A Generalisation of Malliavin Weighted Scheme for Fast Computation of the Greeks

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Abstract

This paper presented a new technique for the simulation of the Greeks (i.e. price sensitivities to parameters), efficient for strongly discontinuous payoff options. The use of Malliavin calculus, by means of an integration by parts, enables to shift the differentiation operator from the payoff function to the diffusion kernel, introducing a weighting function.(Fournie et al. (1999)). Expressing the weighting function as a Skorohod integral, we show how to characterize the integrand with necessary and sufficient conditions, giving a complete description of weighting function solutions. Interestingly, for adapted process, the Skorohod integral turns to be the classical Ito integral.

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1 Introduction

Since price sensitivities are an important measure of risk, growing emphasis on risk management issues has suggested a greater need for their efficient computation. Collectively referred to as "the Greeks", theses sensitivities are mathematically defined as the derivatives of a derivative security's price with respect to various model parameters.

The traditional way to compute the Greeks is to take its finite difference approximation. If we denote by $P(x)$ the price of the option with an initial underlying value of x, one calculate the delta with $\left(P(x+\varepsilon)-P(x)\right)/\varepsilon$. This can produce a significant error since one takes the difference of terms which are already approximations. When looking at Monte Carlo and Quasi Monte Carlo methods, Glynn (1989) showed that the quality of this approximation was depending on the way of approximating the derivative: forward difference $\left(P(x+\varepsilon)-P(x)\right)/\varepsilon$, central difference $\left(P(x+\varepsilon)-P(x-\varepsilon)\right)/2\varepsilon$, or even backward difference scheme $\left(P(x)-P(x-\varepsilon)\right)/\varepsilon$. In the case of the forward and backward difference scheme, if the simulation of the two estimators of $P(x+\epsilon)$ and $P(x)$ or $P(x)$ and $P(x-\varepsilon)$ is drawn independently, he proved that the best theoretical convergence rate is $n^{-1/4}$. As of the central difference scheme, the optimal rate is $n^{-1/3}$. When taking common random numbers, this optimal rate becomes $n^{-1/2}$. This is the best to be expected by standard Monte Carlo simulation as described by Glasserman and Yao (1992), Glynn (1989), and L'Ecuyer and Perron (1994). However, the finite difference method is inefficient when dealing with discontinuous payoffs. This restriction applies to many of the exotic options such as digital, corridor, Asian and lookback options.

To overcome this poor convergence rate, Curran (1994), (1998) and Broadie and Glasserman (1996) suggested to take the differential of the payoff function inside the expectation required to compute a price. This leads to a convergence rate of $n^{-1/2}$. However, this can be applied only to simple payoff functions. Fournie et al. (1999) extended their method to payoffs depending on a finite set of dates, in very general conditions. The original idea comes from a result by Elworthy (1992) which suggests, in a probabilistic framework, to shift the differential operator from the payoff functional to the diffusion kernel, introducing a weighting function. They came to the central result that the common Greeks could be written as an expected value of the payoff times a weighting function.

$$Greek=\mathbb{E}^{Q}\left[e^{-\int_0^T r_s ds}f(X_T).weight\right] \tag{1}$$

The theoretical tool used was the stochastic calculus of variations, traditionally called Malliavin calculus. Their results were given for particular examples of weighting functions. However, a natural question, starting point of this research was to examine all the weighting functions and to determine which conditions a weighting function should satisfy. That is precisely the motivation of this paper.

The contributions of this paper are to characterize by necessary and sufficient conditions the weighting functions in the Malliavin weighted scheme. Expressing weighting functions as Skorohod integral, we introduce the wrighting function generator defined as the Skorohod integrand. We show that these functions can be characterized by necessary and sufficient conditions on their generator. We then examine the different weighting functions and show how to find the one with minimal variance. We then give some key examples of the weighting function generator. We finally discussed the issue of the most appropriate weighted function.

The remainder of this article is organized as follows. In section 2, we explicit the intuition of the methodology with the Black Scholes model as well as some preliminary definitions and results. In section 3, we derive the necessary and sufficient conditions for the weighting function generator. In section 4, we show different example for the weighting function generator. We conclude in section 5. For clarity reason, all the proofs which turned out to be quite involved are given in the appendix section.

2 Mathematical framework and preliminary results

2.1 Intuition

In this subsection, we show by means of the Black Scholes (1973) model, how we derive a formula that reduces the variance of the Greeks when computed by simulation methods. The core of our methodology lies in an integration by parts formula. This allows us to avoid taking the derivative of the payoff functional and to shift the differential operator on the diffusion kernel.

Following Harrison and Kreps (1979), Harrison and Pliska (1981), the price of a contingent claim is traditionally calculated as the expected value of the discounted payoff value in the risk neutral probability measure Q uniquely defined in complete markets with no-arbitrage. We consider a continuous time trading economy with a finite horizon $t \in [0,T]$. The uncertainty in this economy is classically modelized by a complete probability space (Ω, F, Q). The information evolves according to the augmented filtration $\{F_t, t \in [0,T]\}$ generated by a standard one dimensional standard Wiener process $(W_t)_{t\in[0,T]}$. The price $P(x)$ of our contingent claims at time $t=0$ with expiry date T is defined by the expected value of the discounted payoff function at expiry $f(X_T)$ (for a call $f(X_T) = (X_T - K)_+$) conditionally to the present information, described by σ-algebra $F_{t=0}$

$$P(x) = \mathbb{E}^Q \left[f(X_T) e^{-\int_0^T r_s ds} | F_0 \right] \tag{2}$$

$\mathbb{E}^Q[.]$ is the expectation under the risk neutral measure Q, X_t is the underlying price, and r_s is the risk free rate. Following Black Scholes assumptions, the underlying, either an equity, a commodity, an interest rate or an index price, follows a geometric Brownian motion characterized by the following diffusion equation:

$$\frac{dX_t}{X_t} = rdt + \sigma dW_t \tag{3}$$

Let us denote by X_T the unique continuous strong solution of (3) with initial condition x ($X_0 = x$). Replacing in (2) X_T by its probability density function gives us that the price $P(x)$ can be written as an explicit integral:

$$P(x) = \int_{-\infty}^{+\infty} e^{-rT} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

To calculate a Greek based on this formula, and for the clarity of the proof, we chose the delta, sensitivity of the price with respect to the underlying initial price x, traditional methods compute numerically the finite difference between two shifted priced, which leads in the case of a centered scheme to:

$$delta \simeq \frac{P\left(x+\frac{\varepsilon}{2}\right) - P\left(x-\frac{\varepsilon}{2}\right)}{\varepsilon}$$

Its continuous limit leads then to take the derivative of the payoff function since the expression $\frac{f\left(X_T^{x+\frac{\varepsilon}{2}}\right) - f\left(X_T^{x-\frac{\varepsilon}{2}}\right)}{\varepsilon}$ inside the expectation operator in (4) tends to the derivative of the function f as ε tends to zero.

$$\frac{P\left(x+\frac{\varepsilon}{2}\right) - P\left(x-\frac{\varepsilon}{2}\right)}{\varepsilon} = \mathbb{E}^Q\left[e^{-\int_0^T r_s ds}\left(\frac{f\left(X_T^{x+\frac{\varepsilon}{2}}\right) - f\left(X_T^{x-\frac{\varepsilon}{2}}\right)}{\varepsilon} | F_0\right)\right] \tag{4}$$

The driving idea of this article is to avoid taking the derivative of the function, by doing an integration by parts. Assuming that $f(.)$ is a.s. differentiable with derivatives with polynomial growth[1], we can show that the derivative with respect to x is proportional to the derivative with respect to y:

$$\frac{\partial}{\partial x} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}) = \frac{1}{x\sigma\sqrt{T}} \frac{\partial}{\partial y} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T})$$

leading to the following integration by parts:

$$\begin{aligned}
\frac{\partial}{\partial x} P &= \frac{\partial}{\partial x}\left(\int_{-\infty}^{+\infty} e^{-rT} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\right) \\
&= e^{-rT}\left[\frac{1}{x\sigma\sqrt{T}} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right]_{-\infty}^{+\infty} \\
&\quad + e^{-rT} \int_{-\infty}^{+\infty} \frac{1}{x\sigma\sqrt{T}} f(xe^{rT+\sigma\sqrt{T}y-\frac{1}{2}\sigma^2 T}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} y dy
\end{aligned}$$

[1]These are assumptions that justify the interchange of the integration and the differential operator by dominated convergence.

