Consumption and Investment Optimization under Constraints

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Abstract. We analyze a problem of maximization of expected terminal wealth and consumption under constraints in a general financial framework, which includes models with constrained portfolios, labor income and large investor models. By introducing the new finite probability space, as well as a new utility function, the considered problem is converted to the one studied by Pham and Mnif (2002) [48]. By using general optional decomposition under constraints, we can develop a dual formulation under minimal assumption modeled as in Pham and Mnif (2002) [48]. We then are able to prove an existence and uniqueness of an optimal solution to primal problem. Under the assumption that there exists a solution to the corresponding dual problem, an optimal consumption plan can be found by convex duality.

Key words: Stochastic Optimization, Utility Optimization, Duality Theory, Convex and State Constraints, Optional Decomposition

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1 Introduction

A basic problem of mathematical finance is the problem of an economic agent who invest in a financial market so as to maximize the expected utility of his terminal wealth and/or consumption. The problems can be attacked by the stochastic optimal control methods as, for instance, in the papers of Merton (1971) [43], Duffie, Flemming, Soner and Zariphopoulou (1997) [19], or by a modern, more powerful and elegant method: the duality approaches. The difference is that, while the optimal control methods are wedded to the dynamic programming Hamilton-Jacobi-Bellman equation and based on the requirement of Markov state processes, the duality techniques, rather then rely on the Hamilton-Jacobi-Bellman equation, use the stochastic duality theory and permit us to deal with more general and non-markovian processes. The key point in this method is the duality relation of the set of wealth processes provided by the set of martingale measures for state processes.

Duality approaches have been used with success in treating portfolio optimization problems for incomplete financial markets in a continuous-time diffusion model, which are incomplete or impose constraints on portfolio choice, as in Karatzas, Lehoczky, Shreve and Xu (1991) [34], Shreve and Xu (1992) [52], Cvitanic and Karatzas (1992) [8], Cuoco and Cvitanic (1998) [12], Cuoco (1997) [13]. In dealing with a more general framework, where the asset prices are semimartingales, classical references are papers of Kramkov and Schachermayer (1999 - 2001), [41] [42]. Extensions to the case with random initial capital are considered by Cvitanic, Schachermayer and Wang (2001) [11].

Recently, Pham and Mnif (2002) [48] study the case, in which the wealth process is required to be nonnegative over the whole lifetime interval [1], and the family of state processes discounted by the numéraire is a predictably convex set (in the sense of Föllmer and Kramkov (1997) [24]). This financial structure is universal enough to incorporate many financial models, such as with constrained portfolios, random endowment and large investor, as well as reinsurance models.

Since the problem considered in Pham and Mnif (2002) is to optimize the expected utility of

[1] In an incomplete market or when constraints are imposed on proportion, this constraint is guaranteed by the nonnegativity of the terminal wealth. However, this is no more the case in the presence of constraints on the amount or shares and/or with random endowment.

terminal wealth, so the problem is the simplest one in terms of objective. It is undoubtedly an important goal to generalize the study of optimal investment and consumption problems to this semimartingale setting. This paper aims to solve the mentioned problem.

Because we are dealing with the whole path of a consumption process, then we establish a new finite probability space. In a newly defined probability space, we construct a new utility function, which converts the original optimization problem from consumption and terminal wealth into the simplier one considered by Pham and Mnif (2002) [48]. We are then able to prove an existence and uniqueness of an optimal solution to our primal problem. Unfortunately, in our general framework with convex and state constraints, we do not know in general, whether an optimal dual solution does exist. Under the assumption that the optimal dual problem does exist, the nature of the optimal wealth (eventually, consumption plan) is almost the same as in the case considered by Kramkov and Schachermayer (1999) [41] - it equals to the inverse of the marginal utility evaluated at the optimal dual solution. The difference is the additional terms arising from the convex and state space constraints, which leads to a mixed control/singular dual optimization problem.

The outline of the paper is organized as follows. Section 2 recalls the stochastic framework described in Pham and Mnif (2002). After introducing and analyzing the dual set of the state processes in Section 3, in Section 4 we formulate the optimization problem. In this case, since we are dealing with the whole path of consumption process, we have to establish a new finite probability space in Section 5. In this section, we provide the dual and primal sets in an abstract version. In Section 6, we construct a new utility function in a newly defined probability space, which should convert the optimization problem from consumption and terminal wealth into the familiar one considered in Pham and Mnif (2002) [48], that is why many of the ideas and methods of this paper owe much to their paper. In Section 7 we consider the dual problem.

2 The Stochastic Framework

From now on, the stochastic integral of a predictable process H with respect to a semimartingale X can be denoted by $\int HdX$ or $H \bullet X$. We denote by $\mathbf{L}(X)$ the space of all predictable processes integrable with respect to X. The Émery distance between two semimartingales X and Y is defined as:

$$D(X,Y) = \sup_{|H|\le 1}\left(\sum_{n\ge 1} 2^{-n}\mathbf{E}[\min(|(H\bullet(X-Y)_n)|,1)]\right),$$

where the supremum is taken over the set of all predictable processes H bounded by 1. The corresponding topology is called the *semimartingale* topology.

In the sequel, we recall the stochastic framework provided by Pham and Mnif (2002).

Let $\widetilde{S}$ be a $\mathbf{R}^d$-valued semimartingale in $(\Omega,\, \mathcal{F},\, \mathbf{P})$. We prescribe a convex subset $\mathbb{H}$ of $\mathbf{L}(\widetilde{S})$ containing the zero element and convex in the following sense: for any predictable process $\zeta \in [0,1]$ and for all $H^1, H^2 \in \mathbb{H}$ we have:

$$\zeta H^1 + (1-\zeta)H^2 \in \mathbb{H}.$$

We consider a family $\{\widetilde{G}^H : H \in \mathbb{H}\}$ of adapted processes with finite variation, null at 0 and satisfying the concavity property:

$$\widetilde{G}^{\zeta H^1+(1-\zeta)H^2} - \zeta \bullet \widetilde{G}^{H^1} - (1-\zeta)\bullet \widetilde{G}^{H^2} \in \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is the set of all (optional) nondecreasing adapted processes with initial value 0 and null at 0.

Now we consider the following family:

$$\widetilde{\mathbb{X}}_0 = \left\{ H \bullet \widetilde{S} + \widetilde{G}^H \right\}$$

We shall make the following closure property:

Assumption 2.1 *Under the condition (1), the set $\widetilde{\mathbb{X}}_0$ is closed for semimartingale topology.*

Given $\widetilde{X}^0 \in \widetilde{\mathbb{X}}_0$, we define the set

$$\widetilde{\mathbb{X}}^b = \left\{ \widetilde{X} - \widetilde{X}^0 : \widetilde{X} \in \widetilde{\mathbb{X}}_0 \text{ and } X - \widetilde{X}^0 \text{ is locally bounded from below} \right\} \tag{2}$$

so that $\widetilde{\mathbb{X}}^b$ is locally bounded from below, closed for the semimartingale topology null at 0, and containing the constant process 0.

Remark 2.1 *Under the relation (1), the family of semimartingales* $\widetilde{\mathbb{X}}_0$ *is a predictable convex set in the sense of Föllmer and Kramkov (1997)[24], i.e., for* $\widetilde{X}^i \in \widetilde{\mathbb{X}}_0$ *for* $i = 1, 2,$ *and for any predictable process* $\zeta \in [0,1]$ *we have:*

$$\zeta \bullet \widetilde{X}^1 + (1-\zeta) \bullet \widetilde{X}^2 \in \widetilde{\mathbb{X}}_0 - \mathcal{I}$$

For $x \geq 0$, we denote by $\widetilde{\mathbb{X}}_x$ the family:

$$\widetilde{\mathbb{X}}_x \quad = \quad \left\{ x + \widetilde{X} - \widetilde{C};\, \widetilde{X} \in \widetilde{\mathbb{X}}_0,\, \widetilde{C} \in \mathcal{I} \right\}$$

We simply write $\widetilde{\mathbb{X}}$ for $\widetilde{\mathbb{X}}_1$, hence

$$\widetilde{\mathbb{X}}_x = x\widetilde{\mathbb{X}} \triangleq \{x\widetilde{X} \colon \widetilde{X} \in \widetilde{\mathbb{X}}\},$$

Now let us introduce the set $\mathcal{P}(\widetilde{\mathbb{X}}^b)$ of all nonnegative $\mathbf{P}$-local martingales Z with $Z_0 = 1$ such that there exists a process $A \in \mathcal{I}_p$ – the set of nondecreasing predictable processes, null at 0 – satisfying

$$Z(\widetilde{X}^b - A) \text{ is a } \mathbf{P}\text{-local supermartingale for any } \widetilde{X}^b \in \widetilde{\mathbb{X}}^b. \tag{3}$$

The next definition of the upper variation process is adopted from the one in Föllmer and Kramkov (1997) [24].

