A General Characterization of Quadratic Term Structure Models

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Abstract

In this paper, we define a strongly regular quadratic Gaussian process to characterize quadratic term structure models (QTSMs) in a general Markov setting. The key of this definition is to keep the analytical tractability of QTSMs which has the quadratic term structure of the yield curve. In order to keep this property, under the regularity condition, we have proven that no jumps are allowed in the infinitesimal generator of the underlying state process. The coefficient functions defined in the quadratic Gaussian relationship can be decided by the multi-variate Riccati Equations with a unique admissible parameter set. Based on this result, we discuss the pricing problems of QTSMs under default-free and defaultable rates.

1 Introduction

The term structure of the interest rate has long been a topic of concern for economists. Since the ground breaking studies of Vasicek (1977 [21]) and Cox,

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Ingersoll and Ross (1985 [3]), the class of the affine term structure models (ATSMs) of short rate has always been dominant in this area because of their tractability in pricing and estimation. Duffie and Kan (1996 [6]) and Duffie, Pan and Singleton(2000 [7]) proposed a general yield affine model and the affine jump diffusion model which gave almost a complete theory and solutions of the pricing in the "affine" settings in the aspect of Ito's calculus.

Recently quadratic term structure models (QTSMs) have also been studied from both theoretical analysis and empirical testing (e.g. Ahn, Dittmar and Gallant (2002 [1]), Chen and Poor (2002 [2]), Leippold and Wu (2001 [14]) and Leippold and Wu (2002 [15])). It turns out that QTSMs not only empirically outperform the ATSMs in that they are able to capture the nonlinearity of the time series and more flexible for the model design, but also have the nice analytical tractability comparable to ATSMs, namely, the zero-coupon bond price has the exponential-quadratic form of the state variables and the prices of options can be calculated by Fourier transform analysis. Moreover Filipovic (2002 [11]) discovered that under the assumption of linear independence among coefficients of the separable polynomial term structure (SPTS), the maximal degree of SPTS is equal to 2 which means that the identification of ATSMs and QTSMs essentially completes the search for SPTS models.

However, all the literatures mentioned above discuss the problem in the scenario of Ito's calculus which limits the underlying state variables in the diffusion (or jump diffusion) process. Filipovic (2001 [10]) analyzed the one-factor ATSMs from the Markov semigroup theory and Duffie, Filipovic and Schachermayer (2002,[4]) completely characterize the Multi-variable ATSM by proposing the concept of affine process which included a rich and strong structure behind the affine term structure.

In this paper, we define a regular quadratic Gaussian process to characterize QTSMs in a general Markov setting. The key of this definition is to keep the analytical tractability of QTSMs which has the quadratic term structure of the forward rates. In order to keep this property, we have proven that no jumps are allowed in the infinitesimal generator of the underlying state process. The coefficient functions defined in the quadratic Gaussian relationship can be decided by the multi-variate Riccati Equations with a unique admissible parameter set. Based on this result, we also discuss the pricing problems under defaultable rates.

The remainder of the paper is organized as follows. In Section 2, we give the basic notations used in this paper and some preliminary results in the Markov semigroup theory. In Section 3, we provide the definition of the regular quadratic Gaussian process and some straightforward results from the definition. We will propose our main results in Section 4 and discuss a conservative case in Section 5. In Section 6, we give the definition of generalize QTSM and deduce the option pricing formula under both risk-free and defaultable rate circumstances. All the mathematic proofs are included in the Appendix.

2 Basic Notations and Preliminary Results

First we will define some notations used in this paper as shown in Table 1:

Table 1: Summary of Notations

Notations	Implications
D	The state space with $D := \mathbb{R}^d$
D_Δ	One point compactification of D with $D_\Delta := D \cup \{\Delta\}$
$(\mathcal{F}_t)$	The natural filtration generated by X
$\mathcal{F}^\infty$	The σ-algebra equal to $\bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$
$C(D)$	The space of continuous functions on D
$B(D)$	The Banach space of bounded complex-valued Borel-measurable functions on D
$C_b(D)$	The Banach space $C(D) \bigcap B(D)$
$C_0(D)$	The Banach space consisting of $C(D)$ which vanishes at infinity
$C_c(D)$	The Banach space consisting of $C(D)$ with compact support
$C^k(D)$	The space of k-times differentiable functions f on the interior of D such that all partial derivatives of f up to order k are continuous
$C^\infty(D)$	The space $\bigcap_{k\in\mathbb{N}} C^k(D)$
$\nabla f(x)$	The gradient of the function f on D
e_i	The ith base vector on $\mathbb{R}^d$
$\langle\alpha, \beta\rangle$	The inner product on $\mathbb{C}^d$
Sem^d_{--}, Sem^d_-	The collections of $d \times d$ negative and semi-negative definiteness matrices, respectively.
Sem^d_{++}, Sem^d_+	The collections of $d \times d$ positive and semi-positive definiteness matrices, respectively.
∂Sem^d_+, ∂Sem^d_-	The sets are equal to $Sem^d_+ \backslash Sem^d_{++}$ and $Sem^d_- \backslash Sem^d_{--}$, respectively.
$\mathcal{D}(\mathcal{A})$	The space that contains all the functions f such that $\mathcal{A}f$ exists
$A \oplus B$	The set $\{x + y : x \in A, y \in B\}$.

Let's consider a time-homogenous Markov process X starting at $X_0 = x$ with the state space D and a positive contraction semigroup (P_t) on $B(D)$ with

$$P_0 f = f, \quad \text{for each} \quad f \in B(D). \tag{1}$$

Then according to Dynkin (1965, [9], Theorem 2.1), this semigroup corresponds to a transition function $p_t(x, \cdot)$, which satisfies, for each $t \geq 0$ and $x \in D$,

$$P_t(f(x)) = \int_D f(\xi) p_t(x, d\xi). \tag{2}$$

By Kolmogorov's extension theorem, given the above contraction semigroup (P_t), there exists a unique probability law $\mathbb{P}_x$ on the space $(\Omega, \mathcal{F}^\infty)$ such that X is a Markov process with respect to $(\mathcal{F}_t)$ which satisfies:

$$\mathbb{E}_x[f(X_s)|\mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_{s-t})], \quad \mathbb{P}_x - \text{a.s.}, \tag{3}$$

for any $s, t \in \mathbb{R}_+$, such that $s > t$ and for all $f \in B(D)$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

Remark 2.1 *It is not necessary to require the transition function $p_t(x, \cdot)$ to be conservative. Since if $p_t(x, D) < 1$, we can expand D to D_Δ and let*

$$p_t(x, D_\Delta) = 1,\ p_t(\Delta, \{\Delta\}) = 0 \textit{ for each } (t, x) \in \mathbb{R}^+ \times D. \tag{4}$$

3 Definition of the Quadratic Gaussian Process

First let's define two sets that are frequently used in this paper:

$$\begin{aligned}
\mathcal{B} &:= \{(u, V) \in \mathbb{C}^d \times \mathbb{C}^{d\times d} : e^{\langle u,x\rangle + x'Vx} \in B(D)\}, && (5)\\
\mathcal{E} &:= \{(\gamma, \delta, \Phi) : \gamma \in \mathbb{R},\ \delta \in \mathbb{R}^d,\ \Phi \in \mathbb{R}^{d\times d}, \\
&\qquad \gamma + \delta' x + x'\Phi x \geq 0,\ \text{for all } x \in D.\}. && (6)
\end{aligned}$$

Remark 3.1 *It is easy to see that*

i) The set

$$\{(u,V) : u \in \mathbb{C}^d \ , \ V \in Sem^d_{--} \oplus i\mathbb{R}^{d\times d}\} \\ \cup \{(u,V) : u \in i\mathbb{R}^d \ , \ V \in \partial Sem^d_{-} \oplus i\mathbb{R}^{d\times d}\} \in \mathcal{B}. \tag{7}$$

ii) For any $(\gamma, \delta, \Phi) \in \mathcal{E}$*,* $\gamma \in \mathbb{R}_+$ *and* $\Phi \in Sem^d_+$*.*

Then we define the function $f_{u,V}(x)$ as

$$f_{u,V}(x) = e^{\langle u,x\rangle + x'Vx}, \tag{8}$$

where $(u,V) \in \mathcal{B}$. Therefore this definition indicates that $f_{u,V}(x)$ is bounded.