This enables us to write the delta as the expectation of the discounted payoff times a weighting function:

$$\frac{\partial P}{\partial x} = \mathbb{E}^{Q}(\frac{e^{-rT}}{x\sigma T} W_T f(X_T))$$

In the above formula, the differential operator has disappeared. Instead, this methodology has introduced a weighting function $\frac{e^{-rT}}{x\sigma T} W_T$. The weight is not depending on the pay-off function and is easy to simulate. This indicates that the efficiency of this method does not depend on the payoff type. On the contrary, the standard way to compute the Greeks relies on the payoff function since it takes the finite difference approximation of the derivative of the payoff function (4). Since this integration by parts method smoothens the payoff function with a weight independent from the payoff function, it is all the more efficient that the payoff function is discontinuous. This is the case of digital, simple, double barrier and many other exotic options. Furthermore, we can conjecture that this method should be more efficient for second order Greeks, like gamma, than first order one, like delta. Moreover, this methodology should provide us equivalent rate of convergence for the Greeks as for the price. The only difference between the price simulation and the Greek simulation comes from a weighting function to simulate.

2.2 Notations and hypotheses

To avoid heavy notations, and for clarity reason, we present our results in one dimension. However, our results can easily be extended to the multi-dimensional case. Following the traditional literature on continuous time option pricing, the evolution of the underlying price, Ito process $(X_t)_{t\in[0,T]}$, is described by a very general stochastic differential equation (SDE):

$$dX_t = b\,(t, X_t)\,dt + \sigma(t, X_t)dW_t \tag{5}$$

with the initial condition $X_0 = x$, $x \in \mathbb{R}$. The function $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ represents the determinist drift of our process and the function $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ its volatility. The risk free interest rate is denoted by $r\,(t, X_t)$. We assume that:

- the functions b and σ are continuously differentiable with bounded derivatives and verify Lipschitz conditions, i.e., there exists a constant $K < +\infty$ such that

$$|b\,(t,x) - b\,(t,y)| + |\sigma\,(t,x) - \sigma\,(t,y)| \leq K\,|x-y| \tag{6}$$

$$|b\,(t,x)| + |\sigma\,(t,y)| \leq K\,(1+|x|) \tag{7}$$

Inequalities (6) and (7) are classical conditions to ensure the existence and unicity of a continuous, strong solution of the SDE (5) with its initial condition. We denote by X_t^x the continuous, strong solution X_t starting at x.

- the diffusion function $\sigma(t,x)$ is uniformly elliptic[2]:

$$\exists\epsilon > 0, \quad \forall t \in [0,T], \forall x \in \mathbb{R} \quad |\sigma(t,x)| \geq \epsilon \tag{8}$$

We denote by $(Y_t)_{t\in[0,T]}$ the first variation process of $(X_t)_{t\in[0,T]}$, which is characterized as the unique strong continuous solution of the linear stochastic differential equation (9) with initial condition ($Y_{t=0} = 1$):

$$\frac{dY_t}{Y_t} = b'(t, X_t)dt + \sigma'(t, X_t)dW_t \tag{9}$$

where the prime stands for the derivatives with respect to the second variable. We can show that the first variation process is the derivative of $(X_t)_{t\in[0,T]}$ with respect to x, $\left(Y_t = \frac{\partial}{\partial x}X_t\right)$. Malliavin calculus theory proves that the Malliavin derivative can be written as an expression of the first variation process as well as the volatility function:

$$D_s X_t = Y_t Y_s^{-1}\sigma(s, X_s)1_{\{s\leq t\}} a.s. \tag{10}$$

To be as general as possible, we assume that our payoff is depending on a finite set of payment dates: $t_1, t_2, ..., t_m$ with the convention that $t_0 = 0$ and $t_m = T$. The price $P(x)$ of the contingent claim given an initial value of the underlying price x is traditionally computed as the expectation under the risk neutral probability measure of discounted future cash flow:

$$P(x) = \mathbb{E}_x^Q\left[e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right)\right]$$

with the traditional shortcut notation $\mathbb{E}_x^Q[.] = \mathbb{E}^Q[.|X_0 = x]$. The function $f : \mathbb{R}\times\mathbb{R}\times...\times\mathbb{R} \to \mathbb{R}$ denotes the payoff, and is supposed to be first order differentiable with a derivative with polynomial growth. We denote by F the discounted payoff $F = e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1}, X_{t_2}, ..., X_{t_m}\right)$. If we need to specify that the underlying is a function of the initial value x, we denote the discounted payoff by F^x.

2.3 Generalizing Greeks

We take the common definition of the delta and gamma as the first respectively the second order derivative of the price with respect to the underlying process. However for the rho and vega since by assumptions, the drift and volatility terms are functions of the underlying and time, we need to develop a more robust framework than the common Black Scholes one. The meaning of the rho and vega is to quantify the impact of small perturbation, in a specified direction, on

[2] This is to ensure that we can find some solutions for the weighting functions, since it often requires to take the inverse of the volatility function.

either the drift term or the volatility term. We therefore define an "extended" rho as well as an "extended" vega defined as the derivative function of the price along a specified perturbation direction either on the drift term or the volatility term.

Let denote by $\widetilde{b} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ a direction function for the drift term and $\widetilde{\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for the stochastic term. We assume that, for every $\epsilon \in [-1, 1]$, $\widetilde{b}(.,.)$, $\left(b + \varepsilon \widetilde{b}\right)(.,.)$, $\widetilde{\sigma}(.,.)$ and $(\sigma + \varepsilon \widetilde{\sigma})(.,.)$ are continuously differentiable with bounded derivatives and verify Lipstick conditions and moreover that $\widetilde{\sigma}(.,.)$ and $(\sigma + \varepsilon \widetilde{\sigma})(.,.)$ satisfy the uniform ellipticity condition (11). $\forall \varepsilon \in [-1, 1], \forall t \in [0, T]$

$$\exists \eta > 0, |(\sigma + \varepsilon \widetilde{\sigma})(t, x)| \geq \eta \;\; \forall\, x \in \mathbb{R} \tag{11}$$

We then define two different perturbed underlying processes, with their respective prices. The drift-perturbed process is the stochastic process $\left\{X_t^{\varepsilon,rho}, t \in [0, T]\right\}$[3] solution of the perturbed diffusion equation, in the direction $\widetilde{b}$, defined by (12) and the unmodified initial condition ($X_0^{\varepsilon,rho} = x$)

$$dX_t^{\varepsilon,rho} = \left[b\left(t, X_t^{\varepsilon,rho}\right) + \varepsilon \widetilde{b}\left(t, X_t^{\varepsilon,rho}\right)\right] dt + \sigma\left(t, X_t^{\varepsilon,rho}\right) dW_t \tag{12}$$

Similarly, the volatility-perturbed underlying process is the stochastic process $\{X_t^{\varepsilon,vega}, t \in [0, T]\}$ solution of the perturbed diffusion equation in the direction $\widetilde{\sigma}$ defined by (13) and the unmodified initial condition ($X_0^{\varepsilon,vega} = x$)

$$dX_t^{\varepsilon,vega} = b\left(t, X_t^{\varepsilon,vega}\right) dt + \left[\sigma\left(t, X_t^{\varepsilon,vega}\right) + \varepsilon \widetilde{\sigma}\left(t, X_t^{\varepsilon,vega}\right)\right] dW_t \tag{13}$$