Definition 2.1 *The upper variation process of* $\widetilde{\mathbb{X}}^b$ *corresponding to* $Z \in \mathcal{P}(\widetilde{\mathbb{X}}^b)$*, defined as the element* $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ *in* $\mathcal{I}_p$ *satisfying (3) and is minimal with respect to this property, i.e. such that* $(A - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)) \in \mathcal{I}_p$*, for any* $A \in \mathcal{I}_p$ *satisfying (3).*

Throughout this paper we assume

Assumption 2.2 *The upper variation process* $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ *exists.*

On the set $\mathcal{P}(\widetilde{\mathbb{X}}^b)$, we define the subset

$$\mathcal{P}^*(\widetilde{\mathbb{X}}^b) = \left\{ Z \in \mathcal{P}(\widetilde{\mathbb{X}}^b) \colon Z \text{ is a } \mathbf{P}\text{-supermartingale and } \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_T \text{ is bounded a.s.} \right\}$$

and its subset

$$\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) = \left\{ Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b) \colon Z \text{ is a positive and } \mathbf{P}\text{-martingale} \right\}$$

Note that, in fact the set $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ consists of density processes Z of all probability measures $Q \sim \mathbf{P}$ with the property: there exists $A \in \mathcal{I}_p$ such that:

$$\widetilde{X}^b - A, \quad \text{is a } Q\text{-local martingale for } \widetilde{X}^b \in \widetilde{\mathbb{X}}^b \tag{4}$$

and the upper variation process of $\widetilde{\mathbb{X}}^b$ associated with $Z \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ satisfing:

$$\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_T \quad \text{bounded.} \tag{5}$$

By misuse of notation, we shall identify an element $Z \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ with some Q in the set of all equivalent local martingale measures $Q \sim \mathbf{P}$ such that (4) and (5) hold true, and with $Z_t = \mathbf{E}[dQ/d\mathbf{P}|\mathcal{F}_t]$.

We shall make the assumptions that $\widetilde{X}^0$ can be chosen so as:

Assumption 2.3 $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) \neq \emptyset$

Assumption 2.4 $\widetilde{X}^0$ *is a finite variation process with*

$$\mathbf{E}\left[\int_0^T Z_t d|\widetilde{X}_t^0|\right] < \infty, \quad \forall\, Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$$

Let us introduce a strictly positive price process S^0.

We are now interested on the family of state processes:

$$\mathbb{X}_x \triangleq \{S^0\widetilde{X} \colon \widetilde{X} \in \widetilde{\mathbb{X}}_x\}, \tag{6}$$

$$= \left\{S^0\left(x + \widetilde{X} - \widetilde{C}\right) \colon \widetilde{X} \in \widetilde{\mathbb{X}}_0,\, \widetilde{C} \in \mathcal{I}\right\}, \tag{7}$$

with the following assumption:

Assumption 2.5 $\widetilde{C}$ *is a finite variation process.*

In the sequel, let us denote $S^0 \bullet \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) = A^{\widetilde{\mathbb{X}}^b}(Z)$ and $S^0 \bullet \widetilde{X}^0 = X^0$.

It is clear that

$$\mathbb{X}_x = x\mathbb{X}_1 = x\mathbb{X}$$

with $\mathbb{X} \triangleq \mathbb{X}_1$.

In the financial context, any $X \in \mathbb{X}_x$ can be interpreted as a wealth process with an initial nonrandom endowment x of an economic agent, who is taking part in a financial market consisting of $d+1$ assets: one bond with the price process S^0, and d stocks. The vector $\widetilde{S} = (\widetilde{S}^i)_{1 \leq i \leq n}$ is stated as the $\mathbf{R}^d$-valued price process of d stocks, discounted by a numéraire S^0. The nondecreasing process $\widetilde{C}$ appears in (7) is the discounted consumption process. In this framework, the convex set $\mathbb{H}$ models constraints on portfolios H defined by the amounts invested in the risky assets. Process $\widetilde{G}^H$ allows to take into account the term arising from labor income and large investor, where $\widetilde{X}^0$ describes the labor income throughout investment life-time. With $\widetilde{X}^0$ chosen in advance, we state that the income process $\widetilde{X}^0$ is spanned by the market assets and therefore is not a source of new uncertainty. We refer the interested reader to the work of Pham and Mnif (2002) [48] for details of financial applications.

In this paper, we study the case where the liquidity constraint[2]

$$X_t \geq 0, \quad t \in [0, T] \tag{8}$$

The existence of such a constraint implies that the agent cannot borrow against future labor income[3] Given $x \in \mathbf{R}$, we then denote by $\mathbb{X}_x^+$ the setof all processes $X \in \mathbb{X}_x$ satisfying (8).

We now define a consumption process as follows:

Definition 2.2 *A consumption (rate) process is an $\mathcal{F}_t$-adapted nonnegative process $c(\cdot)$, which is related to the process $\widetilde{C}$ appeared in (7) by the formula*

$$\int_0^t c_s ds = \int_0^t S_s^0 d\widetilde{C}_s, \quad 0 \leq t \leq T. \tag{9}$$

Put $\Lambda_t = t$ and suppose that we have the following decomposition $\widetilde{C} = \widetilde{c} \bullet \Lambda$ with some process $\widetilde{c} \geq 0$ a.e.. In the standard notations of stochastic calculus for semmimartingales, (9) can be

[2]The term "liquidity constraint" is attributable to El Karoui and Picqué [20].

[3]The constraint can be formulated as follows:

$$X_t \geq d_t, \quad t \in [0, T]$$

with a given semimartingale $d = (d_t)_{t \in [0,T]}$. However, by considering the family of processes

$$\left\{ \widetilde{X} - \frac{d}{S^0} : \widetilde{X} \in \widetilde{\mathbb{X}}_0 \right\}$$

which still satisfies the predictably convexity property, and assumption 2.1 by the invariance of the Emery distance under translation. Without loss of generality, we may then focus on nonnegativity state constraint (8).

rewritten as follows:

$$c \bullet \Lambda = S^0 \widetilde{c} \bullet \Lambda$$

or we have $c = S^0 \widetilde{c}$.

Given x and $\widetilde{X}^0 \in \widetilde{\mathbb{X}}_0$, we denote by $\mathcal{A}(x)$ the set of the pairs of processes (X, c), where $X \in \mathbb{X}_x^+$ and c satisfying (9).

3 The Dual Sets

We now define the family $\mathbb{Y}_y$ of nonnegative semimartingales Y with $Y_0 = y$ and

$$\mathbb{Y}_y = \left\{ Y = y\frac{Z}{S^0} : Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b) \right\} \tag{10}$$

Lemma 3.1 *For any $x > 0$, $y > 0$, $(X, c) \in \mathcal{A}(x)$, $Y \in \mathbb{Y}_y$, the processes*

$$YX + Yc \bullet \Lambda - Y_- \bullet X^0 - Y_- \bullet A^{\widetilde{\mathbb{X}}^b}(Z)$$

are $\mathbf{P}$-supermartingales.

Proof. From the definition of $\mathcal{P}^*(\widetilde{\mathbb{X}}^b)$, processes

$$Z\left(\widetilde{X} + \widetilde{C} - \widetilde{X}^0 - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)\right), \quad \widetilde{X} \in \widetilde{\mathbb{X}}_0$$

are $\mathbf{P}$-local supermartingales.

Since $\widetilde{C}$, $\widetilde{X}^0$ and $\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)$ have finite variation, Theorem VII.35 in Dellacherie and Meyer [18] implies that the process

$$\begin{aligned} yZ\widetilde{X} + yZ_- \bullet \widetilde{C} - yZ_- \bullet \widetilde{X}^0 - yZ_- \bullet \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) \quad &= \\ = \; YX + Yc \bullet \Lambda - Y_- \bullet X^0 - Y_- \bullet A^{\widetilde{\mathbb{X}}^b}(Z) & \end{aligned}$$

is a $\mathbf{P}$-local supermartingale and bounded from below by a $\mathbf{P}$-integrable random variable and is actually a $\mathbf{P}$-supermartingale from Fatou's lemma. This completes the proof. □

The last Lemma implies that the so-called *budget constraint*

$$\mathbf{E}\left[Y_T X_T + \int_0^T Y_t c_t dt\right] \leq x + \mathbf{E}\left[\int_0^T Y_t dX_t^0 + \int_0^T Y_t dA^{\widetilde{\mathbb{X}}^b}(Z)_t\right] \tag{11}$$

is satisfied for every $(X, c) \in \mathcal{A}(x)$.

4 The Utility Maximization from Terminal Wealth and Consumption

In this section, our goal is to generalize the study of optimal investment and consumption problems to the aforementioned semimartingale setting.

The agent's preferences over consumption and wealth profiles are given by time-additive utility functions U_1 for consumption and U_2 for the terminal wealth. At first, we recall some classical definitions and properties of utility function.

Definition 4.1 *A utility function* $U\colon (0,\infty)\times\Omega \to \mathbf{R}\cup\{-\infty\}$ *is a nondecreasing on its domain, concave, uppersemicontinuous and continuously differentiable function. Moreover, its first derivative function* U' *is continuous, positive, strictly decreasing on the interior of* $\mathrm{dom}(U) = (0,\infty)$ *and satisfies the condition:*

$$U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty \quad \textit{and} \quad U'(\infty) \triangleq \lim_{x\uparrow\infty} U'(\infty) = 0 \quad \textit{a.s.} \tag{12}$$

We shall denote by $I(.)$ the (continuous, strictly decreasing) inverse of the marginal utility function $U'(.)$; this function maps $(0, U'(0+))$ onto $(0,\infty)$, extended by continuity on $(0,\infty)$ by setting $I(y) = 0$ for $U'(0+) \le y \le \infty$.