Now we can define the quadratic Gaussian process as follows:

Definition 3.1 *The Markov Process* $(X, (\mathbb{P}_x)_{x\in D})$ *is said to be quadratic Gaussian if for every* $(t, x, (u,V)) \in \mathbb{R}_+ \times D \times \mathcal{B}$*,* $P_t f_{u,V}(x)$ *has exponential quadratic form of* x*, i.e., there exist functions* $A(t,u,V) \in \mathbb{C}$*,* $B(t,u,V) \in \mathbb{C}^d$ *and* $C(t,u,V) \in \mathbb{C}^{d\times d}$ *such that*

$$P_t f_{u,V}(x) = \exp[A(t,u,V) + \langle B(t,u,V), x\rangle + x'C(t,u,V)x]. \tag{9}$$

The corresponding semigroup $(P_t)_{t\in\mathbb{R}^+}$ *is also called quadratic Gaussian semigroup.*

Without loss of generality, we can assume that the function $C(t,u,V)$ is symmetric. Since $i\mathbb{R}^d \times \{0\} \in \mathcal{B}$, once we derive the functions $A(t,u,V)$, $B(t,u,V)$

and $C(t,u,V)$ for all $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$, the transition function is defined completely, i.e., the law is unique under the definition of (9).

Lemma 3.1 *The Ornstein-Uhlenbeck process is a quadratic Gaussian process.*

From Lemma 3.1, we know that the state variables X_t in QTSMs follow the quadratic Gaussian process.

By the Chapman-Kolmogorov equation we can derive by straightforward calculations, for any $(s,t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times D$,

$$\begin{aligned} A(t+s,u,V) &= A(t,u,V) + A(s,B(t,u,V),C(t,u,V)), \quad (10)\\ B(t+s,u,V) &= B(s,B(t,u,V),C(t,u,V)), \quad (11)\\ \text{and} \quad C(t+s,u,V) &= C(s,B(t,u,V),C(t,u,V)), \quad (12) \end{aligned}$$

and by (1), we obtain the initial conditions as:

$$A(0,u,V) = 0, \quad B(0,u,V) = u \quad \text{and} \quad C(0,u,V) = V. \tag{13}$$

Definition 3.2 *A Markov process $(X,(\mathbb{P}_x)_{x\in D},(P_t))$ is regular, if*

i) The functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ are continuous on $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$.

ii) The weak infinitesimal generator

$$\tilde{\mathcal{A}} f_{u,V}(x) = \partial_t^+ P_t f_{u,V}(x)|_{t=0} = \lim_{t\downarrow 0} \frac{P_t f_{u,V}(x) - f_{u,V}(x)}{t} \tag{14}$$

exists for every $(x,(u,V)) \in D \times \mathcal{B}$*.*

If a Markov process $(X, (\mathbb{P}_x)_{x \in D}, (P_t))$ is regular and quadratic Gaussian, we can define

$$\begin{aligned} F(u,V): &= \partial_t^+ A(t,u,V)|_{t=0}, & (15)\\ R(u,V): &= \partial_t^+ B(t,u,V)|_{t=0}, & (16)\\ T(u,V): &= \partial_t^+ C(t,u,V)|_{t=0} & (17) \end{aligned}$$

and thus we obtain

$$\tilde{\mathcal{A}} f_{u,V}(x) = \left(F(u,V) + \langle R(u,V), x\rangle + x' T(u,V) x\right) f_{u,V}(x), \qquad (18)$$

for all $(x,(u,V)) \in D \times \mathcal{B}$.

Lemma 3.2 *If a Markov process is regular and quadratic Gaussian, we have that*

$$\begin{aligned} \partial_t^+ A(t,u,V) &= F(B(t,u,V), C(t,u,V)), & (19)\\ \partial_t^+ B(t,u,V) &= R(B(t,u,V), C(t,u,V)), & (20)\\ \text{and} \quad \partial_t^+ C(t,u,V) &= T(B(t,u,V), C(t,u,V)), & (21) \end{aligned}$$

for all $t \in \mathbb{R}_+$ *and* $(u, V) \in \mathcal{B}$. *Moreover,*

$$\begin{aligned}
F(u,V) &= \tilde{\mathcal{A}} f_{u,V}(0), && (22)\\
R_i(u,V) &= \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} - \frac{\tilde{\mathcal{A}} f_{u,V}(-e_i)}{f_{u,V}(-e_i)}\right], && (23)\\
T_{ii}(u,V) &= \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} + \frac{\tilde{\mathcal{A}} f_{u,V}(-e_i)}{f_{u,V}(-e_i)}\right] - F(u,V), && (24)\\
\textit{and} \quad T_{ij}(u,V) &= \frac{1}{2}\left[\frac{\tilde{\mathcal{A}} f_{u,V}(e_i+e_j)}{f_{u,V}(e_i+e_j)} - \frac{\tilde{\mathcal{A}} f_{u,V}(e_i)}{f_{u,V}(e_i)} - \frac{\tilde{\mathcal{A}} f_{u,V}(e_j)}{f_{u,V}(e_j)}\right] \\
&\quad + \frac{1}{2} F(u,V), && (25)\\
&\quad \textit{for} \quad i,j \in \{1,2,...,d\} \quad \textit{and} \quad i \neq j.
\end{aligned}$$

Basically Lemma 3.2 gives us a way to find the functions $F(u,V)$, $R(u,V)$ and $T(u,V)$ if we know the explicit formula for $\frac{\tilde{\mathcal{A}} f_{u,V}(\cdot)}{f_{u,V}(\cdot)}$.

4 Infinitesimal Generator of the Regular Quadratic Gaussian Process

Now we try to obtain the analytical expression of the infinitesimal generator for the regular quadratic Gaussian process. We approach this by several steps. First we focus on the function $f_{u,V}(x)$. Here we define a "cut-off" function $\chi(x) = (\chi_1(x), \chi_2(x), ..., \chi_n(x))' : D \to [-1,1]^d$ by

$$\chi_k(x) = \begin{cases} x_k & \text{if} \quad |x_k| \le 1, \\ sgn(x_k) & \text{if} \quad |x_k| > 1, \end{cases} \tag{26}$$

where $sgn(x)$ denotes the sign function. And we define the function $d(x)$ on D as:

$$d(x) = \sum_{k=1}^{d} \chi_k(x)^2. \tag{27}$$

It is easy to see that $0 \leq d(\cdot) \leq d$ and $d(x-y) = 0$ if and only if $x = y$, for $x, y \in D$.

Lemma 4.1 ***(Representation Results for Regular Processes)***

Given $x \in D$, there exist elements

$$\alpha(x) \in Sem_+^d, \quad \beta(x) \in \mathbb{R}^d \quad \textit{and} \quad \gamma(x) \in \mathbb{R}_+ \tag{28}$$

and a positive measure $\nu(x, d\xi)$ on $D\backslash\{x\}$ satisfying

$$\int_{D\backslash\{x\}} d(\xi - x)\nu(x, d\xi) < \infty, \tag{29}$$

such that for all $(u, V) \in \mathcal{B}$,

$$\begin{aligned} \tilde{\mathcal{A}} f_{u,V}(x) &= tr\left(\alpha(x)\frac{\partial^2 f_{u,V}(x)}{\partial x \partial x'}\right) + \langle \beta(x), \nabla f_{u,V}(x)\rangle - \gamma(x) f_{u,V}(x) \\ &\quad + \int_{D\backslash\{x\}} \tilde{h}_{u,V}(x,\xi)\nu(x, d\xi), \end{aligned} \tag{30}$$

where

$$\tilde{h}_{u,V}(x,\xi) = f_{u,V}(\xi) - f_{u,V}(x) - \langle \nabla f_{u,V}(x), \chi(\xi - x)\rangle. \tag{31}$$

Definition 4.1 *A process X is said to be strongly regular if it is regular and the measure $\nu(x,\cdot)$ for each $x \in D$ satisfies:*

i)

$$\int_{D\backslash x} \chi(x-\xi)\nu(x,d\xi) < \infty. \tag{32}$$

ii)

$$\int_{D\backslash\{x\}} ||\xi||^2 \nu(x,d\xi) < \infty, \tag{33}$$

where $||\cdot||$ *denote the Euclidean norm of the d-dimensional vector.*

Definition 4.2 *We call a parameter set* $(\alpha,\beta,b,\gamma,\delta,\Phi)$ *is admissible if and only if*

$$\begin{aligned} & \alpha \in Sem_+^d,\ \beta \in \mathbb{R}^d,\ b \in \mathbb{R}^{d\times d} \\ \textit{and} \qquad & (\gamma,\,\delta,\,\Phi) \in \mathcal{E}. \end{aligned}$$

From Lemma 3.2 and Lemma 4.1, we have the following mappings theorem for functions $F(u,V), R(u,V)$ and $T(u,V)$.