The above definitions of these two perturbed processes lead to perturbed price $P_{rho}^{\varepsilon}(x)$ and $P_{vega}^{\varepsilon}(x)$ defined by

$$\begin{aligned} P_i^{\varepsilon}(x) &= \mathbb{E}_x^Q\left[e^{-\int_0^T r\left(s, X_s^{\varepsilon,i}\right) ds} f\left(X_{t_1}^{\varepsilon,i}, X_{t_2}^{\varepsilon,i}, ..., X_{t_n}^{\varepsilon,i}\right)\right] \\ \text{with } i &= \text{rho or vega} \end{aligned}$$

The physical meaning of the above definitions is to set an appropriate framework so as to see the impact of a structural change of either in the drift or the volatility term on the underlying process as well as on the price function in itself. The extended rho and vega quantify this effect. They are therefore defined by the following definitions:

[3] we put *vega* in upperscript so as to be able to distinguish the two perturbed process $X_t^{\varepsilon,rho}$ and $X_t^{\varepsilon,vega}$. One is corresponding to a perturbation on the drift term whereas the other one on the stochastic term.

Definition 1 *The extended rho is the Gateau derivative of the perturbed price function $P^{\varepsilon}_{rho}(x)$ in the direction given by the function $\widetilde{b}(.)$:*

$$rho = \frac{\partial}{\partial \varepsilon} P^{\varepsilon}_{rho}(x) \bigg|_{\varepsilon=0, \widetilde{b} \text{ given}} \tag{14}$$

Similarly, the extended vega is the Gateau derivative of the perturbed price function $P^{\varepsilon}_{vega}(x)$ in the direction given by the function $\widetilde{\sigma}(.,.)$:

$$vega = \frac{\partial}{\partial \varepsilon} P^{\varepsilon}(x) \bigg|_{\varepsilon=0, \widetilde{\sigma} \text{ given}} \tag{15}$$

2.4 Results on the first variation process

This section shows that the first variation process $(Y_t)_{t\in[0,T]}$ is at the core of the extended Greeks theory. In this section, we introduce Gateau derivatives implied by our the extended Greeks. We show that these two Gateau derivatives can be expressed as simple function of the first variation process Y_t. We denote by $\left(Z^{rho}_t\right)_{t\in[0,T]}$ and $(Z^{vega}_t)_{t\in[0,T]}$ the Gateau derivative of the drift-perturbed underlying process $\left\{X^{\varepsilon,rho}_t, t\in[0,T]\right\}$, respectively the volatility-perturbed underlying process $\{X^{\varepsilon,vega}_t, t\in[0,T]\}$ along the direction $\widetilde{b}$,respectively $\widetilde{\sigma}$. These two quantities are defined as the limit in L^2, uniformly with respect to the time t :

$$Z^{rho}_t = \lim_{L^2, \epsilon\to 0} \frac{X^{\varepsilon,rho}_t - X_t}{\varepsilon} \tag{16}$$

respectively

$$Z^{vega}_t = \lim_{L^2, \epsilon\to 0} \frac{X^{\varepsilon,vega}_t - X_t}{\varepsilon} \tag{17}$$

Interestingly, these two processes can be expressed in terms of the first variation process $(Y_t)_{t\in[0,T]}$ as the following proposition states:

Proposition 1 *The process $\left(Z^{rho}_t\right)_{t\in[0,T]}$ can be expressed in terms of the first variation process by*

$$Z^{rho}_t = \int_{s=0}^{t} \frac{Y_t \widetilde{b}(s, X_s)}{Y_s} ds \tag{18}$$

Similarly, the process $(Z^{vega}_t)_{t\in[0,T]}$ can be expressed in terms of the first variation process $(Y_t)_{t\in[0,T]}$ by

$$Z^{vega}_t = \int_0^t Y_t \frac{\widetilde{\sigma}(s, X_s)}{Y_s} dW_s - \int_0^t Y_t \sigma'(s, X_s) \frac{\widetilde{\sigma}(s, X_s)}{Y_s} ds \tag{19}$$

Proof: in the appendix section, section 6.1, page 17.□

The proposition above explains intuitively why the Malliavin weights for the rho and vega can be expressed in terms of the first variation process. The difference between the volatility-perturbed framework and the drift-perturbed one comes from an additional term in the case of the volatility-perturbed one.

3 A New Method for Computing the Greeks: the Malliavin Weighted scheme

This section shows the necessary and sufficient conditions for a function to serve as a weighting function. We first give the state of the art, then give the necessary and sufficient conditions and finally show how to extend these conditions to models where the risk free interest rate is a function of the underlying as in interest rates models for the spot rate (model of Vasicek, Cox Ingersoll Ross, Black Derman Toy and so on.).

3.1 State of art

Fournie et al. (1999) were the first to suggest that the three Greeks delta, vega and rho could be computed as an expected value of the discounted payoff times a suitable weighting function (20)

$$Greek = \mathbb{E}^Q \left[e^{-\int_0^T r_s ds} f(X_T) weight \right] \tag{20}$$

The paper of Fournie et al. brings to mind many questions. First, a general formula for the gamma is missing. It is only in the particular case of the Black Scholes equation that they are able to provide one. This attempt indicates that the gamma is not a special case. The only difference comes from the fact it is a second order derivatives with respect to the initial underlying level.

Second, it is worth noticing that all their weighting function could be expressed as a Skorohod integral, since Ito integral is only a subset of the Skorohod one. Going the other way round, a new problem is to examine the set of functions expressed as a Skorohod integral and to determine which condition(s) these functions should satisfy to serve as a weighting function. This is precisely the aim of this paper. We restrict ourself to weighting functions that can be expressed as a Skorohod integral. We call the integrand of the Skorohod operator the weighting function generator and denotes it by w_s. The following subsections shows that the weighting function generator should satisfy necessary and sufficient conditions. Interestingly, these conditions are different for each Greek but independent of the payoff function. Therefore, the Malliavin weight is independent from the payoff function.

3.2 Generalization of the method: Exact determination of the Malliavin Weights

Writing the weighting function *weight* as a Skorohod integral, we call weighting function generator w the Skorohod integrand

$$weight = \delta(w) \tag{21}$$

We will assume as well that the weight is L^2 squarable that is

$$\mathbb{E}\left[weight^2\right]^{1/2} < \infty \tag{22}$$

Since the Skorohod integral is at the core of the Malliavin integration by part formula, the weighting function is better characterized by its weighting function generator. We first examine the most common case where we assume that the instantaneous risk-free interest rate does not depend on the underlying process $r'(s, X_s) = 0$. Denoting by $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}$ the conditional expectation with respect to $X_{t_1}, ..., X_{t_m}$, i.e. $\mathbb{E}^Q_{x,X_{t_1},...,X_{t_m}}[.] = \mathbb{E}^Q_x[|X_{t_1}, ..., X_{t_m}]$, we show that this generator should satisfy necessary and sufficient conditions given by the following theorem:

Theorem 1 *Malliavin formula for the Greeks*
There exits necessary and sufficient conditions for a function w to serve as a weighting function generator for the simulation of the Greeks. The first condition is the Skorohod integrability of this function. The second condition, different for each Greeks and summarized in table 1, is depending only on the underlying diffusion characteristics and is independent from the payoff function.