We also introduce the conjugate function of U

$$\widetilde{U}(y) \triangleq \sup_{x>0}[U(x) - xy], \quad y > 0, \tag{13}$$

It is well-known (see e.g. Rockafellar (1970) [45]) that this function is nonincreasing, convex differentiable on $(0,\infty)$ with $\widetilde{U}(0) = U(\infty)$, and satisfies

$$\widetilde{U}'(y) = -I(y), \quad y > 0, \text{ a.s.}, \tag{14}$$

We also know that $I(y)$ attains the supremum in (13), i.e.

$$\widetilde{U}(y) = U(I(y)) - yI(y), \quad y > 0, \text{ a.s.} \tag{15}$$

We shall consider throughout a map $U_1\colon (0,\infty)\times[0,T]\times\Omega \to \mathbf{R}$, such that for any give $(t,\omega) \in [0,T]\times\Omega$ the function $U_1(.,t,\omega)$ is a utility function and, for any $x \in \mathbf{R}_+$ the process

$U_1(x,.)$ is $\mathcal{F}_t$-adapted. The function $U_2\colon \mathbf{R}_+ \times \Omega \to \mathbf{R}$ is such that for any $x > 0$, the random variable $U_2(x,.)$ is $\mathcal{F}_T$-measurable and $U_2(.,\omega)$ is a utility function.

To alleviate notations, we omit the dependence in the state $\omega \in \Omega$ and we write $U_1(x,t)$ and $U_2(x)$.

The agent in our model has time-seperable utility structure as follows

Definition 4.2 *A (time-seperable, von Neumann-Morgenstern) preference structure is a pair of utility functions* $U_1\colon \mathbf{R} \times [0,T] \to [-\infty,\infty)$ *and* $U_2\colon \mathbf{R} \to [-\infty,\infty)$*, which measure the investor's utility from consumption and wealth, respectively.*

Definition 4.3 *Given an initial endowment* $x \in \mathbf{R}$*, the consumption plan* (X_T, c)*, here* X_T *is the terminal wealth, and c the consumption rate process throughout the liftetime interval, is called* x*-affordable if they are financeable from an initial wealth less or equal to* x*, i.e., the pair of a wealth and consumption process* (X,c) *belong to the set* $\mathcal{A}(x^*)$ *with* $0 < x^* \leq x$*.*

Recall that a necessary condition for $(X,c) \in \mathcal{A}(x)$ is the budget constraint (11).

The agent's total expected uility from consumption over the period and expected utility of investment at the end of the period $[0,T]$ is defined as

$$J(x;\, X,c) \triangleq \mathbf{E}\left[\int_0^T U_1(c_t,t)dt + U_2(X_T)\right]. \tag{16}$$

The x-affordable consumption plan is said to be *x-feasible* if it satisfies:

$$J(x;\, X,c)^- = \mathbf{E}\left[\int_0^T U_1(c_t,t)^- dt + U_2(X_T)^-\right] < \infty$$

we denote the set of x-feasible consumption plans (X_T, c) by $\mathcal{A}^*(x)$.

Given an initial endowment x and income stream X^0, an investor wishes to choose a consumption profile and investment policy so as to to maximize his total expected uility from consumption over the period and expected utility of investment at the end of the period $[0,T]$, with the value function:

$$u(x) = \sup_{(X,c)\in\mathcal{A}^*(x)} J(x;\, X,c), \quad x \in \mathbf{R}_+, \tag{17}$$

using feasible policies.

For later use, we recall that the right- and left- derivatives of any convex function $u\colon \mathbf{R} \to \mathbf{R}$ denoted by:

$$\triangle^{\pm} u(x) \triangleq \lim_{\delta \to 0\pm} \frac{1}{\delta}[u(x+\delta) - u(x)],$$

exist, finite and satisfying:

$$\triangle^{+} u(x_1) \leq \triangle^{-} u(x_2) \leq \triangle^{+} u(x_2), \quad x_1 < x_2 \tag{18}$$

For any function u, let us denote by:

$$D^{\pm} u(x) \triangleq \overline{\lim}_{\delta \to 0\pm} \frac{1}{\delta}[u(x+\delta) - u(x)], \tag{19}$$

the upper (right and left) Dini derivatives at x, and by:

$$D_{\pm} u(x) \triangleq \underline{\lim}_{\delta \to 0\pm} \frac{1}{\delta}[u(x+\delta) - u(x)], \tag{20}$$

the lower (right and left) Dini derivatives at x.

5 The Dual Singular Formulation

The main goal of this section is to provide a dual sets and their basic properties. With respect to the classical utility maximization from fixed terminal wealth, we have now to consider the whole path of the consumption process on the support of $\ell[0,T]$, here $\ell[0,T]$ stands for the Lebesgue measure on $[0,T]$.

We now introduce some definitions and notations that will be useful in the rest of the monograph.

Define the finite measure space $(S, \mathcal{S}, \mu)$ as follows:

$$S = [0,T] \times \Omega, \quad \mathcal{S} = \mathcal{B}[0,T] \otimes \mathcal{F}, \quad \mu = (\ell[0,T] + \delta_T) \times \mathbf{P}$$

Let $\mathcal{L}^0_+$ denote the cone of non-negative functions on $\mathcal{L}^0(S, \mathcal{S}, \mu)$, a closed convex set usually abbreviated to $\mathcal{L}^0_+$.

Notice that, for Y^1, $Y^2 \in \mathcal{L}^0_+$, we have:

$$\int (Y^1, Y^2) d\mu = \mathbf{E}\left[\int_0^T Y^1_t Y^2_t dt + Y^1_T Y^2_T\right] \tag{21}$$

Here and in what follows we denote

$$\langle Y^1, Y^2\rangle_{s,t} = \int_s^t Y_u^1 Y_u^2 du + Y_t^1 Y_t^2 \mathbf{1}_{t=T}, \quad t \in [0,T]$$

and let

$$\langle Y^1, Y^2\rangle_t \triangleq \langle Y^1, Y^2\rangle_t, \quad \langle Y^1, Y^2\rangle \triangleq \langle Y^1, Y^2\rangle_{0,T}$$

For Y_1, $Y_2 \in \mathcal{L}_+^0$, we shall say that

$$Y_1 \equiv Y_2, \quad \text{if} \quad Y_1 = Y_2 \quad \mu - a.e.$$

On $\mathcal{L}_+^0$, we define a partial ordering by:

$$Y^1 \preceq Y^2 \Leftrightarrow Y^1 \leq Y^2, \quad \mu - \text{a.e.}$$

We say that a subset $\mathcal{C}$ of $\mathcal{L}_+^0$ is solid if

$$Y_2 \in \mathcal{C},\ Y_1 \preceq Y_2 \Rightarrow Y_1 \in \mathcal{C}$$

We define $\mathcal{L}^1$ as the Banach space of elements $Y = (Y)_t \in \mathcal{L}^0$, equipped with the norm

$$\|Y\|_1 = \mathbf{E}\Big[\int_0^T |Y|_t dt + |Y|_T\Big]$$

We also denote $\mathcal{L}_+^1 = \mathcal{L}_+^0 \cap \mathcal{L}^1$.

The abstract version of the primal and dual sets $\mathcal{A}^*(x)$ and $\mathbb{Y}_y$, are defined as follows:

$$\begin{aligned} \mathbb{C}_x^+ &= \left\{ \begin{array}{c} g \in \mathcal{L}_+^0;\ g\colon S \to \mathbf{R}_+ \quad \text{such that} \quad g \preceq c, \\ \text{and} \quad g_T \leq X_T \quad \text{for some} \quad (X,c) \in \mathcal{A}^*(x) \end{array} \right\} \\ \mathbb{D}_y &= \Big\{ h \in \mathcal{L}_+^0 \colon h = DY, \quad Y \in \mathbb{Y}_y,\ D \in \mathcal{D} \Big\} \end{aligned} \tag{22}$$

We denote by $\mathbb{D}_y^+$ the subset of $\mathbb{D}_y$ consisting of h, that are positve μ-a.e.. In the sequel, $\mathbb{C}$ and $\mathbb{D}$ stand for $\mathbb{C}_1$ and $\mathbb{D}_1$, respectively.

Remark 5.1 *Assumption 2.5 implies that $g \equiv \infty$ does not belong to $\mathbb{C}_x^+$ for any given x.*

We now state some properties on the abstract sets.

Lemma 5.1 *Let $g \in \mathcal{L}_+^0$, then $g \in \mathbb{C}^+$ if and only if*

$$v(g) \triangleq \sup_{h \in \mathbb{D}} \mathbf{E}\Big[\langle g, h\rangle - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \leq 1, \tag{23}$$

Proof. *Necessary condition.* Since D is continuous with finite variation, then an application of Itô product rule to YX leads to

$$hX + hc \bullet \Lambda - h \bullet X^0 - h \bullet A^{\widetilde{\mathbb{X}}^b}(Z) \quad =$$

$$x + D_- \bullet \left(Z\widetilde{X} + Z\widetilde{c} \bullet \Lambda - Z_- \bullet \widetilde{X}^0 - Z_- \bullet \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z) \right) + Z_- \widetilde{X}_- \bullet D \tag{24}$$

Since D is nonnegative and nonincreasing, S^0, Z, $\widetilde{X}$, $\widetilde{c}$ are nonnegative, this shows that the process on the left-hand side of (24) is a $\mathbf{P}$-local supermartingale bounded from below by an $\mathbf{L}^1(\mathbf{P})$-random variable, and hence a $\mathbf{P}$-supermartingale.

By the Doob's Optional Sampling Theorem we have

$$\mathbf{E}\left[\langle h, g \rangle - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \right] \leq 1 \quad \text{a.s.} \tag{25}$$

Because for any $g \in \mathbb{C}^+$, there exists $(X, c) \in \mathcal{A}^*(x)$ that dominate g in a sense of (22). Moreover, since g, h and c are nonnegative, then

$$\langle h, g \rangle - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \quad \leq$$

$$h_T X_T + \int_0^T h_t c_t dt - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \tag{26}$$

for all $h \in \mathbb{D}$.

