T} \\ \sigma_n &= \left(\sigma^2 + \frac{nv^2}{T}\right)^{1/2} \end{aligned}$$

with $i = c$ *or* p *corresponding to a call or put option.*

Proof: Introducing the variable $\varepsilon = 1$ for a call option and $\varepsilon = -1$ for a put option, we treat the two options in the same way. Since $U+1$ follows a lognormal law with mean m and volatility v^2, the expectation of U is given by $\mathbb{E}[U] = e^{m+v^2/2} - 1$. Denoting by $S_0 e^{-Tc\theta}$ and using the pricing formula (10), we can write the price of the option

$$P^i = \sum_{i=0}^{+\infty} \mathbb{E}\left[P^i_{BS}\left(S_0 e^{-Tc\theta} \prod_{j=1}^{n}(1+U_j), K, T, r, \sigma\right)\right] e^{-\theta T} \frac{(\theta T)^n}{n!} \tag{12}$$

The random variables $(1+U_i)_{i\in\mathbb{N}^*}$ are independently distributed, with a lognormal distribution with mean m and volatility v^2. Their product $\prod_{j=1}^{n}(1+U_j)$ follows a lognormal distribution with mean nm and volatility nv^2. Denoting by g a centered normalized normal distribution ($g \sim N(0,1)$), by $S = S_0 e^{-Tc\theta}$, the equation (12) can be rewritten as

$$P^i = \sum_{i=0}^{+\infty} \mathbb{E}\left[\begin{array}{c} \varepsilon S e^{nm+\sqrt{n}vg} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}\right)\right) \\ -\varepsilon K e^{-rT} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}\right)\right) \end{array}\right] e^{-\theta T} \frac{(\theta T)^n}{n!} \tag{13}$$

It can be shown that for $g \sim N(0,1)$ and for every $a, b, c \in \mathbb{R}$,

$$\mathbb{E}\left[e^{ag} N(bg+c)\right] = e^{a^2/2} N\left(\frac{c+ab}{\sqrt{1+b^2}}\right) \tag{14}$$

Using the equation (14), we get

$$\mathbb{E}\left[\begin{array}{c}\left(\varepsilon S e^{nm+\sqrt{n}vg}\right)\\ N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}}+\frac{\sigma}{2}\sqrt{T}\right)\right)\end{array}\right]$$

$$= \varepsilon S e^{nm+\frac{nv^2}{2}} N\left(\varepsilon\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\frac{\sigma^2}{2}T+nv^2}{\sigma\sqrt{T}\sqrt{1+\frac{nv^2}{\sigma^2 T}}}\right)$$

and

$$\mathbb{E}\left[\varepsilon K e^{-rT} N\left(\varepsilon\left(\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm+\sqrt{n}vg}{\sigma\sqrt{T}}-\frac{\sigma}{2}\sqrt{T}\right)\right)\right]$$

$$= \varepsilon K e^{-rT} N\left(\varepsilon\frac{\ln\left(\frac{S}{e^{-rT}K}\right)+nm-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}\sqrt{1+\frac{nv^2}{\sigma^2 T}}}\right)$$

which can be rewritten as

$$P^i = \sum_{i=0}^{+\infty}\left[\begin{array}{c}\varepsilon S_0 e^{\left(\frac{nm+\frac{nv^2}{2}}{T}-\theta c\right)T}\\ N\left(\varepsilon\frac{\ln\left(\frac{S_0}{K}\right)+\left(\frac{n\left(m+\frac{v^2}{2}\right)}{T}+r-\theta c\right)T+\frac{1}{2}\left(\sigma^2+\frac{nv^2}{T}\right)T}{\left(\sigma^2+\frac{nv^2}{T}\right)^{1/2}\sqrt{T}}\right)\\ -\varepsilon K e^{-rT} N\left(\varepsilon\frac{\ln\left(\frac{S_0}{K}\right)+\left(\frac{n\left(m+\frac{v^2}{2}\right)}{T}+r-\theta c\right)T-\frac{1}{2}\left(\sigma^2+\frac{nv^2}{T}\right)T}{\left(\sigma^2+\frac{nv^2}{T}\right)^{1/2}\sqrt{T}}\right)\end{array}\right] e^{-\theta T}\frac{(\theta T)^n}{n!}$$