Greeks	Necessary and Sufficient conditions on the Malliavin Weights
delta	(M1) : $\mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[Y_{t_i}\int_0^{t_i}\frac{\sigma(t,X_t)}{Y_t}w^{delta}(t)\,dt\right] = \mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[Y_{t_i}\right]$
gamma	(M2) : $\mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[\delta\left(w^{gamma}\right)\right]$ $\mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[\delta\left(w^{delta}\delta\left(w^{delta}\right)+\frac{\partial}{\partial x}w^{delta}\right)\right]$
"extended" rho	(M3) : $\mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[Y_{t_i}\int_0^{t_i}\frac{\sigma(t,X_t)}{Y_t}w^{rho}(t)\,dt\right] = \mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[Y_{t_i}\int_0^{t_i}\frac{\widetilde{b}(t,X_t)}{Y_t}dt\right]$
"extended" vega	(M4) : $\mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[Y_{t_i}\int_0^{t_i}\frac{\sigma(t,X_t)}{Y_t}w^{vega}(t)\,dt\right] = \mathbb{E}^{Q}_{x,X_{t_1},\ldots,X_{t_m}}\left[\begin{array}{c}\int_0^{t_i}\frac{\widetilde{\sigma}(t,X_t)Y_{t_i}}{Y_t}dW_t \\ -\int_0^{t_i}\sigma'(s,X_s)\frac{\widetilde{\sigma}(s,X_s)Y_{t_i}}{Y_t}ds\end{array}\right]$

Table 1: Necessary and Sufficient conditions for the Weighting Function Generators in a model with interest rates independent of the underlying. The proof for the equations (M1), (M2), (M3) and (M4) are given in the appendix section, respectively in section 6.2, 6.3, 6.4 (skeleton of the proof for (M3) and (M4))

3.3 Extension to models with stochastic interest rates

When we assume that the risk free interest rate is a function of the underlying, we need to take into account the dependency of the risk free rate from the underlying process. The necessary and sufficient conditions given in table 1 are not valid any more. We need to include in the expectation operator the discount factor $e^{-\int_0^T r(s,X_s^x)ds}$, term which is stochastic and complete them by a second condition. The second conditions are obtained by the same way the first one were derived. However, since these expressions does not bring any new intuition and are tedious rephrasing of the simpler results of table 1, we have put them in the appendix section, section 6.6, page 25 as table 6.

3.4 The minimal variance weighting function

If we want the weighting function with the minimal variance, we have to understand the way the Greeks are calculated. We have found that the Greeks are expressed as the expectation of a weighting function times the discounted payoff.

The only information we have about the payoff function is its measurability with respect to the filtration F_T. It means that the product inside our expectation can be seen as the scalar product of the weighting function with any function F_T measurable. It is then pretty intuitive that the weighting function with minimal variance is the conditional expectation of any weighting function with respect to the filtration F_T by means of the theorem of projection. More precisely, we have the following proposition

Proposition 2 *The weighting function with minimal variance denoted by π_0 is the conditional expectation of any weighting function with respect to the filtration F_T*

$$\pi_0 = \mathbb{E}\left[weight|F_T\right]$$

Proof: Let π be a weighting function. The Greek ratio can be expressed as the expected value of the scalar product of the discounted payoff denoted by F with this weighting function time $Greek = \mathbb{E}\left[F.\pi\right]$. The variance V of this estimator is given by the quadratic variation of our estimator of the Greek minus the true value of the Greek.

$$V = \mathbb{E}\left[(F.\pi - Greek)^2\right]$$

We can introduce the conditional expectation π_0, leading to

$$\begin{aligned} V &= \mathbb{E}\left[\left(F.\left(\pi - \pi_0\right) + F.\pi_0 - Greek\right)^2\right] \\ &= \mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right)^2\right] + \mathbb{E}\left[\left(F.\pi_0 - Greek\right)^2\right] \\ &\quad +2\mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right).\left(F.\pi_0 - Greek\right)\right] \end{aligned}$$

But indeed the last term in the equation above is equal to zero since

$$\begin{aligned} \mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right).\left(F.\pi_0 - Greek\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right).\left(F.\pi_0 - Greek\right)|F_T\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right)|F_T\right].\left(F.\pi_0 - Greek\right)\right] \\ &= 0 \end{aligned}$$

where we have used first the fact that $(F.\pi_0 - Greek)$ and F are F_T measurable and therefore $\mathbb{E}\left[\left(F.\left(\pi - \pi_0\right)\right)|F_T\right] = 0.\Box$

This is a strong result. It indicates that the best weighting function should always be the one F_T measurable. It indicates as well that without any more specification on the pay-off function, the variance is lower-bounded by the variance of the particular weighting function π_0. This indicates as well that with more information on the pay-off function, we can have more efficient estimators. This is the case when for example, we have a payoff-function which can be expressed in terms of some particular points of the Brownian motion trajectory. In this case, the best weighting function would be the one expressed in terms of these particulars points.

4 Examples of Malliavin weights

In this section, we give examples of weighting functions generator. Instead of using the necessary and sufficient conditions derived above, expressed as an equality of conditional expectations, we look for solutions that satisfy the equality of the two terms inside the expectation. Of course, these conditions are stronger and are only sufficient but not necessary.

We show that the solutions given by Fournie et al. (1999) are particular solutions for generator functions. But we exhibit other solutions. This raises the interesting question of the choice of the weighting function generator.

4.1 Fournie et al. solutions

Let us define $T_m = \left\{ a \in L^2[0,T] \mid \int_0^{t_i} a(t)\, dt = 1 \ \forall i = 1...m \right\}$ and

$\widetilde{T}_m = \left\{ \widetilde{a} \in L^2[0,T] \mid \int_{t_{i-1}}^{t_i} \widetilde{a}(t)\, dt = 1 \ \forall i = 1...m \right\}$. Rewriting all the weighting functions of Founie et al. (1999) as Skorohod integral, we can see that of course these functions satisfies the necessary and sufficient conditions. Indeed, an easy way to check that the conditional expectations of the equations (M1), (M3) and (M4) are equal is to verify that the terms inside the expectations are equal. That is precisely what we can do in the case of the Fournie et al. weights. The table 2 summarizes the different weighting function generator of Fournie et al.

Greeks	**Weighting Function Generators of Founie et al.**
delta	$a(t)\frac{Y_t}{\sigma(t,X_t)}$
"extended" rho	$\frac{1}{\sigma(t,X_t)}\widetilde{b}(t,X_t)$
"extended" vega	$\frac{1}{\sigma(t,X_t)}\widetilde{a}(t)\sum_{i=1}^{m}\left(Z_{t_i}^{vega} - Z_{t_{i-1}}^{vega}\right)1_{\{t_{i-1}\leq t<t_i\}}$

Table 2: Summary of Particular Malliavin Weights given by Fournie et al.

4.2 Other examples

In fact, there many other judicious weighting function generators that can be used. We only need to find functions that satisfies the necessary and sufficient conditions and are element of the Skorohod operator domain denoted by $D^{1,2}$.

We can prove that the piecewise constant solution given below satisfies the necessary and sufficient conditions and is an element of the Skorohod operator domain denoted by $D^{1,2}$. It is therefore an other example of a weighting function generator.

$$\begin{aligned} w^{delta}(t) &= \sum_{i=1}^{m} \alpha_i^{delta} 1_{t_{i-1}\le t\le t_i} && (23) \\ \text{with} \quad \sum_{j=1}^{i} \alpha_j^{delta} \int_{t_{j-1}}^{t_j} \frac{\sigma(t,X_t)}{Y_t} dt &= 1 \quad \forall i = 1...m && (24) \end{aligned}$$