(25) and (26) lead immediately to the "if part" of the relation (23).

Sufficient condition. Let us fix some $g \in \mathcal{L}^0_+$, and consider an $\mathcal{F}_T$-measurable random variable A, defined as

$$A = \langle g, \frac{1}{S^0} \rangle - \widetilde{X}^0_T,$$

Using successively the definition of $\mathbb{Y}$, (10), Bayes formula, Remarks VI.45.a and Theorem VI.57 of [18], we obtain

$$\begin{aligned}
&\sup_{Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \mathbf{E}^Q \left[A - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_T \right] \leq \\
&\leq \sup_{Z \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)} \mathbf{E}\left[\langle g, \frac{Z}{S^0} \rangle - \int_0^T Z_t d\widetilde{X}^0_t - \int_0^T Z_t d\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Z)_t \right] \\
&= \sup_{Y \in \mathbb{Y}} \mathbf{E}\left[\langle g, Y \rangle - \int_0^T Y_t dX^0_t - \int_0^T Y_t dA^{\widetilde{\mathbb{X}}^b}(Z)_t \right] \\
&\leq \sup_{h \in \mathbb{D}} \mathbf{E}\left[\langle g, h \rangle - \int_0^T h_t dX^0_t - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \right] \leq 1
\end{aligned} \tag{27}$$

where the second inequality follow from the inclusions $\mathbb{Y}_y \subseteq \mathbb{D}$ and $\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b) \subseteq \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$.

By the stochastic control lemma A.1 and Proposition 4.1 in Föllmer and Kramkov [24], there exists a *càdlàg* version of the nonnegative process:

$$\widetilde{X}_t^b = \operatorname*{ess\,sup}_{Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \Big(\mathbf{E}^Q \big[A - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_T \,|\mathcal{F}_t \big] + \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_t \Big), \quad 0 \le t \le T \tag{28}$$

Moreover, for any $Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$, the process $\widetilde{X}^b - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)$ is a Q-local supermartingale. By the optional decomposition under constraints, the process $\widetilde{X}^b$ admits a decomposition:

$$\widetilde{X}^b = v(g) + \widetilde{X}_b - \widetilde{C}^b \tag{29}$$

where $\widetilde{X}_b \in \widetilde{\mathbb{X}}^b$ defined as in (2) and $\widetilde{C}^b$ is an optional nondecreasing process.

Now let us consider process $X \triangleq S^0 \Big(\widetilde{X}^b + \widetilde{X}^0 - \frac{g}{S^0} \bullet \Lambda \Big)$.

From the definition of $\widetilde{\mathbb{X}}_0$ and (29) we have

$$\begin{aligned} X &= S^0 \Big(v(g) + \widetilde{X}^0 + \widetilde{X}_b - \widetilde{C}^b - \frac{g}{S^0} \bullet \Lambda \Big) \\ &= S^0 \Big(1 + \widetilde{X} - \Big(\widetilde{C}^b + 1 - v(g) + \frac{g}{S^0} \bullet \Lambda \Big) \Big) \end{aligned}$$

where $\widetilde{X} = \widetilde{X}^0 + \widetilde{X}_b \in \widetilde{\mathbb{X}}_0$. It is obvious that $X_T \ge g_T$. Moreover, since $v(g) \le 1$, and $\widetilde{C}^b \in \mathcal{I}$ then $\widetilde{C}^b + 1 - v(g) + \frac{g}{S^0} \bullet \Lambda \in \mathcal{I}$, hence X belongs to the set $\mathbb{X}^+$ with the corresponding consumption process c satisfying:

$$\frac{g}{S^0} \bullet \Lambda \preceq \frac{c}{S^0} \bullet \Lambda \triangleq \widetilde{C}^b + 1 - v(g) + \frac{g}{S^0} \bullet \Lambda$$

or equivalently, $g_t \le c_t$, $\forall\, t \in [0, T]$. Hence, the obtained pair of process (X, c) is a pair of wealth and consumption process that dominates g in a sense of (22). □

Clearly, the bipolar relation (23) implies the budget constraint (10) and the "sufficient part" of the last Lemma can be stated as a sufficient condition for the budget constraint (10).

Remark 5.2 *The last Lemma implies that the sets $\mathbb{X}_x^+$ and $\mathbb{C}_x^+$ are nonempty if and only if:*

$$v(0) \triangleq \sup_{h \in \mathbb{D}} \mathbf{E} \Big[- \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \Big] \le x \tag{30}$$

The following Lemma is taken from Pham and Mnif (2002) [48]. We refer the interested reader to their work for the proof.

Lemma 5.2 *Given $y > 0$, for all $h^1 \in \mathbb{D}_y$, $h^2 \in \mathbb{D}_y^+$ and $\zeta \in (0,1)$, there exist $h^\zeta \in \mathbb{D}_y$ such that:*

$$(1-\zeta)h^1 + \zeta h^2 = h^\zeta.$$

Moreover, we have:

$$\mathbf{E}\Big[\int_0^t h_s^\zeta dA^{\widetilde{\mathbb{X}}^b}(h^\zeta)_s\Big] \leq (1-\zeta)\mathbf{E}\Big[\int_0^t h_s^1 dA^{\widetilde{\mathbb{X}}^b}(h^1)_s\Big] + \zeta\mathbf{E}\Big[\int_0^t h_s^2 dA^{\widetilde{\mathbb{X}}^b}(h^2)_s\Big] \tag{31}$$

for all $t \in [0,T]$.

Lemma 5.3 *Given $x > 0$, the set $\mathbb{C}_x^+$ is convex, solid and closed under convergence in μ-measure.*

Proof. Note that the solidity of $\mathbb{C}_x^+$ is rather obvious. It remains now to prove its convexity.

Let $(S^0\widetilde{X}_1, S^0\widetilde{c}_1)$ and $(S^0\widetilde{X}_2, S^0\widetilde{c}_2)$ are two pairs of processes in $\mathcal{A}^*(x)$. Taking any real number $\zeta_1 = 1-\zeta_2 \in (0,1)$ and defining the convex combinations

$$\begin{aligned}\widetilde{X}^* &= \zeta_1\widetilde{X}_1 + \zeta_2\widetilde{X}_2\\ \widetilde{c}^* &= \zeta_1\widetilde{c}_1 + \zeta_2\widetilde{c}_2\end{aligned}$$

By the predictable convexity property on the set $\widetilde{\mathbb{X}}_0$ we find immediately that:

$$\begin{aligned}\widetilde{X}^* &= x + (\zeta_1\widetilde{X}_1 + \zeta_2\widetilde{X}_2) - \widetilde{c}^* \bullet \Lambda\\ &= x + \overline{X} - (\widetilde{c}^* \bullet \Lambda + \overline{C})\end{aligned}$$

where $\overline{X} \in \widetilde{\mathbb{X}}_0$, $\overline{C} \in \mathcal{I}$. Clearly, $\widetilde{c}^* \bullet \Lambda \preceq c^* \bullet \Lambda \triangleq \widetilde{c}^* \bullet \Lambda + \overline{C}$, hence $\widetilde{c}^* \preceq c^*$. Moreover, since $U_1,\ U_2$ are nondecreasing functions, then $(S_T^0\widetilde{X}_T^*, S^0c^*)$ is a pair of a wealth and a consumption process in $\mathcal{A}_x^*$. By the definition of $\mathbb{C}_x^+$, the convex combination $(S^0\widetilde{X}^*, S^0\widetilde{c}^*)$ is also in $\mathbb{C}_x^+$, hence $\mathbb{C}_x^+$ is convex.

Now, let $(g^n)_{n\in\mathbf{N}} \in \mathcal{L}_+^0$ be a sequence in $\mathbb{C}_x^+$ convergering in μ-measure; we may (and shall) by passing to a subsequence (still denoted by $(g^n)_{n\in\mathbf{N}}$) and suppose that this sequence converges μ-almost everywhere to limit g. Taking any $h \in \mathbb{D}$, by Fatou's lemma we get:

$$\begin{aligned}&\mathbf{E}\Big[\langle g,h\rangle - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big]\\ &\leq \liminf_{n\to\infty}\mathbf{E}\Big[\langle g^n,h\rangle - \int_0^T h_t dX_t^0 - \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \leq x\end{aligned}$$

and so $v(g) \leq x$, or equivalently $g \in \mathbb{C}_x^+$. This proves the closeness property of $\mathbb{C}_x^+$. □

Remark 5.3 *If we let the initial capital value in the market setting also change, then we shall have:*

$$\zeta \times \mathbb{C}^+_{x_1} \oplus (1-\zeta) \times \mathbb{C}^+_{x_2} \subseteq \mathbb{C}^+_{\zeta x_1 + (1-\zeta)x_2}$$

i.e.,

$$\zeta g_{x_1} + (1-\zeta) g_{x_2} \in \mathbb{C}^+_{\zeta x_1 + (1-\zeta)x_2}$$

where $g_{x_i} \in \mathbb{C}^+_{x_i}$, $i = 1, 2$, *and* $\zeta \in (0, 1)$.