which leads to the result (11).□

Corollary 11 *The price of a binary or a range option is given by*

$$P^i = e^{-\theta(1+c)T}\sum_{i=1}^{+\infty}\frac{(\theta(1+k)T)^n}{n!}P^i_{BS}(S_0,K,T,r_n,\sigma_n) \quad (15)$$

with

$$\begin{aligned} c &= \mathbb{E}[U] = e^{m+v^2/2}-1\\ r_n &= r-\theta c+\frac{n\left(m+v^2/2\right)}{T}\\ \sigma_n &= \left(\sigma^2+\frac{nv^2}{T}\right)^{1/2}\end{aligned}$$

with $i = Bin$ or $Range$ according to the option type

Proof: same as above.□

3.2 Fourier transform and convolution product

3.2.1 Preliminary results

We denote by $\mathcal{L}^1(\mathbb{R})$ the linear space of integrable function de&ned on $\mathbb{R}$, and by $F(f)$ the Fourier transform of the function f. It is interesting to see that we can interpret any expectation as the convolution product of a function with the density function of our stochastic variable. Let us assume that our underlying security can be written as the exponential of a Levy process with some Brownian part and drift term

$$S_t = e^{X_t}$$

Let $dp_{X_T}(x)$ be the probability measure of the process X_T. Any option price which can be written as the expectation of some discounted payoff can be reinterpreted as a convolution product as stated by the following proposition

Proposition 12 *Let f be a function $f : \mathbb{R} \to \mathbb{R}$, continuous, bounded, element of $\mathcal{L}^1(\mathbb{R})$ so that $F(f)$ belongs to $\mathcal{L}^1(\mathbb{R})$. We have*

$$\mathbb{E}[f(X_t)] = \left(F^{-1}\left(F\widetilde{f}(.)\,e^{T\psi(.)}\right)\right)(0) \tag{16}$$

with ψ the Fourier exponent as de&ned in proposition (3) and $F\widetilde{f}(.)$ the Fourier transform of the function $\widetilde{f} : \widetilde{f}(x) = f(-x)$

Proof: Let g denote the convolution product of $\widetilde{f}$ with dp_{X_T}

$$g(y) = \int_{-\infty}^{+\infty} \widetilde{f}(y-x)\,dp_{X_T}(x)$$

g is well de&ned since f and $\widetilde{f}$ is bounded. g is integrable since

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g(y)|\,dy &\leq \int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} \left|\widetilde{f}(y-x)\right| dp_{X_T}(x)\,dy \\
&\leq \int_{x=-\infty}^{+\infty}\left(\int_{y=-\infty}^{+\infty}\left|\widetilde{f}(y-x)\right| dy\right) dp_{X_T}(x) \\
&\leq \left\|\widetilde{f}\right\|_{\mathcal{L}^1(\mathbb{R})}\int_{x=-\infty}^{+\infty} dp_{X_T}(x) \\
&\leq \|f\|_{\mathcal{L}^1(\mathbb{R})}
\end{aligned}$$

The expectation of $f(X_t)$ can be seen as the value in zero of this convolution product since

$$\begin{aligned}\mathbb{E}[f(X_t)] &= \int_{-\infty}^{+\infty} f(x)\, dp_{X_T}(x) \\ &= \int_{-\infty}^{+\infty} \widetilde{f}(-x)\, dp_{X_T}(x) \\ &= g(0)\end{aligned}$$

The efficient way of calculating convolution product (see Benhamou [2000] is to use the property of Fourier transform. The Fourier transform of a convolution product is simply the product of the Fourier transform. This leads to multiply the two Fourier transform and invert the Fourier transform of the &nal result. We verify that $g(y)$ is a continuous function since by assumption f is continuous and bounded. We have veri&ed as well that g belongs to $\mathcal{L}^1(\mathbb{R})$. Using the fact that the Fourier transform of a convolution product is simply the product of the Fourier transform and that $F(f)$ belongs to $\mathcal{L}^1(\mathbb{R})$ as well as the characteristic function of the Levy process $F(p_{X_T})$, we get the $F(g)$ belongs as well to $\mathcal{L}^1(\mathbb{R})$. The validity of the Fourier transform and its inversion is then given by the lemma below.□