It is interesting as well to examine the case of the gamma Greek. Even if in some special cases, there is a link between gamma and vega when the first variation process is proportional to the underlying which leads to the Geometric Brownian motion (Benhamou (2000c)), for a general model the calculation of the formula for gamma cannot be avoided. Without an abstract framework, formulae becomes soon complicated. This might be the reason why gamma calculation is missing in previous works like Broadie and Glasserman (1996) and Fournie et al. (1999). We needs to assume for this calculation that b and σ are continuously differentiable up to the second order with bounded first and second order derivatives. These conditions are to justify the existence of the weighting function. We can then show that one particular solution of the weighting function of the gamma is given by:

$$weight_\Gamma = \left[\begin{array}{c} \left(\int_0^T a(t) \frac{Y_t}{\sigma(t,X_t)} dW_t\right)^2 - \int_0^T \mathbb{E}\left[\left(a(t)\frac{Y_t}{\sigma(t,X_t)}\right)^2\right] dt \\ - \int_0^T a(t) \frac{Y_t \sigma'(t,X_t)}{\sigma^2(t,X_t)} dW_t \\ \int_{0=s\le s_2}^T \frac{a(s_2)Y_{s_2}Y_{s_1}}{\sigma(s_2,X_{s_2})} \left(b''(s_1,X_{s_1}) - \sigma'(s_1,X_{s_1})\sigma''(s_1,X_{s_1})\right) ds_1 dW_{s_2} \\ + \int_{0=s_1\le s_2}^T \frac{a(s_2)Y_{s_2}Y_{s_1}}{\sigma(s_2,X_{s_2})} \sigma''(s_1,X_{s_1}) dW_{s_1} dW_{s_2} \end{array} \right] \quad (25)$$

Proof: given in the appendix section, section 6.3.2 page 23.□

We can as well define piecewise solutions for the other Greeks : rho and vega:

$$w^{rho}(t) = \sum_{i=1}^{m} \alpha_i^{rho} 1_{t_{i-1}\le t\le t_i}, \qquad w^{vega}(t) = \sum_{i=1}^{m} \alpha_i^{vega} 1_{t_{i-1}\le t\le t_i}$$

We have seen that the generator has to satisfy some necessary and sufficient conditions. Indeed, when taking a stronger assumption of these conditions which is the equality of the terms inside the conditional expectations, we get that the generator satisfy some technical conditions, which can be expressed in terms of the different elements α_i^{delta}, α_i^{vega}, α_i^{rho}. We have summarized these conditions fin the table 3.

Greeks	**Conditions for the generator in terms of the elements** α_j^{greek}
delta	$\sum_{j=1}^{i} \alpha_j^{delta} \int_{t_{j-1}}^{t_j} \frac{\sigma(t,X_t)}{Y_t} dt = 1$
"extended" rho	$\sum_{j=1}^{i} \alpha_j^{rho} \int_{t_{j-1}}^{t_j} \frac{\sigma(t,X_t)}{Y_t} dt = \int_0^{t_i} \frac{\widetilde{b}(t,X_t)}{Y_t} dt$
"extended" vega	$\sum_{j=1}^{i} \alpha_j^{vega} \int_{t_{j-1}}^{t_j} \frac{\sigma(t,X_t)}{Y_t} dt = \frac{Z_{t_i}^{vega}}{Y_{t_i}}$

Table 3: Conditions for piecewise constant generator

We can as well define weights which emphasizes the role of the first variation process by writting it as a linear combination of the first variation process, where the linear components β_i^{greek} are stochastic:

$$w^{greek}(t) = \sum_{i=1}^{m} \beta_i^{greek} Y_t 1_{\{t_{i-1} \leq t < t_i\}}$$

where the index *greek* stands for either delta, vega or rho. Like in the previous case, we can express the sufficient conditions of the generator in terms of these elements. Like in the previous case, we have summarized all these results in the table 4.

Greeks	**Conditions for the generator in terms of the elements** β_j^{greek}
delta	$\sum_{j=1}^{i} \beta_j^{delta} \int_{t_{j-1}}^{t_j} \sigma(t,X_t) dt = 1$
"extended" rho	$\sum_{j=1}^{i} \beta_j^{rho} \int_{t_{j-1}}^{t_j} \sigma(t,X_t) dt = \int_0^{t_i} \frac{\widetilde{b}(t,X_t)}{Y_t} dt$
"extended" vega	$\sum_{j=1}^{i} \beta_j^{vega} \int_{t_{j-1}}^{t_j} \sigma(t,X_t) dt = \frac{Z_{t_i}^{vega}}{Y_{t_i}}$

Table 4: Conditions for piecewise constant generator

4.3 Choice of the generator

When dealing with Malliavin weight, the true question is the choice of the best generator. Since the Skorohod integral coincides with the Ito integral for adapted processes, it is very interested to find an adapted generator. A second feature is to base the choice on a variance minimization criterium as well. However, this problem is extremely difficult to treat in its general framework. To tackle this issue, one needs to specify our diffusion parameters : drift and volatility term. The problem is then to determine the adapted generator with the lowest formula variance. However, this problem cannont be solved in this too general framework. We need stronger assumptions on the diffusion of the underlying for a fruitful discussion about the choice of the generator.

5 Conclusion

This article gives the theoretical skeleton for many future research for the simulation of the Greeks with no differentiation of the payoff function. Its innovation can be classified into two parts:

- We have taken very general model hypotheses. We have broadened the assumptions of Fournie et al. to a stochastic risk-free interest rate, function of the time and the underlying process. We have given proper definitions for extended Greeks and examined the particular case of the gamma, which was missing in works like Broadie and Glasserman (1996) and Fournie et al. (1999).
- However, the main interest of the paper lies in its second innovation. We have seen that introducing the weighting function generator as the Skorohod integrand of the weighting function, we can characterize this generator. The conditions hereby derived enables us to find the ones given by Fournie et al. (1999) as particular solutions of our general conditions. We have shown that there exists many more solutions.

There are many possible extensions and applications of this theoretical article. One area of research is to extend the previous results to other option types (Asian and lookback options Benhamou (2000a)). Another domain of interest is to find specific examples of weighting function, according to a certain criterium. The question of the choice of generator needs to refer to stronger hypotheses on the diffusion of the underlying. Another question is a comparison study of the efficiency of Malliavin weights compared to traditional methods. Fournie et al. (1999) and Benhamou (2000b) examined the particular case of the Black diffusion. They concluded that Malliavin formulas are very efficient for non-linear payoffs but not for vanilla options. Their main conclusion is that one should be cautious when using the Malliavin formulae. As a suggestion, one should use locally the Malliavin formulae at region of discontinuity and the finite difference method elsewhere as suggested by Fournie et al. (1999) and Benhamou (2000b).

6 Appendix

6.1 Proof of proposition (1)

The proof is only given for the Z_t^{rho} process. It is identical for Z_t^{vega}. To prove proposition (1), we first show that the process $\left(Z_t^{rho}\right)_{t\in[0,T]}$ verifies a stochastic differential equation (26). Since the two process $\left(Z_t^{rho}\right)_{t\in[0,T]}$ and $\left(\int_0^t Y_t Y_s^{-1}\widetilde{b}\left(s,X_s\right)ds\right)_{t\in[0,T]}$ verify the same SDE (26) and have the same initial conditions, they are equal according to the stochastic version of the Cauchy Lipschitz theorem.□

We now prove the lemma about the stochastic differential equation (26):

Lemma 1 *Under the assumption of continuous differentiability of* b,σ *with bounded derivatives, the process* $\left(Z_t^{rho}\right)_{t\in[0,T]}$ *defined by (16) is the unique solution of the following stochastic differential equation*

$$dZ_t = \left(\widetilde{b}\left(t,X_t\right) + Z_t b'\left(t,X_t\right)\right)dt + Z_t\sigma'\left(t,X_t\right)dW_t \tag{26}$$

with initial condition $Z_0 = 0_n$.