6 The Primal Problem - Existence and Uniqueness

In the newly established finite measure $(S, \mathcal{S}, \mu)$, we define a $\mathcal{S}$-measurable function $U \colon S \times \mathbf{R}^+ \to \mathbf{R} \cup \{-\infty\}$ as follows:

$$U((t,\omega),x) = U_1(t,x), \quad t \in [0,T], \quad U((T,\omega),x) = U_2(x), \quad a.s.$$

Quite clearly, U has the following properties:

1. $s \mapsto U(s,x)$ is $\mathcal{S}$-measurable for all $x \geq 0$;
2. $x \mapsto U(s,x)$ is again a utility function for any $s \in S$.

We slightly abuse notation and omit the dependence in the state $s \in \mathcal{S}$ and write $U(x)$ in place of $U(s,x)$ henceforth.

Notice that

$$\langle \mathbf{1}, U(g) \rangle = \int_0^T U_1(g_t, t)dt + U_2(g_T), \quad g \in \mathcal{L}^0_+ \tag{32}$$

We shall denote by $I(.) \colon (0, U'(0)) \to (0, \infty)$ the inverse of the marginal utility function $U'(.)$. We set $I(y) = 0$ for $y > U'(0)$. From (32) it follows that

$$\langle \mathbf{1}, I(g) \rangle = \int_0^T I_1(g_t, t)dt + I_2(g_T), \quad g \in \mathcal{L}^0_+ \tag{33}$$

Following Pham and Mnif (2002), we formulate the next lemma.

Lemma 6.1

$$u(x) = \sup_{g \in \mathbb{C}^+_x} \int U(g)d\mu = \sup_{g \in \mathbb{C}^+_x} \mathbf{E}\Big[\int_0^T U_1(g_t,t)dt + U_2(g_T)\Big] \quad x > 0 \tag{34}$$

1. *If* $(X^*, c^*) \in \mathcal{A}^*(x)$ *solves (17), then*

$$g_t = c_t^*, \quad t \in [0,T], \quad g_T = X_T^* \in \mathbb{C}_x^+$$

solves (34),

2. *Conversely, if* $g^* \in \mathbb{C}_x^+$ *solves (34), then* $(X, c) \in \mathbb{C}_x^+$*, such that*

$$g_t^* \le c_t, \quad t \in [0,T], \quad g_T^* \le X_T$$

solves (17).

Proof. Since $\mathcal{A}^*(x) \subseteq \mathbb{C}_x^+$. Then we shall have:

$$u(x) = \sup_{(X,c)\in\mathcal{A}^*(x)} \int U((c + X\mathbf{1}_T))d\mu \le \sup_{g\in\mathbb{C}_x^+} \int U(g)d\mu \tag{35}$$

where

$$\int U((c + X\mathbf{1}_T))d\mu \triangleq \mathbf{E}\Big[\int_0^T U_1(c_t, t)dt + U_2(X_T)\Big]$$

On the other hand, by definition, for any $g \in \mathbb{C}_x^+$ there exists some $(X, c) \in \mathcal{A}^*(x)$ that dominates g in the sense of (22). Therefore, by the nondecrease of U (in a $\preceq$ sense), we have

$$\sup_{g\in\mathbb{C}_x^+} \int U(g)d\mu \le u(x) \tag{36}$$

From (35) and (36) we obtain (34).

(1) Now suppose that $(X^*, c^*) \in \mathcal{A}^*(x)$ solves (17). Then clearly,

$$g_t = c_t^*, \quad t \in [0,T], \quad g_T = X_T^* \in \mathbb{C}_x^+$$

solves (34):

$$u(x) = \mathbf{E}\Big[\int_0^T U_1(c_t^*, t)dt + U_2(X_T^*)\Big] = \int U(g)d\mu$$

(2) Now suppose that $g^* \in \mathbb{C}_x^+$ solves (34), then for any $(X, c) \in \mathcal{A}^*(x)$ that dominates g^* in the sense of (22) we have:

$$u(x) = \int U(g^*)d\mu \le \int U(c + X\mathbf{1}_T)d\mu$$

which shows that (X, c) solves (17). □

To exclude the trivial case, we shall assume

Assumption 6.1

$$u(x) < \infty, \quad \textit{for all } x \geq v(0)$$

Hereafter, we need the following technical condition on the numéraire S^0:

Assumption 6.2 *S_t^0 is bounded, for all $t \in [0, T]$.*

Assumption 6.3 *There exists $\lambda \in (0,1)$, $\overline{Z} \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$ satisfying*

$$\overline{Z}^{-1} \in \mathcal{L}^{\overline{p}}(\mu),$$

for some $\overline{p} \geq \frac{\lambda}{1-\lambda}$ and $x_0 \in dom(U)$, $U(x_0) \in \mathcal{L}^p(\mu)$ where $p = \frac{\overline{p}}{\lambda(1+\overline{p})}$, $\Upsilon \in \mathcal{L}^p(\mu)$ and $k \in \mathcal{L}^\infty(\mu)$ such that

$$\langle \mathbf{1}, U^+(x) \rangle \leq \langle k, x^\lambda \rangle + \Upsilon \quad \forall\, x \in dom(U) \cap [x_0, \infty)$$

We now closely follow Pham and Mnif (2002).

Lemma 6.2 *Under Assumption 6.2 and 6.3 (2) the family $\{U^+(g),\ g \in \mathbb{C}_x^+\}$ is uniformly integrable under μ, or equivalently the family*

$$\left\{ \left(\int_0^T U_1(g_t, t)dt + U_2(g_T) \right)^+, \quad g \in \mathbb{C}_x^+ \right\}$$

is uniformly **P***-integrable.*

Proof. Let us fix an arbitrary $g \in \mathbb{C}_x^+$. We then have

$$\begin{aligned} \int g^{\lambda p} d\mu \quad &\leq \quad \left(\int \langle g, \overline{Z} \rangle d\mu \right)^{\lambda p} \left(\int \overline{Z}^{-\frac{\lambda p}{1-\lambda p}} d\mu \right)^{1-\lambda p} \\ &\quad\ \text{const.}\left(x + \mathbf{E}\left[\int_0^T \overline{Z}_t d\widetilde{A}^{\widetilde{\mathbb{X}}^b}(\overline{Z})_t + \int_0^T \overline{Z}_t d\widetilde{X}_t^0 \right] \right)^{\lambda p} \times \\ &\quad\ \times \left(\int \overline{Z}^{-\overline{p}} d\mu \right)^{1-\lambda p} \\ &< \quad \infty \end{aligned} \tag{37}$$

where the first inequality is followed by applying Hölder's inequality. The second follows from Lemma 5.1, and the last by the assumption 6.3, and the boundness of S^0. Since U is $\preceq$-nondecreasing on its domains, by Assumption 6.3 there exists some $x_0 \in \text{dom}(U)$ such that

$$\begin{aligned} \int (U^+(g))^p d\mu \quad &\leq \quad \int (U^+(g))^p \mathbf{1}_{g \prec x_0} d\mu + \int (U^+(g))^p \mathbf{1}_{g \succeq x_0} d\mu \\ &\leq \quad \int (U^+(x_0))^p d\mu + \text{const.}\left(\int (kg^\lambda)^p d\mu + \int \Upsilon^p d\mu \right) \end{aligned}$$

By (37) and assumptions on k and Λ, this proves the $\mathcal{L}^p(\mu)$-boundedness of the family $\Big\{U^+(g);\ g \in \mathbb{C}_x^+\Big\}$ and therefore its uniform integrability under $\mathbf{P}$. □

Theorem 6.1 *Under Assumptions 6.1, 6.2 and 6.3.*

1. *The optimal solution $g^* \in \mathbb{C}_x^+$ to problem (34) exits for all $x \geq v(0)$, and is unique if U_1 or U_2 are strictly concave on their domains a.s..*
2. *The function u is nondecreasing and concave on $[v(0), \infty)$ and if U_1 or U_2 are strictly concave on their domains, then u is strictly concave.*

Proof.

(1) Now let $x \geq v(0)$ and $(g^n)_{n\in\mathbf{N}} \in \mathbb{C}_x^+$ be a maximizing sequence of the problem (34), i.e.

$$\lim_{n\to\infty} \int U(g^n)d\mu = \lim_{n\to\infty} \mathbf{E}\Big[\int_0^T U_1(g_t^n, t)dt + U_2(g_T^n)\Big] = u(x)$$

Since g^n are nonnegative, by applying Lemma A.1 in Delbaen and Schachermayer (1994) [15] [4] we can find a sequence (possiby up to a subsequence) $g_1^n \in \text{conv}(g^n, g^{n+1}, \cdots)$, for $n \in \mathbf{N}$, which converges μ-almost everywhere to a limit g^*. By lemma 5.3, g^* belongs to the set $\mathbb{C}_x^+$, and by the concavity and upper-semicontinuity of U_1, U_2 we have:

$$\begin{aligned}
u(x) &\leq \limsup_{n\to\infty} \mathbf{E}\Big[\int_0^T U_1(g_{1_t}^n, t)dt + U_2(g_{1_T}^n)\Big] \\
&= \limsup_{n\to\infty} \mathbf{E}\Big[\Big(\int_0^T U_1(g_{1_t}^n, t)dt + U_2(g_{1_T}^n)\Big)^+\Big] - \\
&\quad - \liminf_{n\to\infty} \mathbf{E}\Big[\Big(\int_0^T U_1(g_{1_t}^n, t)dt + U_2(g_{1_T}^n)\Big)^-\Big] \\
&\leq \mathbf{E}\Big[\Big(\int_0^T U_1(g_t^*, t)dt + U_2(g_T^*)\Big)^+ - \Big(\int_0^T U_1(g_t^*, t)dt + U_2(g_T^*)\Big)^-\Big] \\
&= \mathbf{E}\Big[\int_0^T U_1(g_t^*, t)dt + U_2(g_T^*)\Big]
\end{aligned}$$

where the second inequality follows from Lemma 6.2 and Fatou's lemma.