Lemma 13 *If f belongs to $\mathcal{L}^1(\mathbb{R})$ is a continuous function so that $F(f)$ belongs to $\mathcal{L}^1(\mathbb{R})$, then $F^{-1}(F(f)) = f$*

Proof: standard in the Fourier theory (see Bracewell [1965].□

We cannot apply the result of proposition 12 equation (16) straightforward to the call or put option. This is due to the fact that the payoff function equal to $(e^x - K)^+$ for a call or $(K - e^x)^+$ does not belong to $\mathcal{L}^1(\mathbb{R})$. However, it is possible to &nd a solution to this problem.

3.2.2 General formula

One solution is to use a truncated version of the payoff function. Let M_c, M_p be two real numbers satisfying

$$M_c > \ln K > M_p \tag{17}$$

Let α be a real number so that $\alpha > 1$ Let us de&ne the standard payoff of a vanilla option

$$f^c(x) = (e^x - K)^+$$

for a call option and

$$f^p(x) = (K - e^x)^+$$

the truncated payoff de&ned as

$$f^c_{M_c}(x) = (e^x - K)^+ \left(1_{\{x \leq M_c\}} + e^{-\alpha(x - M_c)} 1_{\{x > M_c\}}\right)$$

for a call option and

$$f^p_{M_p}(x) = (K - e^x)^+ \left(1_{\{x \geq M_p\}} + e^{-\alpha(x - M_p)} 1_{\{x < M_p\}}\right)$$

Proposition 14 *These two functions are continuous, with positive values. They belongs as well to $\mathcal{L}^1(\mathbb{R})$ and are bounded. They satisfy as well that their Fourier transform belongs to $\mathcal{L}^1(\mathbb{R})$.*

Proof: We examine the case of the call truncated payoff. The proof goes along the same line for the second function. The function $f^c_{M_c}(x)$ is continuous as the product of continuous functions. It is positive as the product of two positive functions. It is integrable since for values of x smaller than $\ln K$, it is equal to zero and for large values of x it is equivalent to the function $e^{-(\alpha-1)x} e^{\alpha M_p}$. Moreover, this implies that this function is bounded since it is a continuous function with asymptotic limits equal to zero. Its Fourier transform can be written as

$$\begin{aligned}
& \int_{-\infty}^{+\infty} f^c_{M_c}(x) e^{i\lambda x} dx \\
= & \int_{\ln K}^{M_c} (e^x - K) e^{i\lambda x} dx + \int_{M_c}^{+\infty} (e^x - K) e^{-\alpha(x - M_c)} e^{i\lambda x} dx \\
= & -\frac{K e^{i\lambda \ln K} (\lambda + i)}{(1 + \lambda^2) \lambda} - K\alpha e^{i\lambda M_c} \frac{\lambda - i\alpha}{\lambda (\alpha^2 + \lambda^2)} \\
& + \alpha e^{(1 + i\lambda) M_c} \frac{\lambda^2 + (\alpha - 1) + i\lambda (2 - \alpha)}{(1 + \lambda^2) \left((\alpha - 1)^2 + \lambda^2\right)}
\end{aligned}$$

For large values of λ, the three different terms are equivalent to $\frac{1}{\lambda^2}$, which proves the absolute integrability of the Fourier transform of $f^c_{M_c}$.□

We can then prove that these functions converges to the call and put option when $|M|$ tends to in&nity.