Proof: Solving the diffusion equation (12) with the initial condition $\left(X_{t=0}^{\varepsilon,rho} = x\right)$ gives

$$X_t^{\varepsilon,rho} = x + \int_0^t \left[b\left(s,X_s^{\varepsilon,rho}\right) + \varepsilon\widetilde{b}\left(s,X_s^{\varepsilon,rho}\right)\right]ds + \int_0^t \sigma\left(s,X_s^{\varepsilon,rho}\right)dW_s$$

For $\varepsilon \neq 0$

$$\frac{X_t^{\varepsilon,rho} - X_t}{\varepsilon}$$

$$= \int_0^t \left(\widetilde{b}\left(s,X_s^{\varepsilon,rho}\right) + \frac{b\left(s,X_s^{\varepsilon,rho}\right) - b\left(s,X_s\right)}{\varepsilon}\right)ds + \int_0^t \frac{\sigma\left(s,X_s^{\varepsilon,rho}\right) - \sigma\left(s,X_s\right)}{\varepsilon}dW_s$$

Using the hypothesis that b,σ are continuously differentiable with bounded derivative, as well as the continuity of $X_s^{\varepsilon,rho}$ in ε with its limit equal to the non-perturbed process $\left(X_t\right)_{t\in[0,T]}$, we can show that the Gateau derivative $\left(Z_t^{rho}\right)_{t\in[0,T]}$ of the drift-perturbed underlying process $\left\{X_t^{\varepsilon,rho}, t\in[0,T]\right\}$along the direction $\widetilde{b}$ can be expressed as:

$$Z_t^{rho} = \int_0^t \left(\widetilde{b}\left(s,X_s\right) + Z_s^{rho}b'\left(s,X_s\right)\right)ds + \int_0^t Z_s^{rho}\sigma'\left(s,X_s\right)dW_s$$

which in its differential form is exactly equal to the result. The unicity is then given by the stochastic version of the Cauchy Lipschitz theorem.□

6.2 Proof of the delta formula (M1)

In this section, we prove that the weighting function for the delta should satisfy necessary and sufficient conditions. The proof is given for the case of a stochastic interest rate depending both on time and the underlying. As a special case, we derive the necessary and sufficient condition given in table 1 when the interest rate is only a function of time. For the sake of simplicity, we denote in this section w^{delta} by w,and **d**enote by a prime the derivative with respect to the second variable. The part of the proof based on integration by parts is quite short and follows the one of Elworthy (1992). The technical difficulty here is to justify rigorously the use of weaker assumptions. It can be divided into three major steps:

1. first preliminary: weaker conditions on the payoff function f: show that if the result holds for any function of C_K^∞ (set of infinitely differentiable functions with compact support), it also holds for any element of L^2.
2. second preliminary: interchange of the order of differentiation and expectation: show that one can interchange the order of differentiation and expectation.
3. integration by parts:
   (a) necessary condition.
   (b) sufficient condition.

6.2.1 First preliminary: Weaker assumptions

The idea of the first technical point is the following: taking f as an element of L^2 is the same as assuming f infinitely differentiable with a compact support. It is based on a density argument using Cauchy Schwartz inequality and the continuity of the expectation operator.

More precisely, let assume the result is true for any function of C_K^∞ (set of infinitely differentiable functions with compact support). Let f be now only in L^2. Using the density of $C_K^\infty [0,T]$ in L^2, there exits a sequence $(f_n)_{n\in N}$ of C_K^∞ elements that converges to f in L^2. Let 's denote $u(x) = \mathbb{E}_x^Q[F]$ and $u_n = \mathbb{E}_x^Q[F_n]$ the prices associated with the discounted payoff functions F and F_n and x as the starting point of the underlying security price. Since L^2 convergence implies L^1 convergence, we know that the set of functions u_n converges simply to the function u.

$$\forall x \in \mathbb{R} \quad u_n(x) \underset{n\to\infty}{\to} u(x)$$

Since the result is true for payoff functions element of C_K^∞, the derivative of the u_n function can be written as the expectation of the discounted payoff function

f_n times a suitable "Malliavin" weight $\delta(w)$ defined as the Skorohod integral of a function w:

$$\frac{\partial}{\partial x} u_n(x) = \mathbb{E}_x^Q [F_n \delta(w)]$$

Let 's denote by g the function obtained as the expectation of the discounted payoff function f times the Malliavin weight $\delta(w)$: $g(x) = \mathbb{E}_x^Q [F\delta(w)]$. By Cauchy Schwartz inequality

$$\left| g(x) - \frac{\partial}{\partial x} u_n(x) \right| = \left| \mathbb{E}_x^Q [(F - F_n) \delta(w)] \right| \leq h(x) \epsilon_n(x) \tag{27}$$

with

$$h(x) = E_x^Q \left[(\delta(w))^2 \right]^{1/2} \quad \epsilon_n(x) = E_x^Q \left[(F - F_n)^2 \right]^{1/2}$$

By definition, the L^2 convergence of u_n means $\epsilon_n(x)$ converges simply to zero as n tends to infinity. Therefore we already know that the function sequence $\left(\frac{\partial}{\partial x} u_n\right)_{n \in N}$ converges simply to the function g. By property of Lebesgue compacity and the fact that the functions F and F_n are continuous and that $h(x)$ is bounded (non-explosive condition (22)), inequality (27) proves that this convergence is uniform on any compact subsets K of $\mathbb{R}$..

We conclude using the fact that if a sequence of functions $(u_n)_{n \in \mathbb{N}}$ converges simply to a function u and the sequence of function's derivative $\left(\frac{\partial}{\partial x} u_n\right)_{n \in \mathbb{N}}$ converges uniformly to a function g on any compact subsets of $\mathbb{R}$, the limit function u is continuously differentiable with its derivative equal to the limit function of the sequence of function's derivative $\left(\frac{\partial}{\partial x} u_n\right)_{n \in \mathbb{N}}$ leading to the final result:

$$\frac{\partial}{\partial x} E_x^Q [F] = \mathbb{E}_x^Q [F\delta(w)]$$

□

6.2.2 Second preliminary: Interchanging the order of expectation and differentiation

The second technical point is to show that we can interchange the order of expectation and differentiation (using the dominated convergence theorem).

More precisely, since because of the first preliminary, f is assumed to be element of C_K^∞ and therefore is continuously differentiable with bounded derivative, we have

$$\frac{F^{x+h} - F^x}{\|h\|} - \frac{\left\langle \frac{\partial}{\partial x} F, h \right\rangle}{\|h\|} \underset{\|h\| \to 0.}{\to} 0 \quad a.s.$$

An elementary calculation gives us

$$\frac{\partial}{\partial x}F = \begin{pmatrix} \sum_{i=1}^{m} e^{-\int_0^T r(s,X_s^x)ds}\partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right)\frac{\partial}{\partial x}X_{t_i}^x \\ -F\int_0^T r'\left(s, X_s^x\right)\frac{\partial}{\partial x}X_s^x ds \end{pmatrix}$$

Since f has bounded derivative,first, $\frac{\left\langle \frac{\partial}{\partial x}F,h\right\rangle}{\|h\|}$ is uniformly integrable in h and second, by Taylor Lagrange theorem,

$$\left\|\frac{F^{x+h}-F^x}{\|h\|}\right\| \leq M\sum_{i=1}^{m}\frac{\left\|X_{t_i}^{x+h}-X_{t_i}^x\right\|}{\|h\|}$$

Using the result that $\sum_{i=1}^{m}\frac{\left\|X_{t_i}^{x+h}-X_{t_i}^x\right\|}{\|h\|}$ is uniformly integrable in h (See Theorem 2.4 pp 362 Chapter IX Stochastic Differential Equations, Revuz and Yor (1994)) leads to the uniform integrability in h of $\left\|\frac{F^{x+h}-F^x}{\|h\|}\right\|$

This in turn tells us that $\frac{F^{x+h}-F^x}{\|h\|} - \frac{\left\langle \frac{\partial}{\partial x}F,h\right\rangle}{\|h\|}$ is uniformly integrable in h. Since it converges to zero a.s., the dominated convergence theorem gives us that it converges also to zero in L^1. We conclude that

$$\frac{\partial}{\partial x}u\left(X^x\right) = \mathbb{E}_x^Q\left[\frac{\partial}{\partial x}F\right] \tag{28}$$