Proposition 4.1 ***(Mappings Theorem)***

For a strongly regular Gaussian quadratic process X*, there exists an admissible parameter set* $(\alpha,\beta,b,\gamma,\delta,\Phi)$ *such that for all* $(x,(u,V)) \in D\times\mathcal{B}$*,*

$$\tilde{\mathcal{A}}f_{u,V}(x) = tr\left(\alpha\frac{\partial^2 f_{u,V}(x)}{\partial x\partial x'}\right) + \langle\beta+bx,\nabla f_{u,V}(x)\rangle - (\gamma+\delta' x + x'\Phi x)f_{u,V}(x). \tag{34}$$

Moreover, the functions $A(t,u,V)$*,* $B(t,u,V)$ *and* $C(t,u,V)$ *satisfy the following*

Riccati equations:

$$\begin{aligned}\partial_t A(t,u,V) &= F(B(t,u,V),C(t,u,V)), \quad A(0,u,V)=0, && (35)\\ \partial_t B(t,u,V) &= R(B(t,u,V),C(t,u,V)), \quad B(0,u,V)=u && (36)\\ \textit{and} \quad \partial_t C(t,u,V) &= T(B(t,u,V),C(t,u,V)), \quad C(0,u,V)=V, && (37)\end{aligned}$$

with

$$\begin{aligned}F(u,V) &= \langle \alpha u,u\rangle + 2tr(\alpha V) + \langle \beta,u\rangle - \gamma, && (38)\\ R(u,V) &= 4V'\alpha u + b'u + 2V\beta - \delta, && (39)\\ T(u,V) &= 4V'\alpha V + b'V + V'b - \Phi,. && (40)\end{aligned}$$

Since from the above proposition, we find that $T(u,V)$ does not depend on u, it means that the function $C(t,u,V)$ can be rewritten as $C(t,V)$. In particular, if the initial value V is zero, then $u \in i\mathbb{R}^d$. Hence we will have

$$\begin{aligned}C(t,u,0) &= -\Phi t && (41)\\ \text{and} \quad B(t,u,0) &= e^{bt}u - \int_0^t e^{b(t-s)}\delta ds, \quad t \in \mathbb{R}_+. && (42)\end{aligned}$$

If $\Phi \neq 0$ and since $\Phi \in Sem_+^d$, we have $C(t,u,0)) \in Sem_{--}^d$, for $t > 0$. If $\Phi = 0$, then $C(t,u,0) = 0$. Since $(\gamma,\delta,\Phi) \in \mathcal{E}$, we have $\delta = 0$. By (42), we derive that $B(t,u,0) \in i\mathbb{R}^d$. Thus $(B(t,u,0),C(t,u,0)) \in \mathcal{B}$ for all $(t,u) \in \mathbb{R}_+ \times i\mathbb{R}^d$.

This can be also deduced in another way. Since P_t is a contraction semigroup and $f_{u,V}(x) \in B(D)$ for each $(u,V) \in \mathcal{B}$, thus $P_t f_{u,V}(x) \in B(D)$ which just means that $(B(t,u,V), C(t,u,V)) \in \mathcal{B}$, for all $(t,(u,V)) \in \mathbb{R}_+ \times \mathcal{B}$.

Theorem 4.1 *Suppose X is a strongly regular quadratic Gaussian process. Then X is a Feller process. Let $\mathcal{A}$ be its infinitesimal generator. Then there exists an admissible parameter set $(\alpha, \beta, b, \gamma, \delta, \Phi)$ such that, for all $f \in C_c^2(D)$,*

$$\mathcal{A}f(x) = tr\left(\alpha \frac{\partial^2 f(x)}{\partial x \partial x'}\right) + \langle \beta + bx, \nabla f(x) \rangle - (\gamma + \delta' x + x'\Phi x) f(x). \tag{43}$$

Conversely, given an admissible parameter set $(\alpha, \beta, b, \gamma, \delta, \Phi)$, there exists a regular quadratic Gaussian semigroup (P_t) with infinitesimal generator (43) and (9) holds for all $(u,V) \in \mathcal{B}$. The functions $A(\cdot,u,V)$, $B(\cdot,u,V)$ and $C(\cdot,u,V)$ are given by (35) to (40).

Remark 4.1 *From Theorem 4.1, we find in order to hold (9), there does not exist jump parts in the infinitesimal generator of the strongly regular quadratic Gaussian process. It is worth to mention that the poisson jump measure which is frequently in the jump diffusion process is ruled out of the quadratic term structure models in order to keep the analytical tractability.*

5 Conservative Regular Quadratic Gaussian Process

By Dynkin (1965 [9], Lemma 2.3), we have a quadratic Gaussian process X with the parameter set $(\alpha, \beta, b, \gamma, \delta, \Phi)$ is conservative if and only if $\gamma = 0$, $\delta = 0$ and $\Phi = 0$.

We know if we set $V = 0$, then $C(t, u, 0) = 0$ for all $t \in \mathbb{R}^+$ and $u \in i\mathbb{R}^d$, thus

$$P_t e^{\langle u,x \rangle} = e^{A(t,u,0)+\langle B(t,u,0),x \rangle}, \quad x \in D. \tag{44}$$

It gives us the following proposition:

Proposition 5.1 *The conservative strongly regular quadratic Gaussian process is a regular affine process and moreover it is Ornstein-Uhlenbeck process.*

6 Generalized Quadratic Term Structure Models

Here we can define the generalized quadratic term structure model (GQTSM) as follows:

Definition 6.1 *The term structure model is a GQTSM with the parameters* $(\alpha, \beta, b, \gamma, \delta, \Phi, r_0, r_1, r_2)$ *if*

i) $(\alpha, \beta, b, \gamma, \delta, \Phi)$ *is admissible and a d-dimensional state vector* X *is a strongly regular quadratic Gaussian process under the risk-neutral measure* $\mathbb{P}_x$

with this parameter set.

ii) $(r_0, r_1, r_2) \in \mathcal{E}$ *and the short rate* r_t *is a quadratic function of* X*:*

$$r(X_t) = r_0 + \langle r_1, X_t \rangle + X_t' r_2 X_t, \quad (r_0, r_1, r_2) \in \mathcal{E}. \tag{45}$$

(iii) For any fixed $x \in D$ *and* $t \in \mathbb{R}^+$*,*

$$\mathbb{P}_x \left[e^{\int_0^t r(X_s) ds} \right] < \infty. \tag{46}$$

Remark 6.1 *The condition* $(r_0, r_1, r_2) \in \mathcal{E}$ *is necessary for keeping the short rate* $r(X_t)$ *nonnegative.*

6.1 Bond Pricing and Quadratic Pricing Class

First we define (Q_t) as

$$Q_t f(x) := \mathbb{E}_x \left[e^{-\int_0^t r(X_s) ds} f(X_t) \right], \quad x \in D, \ t \in \mathbb{R}^+. \tag{47}$$

then the following proposition proves that Q_t is a pricing semigroup.