(2) The uniqueness of the optimal solution g^*, when U_1, U_2 are strictly concave, is trivial and is omitted here.

[4] This Lemma states that for any sequence of nonnegative random variables $(f^n)_{n\in\mathbf{N}}$, there always exists a sequence $g^n \in\text{conv}(f^n, f^{n+1}, \cdots)$, for $n \in \mathbf{N}$, which converges almost surely to a variable g with values in $[0, \infty]$.

Let $x_1 \leq x_2$. Since $\mathbb{C}_x^+$ is a solid set then $\mathbb{C}_{x_1}^+ \subseteq \mathbb{C}_{x_2}^+$ and we deduce that $u_{x_1} \leq u_{x_2}$, i.e. u is nondecreasing on $(0, \infty)$.

Now let $\zeta \in (0, 1)$ and from Remark 5.3 we see also that

$$\zeta g_{x_1} + (1-\zeta) g_{x_2} \in \mathbb{C}^+_{\zeta x_1 + (1-\zeta) x_2}$$

where $g_{x_i} \in \mathbb{C}^+_{x_i},\ i = 1,\, 2$ Then by the concavity of functions U_1, U_2, we have:

$$\begin{aligned} u(\zeta x_1 + (1-\zeta)x_2) &\geq \mathbf{E}\Big[\int_0^T U_1(\zeta g^*_{x_1,\,t} + (1-\zeta) g^*_{x_2,\,t}, t)dt + \\ &\quad + U_2(\zeta g^*_{x_{1_T}} + (1-\zeta) g^*_{x_{2T}})\Big] \\ &\geq \mathbf{E}\Big[\int_0^T \Big(\zeta U_1(g^*_{x_1,\,t}, t) + (1-\zeta) U_1(g^*_{x_2,\,t}, t)\Big)dt + \\ &\quad + \zeta U_2(g^*_{x_{1_T}}) + (1-\zeta) U_2(g^*_{x_{2_T}})\Big] \\ &= \zeta u(x_1) + (1-\zeta) u(x_2) \end{aligned}$$

this implies the concavity of u. Moreover, if U_1 or U_2 are strictly concave then it is obvious that u is a strictly concave. $\square$

7 The Dual Problem

On the dual side, we define the conjugate function $\widetilde{U} \colon S \times \mathbf{R}_+ \to \mathbf{R} \cup \{\infty\}$:

$$\widetilde{U}(s, h) = \sup_{g>0} [U(s, g) - \langle g, h \rangle], \quad h \in \mathcal{L}^0_+$$

To alleviate notations, we omit the dependence in the state $s \in S$ and write $\widetilde{U}(y)$ in place of $\widetilde{U}(s, y)$ henceforth.

Clearly, $\widetilde{U}$ is decreasing (in a $\prec$ sense) and

$$\langle 1, \widetilde{U}(h) \rangle = \int_0^T \widetilde{U}_1(h_t, t)dt + \widetilde{U}_2(h_T)$$

where

$$\begin{aligned} \widetilde{U}_1(y, t) &= \sup_{x>0} [U_1(x, t) - xy], \quad y > 0 \\ \widetilde{U}_2(y) &= \sup_{x>0} [U_2(x) - xy], \quad y > 0 \end{aligned}$$

We now consider the following optimization problem:

$$\begin{aligned} \widetilde{u}(y) &= \inf_{h\in\mathbb{D}} \widetilde{J}(y;h) \\ &\triangleq \inf_{h\in\mathbb{D}} \Big(\int \widetilde{U}(yh)d\mu + \mathbf{E}\Big[\int_0^T yh_t dX_t^0 + \int_0^T yh_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \Big] \Big) \\ &= \inf_{h\in\mathbb{D}} \mathbf{E}\Big[\int_0^T \widetilde{U}_1(yh_t,t)dt + \widetilde{U}_2(yh_T) + \int_0^T yh_t dX_t^0 + \int_0^T yh_t dA^{\widetilde{\mathbb{X}}^b}(h)_t \Big] \end{aligned} \tag{38}$$

In order to proceed, we shall need the following standing assumptions

Assumption 7.1

$$\widetilde{u}(y) < \infty, \quad \textit{for all } y > 0$$

Assumption 7.2 *(i) There exists $x_0 \in dom(U)$, with $x_0 \in \mathcal{L}^\infty(\mu)$ and $U(x_0) \in \mathcal{L}^1(\mu)$ such that:*

$$\langle x, U'(x)\rangle \leq \langle \lambda, U(x)\rangle + \Upsilon, \quad \mu - a.s. \ , \ \forall\, x \in dom(U) \cap [x_0, \infty)$$

(ii) For any $\zeta \geq 0$, there exists a real number $\delta_\zeta \in [0, \frac{\zeta}{T+1})$ such that

$$\delta_\zeta S^0 \in dom\,(U), \quad and \quad U(\delta_\zeta S^0) \in \mathcal{L}^1(\mu)$$

Lemma 7.1 *Under Assumptions 7.1 and 7.2 (ii), for all $x \in (v(0), \infty)$, there exists an optimal solution $y^* > 0$ to the optimization problem*

$$\inf_{y>0} [\widetilde{u}(y) + xy].$$

Proof. The argument here is slight modification of that of Pham and Mnif (2002); we include it for completeness. Fix any $x \in (v(0), \infty)$. Under assumption 7.2 (2), there exists δ_x real-valued in $\Big[0, \frac{x-v(0)}{T+1}\Big)$ such that:

$$\begin{aligned} \delta_x S^0 &\in \mathrm{dom}\,(U) \\ U(\delta_x S^0) &\in \mathcal{L}^1(\mu), \end{aligned}$$

By definitions of $\widetilde{U}$, and from (10), we have

$$\widetilde{U}(yh) \geq U(\delta_x S^0) - y\delta_x ZD, \quad \forall\, y > 0$$

By definition of $\widetilde{J}(y,h)$, Remark (5.2) we get

$$\begin{aligned}\int \widetilde{U}(yh)d\mu = \widetilde{J}(y;h) &\geq \int U(\delta_x S^0)d\mu - \int y\delta_x ZDd\mu \quad \forall\ y>0,\ h\in\mathbb{D}\\ &= \int U(\delta_x S^0)d\mu - y\mathbf{E}\Big[\int_0^T \delta_x Z_t D_t dt + \delta_x Z_T D_T\Big]\\ &= \int U(\delta_x S^0)d\mu + y(x - v(0) - (T+1)\delta_x),\end{aligned}$$

where the second equality follows from (21), the third follows from Theorem VI.57 of Dellacherie and Mayer (1982) [18], by Fubini's theorem and the fact that

$$\mathbf{E}[Z_t D_t] \leq \mathbf{E}[Z_t] \leq 1, \quad \forall\ t\in[0,T]$$

by the definition of $\mathcal{P}^*(\widetilde{\mathbb{X}}^b)$.

Taking infimum in this last inequality over $h\in\mathbb{D}$ implies:

$$\widetilde{u}(y) + xy \geq \int U(\delta_x S^0)d\mu + y(x - v(0) - (T+1)\delta_x),$$

for all $y>0$ and $h\in\mathbb{D}$.

Since

$$\int U(\delta_x S^0)d\mu > -\infty$$

and $(x-v(0)-(T+1)\delta_x) > 0$, we deduce that $y\mapsto \widetilde{u}(y)+xy$ is a proper convex function. Moreover, $\widetilde{u}(y)+xy\to\infty$ as $y\to\infty$, this shows that the infimum $\widetilde{u}(y)+xy$ is attained in $y^*\in\mathbf{R}_+$.

To prove that $y^*>0$, we assume the contrary, then:

$$\widetilde{u}(0) \leq xy + \int \widetilde{U}(yh)d\mu + \mathbf{E}\Big[\int_0^T yh_t dX_t^0 + \int_0^T yh_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \tag{39}$$

for all $y>0$, $h\in\mathbb{D}$. By the properties of utility functions, we have:

$$\widetilde{U}(yh) + yhI(yh) \leq \widetilde{u}(0) \tag{40}$$

Plugging (40) into (39) and dividing by $y>0$, we obtain for all $y>0$, $h\in\mathbb{D}$:

$$\int hI(yh)d\mu - \mathbf{E}\Big[\int_0^T h_t dX_t^0 + \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \leq x$$

By the properties of utility functions (see (12)), and since I_i, $i=1,2$ are nonnegative, and as $y\to 0$, $I_1(yh_t,t)\to\infty$ $I_2(yh_T)\to\infty$,hence $I(yh)\to\infty$ as $h\equiv\infty$. Then we get by Fatou's lemma $v(\infty)\leq x$. This implies the contradiction, since either $x<\infty$ or $\infty\notin\mathbb{C}_x^+$ (see Remark 5.1). □

Lemma 7.2 *Under Assumption 7.2 (i), there exist* $\alpha \in (0,1)$, $c \geq 0$ *and* $\Gamma \in \mathbf{L}^1(\mathbf{P})$, *such that:*

$$\langle h, I(\alpha h)\rangle \leq \langle c, \widetilde{U}(yh)\mathbf{1}_{h \preceq U'(x_0)} + U'(x_0)\mathbf{1}_{h \succ U'(x_0)}\rangle + \Gamma$$

μ*-a.e. for all* $h \in \mathcal{L}^0_+$.