□

6.2.3 Integration by parts:

Necessary condition: In this subsection, we examine the necessary condition to be satisfied by the weighting function. The delta is defined as the derivative of the price function with respect to the initial condition x

$$delta = \frac{\partial}{\partial x}\mathbb{E}_x^Q\left[e^{-\int_0^T r(s,X_s^x)ds}f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right)\right] \tag{29}$$

Assuming the delta can be written with a weighting function leads to

$$\begin{aligned} delta &= \mathbb{E}_x^Q\left[e^{-\int_0^T r(s,X_s^x)ds}f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right)\delta\left(w\right)\right] \\ &= E_x^Q\left[\left\langle D_t\left(e^{-\int_0^T r(s,X_s^x)ds}f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right)\right), w\left(t\right)\right\rangle\right] \end{aligned}$$

Using the property of Malliavin derivatives for compound functions, this can be written as:

$$\mathbb{E}_x^Q\begin{bmatrix} e^{-\int_0^T r(s,X_s^x)ds}\sum_{i=1}^{m}\partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right)\int_{t=0}^{T}D_t X_{t_i}^x w\left(t\right)dt \\ -F\int_{t=0}^{T}\int_{s=0}^{T}\frac{\partial}{\partial X}r\left(s, X_s^x\right)D_t X_s w\left(t\right)dsdt \end{bmatrix}$$

Using the relationship between the Malliavin derivative and the first variation process (10), we can replace the expression of $D_t X_u$ $u \geq t$ in the equation above, leading to

$$\mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s,X_s^x)ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) \\ \int_0^T Y_{t_i} Y_t^{-1} \sigma(t, X_t^x) w(t) 1_{\{t \leq t_i\}} dt \\ -F \int_{s=0}^T \int_{t=0}^T \frac{\partial}{\partial X} r(s, X_s^x) Y_s Y_t^{-1} \sigma(t, X_t^x) w(t) t 1_{\{t \leq s\}} dt ds \end{array} \right]$$

On the other hand, the delta is defined as the derivative of the price function with respect to the initial condition x. Using (10) and the second preliminary's results (28), we can change the LHS of (29)

$$\begin{array}{rl} & delta \\ = & \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s,X_s^x)ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) \frac{\partial}{\partial x} X_{t_i} \\ -F \int_0^T r'(s, X_s^x) \frac{\partial}{\partial x} X_s ds \end{array} \right] \\ = & \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s,X_s^x)ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) Y_{t_i} \\ -F \int_0^T r'(s, X_s^x) Y_s ds \end{array} \right] \end{array}$$

At this stage, equalling the two expressions of delta gives us:

$$\begin{array}{rl} & \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s,X_s^x)ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) \\ \int_0^T Y_{t_i} Y_t^{-1} \sigma(t, X_t) w(t) 1_{\{t \leq t_i\}} dt \\ -F \int_{s=0}^T \int_{t=0}^T r'(s, X_s^x) Y_s Y_t^{-1} \sigma(t, X_t) w(t) t 1_{\{t \leq s\}} dt ds \end{array} \right] \\ = & \mathbb{E}_x^Q \left[\begin{array}{c} e^{-\int_0^T r(s,X_s^x)ds} \sum_{i=1}^m \partial_i f\left(X_{t_1}^x, X_{t_2}^x, ..., X_{t_m}^x\right) Y_{t_i} \\ -F \int_0^T r'(s, X_s^x) Y_s \, ds \end{array} \right] \end{array}$$

Using the fact that this should hold for any f and any function $r(.,.)$, we get that the following two quantities should be equal on any functions measurable,leading to conditions expressed with conditional expectations (where to simplify notations the x in superscript have been omitted):

$$\begin{array}{rl} & \mathbb{E}_x^Q \left[e^{-\int_0^T r(s,X_s^x)ds} \int_0^{t_i} \frac{Y_{t_i} \sigma(t, X_t)}{Y_t} w(t) \, dt | X_{t_1}, ..., X_{t_m} \right] \\ = & \mathbb{E}_x^Q \left[e^{-\int_0^T r(s,X_s^x)ds} Y_{t_i} | X_{t_1}, ..., X_{t_m} \right] \qquad \forall i = 1...m \end{array} \tag{30}$$

$$\begin{array}{rl} & \mathbb{E}_x^Q \left[\int_{s=0}^T \int_{t=0}^T r'(s, X_s) Y_s Y_t^{-1} \sigma(t, X_t) w(t) t 1_{\{t \leq s\}} dt ds | X_{t_1}, ..., X_{t_m} \right] \\ = & \mathbb{E}_x^Q \left[\int_0^T r'(s, X_s) Y_s ds | X_{t_1}, ..., X_{t_m} \right] \end{array} \tag{31}$$

this is exactly (M1) when the interest rate is a only function of the time □

Sufficient condition: If we know a function w that verifies two equations (30) and (31) and its Skorohod integral; is L_2 squarable, the above proof can be conducted backwards:

$$\begin{aligned} delta &= \frac{\partial}{\partial x}\mathbb{E}_x^Q\left[\left(e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right)\right] \\ &= \mathbb{E}_x^Q\left[\begin{array}{c} \sum_{i=1}^m e^{-\int_0^T r(s,X_s)ds}\partial_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\frac{\partial}{\partial x}X_{t_i} \\ -\left(e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right)\int_0^T r'\left(s,X_s\right)\frac{\partial}{\partial x}X_s ds \end{array}\right] \end{aligned}$$

then using the necessary conditions, we get

$$\begin{aligned} &= \mathbb{E}_x^Q\left[\begin{array}{c} \sum_{i=1}^m e^{-\int_0^T r(s,X_s)ds}\partial_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right) \\ \int_0^T Y_{t_i}Y_t^{-1}\sigma(t,X_t)w\left(t\right)1_{\{t\le t_i\}}dt \\ -F\int_{s=0}^T\int_{t=0}^T r'\left(s,X_s\right)Y_sY_t^{-1}\sigma(t,X_t)\,w\left(t\right)t1_{\{t\le s\}}dtds \end{array}\right] \\ &= \mathbb{E}_x^Q\left[\begin{array}{c} e^{-\int_0^T r(s,X_s)ds}\sum_{i=1}^m \nabla_i f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\int_{t=0}^T D_tX_{t_i}w\left(t\right)dt \\ -F\int_{t=0}^T\int_{s=0}^T r'\left(s,X_s\right)D_tX_sw\left(t\right)dsdt \end{array}\right] \end{aligned}$$

which then using the expression of the Malliavin derivative in terms of the first variation process, leads to

$$\mathbb{E}_x^Q\left[\left\langle D_t\left(e^{-\int_0^T r(s,X_s)ds} f\left(X_{t_1},X_{t_2},...,X_{t_m}\right)\right),w\left(t\right)\right\rangle\right]$$

leading to the final result:

$$delta = \mathbb{E}_x^Q\left[F\delta\left(w\right)\right]$$

where in the last step, we made use of the integration by parts formula.□

6.3 Proof of the gamma formula (M2)

6.3.1 Necessary and sufficient condition

The proof goes along the same lines as for the delta case, so we omit to give all details of it. We assume that f is continuously twice differentiable with bounded first and second order derivatives. To remind that the generator w^{delta} does depend on x, we adopt an explicit notation w_x^{delta}.