Proposition 6.1 ***(The Feynman_Kac Formula)***

Given a quadratic Gaussian term structure model with the parameters $(\alpha, \beta, b, \gamma, \delta, \Phi, r_0, r_1, r_2)$*, the family* (Q_t) *forms a regular quadratic Gaussian semigroup with the parameter set* $(\alpha, \beta, b, \gamma + r_0, \delta + r_1, \Phi + r_2)$*.*

Remark 6.2 *Since* $(r_0, r_1, r_2) \in \mathcal{E}$*,* $(\gamma + r_0, \delta + r_1, \Phi + r_2) \in \mathcal{E}$ *and thus* $(\alpha, \beta, b, \gamma + r_0, \delta + r_1, \Phi + r_2)$ *is admissible.*

Then it is easy to derive the price $\pi(0,T)$ of a zero-coupon bond with maturity T at time 0 as

$$\pi(0,T) = Q_T 1 = e^{A(T,0,0)+\langle B(T,0,0),x\rangle + x' C(T,0,0)x}, \tag{48}$$

where $A(T,0,0)$, $B(T,0,0)$ and $C(T,0,0)$ can be solved from (35) to (40) with the initial value $(u,V)=(0,0)$ and the parameter set $(\alpha,\beta,b,\gamma+r_0,\delta+r_1,\Phi+r_2)$. By the Markov property we also have

$$\begin{aligned}
\pi(t,T) &= \mathbb{E}_x[e^{-\int_t^T r(X_s)ds}1|\mathcal{F}_t] && (49)\\
&= \mathbb{E}_x[e^{-\int_0^{T-t}(r_0+r_1'X_s+X_s'r_2X_s)ds}\circ\theta_t 1|X_t] && (50)\\
&= \mathbb{E}_{X_t}[e^{-\int_0^{T-t}(r_0+r_1'X_s+X_s'r_2X_s)ds}] && (51)\\
&= e^{A(T-t,0,0)+\langle B(T-t,0,0),X(t)\rangle + X_t'C(T-t,0,0)X_t}, && (52)
\end{aligned}$$

where $\theta_t(\omega)(s) = \omega(t+s)$ which is a shift operator: $\Omega \mapsto \Omega$.

Definition 6.2 [1] *A model is said to be in the Quadratic Pricing Class (QPC) if the prices of zero-coupon bonds, $\pi(t,T)$, are exponential-quadratic functions of a Markov process X:*

$$\pi(t,T) = e^{A(T-t)+\langle B(T-t),X_t\rangle + X_t'C(T-t)X_t}. \tag{53}$$

Under Definition 6.2, we have the following proposition:

Proposition 6.2 *A model belongs to QPC if it is a GQTSM.*

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[1] Here we follow the same definition as shown in Leippold and Wu (2002, [15])

6.2 Option Pricing

For a European zero-coupon bond option, the price can be easily obtained by taking advantage of the quadratic Gaussian pricing semigroup. Let the payoff $h(X_t) = (K - \pi(t,T))^+$, where $K(K > 0)$ is the strike price.

$$\begin{aligned} Q_t h(x) &= \mathbb{E}_x \left[e^{-\int_0^t r(X_s)ds} h(X_t) \right] \qquad (54) \\ &= \mathbb{E}_x \left[e^{-\int_0^t r(X_s)ds} \pi(t,T) 1_{\pi(t,T)\le K} \right] \\ &\quad -K\mathbb{E}_x \left[e^{-\int_0^t r(X_s)ds} 1_{\pi(t,T)\le K} \right]. \qquad (55) \end{aligned}$$

Since $\pi(t,T)$ has an exponential-quadratic form on X_t and let

$$G_{u_1,V_1;u_2,V_2}(y,x,t) = \mathbb{E}_x [e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} 1_{u_2' X_t + X_t' V_2 X_t \le y}], \qquad (56)$$

where u_1, u_2 are two d-dimensional vectors and two V_1, V_2 are $d \times d$ matrices, such that $(u_1, V_1) \in \mathcal{B} \cap \left(\mathbb{R}^d \times \mathbb{R}^{d\times d}\right)$.

Remark 6.3 *Since the price of a zero-coupon bond is a positive real value bounded by 1 for any initial state x, reasonably we can assume that* $(u_1, V_1) \in \mathcal{B} \cap \left(\mathbb{R}^d \times \mathbb{R}^{d\times d}\right)$.

Since from (56), we know that $G_{u_1,V_1;u_2,V_2}(y,x,t)$ is the cumulative distribution function of $u_2' X_t + X_t' V_2 X_t$ under the measure $e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} \mathbb{P}_x$, necessarily we only need to calculate the Fourier transform of the function G.

Let $\mathcal{G}$ be its Fourier transform, we have

$$\begin{aligned}
\mathcal{G}_{u_1,V_1;u_2,V_2}(z,x,t) &= \int_{\mathbb{R}} e^{izy} dG_{u_1,V_1;u_2,V_2}(y,x,t) \\
&= \int_{\mathbb{R}} e^{izy} d\mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} 1_{u_2' X_t + X_t' V_2 X_t \leq y}] \\
&= \int_{\mathbb{R}} e^{izy} \mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{u_1' X_t + X_t' V_1 X_t} \delta(y - u_2' X_t + X_t' V_2 X_t)] \\
&= \mathbb{E}_x[e^{-\int_0^t r(X_s)ds} e^{(u_1 + izu_2)' X_t + X_t' (V_1 + izV_2) X_t}] \\
&= Q_t f_{u_1+izu_2, V_1+izV_2}(x). \qquad (57)
\end{aligned}$$

Because $(u_1 + izu_2, V_1 + izV_2) \in \mathcal{B}$, by Proposition 6.1, we can calculate the Fourier Transform of (56). This technique was originally proposed by Heston (1993, [12]), generalized by Duffie, Pan and Singleton (2000 [7]) to the affine jump-diffusion model and by Leippold and Wu (2002, [15]) to the quadratic term structure model.

6.3 Pricing Under the Defaultable Rate

There are two approaches to evaluating the defaultable corporate bonds: One is to specify the dynamics of the process that triggers the default (e.g. Jarrow and Turnbull (1995 [13]), Madam and Unal (1996 [16]), Duffie and Huang (1996 [5]), Duffie and Singleton (1999 [8])); The other one is to model the dynamics of the defaultable bond prices and forward rates directly (e.g. Schonbucher (1996 [20])). Here we follow the first approach since the results established in the previous sections can be applied. First we give some underlying assumptions:

Assumption 6.1 *The default time is a stopping time* τ *and* $I(t) = 1_{\tau \geq t}$ *is a* *default indicator function.*

Assumption 6.2 *The occurrence of the default is modelled as an Poisson Jump Process with the arrival intensity $h(t)$ which is the intensity conditional on the default not happening prior to time t.*

Assumption 6.3 *It is assumed that the recovery from the default is the recovery of market value (RMV), which means that recovery from the default at time t is equal to $(1 - l(t))\pi^d(t, T)$, where $l(t)$ is the loss rate and $\pi^d(t, T)$ denotes the price of the defaultable bond which the will payoff a dollar at the maturity T conditioned on that it hasn't been defaulted at time t .*

Assumption 6.4 *Both the intensity of the default $h(X_t)$ and the loss rate $l(X_t)$ depend on exogenous state variables.*

Proposition 6.3 *Given the the assumptions (6.1)-(6.4), for any any traded asset S with the value function $f(X_t)$ at time t and the maturity T, such that $f(\cdot) \in B(D)$, we have*

$$f(X_t) = \mathbb{E}_x \left[e^{-\int_0^T \rho(X_s)ds} f(X_T) \right], \tag{58}$$

where

$$\rho(X_t) = l(X_t)h(X_t) + r(X_t). \tag{59}$$

Also as mentioned in Duffie and Singleton (1999 [8]), using the prices of the defaultable bonds is not sufficient to separately specify the dynamics of $l(t)$ and $h(t)$. So here we only make an assumption on the product of the two variables:

Assumption 6.5 *At any time t, $l(X_t)h(X_t)$ is a quadratic function of the state variables X_t and so is $\rho(X_t)$:*

$$\rho(X_t) = r(X_t) + l(X_t)h(X_t) \tag{60}$$

$$= \rho_0 + \langle \rho_1, X_t \rangle + X_t' \rho_2 X_t, \tag{61}$$

where $(\rho_0, \rho_1, \rho_2) \in \mathcal{E}$.