Proposition 15 *for i representing either a call or a put, the expectation of the truncated payoff converges to the standard payoff*

$$\lim_{|M|\to+\infty} \mathbb{E}\left[e^{-rT} f^i_{M_i}\right] = \mathbb{E}\left[e^{-rT} f^i\right]$$

with $i = c$ or p representing either a call or a put option

Proof: Let ε be equal to 1 (call) or -1 (put) according to the option type. We have for every $M_i \in \mathbb{R}$ satisfying the conditions (17)

$$\begin{aligned} \int \left|e^{-rT} f^i_{M_i}\right| dp_{X_T}(x) &\leq \int \left|e^{-rT} f^i\right| dp_{X_T}(x) \\ &\leq +\infty \end{aligned}$$

as well as

$$\lim_{|M|\to+\infty} f^i_{M_i} = f^i$$

This gives the result by dominated convergence.□

The same methodology can be applied to the case of a binary option. The truncated function can be the following

$$f^{Range}_{M_{Range}}(x) = 1_{\{e^x \geq K\}} \left(1_{\{x\leq M_c\}} + e^{-\alpha(x-M_c)} 1_{\{x>M_c\}}\right)$$

with $\alpha > 0$.

3.2.3 Methodology for option pricing with the Fourier transform

We have seen that when we know the Fourier exponent of the Levy process as well as the Fourier transform of the truncated payoff, using the proposition 12, we need to multiply these two function and invert their Fourier transform. The proposition 15 shows us that the limit of these truncated payoff option converges to the standard option. This gives us a methodology for pricing option with the Fourier transform.

4 Volatility Smile

The true motivation of Levy process is to infer some characteristic of our process that take account for the volatility smile. Let us remind the result of static replication (see Carr [1997]), which is used for the static hedging of derivatives product

Proposition 16 *If f belongs to $C^2(\mathbb{R}^{+*})$ (set of functions twice differentiable with continuous second order derivative function), then for every $x \in \mathbb{R}^+$, for every $\kappa \in \mathbb{R}^+$*

$$f(x) = \int_0^\kappa (x-u)^+ f''(u)\, du + \int_\kappa^{+\infty} (u-x)^+ f''(u)\, du$$

Proof: see Car [1997].□

4.1 Estimation of the Levy process

Let $\lambda \leq 0$. If we want to estimate the Laplace exponent, an interesting property is to use the proposition 16, leading to the following proposition:

Proposition 17 *Denoting by $Put(u)$ (respectively $Call(u)$) the price of a put (respectively a call) on S_T with strike u, we can calculate the Levy-Laplace exponent as*

$$\begin{aligned} & \phi(\lambda) \\ = & \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty} e^{rT} Put(u)\, u^{(\lambda-2)} du\right) - \lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2} \\ = & \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty} e^{rT} Call(u)\, u^{(\lambda-2)} du\right) - \lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2} \end{aligned}$$

Proof: The function $f : x \to x^\lambda$ belongs to $C^2(\mathbb{R}^{+*})$. The proposition 16 leads to

$$\mathbb{E}\left[\frac{S_T^\lambda}{\lambda(\lambda-1)}\right] = \int_0^\kappa \mathbb{E}\left[(S_T-u)^+ u^{(\lambda-2)}\right] du + \int_\kappa^{+\infty} \mathbb{E}\left[(u-S_T)^+ u^{(\lambda-2)}\right] du$$

using the two limiting case $\kappa = 0$ and $\kappa = +\infty$, we get

$$\begin{aligned} \mathbb{E}\left[\frac{S_T^\lambda}{\lambda(\lambda-1)}\right] &= \int_0^{+\infty} e^{rT} Put(u)\, u^{(\lambda-2)} du \\ &= \int_0^{+\infty} e^{rT} Call(u)\, u^{(\lambda-2)} du \end{aligned}$$

Using the de&nition of the Levy exponent (equation (4)) , we get the &nal result.□

4.2 Particular case

In the case of the process given in the section 2.2.2 by equation (5), we get that for the Levy-Laplace exponent the following formula

$$\begin{aligned} \phi(\lambda) &= \theta\left(\mathbb{E}\left((1+U)^\lambda\right) - 1\right) \\ &= \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty} e^{rT} Put(u)\, u^{(\lambda-2)} du\right) - \lambda\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2\lambda^2}{2} \end{aligned}$$

which gives us a way of calibrating our model by means of call and put prices of the markets. If we assume furthermore that $1+U$ follows a lognormal law with mean m and volatility v^2, we get the closed formula

$$\begin{aligned} & \theta\left(e^{\lambda m+\frac{\lambda v^2}{2}}-1\right) \\ = & \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty}e^{rT}Put(u)\,u^{(\lambda-2)}du\right)-\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2} \\ = & \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty}e^{rT}Call(u)\,u^{(\lambda-2)}du\right)-\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2}\end{aligned}$$