Proof. We follow the arguments for Lemma 7.2 in Pham and Mnif [48]. Take $\alpha \in (\lambda, 1)$ and suppose that $(t_i)_{i=1,2,\cdots} \subset [0,T]$ are the intervals, in which we have $h \preceq U'(x^0)$. By assumption 7.2 (1), and by properties of utility function (see (13), (14) and (15)) we have:

$$\begin{aligned}
\sum_{i=0}^{\infty}\langle \alpha h, I(\alpha h)\rangle_{t_i,t_{i+1}} &\leq \langle \alpha h, I(\alpha h)\rangle \\
&= \langle I(\alpha h), U'(I(\alpha h))\rangle \\
&\leq \langle \lambda, U(I(\alpha h))\rangle + \Upsilon \\
&\leq \langle \lambda, (\widetilde{U}(h) + hI(\alpha h))\rangle + \Upsilon
\end{aligned}$$

where the first inequality follows from the fact that I_1 and h are nonnegative functions. The third inequality follows from Assumption 7.2 (1), the last follows from the property of a dual function. Therefore we obtain:

$$\sum_{i=0}^{\infty}\langle \alpha h, I(\alpha h)\rangle \leq \langle \frac{\lambda}{\alpha - \lambda}, \widetilde{U}_1(h)\rangle + \frac{\Upsilon}{\alpha - \lambda} \tag{41}$$

Consider now the intervals, in which $h \succ U(x_0)$, since I are nonincreasing, by using (41) we get

$$\begin{aligned}
\sum_{i=0}^{\infty}\langle h, I(\alpha h)\rangle_{t_i,t_{i+1}} &\leq \langle h, I(\alpha h)\rangle \\
&\leq \langle h, I(\alpha U'(x_0))\rangle \\
&\leq \langle c, \widetilde{U}(U'(x_0))\rangle + \Gamma \\
&\leq \langle c, U(x_0)\rangle + \Gamma
\end{aligned} \tag{42}$$

with $c = \frac{\lambda}{\alpha - \lambda}$ and $\Gamma = \frac{\Upsilon}{\alpha - \lambda}$, where the last inequality comes from the following properties of utility function:

$$\begin{aligned}\widetilde{U}(U'(x)) &= U(I(U'(x))) - U'(x)I(U'(x)) \\ &\leq U(I(U'(x))) \\ &= U(x)\end{aligned}$$

From (41) and (42) we get the desired result. □

Lemma 7.3 *Let Assumption 7.2 (1) hold and suppose that there exists a solution h^* to the problem (38), for some $y > 0$. Then $\widetilde{u}(y)$ is differentiable in y and we have:*

$$\begin{aligned}\widetilde{u}'(y) &= -\int h^* I(yh^*)d\mu + \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\ &= \mathbf{E}\Big[-\int_0^T h_t^* I_1(yh_t^*, t)dt - h_T^* I_2(yh_T^*)\Big] + \\ &\quad + \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big]\end{aligned} \tag{43}$$

Moreover, if in addition $D^* \in \mathbb{D}^+$ then

$$\begin{aligned}\widetilde{u}'(y) &= -\int \frac{Z^*}{S^0} I(yh^*)d\mu + \mathbf{E}\Big[\int_0^T \frac{Z_t^*}{S_t^0} dX_t^0 + \int_0^T \frac{Z_t^*}{S_t^0} dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \\ &= \mathbf{E}\Big[-\int_0^T \frac{Z_t^*}{S_t^0} I_1(yh_t^*, t)dt - \frac{Z_T^*}{S_T^0} I_2(yh_T^*)\Big] + \\ &\quad + \mathbf{E}\Big[\int_0^T \frac{Z_t^*}{S_t^0} dX_t^0 + \int_0^T \frac{Z_t^*}{S_t^0} dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big]\end{aligned} \tag{44}$$

Proof. Fix any δ sufficiently small, we shall show that

$$D^+\widetilde{u}(y) \leq -\int h^* I(yh^*)d\mu + \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \tag{45}$$

and

$$D_-\widetilde{u}(y) \geq -\int h^* I(yh^*)d\mu + \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \tag{46}$$

Let $\delta > 0$. By using successively the definition of $\widetilde{u}(y)$, the convexity of $\widetilde{U}$ and its properties, we obtain:

$$\begin{aligned}-\frac{\widetilde{u}(y+\delta)-\widetilde{u}(y)}{\delta} &\geq \int \frac{\widetilde{U}((y+\delta)h^*)-\widetilde{U}(yh^*)}{-\delta}d\mu - \\ &\quad -\mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\ &\geq \int h^* I((y+\delta)h^*)d\mu - \\ &\quad -\mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big],\end{aligned}$$

which implies (45) by the monotone convergence theorem.

By the same arguments in the case $\delta < 0$, with $y+\delta > 0$, we obtain:

$$\begin{aligned}-\frac{\widetilde{u}(y+\delta)-\widetilde{u}(y)}{\delta} &\leq \int h^* I((y+\delta)h^*)d\mu - \\ &\quad -\mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \qquad (47)\end{aligned}$$

Under Assumption 7.2 (1) and by lemma 7.2, we have for $\delta < 0$ sufficiently small:

$$\begin{aligned}&-\int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t \\ &\leq \langle h^*, I(h^*)\rangle - \int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t \\ &\leq \Gamma + \langle c, \widetilde{U}(yh^*)\mathbf{1}_{yh^*\preceq U'(x_0)} + |U(x_0)|\rangle - \\ &\quad -\int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t \qquad (48)\end{aligned}$$

We first show that the right-hand side in (48) is integrable under $\mathbf{P}$.

Since $\widetilde{u}(y) < \infty$, we already have:

$$\int \widetilde{U}(yh^*)d\mu - \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] < \infty \qquad (49)$$

By definition of $\widetilde{U}$ we have:

$$\widetilde{U}(yh^*) \geq U(x_0) - \langle yh^*, x_0\rangle \qquad (50)$$

Notice that by Hölder inequality we have:

$$\begin{aligned}\int |yh^* x_0| d\mu &\leq y\|x_0\|_\infty \|h^*\|_1 \\ &= y\|x_0\|_\infty \mathbf{E}\Big[\int_0^T \frac{Z_t D_t}{S_t^0}dt + \frac{Z_T D_T}{S_T^0}\Big] < \infty \qquad (51)\end{aligned}$$

where the last inequality follows from Assumption 7.2 (1), 6.2, the fact that $|D| < 1$ and Z is a **P**-supermartingale.

From (50), (51) we deduce that

$$\int \widetilde{U}(yh^*)d\mu - \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] > -\infty \tag{52}$$

By (49), (52) and Assumption 7.2 (1) we deduce that the right-hand side in (48) is integrable. Therefore we can apply the dominated convergence theorem to (47) and obtain (46).

From (45), (46) and the property of the convex function $\widetilde{u}(y)$ we get (43).

To prove (44), firstly we take an arbitrary element $h \in \mathbb{D}^+$ and let $\zeta \in (0,1)$. Lemma 5.2 implies that there exists $h^\zeta \in \mathbb{D}$ such that:

$$(1-\zeta)h^* + \zeta h = h^\zeta$$

Since yh^* solves (38), we have:

$$\begin{aligned}\int \widetilde{U}(yh^*)d\mu + \mathbf{E}\Big[\int_0^T yh_t^* dX_t^0 + \int_0^T yh_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \leq \\ \int \widetilde{U}(yh^\zeta)d\mu + \mathbf{E}\Big[\int_0^T yh_t^\zeta dX_t^0 + \int_0^T yh_t^\zeta dA^{\widetilde{\mathbb{X}}^b}(h^\zeta)_t\Big]\end{aligned} \tag{53}$$

By convexity of $\widetilde{U}$ and noting that $h^\zeta - h^* = \zeta(h - h^*)$, we have:

$$\widetilde{U}(yh^*) \geq \widetilde{U}(yh^\zeta) + \langle \zeta y(h-h^*), I(yh^\zeta)\rangle \tag{54}$$

By Lemma (5.2) we have:

$$\mathbf{E}\Big[\int_0^T h_t^\zeta dA^{\widetilde{\mathbb{X}}^b}(h^\zeta)_t\Big] \leq (1-\zeta)\mathbf{E}\Big[\int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] + \zeta\mathbf{E}\Big[\int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \tag{55}$$

Plugging (54 - 55) into (53) and dividing by ζ, we obtain:

$$\begin{aligned}\int hI(yh^\zeta)d\mu - \mathbf{E}\Big[\int_0^T h_t dX_t^0 + \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big] \leq \\ \int h^* I(yh^\zeta)d\mu - \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big]\end{aligned} \tag{56}$$

Since $h^\zeta \geq (1-\zeta)h^*$ and I is nonincreasing, we have:

$$\begin{aligned}
& - \int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t \\
\leq\ & \langle h^*, I(yh^\zeta)\rangle - \int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t \\
\leq\ & \Gamma + \langle c, \widetilde{U}(yh^*)\mathbf{1}_{yh^* \preceq U'(x_0)} + |U|(x_0)\rangle - \\
& - \int_0^T h_t^* dX_t^0 - \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t
\end{aligned}$$

By the same arguments as in (48) we apply the dominated convergence theorem to the right-hand side of (56) and obtain

$$\begin{aligned}
-\widetilde{u}'(y) &= \int h^* I(yh^*)d\mu - \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\
&\geq \int hI(yh^*)d\mu - \mathbf{E}\Big[\int_0^T h_t dX_t^0 + \int_0^T h_t dA^{\widetilde{\mathbb{X}}^b}(h)_t\Big], \qquad (57)\\
&\geq \int \frac{Z^*}{S^0} I(yh^*)d\mu - \mathbf{E}\Big[\int_0^T \frac{Z_t^*}{S_t^0} dX_t^0 + \int_0^T \frac{Z_t^*}{S_t^0} dA^{\widetilde{\mathbb{X}}^b}(Z^*)_t\Big] \qquad (58)
\end{aligned}$$

where the last inequality follows from the fact that $\mathbf{1} \in \mathbb{D}$ and $Z^* \in \mathcal{P}^*(\widetilde{\mathbb{X}}^b)$.