Now let $Q_t^d f(x) = \mathbb{E}_x \left[e^{-\int_0^t (\rho_0 + \langle \rho_1, X_s \rangle + X_s' \rho_2 X_s)} f(X_s) ds) \right]$. By Proposition 6.2, we know that Q_t^d is also a quadratic Gaussian semigroup with the parameter set $(\alpha, \beta, b, \gamma + \rho_0, \delta + \rho_1, \Phi + \rho_2)$. Therefore, we can price the defaultable bond and using the same arguments in the previous section, we can price the options on the defaultable bonds.

7 Acknowledgement

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8 Appendix

8.1 Proof of Lemma 3.1

Our proof is based on the definition of Orstein-Uhlenbeck process by Sato(1999 [19], Definition 17.2). If X is a Orstein-Uhlenbeck process, then there exists

$c > 0$ and a time-homogenous transition function $p_t(x, \cdot)$ such that

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{i\langle z,\xi\rangle} p_t(x, d\xi) &= \exp\left[ie^{-ct}\langle x, z\rangle - \int_0^t \frac{e^{-2cs} z'z}{2} ds\right] \\
&= \exp\left[ie^{-ct}\langle x, z\rangle - \frac{(1-e^{-2ct})z'z}{4c}\right], \quad z \in \mathbb{R}^d. \quad (62)
\end{aligned}$$

From (133), we know that for every $t \in \mathbb{R}_+$, X_t is a d-dimensional Gaussian random vector with mean $e^{-ct}x$, and covariance $\frac{(1-e^{-2ct})}{4c} I$. Therefore we have, for all $(u, V) \in \mathcal{B}$,

$$\begin{aligned}
P_t f_{u,V}(x) &= \int_{\mathbb{R}^d} e^{\langle u,\xi\rangle + \xi' V \xi} p_t(x, d\xi) \\
&= \int_{\mathbb{R}^d} e^{\langle u,\xi\rangle + \xi' V \xi} \left(\frac{4c}{2\pi(1-e^{-2ct})}\right)^{\frac{d}{2}} e^{-\frac{2c(\xi - e^{-ct}x)'(\xi - e^{-ct}x)}{(1-e^{-2ct})}} d\xi \\
&= \int_{\mathbb{R}^d} e^{A(t,u,V) + \langle B(t,u,V), x\rangle + x' C(t,u,V) x}, \quad (63)
\end{aligned}$$

where

$$\begin{aligned}
A(t, u, V) &= \left|I - \frac{1-e^{-2ct}}{2c} V\right|^{-1} \frac{u'\Sigma(t)u}{2}, \quad (64) \\
B(t, u, V) &= \left|I - \frac{1-e^{-2ct}}{2c} V\right|^{-1} \frac{4c}{1-e^{-2ct}} \Sigma(t) u, \quad (65) \\
C(t, u, V) &= \left|I - \frac{1-e^{-2ct}}{2c} V\right|^{-1} \frac{4ce^{-2ct}}{1-e^{-2ct}} \left(\frac{4c}{1-e^{-2ct}} \Sigma(t) - I\right), \quad (66) \\
\text{with} \quad \Sigma(t) &= \left(\frac{4c}{1-e^{-2ct}} I - 2V\right)^{-1}. \quad (67)
\end{aligned}$$

Since $V \in Sem_-^d$, $I - \frac{1-e^{-2ct}}{2c} V \in Sem_{++}^d$ and $\left|I - \frac{1-e^{-2ct}}{2c} V\right|^{-1}$ is well defined. This completes the proof.

8.2 Proof of Lemma 3.2

Given (10) to (12) and (3), we have

$$A(t+s,u,V) - A(t,u,V) = A(s,B(t,u,V),C(t,u,V)) - A(0,B(t,u,V),C(t,u,V)) \tag{68}$$

$$B(t+s,u,V) - B(t,u,V) = B(s,B(t,u,V),C(t,u,V)) - B(0,B(t,u,V),C(t,u,V)) \tag{69}$$

and

$$C(t+s,u,V) - C(t,u,V) = C(s,B(t,u,V),C(t,u,V)) - C(0,B(t,u,V),C(t,u,V)) \tag{70}$$

Now let $s \to 0^+$, we prove (19) to (21). By using (18), we can easily derive (22) to (25). This completes the proof.

8.3 Proof of Lemma 4.1

A more general proof can be found in Duffie, Filipovic and Schachermayer (2002 [4], Section 4).

8.4 Proof of Proposition 4.1

In order to prove Proposition 4.1, first we give two lemmas for the unique representations.

Lemma 8.1 *Given (A,β,γ,ν), where $A \in \mathbb{R}^{d\times d}$, $\beta \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$ and ν is a measure on D, then the representation*

$$\begin{aligned}\hat{\mu}(u) &= \langle u, Au\rangle + \langle \beta, u\rangle + \gamma \\ &\quad + \int_{D\setminus\{0\}} \left[e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\right] \nu(d\xi), \quad u \in i\mathbb{R}^d \end{aligned} \tag{71}$$

by A, β, γ, ν is unique.

Proof. see Sato(1999 [19], Theorem 8.1).

Lemma 8.2 *Given functions $(A(x), \beta(x), \gamma(x))$, where $A : D \mapsto \mathbb{R}^{d\times d}$, $\beta : D \mapsto \mathbb{R}^d$ and $\gamma : D \mapsto \mathbb{R}$, then the representation*

$$\begin{aligned} \bar{\mu}(x,u,V) &= (u+2Vx)'A(x)(u+2Vx) + 2tr(A(x)V) + \langle \beta(x), u+2Vx\rangle \\ &\quad +\gamma(x), \quad \textit{for each} \quad (x,(u,V)) \in D \times \mathcal{B} \end{aligned} \tag{72}$$

by A, β, γ is unique.

Proof. For any fixed $x_0 \in D$. Let $V = 0$ by Lemma 8.1, we know that $A(x_0)$, $\beta(x_0)$ and $\gamma(x_0)$ is unique, so the functions (A, β, γ) are uniquely defined on D given the representation $\bar{\mu}(x,u,V)$.

Now we start to prove Proposition 4.1.

By simply plug (31) into (30), we have

$$\begin{aligned} &\frac{\mathcal{A}f_{u,V}(x)}{f_{u,V}(x)} = (u+2Vx)'\alpha(x)(u+2Vx) + 2tr(\alpha(x)V) + \langle \beta(x), u+2Vx\rangle + \gamma(x) \\ &+\int_{D\setminus\{x\}} \left(e^{\langle u,\xi-x\rangle + \xi'V\xi - x'Vx} - 1 - \langle u+2Vx, \chi(\xi-x)\rangle\right) \nu(x,d\xi). \end{aligned} \tag{73}$$

By applying (22) and (73), we obtain

$$\begin{aligned} &F(u,V) = \langle \alpha u, u\rangle + 2tr(\alpha V) + \langle \beta, u\rangle + \gamma \\ &+\int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\right) m(d\xi), \end{aligned} \tag{74}$$

where

$$\alpha = \alpha(0),\ \beta = \beta(0),\ \gamma = \gamma(0) \text{ and } m(d\xi) = \nu(0, d\xi). \tag{75}$$