$$\begin{aligned} \Gamma &= \Delta_x\mathbb{E}_x^Q\left[F\right] \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\mathbb{E}_x^Q\left[F\right]\right) \\ &= \frac{\partial}{\partial x}\left(\mathbb{E}_x^Q\left[F\delta\left(w_x^{delta}\right)\right]\right) \\ &= \mathbb{E}_x^Q\left[\frac{\partial}{\partial x}F\delta\left(w_x^{delta}\right)\right]+\mathbb{E}_x^Q\left[F\frac{\partial}{\partial x}\delta\left(w_x^{delta}\right)\right] \end{aligned}$$

using the fact that one could invert the Skorohod integral operator $\delta(.)$ and the differential operator $\frac{\partial}{\partial x}$ (thanks to a mathematical argument based on dominated convergence theorem), we get

$$\begin{aligned}\Gamma &= \mathbb{E}_x^Q\left[F\left(\delta\left(w_x^{delta}\right)\delta\left(w_x^{delta}\right)+\delta\left(\frac{\partial}{\partial x}w_x^{delta}\right)\right)\right] \\ &= \mathbb{E}_x^Q\left[F\delta\left(w^{delta}\delta\left(w_x^{delta}\right)+\frac{\partial}{\partial x}w_x^{delta}\right)\right]\end{aligned}$$

where in the last inequality we used the linearity of the Skorohod integral operator. Since this should hold for any F, the necessary and sufficient condition is

$$\mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q\left[\delta\left(w^{gamma}\right)\right]=\mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q\left[\delta\left(w^{delta}\delta\left(w^{delta}\right)+\frac{\partial}{\partial x}w^{delta}\right)\right]$$

□

6.3.2 Particular solution

In this section, we prove the formula for the particular form of weight, we have already encountered, namely:

$$w_t^{delta}=Y_t\sigma^{-1}(t,X_t)\lambda_t$$

with

$$\int_0^T \lambda_t\, 1_{\{t\le t_i\}}dt=1 \qquad \forall i=1...m$$

Using result on the Gamma weighting function, a sufficient condition on the Malliavin weight is the equality:

$$\delta\left(w^{gamma}\right)=\delta\left(\frac{\partial}{\partial x}w^{delta}\right)+\delta\left(w^{delta}\right)\delta\left(w^{delta}\right)$$

with

$$\delta\left(w^{delta}\right)\delta\left(w^{delta}\right)=\left(\int_{t=0}^T\lambda_t\sigma^{-1}(t,X_t)Y_tdW_t\right)\left(\int_{u=0}^T\lambda_u\sigma^{-1}\left(X_u\right)Y_udW_u\right)$$

which can be expressed in terms of the square of the simple integral:

$$\delta\left(w^{delta}\right)\delta\left(w^{delta}\right)=\left(\int_0^T\lambda_t\sigma^{-1}(t,X_t)Y_tdW_t\right)^2-\int_0^T\mathbb{E}\left[\left(\lambda_t\sigma^{-1}(t,X_t)Y_t\right)^2\right]ds$$

The term $\frac{\partial}{\partial x} w^{delta}$ can be calculated as the sum of two terms:

$$\frac{\partial}{\partial x} w^{delta} = \lambda_t \left(\partial_2 \left(\sigma^{-1} \right) (t, X_t) Y_t + \sigma^{-1}(t, X_t) \frac{\partial}{\partial x} Y_t \right)$$

we use then the following equation:

$$\partial_2 \left(\sigma^{-1} \right) (t, X_t) = -\sigma^{-2}(t, X_t) \sigma' (t, X_t) Y_t$$

and we use for the second term that

$$\begin{aligned} \frac{\partial}{\partial x} Y_t &= \int_0^t Y_t Y_s b'' (s, X_s) \, ds + \int_0^t Y_t Y_s \sigma'' (s_1, X_{s_1}) \, dW_s \\ &\quad - \int_0^t Y_t \sigma' (s, X_s) Y_s \sigma'' (s, X_s) \, ds \end{aligned}$$

giving then

$$\begin{aligned} &\partial_x \left(w^{delta} \right) \\ = &\int_{s_2 0}^T \int_{s=0}^{s_2} \lambda_{s_2} \sigma^{-1}(s_2, X_{s_2}) Y_{s_2} Y_{s_1} \left(b'' (s_1, X_{s_1}) - \sigma' (s_1, X_{s_1}) \sigma'' (s_1, X_{s_1}) \right) ds_1 dW_{s_2} \\ &+ \int_{s_2=0}^T \lambda_{s_2} \sigma^{-1}(s_2, X_{s_2}) Y_{s_2} \int_{s_1=0}^{s_2} Y_{s_1} \sigma'' (s_1, X_{s_1}) \, dW_{s_1} dW_{s_2} \end{aligned}$$

we conclude that the Malliavin weight is given by (25).□

6.4 Proof of the rho and the vega formulae (M3) and (M4)

These proofs are similar to the one given for the delta (M1) and are available upon request. The only difference between the delta and the rho or vega formula lies in the fact that we derive with respect to x in the case of the delta, and with respect to the ε associated with the drift-perturbed or volatility-perturbed in the case of the rho or vega. We therefore need to change in the proofs that the derivative of X_t with respect to the initial condition x, $\frac{\partial}{\partial x} X_t = Y_t$, is changed by $\frac{\partial}{\partial \varepsilon} X_t^{rho} = Z_t^{rho}$ and $\frac{\partial}{\partial \varepsilon} X_t^{vega} = Z_t^{vega}$ and then to use the proposition (1).□

6.5 Summary of Fournie et al. particular solutions

Founie et al. proved that the weighting function could be writen in the case of adapted processes as some Ito i tegral. Let us define

$T_m = \left\{ a \in L^2 [0, T] \mid \int_0^{t_i} a(t) \, dt = 1 \; \forall i = 1...m \right\}$

and $\widetilde{T}_m = \left\{ \widetilde{a} \in L^2 [0, T] \mid \int_{t_{i-1}}^{t_i} \widetilde{a}(t) \, dt = 1 \; \forall i = 1...m \right\}$. Their results are summarized in the table 1, where the symbol δ stands for the Skorohod integral and a is an element of T_m, $\widetilde{a}$ an element of $\widetilde{T}_m$.

Greeks	Weighting Function
delta	$\int_0^T a(t) \frac{Y_t}{\sigma(t,X_t)} dW_t$
"extended" rho	$\int_0^T \frac{1}{\sigma(t,X_t)} \widetilde{b}(t,X_t)\, dW_t$
"extended" vega	$\delta\left(\frac{Y_t}{\sigma(t,X_t)} \widetilde{a}(t) \sum_{i=1}^m \left(\frac{Z_{t_i}^{vega}}{Y_{t_i}} - \frac{Z_{t_{i-1}}^{vega}}{Y_{t_{i-1}}} \right) 1_{\{t_{i-1} \leq t < t_i\}} \right)$

Table 5: Summary of Fournie et al. Results.

6.6 Second conditions for interest rate dependent solutions

Greeks	Supplementary conditons
delta	$\mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\int_{0=t\leq s}^T r'(s,X_s) \frac{Y_s \sigma(t,X_t)}{Y_t} w^{delta}(t)\, dtds \right] = \mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\int_{0=s}^T r'(s,X_s) Y_s ds \right]$
gamma	the extension to this case is included in the equality of the delta
"extended" rho	$\mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\int_{0=t\leq s}^T r'(s,X_s) \frac{Y_s \sigma(t,X_t)}{Y_t} w^{rho}(t)\, dt\, ds \right] = \mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\int_{0=t\leq s}^T r'(s,X_s) \frac{Y_s \widetilde{b}(t,X_t)}{Y_t} dtds \right]$
"extended" vega	$\mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\int_{0=t\leq s}^T r'(s,X_s) \frac{Y_s \sigma(t,X_t)}{Y_t} w^{vega}(t)\, dtds \right] = \mathbb{E}_{x,X_{t_1},\ldots,X_{t_m}}^Q \left[\begin{array}{c} \int_{0=t\leq s}^T r'(s,X_s) \\ \left(\begin{array}{c} \frac{Y_s}{Y_t} \widetilde{\sigma}(t,X_t)\, dW_t \\ -\frac{Y_s}{Y_t} \sigma'(t,X_t)\, \widetilde{\sigma}(t,X_t)\, ds \end{array} \right) ds \end{array} \right]$

Table 6: Supplementary conditions for models with risk free rate depending on the underlying

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