The same case provides us as well an explicit formula for the Levy-Khintchine exponent

$$\psi(\lambda)=i\mu\lambda-\frac{\sigma^2}{2}\lambda^2+\theta\left(e^{im\lambda-\frac{v^2}{2}\lambda^2}-1\right)$$

we need to estimate the different parameters so that the price of the different options assuming this Levy process is consistent with market data.

4.3 Implication

We have implemented this for different level of λ, taking σ equal to the implied Black Scholes volatility, μ satisfying the no-arbitrage condition (6)

$$\mu=r-\theta\left(e^{m+\frac{v^2}{2}}-1\right)$$

If the model above is realistic, we should get that the following function

$$\begin{aligned} g(\lambda) &= \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty}e^{rT}Call(u)\,u^{(\lambda-2)}du\right)-\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2} \\ &= \frac{1}{T}\ln\left(\frac{\lambda(\lambda-1)}{S_0^\lambda}\int_0^{+\infty}e^{rT}Put(u)\,u^{(\lambda-2)}du\right)-\lambda\left(\mu-\frac{\sigma^2}{2}\right)+\frac{\sigma^2\lambda^2}{2}\end{aligned}$$

should be equal to a polynomial expression in λ since it is equal to $\theta\left(e^{m+\frac{v^2}{2}}\right)^\lambda-\theta$.

4.4 Impact on the smile

This leads to the complicated issue of calibrating the model. This is a very complicated issue and no simple answer exits. However, we have taken different values of parameters for the Levy process and we have obtained realistic form of smiles. This is summarized by the &gure 1, which shows the evolution of the implied volatility with the time to maturity for the set of parameters $\theta=1$, $m=-0.15$, $v^2=0.20$

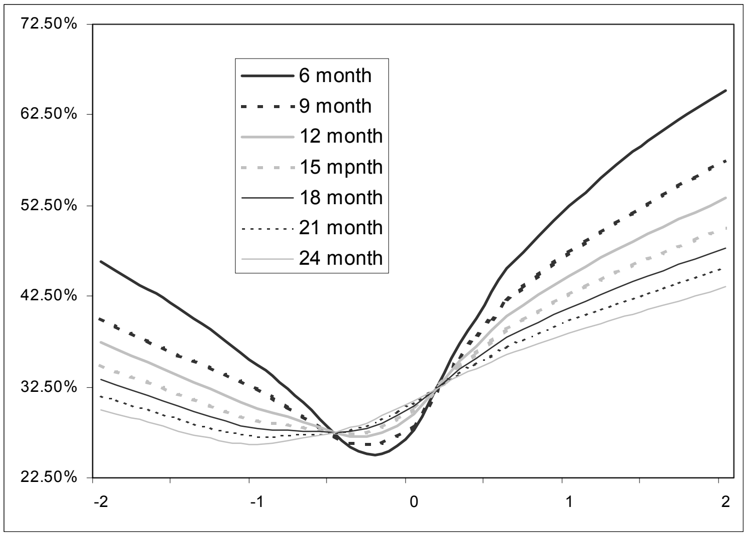

Figure 1: Volatility Smile implied by a Levy process with jumps modelled as a Poisson process with a lognormal intensity

5 Conclusion

In this paper, we have seen that the use of Levy processes enables us to take account for the volatility smile. The approach adopted here is a semi-parametric one based on a Levy process description of our economy. This has the great advantage to encompass many previous works since Levy process includes Brownian motion as well as many jump processes. We show that the Fourier transform can lead to an efficient way of getting prices after inferring the Levy-Khintchine exponent.

There are many possible extensions to this work. The &rst one concerns some empirical studies to quantify the &t of this model with market data. This is a complicated issue like all calibration procedure and test of goodness of &t. The second one is to develop more efficient numerical procedure based on Fast Fourier Transform algorithm that take account of the particular situation developed here.

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