In the same way, by applying (23), (24) and (73), we find that

$$\begin{aligned} R_i(u,V) &= \langle \hat{\alpha}_i u, u\rangle + 4\langle \bar{\alpha}_i u + \hat{\alpha}_i V^i, V^i\rangle + 2tr(\hat{\alpha}_i V) + \langle b_i, u\rangle + 2\langle \bar{\beta}_i, V^i\rangle \\ &+\delta_i + \frac{1}{2}\int_{D\setminus\{0\}} \left(e^{\langle u+2V^i,\xi\rangle + \xi' V\xi} - 1 - \langle u+2V^i, \chi(\xi)\right) \nu_i(d\xi) \\ &-\frac{1}{2}\int_{D\setminus\{0\}} \left(e^{\langle u-2V^i,\xi\rangle + \xi' V\xi} - 1 - \langle u-2V^i, \chi(\xi)\right) \nu_{-i}(d\xi), \end{aligned} \tag{76}$$

and

$$\begin{aligned} T_{ii}(u,V) &= \langle (\bar{\alpha}_i - \alpha)u, u\rangle + 4\langle \hat{\alpha}_i u + \bar{\alpha}_i V^i, V^i\rangle + 2tr((\bar{\alpha}_i - \alpha)V) + \langle (\bar{\beta}_i - \beta), u\rangle + 2\langle b_i, V^i\rangle \\ &+\Phi_{ii} + \frac{1}{2}\int_{D\setminus\{0\}} \left(e^{\langle u+2V^i,\xi\rangle + \xi' V\xi} - 1 - \langle u+2V^i, \chi(\xi)\right) \nu_i(d\xi) \\ &+\frac{1}{2}\int_{D\setminus\{0\}} \left(e^{\langle u-2V^i,\xi\rangle + \xi' V\xi} - 1 - \langle u-2V^i, \chi(\xi)\right) \nu_{-i}(d\xi) \\ &+\int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\right) m(d\xi), \end{aligned} \tag{77}$$

for each $1 \leq i \leq d$, where

$$\begin{aligned}
\hat{\alpha}_i &= \frac{1}{2}(\alpha(e_i) - \alpha(-e_i)), && (78)\\
\bar{\alpha}_i &= \frac{1}{2}(\alpha(e_i) + \alpha(-e_i)), && (79)\\
b_i &= \frac{1}{2}(\beta(e_i) - \beta(-e_i)), && (80)\\
\bar{\beta}_i &= \frac{1}{2}(\beta(e_i) + \beta(-e_i)), && (81)\\
\delta_i &= \frac{1}{2}(\gamma(e_i) - \gamma(-e_i)), && (82)\\
\Phi_{ii} &= \frac{1}{2}(\gamma(e_i) + \gamma(-e_i)) - \gamma, && (83)\\
\nu_i(\cdot) &= \nu(e_i, \cdot), && (84)\\
\text{and} \quad \nu_{-i}(\cdot) &= \nu(-e_i, \cdot). && (85)
\end{aligned}$$

Now by (18), we have, for each $s \in \mathbb{R}$,

$$\frac{\tilde{\mathcal{A}} f_{u,V}(se_i)}{f_{u,V}(se_i)} = F(u,V) + sR_i(u,V) + s^2 T_{ii}(u,V), \tag{86}$$

$$\text{for each } (i,(u,V)) \in \{1,2,...,d\} \times \mathcal{B}. \tag{87}$$

We approach the proof in two steps. First we let $V = 0$, according to Lemma 8.1, we have, for each $s \in \mathbb{R}$,

$$\begin{aligned}
\alpha(se_i) &= \alpha + s\hat{\alpha}_i + s^2(\bar{\alpha}_i - \alpha), & (88)\\
\beta(se_i) &= \beta + sb_i + s^2(\bar{\beta}_i - \beta), & (89)\\
\gamma(se_i) &= \gamma + s\delta_i + s^2\Phi_{ii}, & (90)\\
\text{and} \quad \nu(se_i, \cdot) &= \left(\frac{1}{2}[\nu_i(\cdot) + \nu_{-i}(\cdot)] - m(\cdot)]\right) s^2 \\
&\quad \frac{1}{2}(\nu_i(\cdot) - \nu_{-i}(\cdot))s + m(\cdot). & (91)
\end{aligned}$$

Since $\nu(se_i, \cdot)$ is a nonnegative measure for each $s \in \mathbb{R}$, we can deduce the following constraints among the measures $m(\cdot)$, $\nu_i(\cdot)$ and $\nu_{-i}(\cdot)$:

i)

$$\nu_i(\cdot) + \nu_{-i}(\cdot)] - 2m(\cdot) \geq 0. \tag{92}$$

ii)If

$$\nu_i(\cdot) + \nu_{-i}(\cdot) - 2m(\cdot) = 0, \tag{93}$$

then

$$\nu_i(\cdot) = \nu_{-i}(\cdot) = m(\cdot). \tag{94}$$

Secondly, we let $u = 0$. By applying (88) and (89), (86) can be rewritten, for each $s \in \mathbb{R}$, as:

$$\begin{aligned}
& 4(s^2-1)\langle(\alpha(se_i)-\alpha)V^i, V^i\rangle + 2(s^3-s)\langle(\bar{\beta}_i-\beta), V^i\rangle \\
+ & \int_{D\backslash\{0\}} \left[e^{\xi' V\xi + 2s\langle V^i,\xi\rangle} - 1 - 2s\langle V^i, \chi(\xi)\rangle\right] \nu(se_i, d\xi) \\
- & \int_{D\backslash\{0\}} \left[e^{\xi' V\xi} - 1\right] m(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s}{2} \left[e^{\xi' V\xi + 2\langle V^i,\xi\rangle} - 1 - 2\langle V^i, \chi(\xi)\rangle\right] \nu_i(d\xi) \\
+ & \int_{D\backslash\{0\}} \frac{s}{2} \left[e^{\xi' V\xi - 2\langle V^i,\xi\rangle} - 1 + 2\langle V^i, \chi(\xi)\rangle\right] \nu_{-i}(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s^2}{2} \left[e^{\xi' V\xi + 2\langle V^i,\xi\rangle} - 1 - 2\langle V^i, \chi(\xi)\rangle\right] \nu_i(d\xi) \\
- & \int_{D\backslash\{0\}} \frac{s^2}{2} \left[e^{\xi' V\xi - 2\langle V^i,\xi\rangle} - 1 + 2\langle V^i, \chi(\xi)\rangle\right] \nu_{-i}(d\xi) \\
+ & \int_{D\backslash\{0\}} s^2 \left[e^{\xi' V\xi} - 1\right] m(d\xi) = 0. \qquad (95)
\end{aligned}$$

Since the equation (95) is true for all $V \in Sem^d_-$. In particular, for any given d-dimensional vector ϕ which is on the unit hyper-sphere and $r \in \mathbb{R}_+$, when we apply such a V that the ith column and row of V are equal to $ir\phi$ and $ir\phi'$, respectively and all the other entries are zero. Note that

$$\begin{aligned}
e^{\xi' V\xi + 2s\langle V^i,\xi\rangle} - 1 - 2s\langle V^i, \chi(\xi)\rangle &\leq (2||V|| + 2s^2||V^i||^2)||\xi||^2 + ||V||^2(2|s|||\xi||^3 + ||\xi||^4) \\
&\leq 4r||\xi||^2 + 2r^2(s^2||\xi||^2 + 4|s|||\xi||^3 + 2||\xi||^4) \text{ for } \xi \leq 1.
\end{aligned}$$

and bounded when $\xi > 1$. Therefore by Lemma 4.1 and dominated convergence theorem, after we divide both sides by r^2 and let $r \to +\infty$, we can derive that

$$4(s^2-1)\langle(\alpha(se_i)-\alpha)V^i, V^i\rangle = 0. \tag{96}$$

Because this is true for each ϕ on the unit hyper-sphere and $s \in \mathbb{R}$, we have

$$\alpha(se_i) = \alpha, \quad \text{for each} \quad (i,s) \in \{1,2,...,d\} \times \mathbb{R}. \tag{97}$$

Therefore the first item in the left-side of (95) vanishes.

Using the same strategy as the above, we divide both sides by r and let $r \to +\infty$, by dominated convergence theorem, we can obtain:

$$\bar{\beta}_i - \beta = \int_{D\backslash\{0\}} \chi(\xi)\left[\frac{1}{2}(\nu_i(d\xi)+\nu_{-i}(d\xi)-2m(d\xi))\right]. \tag{98}$$

Now the equation (86) remains as

$$\begin{aligned} 0 &= (1-s^2)\int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-1\right]m(d\xi) \\ &+ \frac{1}{2}(s^2+s)\int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-e^{2\langle V^i,\xi\rangle}\right]\nu_i(d\xi) \\ &+ \frac{1}{2}(s^2-s)\int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left[e^{2s\langle V^i,\xi\rangle}-e^{-2\langle V^i,\xi\rangle}\right]\nu_{-i}(d\xi), \qquad (99) \\ &\quad \text{for each} \quad (i,s,(u,V)) \in \{1,2,...,d\}\times\mathbb{R}\times\mathcal{B}. \end{aligned}$$

Let $\mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i})$ be equal to the right-side of the above equation and therefore for each $(i,s,(u,V)) \in \{1,2,...,d\}\times\mathbb{R}\times\mathcal{B}$, $\mathcal{G}_i = 0$, which will yield the following equations:

$$
\begin{aligned}
\nabla_u \mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) &= 0, && (100)\\
\nabla_{V^i} \mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) &= 0, && (101)\\
\text{and} \quad \nabla_u \nabla_{V^i} \mathcal{G}_i(s,u,V,m,\nu_i,\nu_{-i}) &= 0, && (102)\\
\text{for each} \quad (i,s,(u,V)) &\in \{1,2,...,d\} \times \mathbb{R} \times \mathcal{B}. && (103)
\end{aligned}
$$

Let $V^i = 0$, thus (101) can be rewritten as

$$
(s^3 - s) \int_{D \backslash \{0\}} e^{\langle u,\xi \rangle + \xi' V \xi} \xi [\nu_i(d\xi) + \nu_{-i}(d\xi) - 2m(d\xi)] = 0, \tag{104}
$$

$$
\text{for each} \quad (u,V) \in \mathcal{B} \quad \text{with} \quad V^i = 0.
$$

Then let $u = 0$ and $V = 0$, by the definition of strongly regular, the equation (102) yields

$$
(s^3 - s) \int_{D \backslash \{0\}} \xi \xi' [\nu_i(d\xi) + \nu_{-i}(d\xi) - 2m(d\xi)] = 0, \tag{105}
$$

$$
\text{for each} \quad s \in \mathbb{R}.
$$

By (92), the above equation just tells us that the measure $\nu_i(\cdot) + \nu_{-i}(\cdot) - 2m(\cdot)$ is zero. By (93) and (94), we have

$$
\nu_i(\cdot) = \nu_{-i}(\cdot) = m(\cdot), \quad \text{for each} \quad i \in \{1,2,...,d\}. \tag{106}
$$

Then we can rewrite (99) as:

$$\begin{aligned}\int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi} e^{2s\langle V^i,\xi\rangle} m(d\xi) &= \frac{1}{2}s^2 \int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left(e^{\langle V^i,\xi\rangle}-e^{-\langle V^i,\xi\rangle}\right)^2 m(d\xi) \\ &+ \frac{1}{2}s \int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi}\left(e^{2\langle V^i,\xi\rangle}-e^{-2\langle V^i,\xi\rangle}\right) m(d\xi) \\ &+ \int_{D\backslash\{0\}} e^{\langle u,\xi\rangle+\xi' V\xi} m(d\xi). \qquad (107)\end{aligned}$$

Now let $u = 0$ and $V = 0$ except that V_{ii} is set to be -1. After differentiating both sides of (107) twice on s and let $s = 0$, we have

$$\int_{D\backslash\{0\}} e^{-\xi_i^2} 4\xi_i^2 m(d\xi) = \int_{D\backslash\{0\}} e^{-\xi_i^2}\left(e^{-\xi_i}-e^{\xi_i}\right)^2 m(d\xi). \qquad (108)$$

Since we have the following inequality holds for all $\xi \neq 0$

$$4\xi_i^2 < \left(e^{-\xi_i}-e^{\xi_i}\right)^2, \qquad (109)$$

and (108), (109) are true for all $i \in \{1, 2, ..., d\}$, this results in that $m(\cdot)$ is a zero measure. From (98), we have

$$\bar{\beta}_i = \beta, \quad \text{for each} \quad i \in \{1, 2, ..., d\}. \qquad (110)$$

Thus we have proved (38) and (39) by simply letting $b = [b_1, b_2, ..., b_d]$. We can rewrite (73) as

$$\begin{aligned} \frac{\tilde{\mathcal{A}} f_{u,V}(x)}{f_{u,V}(x)} &= \langle \alpha(x)(u+2Vx), u+2Vx\rangle + 2tr(\alpha(x)V) \\ &\quad +\langle \beta(x), u+2Vx\rangle + \gamma(x). \end{aligned} \tag{111}$$

As to (40), we can use the same arguments on the equation

$$\frac{\tilde{\mathcal{A}} f_{u,V}(se_i+te_j)}{f_{u,V}(se_i+te_j)} = \frac{\tilde{\mathcal{A}} f_{u,V}(se_i)}{f_{u,V}(se_i)} + \frac{\tilde{\mathcal{A}} f_{u,V}(te_j)}{f_{u,V}(te_j)} + 2stT_{ij}(u,V) - F(u,V),$$
$$\text{for all} \quad (s,t) \in \mathbb{R}^2. \tag{112}$$

Then we can obtain:

$$\begin{aligned} \alpha(se_i+te_j) &= \alpha, && (113)\\ \beta(se_i+te_j) &= \beta + sb_i + tb_j, && (114)\\ \gamma(se_j+te_j) &= \gamma + s\delta_i + t\delta_j + s^2\Phi_{ii} + t^2\Phi_{jj} + 2st\Phi_{ij}, && (115)\\ \nu(se_i+te_j,\cdot) &= 0, && (116)\\ \text{for each} \quad & i,j \in \{1,2,...,d\}. && (117) \end{aligned}$$

This completes the proof of (40) by letting $\Phi = [\Phi_{ij}]$.

Now by (18) and (38) to (40), we derive that

$$\begin{aligned}\frac{\tilde{\mathcal{A}} f_{u,V}(x)}{f_{u,V}(x)} &= F(u,V) + \langle R(u,V), x\rangle + x'T(u,V)x \qquad (118)\\ &= (u+2Vx)'\alpha(u+2Vx) + 2tr(\alpha V)\\ &\quad +\langle \beta + bx, u+2Vx\rangle + \gamma + \delta' x + x'\Phi x. \qquad (119)\end{aligned}$$

Thus comparing it with (111), by Lemma 8.2, we have proved that

$$\alpha(x) = \alpha, \quad \beta(x) = \beta + bx \quad \text{and} \quad \gamma(x) = \gamma + \delta' x + x'\Phi x. \qquad (120)$$

Since for each $x \in D$, $\gamma(x)$ is nonnegative, thus $(\gamma, \delta, \Phi) \in \mathcal{E}$. This completes the proof of (34).

By the properties of Riccati equations, we know that there exist unique continuous solutions $A(\cdot, u, V)$, $B(\cdot, u, V)$ and $C(\cdot, u, V)$ for all initial values $(u, V) \in \mathcal{B}$ and all admissible parameter sets and since $F(u,V)$, $R(u,V)$ and $T(u,V)$ are also continuous functions, we have $\partial_t A(t,u,V)$, $\partial_t B(t,u,V)$ and $\partial_t C(t,u,V)$ are continuous. Then by Lemma 3.1 and (13), we derive (35), (36) and (37). This completes the proof of Proposition 4.1.

8.5 Proof of Theorem 4.1

(The proof in more general case can be found in Duffie, Filipovic and Schachermayer (2002 [4], Section 8).)

Proof. Let $\mathcal{A}^\sharp f(x)$ be equal to the right-side of (43) for every $f \in C_c^2(D)$ and

$\mathcal{T}$ denote the Fréchet space of rapidly decreasing C^{∞}-functions on D defined by Rudin (1991 [18], Definition 7.1) and denote its complex linear hull by $\mathcal{L}(\mathcal{T})$.

Since by Rudin (1991 [18], Theorem 7.15), we know that $C_c^{\infty}(D)$ is dense in $\mathcal{T}$ and the Fourier Transform is a linear, continuous, one to one mapping on $\mathcal{T}$. Therefore there exists a subset $\mathcal{T}_0$ of $\mathcal{T}$ such that its complex linear hull $\mathcal{L}(\mathcal{T}_0)$ is also dense in $\mathcal{L}(\mathcal{T})$ and for every $g \in \mathcal{T}_0$

$$\begin{aligned} g(x) &= \int_D e^{i\langle q,x\rangle}\tilde{g}(q)dq \\ &= \int_D f_{iq,0}(x)\tilde{g}(q)dq, \end{aligned} \tag{121}$$

for some $\tilde{g} \in C_c^{\infty}(D)$. Now let $t \in \mathbb{R}_+$, we have

$$\begin{aligned} P_t g(x) &= \int_D P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \int_D e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x}\tilde{g}(q)dq. \end{aligned} \tag{122}$$

Because $B(t,iq,0)$ and $C(t,iq,0)$ are continuous functions on $\mathbb{R}_+$, we obtain that $P_t g \in C^{\infty}(D)$, for $g \in \mathcal{L}(\mathcal{T})$. Moreover, since it is easy to check for all $(u,V) \in \mathcal{B}$,

$$\mathcal{A}^{\sharp} P_t f_{u,V}(x) = \partial_t P_t f_{u,V}(x), \tag{123}$$

then we have

$$\begin{aligned}\partial_t P_t g(x) &= \int_D \partial_t P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \int_D \mathcal{A}^\sharp P_t f_{iq,0}(x)\tilde{g}(q)dq \\ &= \mathcal{A}^\sharp P_t g(x), \text{ for each } (t,x) \in \mathbb{R}_+ \times D. \end{aligned} \tag{124}$$

In particular,

$$\lim_{t\downarrow 0} P_t g(x) = g(x), \tag{125}$$

for each $x \in D$ and $g \in \mathcal{L}(\mathcal{T}_0)$.

We know that if $\Phi \neq 0$, then

$$\lim_{|x|\to\infty} e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x} = 0, \text{ for each } q \in \mathbb{R}^d. \tag{126}$$

and also because $\tilde{g} \in C_c^\infty(D)$, by the dominated convergence theorem, we have

$$P_t g(x) = \int_D e^{A(t,iq,0)+\langle B(t,iq,0),x\rangle + x'C(t,iq,0)x}\tilde{g}(q)dq \in C_0(D). \tag{127}$$

If $\Phi = 0$, we have $C(t,iq,0) = 0$ and $B(t,iq,0) = ie^{bt}q \in i\mathbb{R}^d$. Thus by the Riemann-Lebesgue lemma, we derive that

$$P_t g(x) = \int_D e^{\langle ie^{bt}q,x\rangle} e^{A(t,iq,0)}\tilde{g}(q)dq \in C_0(D). \tag{128}$$

Therefore we prove that

$$P_t g \in C_0(D) \text{ for each } g \in \mathcal{L}(\mathcal{T}_0). \tag{129}$$

From Duffie, Filipovic and Schachermayer (2002 [4], Lemma 8.4), we can extend (125) and (129) to all the functions $g \in C_0(D)$, by Revuz and Yor (1994 [17], Proposition III 2.4), we have proven that X is a Feller process.

Because X is a Feller process, if we can prove (124) is true for all $g \in C_c^2(D)$, by Sato(1999 [19], Lemma 31.7), we just finish the proof. At this time we only obtain that (124) holds for $g \in \mathcal{L}(\mathcal{T}_0)$. By the closeness of $\mathcal{A}$, we have $\mathcal{L}(\mathcal{T}) \subset \mathcal{D}(\mathcal{A})$ and (124) holds for $g \in \mathcal{L}(\mathcal{T})$. By Duffie, Filipovic and Schachermayer (2002 [4], Lemma 8.4), it is easy to see that actually (43) holds for all $g \in C_c^2(D)$. This completes the proof of the first part.

It is easy to see the converse part. Since given an admissible parameter set $(\alpha, \beta, b, \gamma, \delta, \Phi)$, there exist unique continuous solutions $A(\cdot, u, V)$, $B(\cdot, u, V)$ and $C(\cdot, u, V)$ for (35) to (37) which means that we have well defined (9). Because the law is unique under Definition 3.1, we finish the proof of Theorem 4.1.

8.6 Proof of Proposition 6.1

By the Markov property it is straightforward to prove that Q_t satisfy the Chapman-Kolmogorov equation such that, for any $f \in B(D)$,

$$Q_{t+s} f(x) = Q_t Q_s f(x). \tag{130}$$

and since $Q_0 f = f$ and $Q_t 1 \in [0,1]$, we proved that (Q_t) is a positive contraction semigroup on $B(D)$. Now since X is a Feller process and thus admits a cadlag modification, then by the definition of Q_t, we know that $(t,x) \mapsto Q_t f(x)$ is measurable with respect to $\mathbb{R}_+ \times D$. By the , for every $\lambda > 0$, the resolvent of (Q_t),

$$R^Q_\lambda g(x) = \int_{\mathbb{R}_+} e^{-\lambda t} Q_t g(x) dt, \tag{131}$$

is a linear operator mapping: $B(D) \mapsto B(D)$. The following deductions are proceeded same as Rogers and Williams (1994 [?]):

$$\begin{aligned}
R_\lambda g(x) - R^Q_\lambda g(x) &= \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_t) \left(1 - e^{-\int_0^t r(X_s)ds}\right) dt \right] \\
&= \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_t) \left(\int_0^t r(X_u) e^{-\int_0^u r(X_s)ds} du \right) dt \right] \\
&= \mathbb{E}_x \left[\int_{\mathbb{R}_+} r(X_u) e^{-\int_0^u r(X_s)ds} \left(\int_{\mathbb{R}_+} e^{-\lambda(t+u)} P_{t+u} g(X_{t+u}) dt \right) du \right] \\
&= \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda u} r(X_u) e^{-\int_0^u r(X_s)ds} \left(\int_{\mathbb{R}_+} e^{-\lambda t} P_t g(X_u) dt \right) du \right] \\
&= \mathbb{E}_x \left[\int_{\mathbb{R}_+} e^{-\lambda u} e^{-\int_0^u r(X_s)ds} r(X_u) R_\lambda g \right] \\
&= R^Q_\lambda (r(x) R_\lambda g). \qquad (132)
\end{aligned}$$

By Revuz and Yor (1994 [17], Proposition VII 1.4), let $f \in C^2_c(D) \subset \mathcal{D}(\mathcal{A})$, there exists a unique $g \in C_0(D)$ such that $R_\lambda g = f$, therefore $g = (\lambda I - \mathcal{A}) f$. Then by (132), we derive that

$$f = R^Q_\lambda (g(x) + r(x) f(x)). \tag{133}$$

Since $g + rf \in C_0(D)$, $f \in \mathcal{D}(\mathcal{A}_Q)$ and thus $C_c^2(D) \subset \mathcal{D}(\mathcal{A}_Q)$. Now by (133), we have

$$(\lambda I - \mathcal{A}_Q)f = g + rf = (\lambda I - \mathcal{A})f + rf, \tag{134}$$

where $\mathcal{A}_Q$ denote the infinitesimal generator of the semigroup (Q_t). Hence

$$\mathcal{A}_Q f(x) = \mathcal{A}f(x) - r(x)f(x), \quad f \in C_c^2(D). \tag{135}$$

By using Theorem 4.1, we finish the proof.

8.7 Proof of Proposition 6.3

Proof. Let's consider a discounted gain process $U(t)$ for holding the traded asset S:

$$U(t) = e^{-\int_0^t r(X_s)ds} f(X_t)I(t) + \int_0^t e^{-\int_0^s r(X_u)du}(1 - l(X_s))f(X_s)h(X_s)I(s)ds. \tag{136}$$

Since it is a martingale under the risk neutral measure, where $f(X_t)$ is the value of S at time t, we have

$$\begin{aligned} 0 \;=\; & -r(X_t)e^{-\int_0^t r(X_s)ds} f(X_t)I(t) \\ & + e^{-\int_0^t r(X_s)ds}\mathcal{A}f(X_t)I(t) - e^{-\int_0^t r(X_s)ds} f(X_t)I(t)h(X_t) \\ & + e^{-\int_0^t r(X_u)du}(1 - l(X_t))f(X_t)h(X_t)I(t). \end{aligned} \tag{137}$$

From (137), we can derive

$$\begin{aligned} \mathcal{A}f(X_t) &= -(r(X_t) + l(X_t)h(X_t))f(X_t) \qquad (138) \\ &= -\rho(X_t)f(X_t). \qquad (139) \end{aligned}$$

By applying the Feynman-Kac Theorem we have that

$$f(X_t) = \mathbb{E}_x \left[e^{-\int_t^T \rho(s)ds} f(X_T) \right\} . \qquad (140)$$

This completes the proof.

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