Now we suppose that $D^* \in \mathbb{D}^+$, we will show the converse inequality.

Fix $\delta \in (0, y)$ and consider the process

$$D_t^\delta = \left(D_t^* - \frac{\delta}{y}(1 - D_t^*)\right)\mathbf{1}_{D_t^* \geq \frac{\delta}{y+\delta}}, \quad t \in [0, T] \qquad (59)$$

and we see that $\lim_{\delta\to 0} D_t^\delta = D_t^*$ a.s.. By definition of $\widetilde{u}$ and convexity of $\widetilde{U}$ we have:

$$\begin{aligned}
\frac{\widetilde{u}(y) - \widetilde{u}(y-\delta)}{\delta} &\geq \int \frac{\widetilde{U}(yh^*) - \widetilde{U}((y-\delta)h^\delta)}{\delta} d\mu \\
&+ \int_0^T \frac{(yD_t^* - (y-\delta)D_t^\delta)Z_t^*}{\delta S_t^0} dX_t^0 + \int_0^T \frac{(yD_t^* - (y-\delta)D_t^\delta)Z_t^*}{\delta S_t^0} dA^{\widetilde{\mathbb{X}}^b}(Z^*)_t \\
\geq\ & - \int \frac{(yD^* - (y-\delta)D^\delta)Z^*}{\delta S^0} I(yh^\delta)d\mu + \\
&+ \int_0^T \frac{(yD_t^* - (y-\delta)D_t^\delta)Z_t^*}{\delta S_t^0} dX_t^0 + \int_0^T \frac{(yD_t^* - (y-\delta)D_t^\delta)Z_t^*}{\delta S_t^0} dA^{\widetilde{\mathbb{X}}^b}(Z^*)_t
\end{aligned} \qquad (60)$$

On the other hand, from (59) and since $D^* > 0$ a.s., we have:

$$\begin{aligned}
\lim_{\delta\to 0} -\frac{yD_t^* - (y-\delta)D_t^\delta}{\delta} &= \lim_{\delta\to 0}\left(-1 + \frac{\delta}{y}(1 - D_t^*) + \frac{y-\delta}{\delta}\Big[1 - \frac{y+\delta}{\delta}D^*\Big]\mathbf{1}_{D_t^* < \frac{\delta}{y+\delta}}\right) \\
&= -1 \quad a.s. \qquad (61)
\end{aligned}$$

Sending δ in (60) to zero, by Fatou's lemma and using (61) we obtain:

$$-\widetilde{u}'(y) \leq \int \frac{Z^*}{S^0} I(yh^*) d\mu - \mathbf{E}\Big[\int_0^T \frac{Z_t^*}{S_t^0} dX_t^0 + \int_0^T \frac{Z_t^*}{S_t^0} dA^{\widetilde{\mathbb{X}}^b}(Z^*)_t\Big] \tag{62}$$

From (58) and (62) we get the desired result. □

We now state the main theorem of this section.

Theorem 7.1 *Under Assumptions 6.1, 6.2, 6.3, 7.1 and 7.2.*

1. *For any $x \in (v(0), \infty)$, there always exists an optimal solution y^* to the problem* $\inf_{y \geq 0}[\widetilde{u}(y) + xy]$.
2. *Suppose that there exists an optimal solution $y^* h^*$ to problem $\widetilde{u}(y^*)$. Then*

   (a) *The unique solution of (34) is given by:*

$$g_t^* = I_1(y_t^* h_t^*, t), \quad t \in [0, T], \quad g_T^* = I_2(y^* h_T^*), \tag{63}$$

   and the solution to (17) (X^*, c^*) satisfies:

$$\begin{aligned} h_t^* X_t^* + \int_0^t h_s^* c_s^* ds = \\ \mathbf{E}\Big[\langle g^*, h^* \rangle_{t,T} - \int_t^T h_s^* dX_s^0 - \int_t^T h_s^* dA^{\widetilde{\mathbb{X}}^b}(Z^*)_s \Big| \mathcal{F}_t\Big] \end{aligned} \tag{64}$$

   (b) *If in addition $D^* \in \mathbb{D}^+$ a.s., then we also have*

$$\begin{aligned} \frac{Z_t^* X_t^*}{S_t^0} + \int_0^t \frac{Z_s^* c_s^*}{S_s^0} ds = \\ \mathbf{E}\Big[\langle g^*, \frac{Z^*}{S^0} \rangle_{t,T} - \int_t^T \frac{Z_s^*}{S_s^0} dX_s^0 - \int_t^T \frac{Z_s^*}{S_s^0} dA^{\widetilde{\mathbb{X}}^b}(Z^*)_s \Big| \mathcal{F}_t\Big] \end{aligned} \tag{65}$$

   for all $t \in [0, T]$.

   (c) *Furthermore, if $Z^* \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)$, then the pair of wealth/consumption processes (X^*, c^*) can be determined as follows:*

$$\begin{aligned} X_t^* &= S_t^0 \Big(\operatorname*{ess\,sup}_{Q \in \overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \Big(\mathbf{E}^Q \big[A^* - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_T \,|\mathcal{F}_t\big] + \widetilde{A}^{\widetilde{\mathbb{X}}^b}()Q_t \Big) + \widetilde{X}_t^0 - \\ &\quad - \int_0^t \frac{I_1(y_s^* h_s^*, s)}{S_s^0} ds \\ &= S_t^0 \Big(\Big(\mathbf{E}^{Q^*} \big[A^* - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q^*)_T \,|\mathcal{F}_t\big] + +\widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q^*)_t \Big) + \widetilde{X}_t^0 - \\ &\quad - \int_0^t \frac{I_1(y_s^* h_s^*, s)}{S_s^0} ds \Big) \end{aligned} \tag{66}$$

and

$$c_t^* \triangleq \widetilde{c}_t S_t^0 + I_1(y_t^* h_t^*, t) \tag{67}$$

where $\widetilde{c}$ *is a nonnegative process that appears in the optional decomposition of process* $\widetilde{X}$*:*

$$\begin{aligned} \widetilde{X}_t &\triangleq \operatorname*{ess\,sup}_{Q\in\overline{\mathcal{P}^*}(\widetilde{\mathbb{X}}^b)} \Big(\mathbf{E}^Q\big[A^* - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_T\,|\mathcal{F}_t\big] + \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q)_t\Big) \\ &= \Big(\mathbf{E}^{Q^*}\big[A^* - \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q^*)_s\,|\mathcal{F}_t\big] + \widetilde{A}^{\widetilde{\mathbb{X}}^b}(Q^*)_t\Big) \\ &= x + X_b^* - \widetilde{c} \bullet \Lambda, \quad X_b^* \in \widetilde{\mathcal{X}}_b \end{aligned} \tag{68}$$

A^* *is an* $\mathcal{F}_T$*-measurable random variable defined as*

$$A^* = \langle g^*, \frac{1}{S^0}\rangle - \widetilde{X}_T^0$$

and Q^* *is equivalent probability to* $\mathbf{P}$*, whose density process is* $\frac{dQ^*}{d\mathbf{P}} = Z^*$*.*

3. *Suppose that for all* $y > 0$*, there exists a solution to the dual problem* $\widetilde{u}(y)$*, then the value functions* u *and* $\widetilde{u}$ *are conjugate,*

$$\widetilde{u}(y) = \max_{x\geq v(0)}[u(x) - xy], \quad y > 0, \tag{69}$$

$$u(x) = \min_{y>0}[\widetilde{u}(y) + xy], \quad x \geq v(0),. \tag{70}$$

Proof.

(1) The assertion is a result of Lemma 7.1

As a result of the last lemma, $\widetilde{u}$ is differentiable at y^* and:

$$-\widetilde{u}'(y^*) = \int h^* I(y^*h^*)d\mu - \mathbf{E}\Big[\int_0^T h_t^* dX_t^0 + \int_0^T h_t^* dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] = x \tag{71}$$

Let us define

$$g_t^* \triangleq I_1(y_t^* h_t^*, t), \quad t \in [0,T], \quad g_T^* \triangleq I_2(y^* h_T^*)$$

Remark 5.3, (57) and (71) imply that $g^* \in \mathbb{C}_x^+$. Now, for any $g \in \mathbb{C}_x^+$ we have:

$$\begin{aligned} U(g) &\leq \widetilde{U}(y^*h^*) + \langle g, y^*h^*\rangle \\ &\leq U(I(y^*h^*)) + \langle g, y^*h^*\rangle - \langle y^*h^*, I(y^*h^*)\rangle \\ &\leq U(g^* + \langle y^*h^*, g\rangle - \langle y^*h^*, g^*\rangle \end{aligned}$$

where the second inequality follows from the definition of $\widetilde{U}$.

Taking expectation under μ and using (71), we obtain:

$$
\begin{aligned}
\int U(g)d\mu &\leq \int U(g^*)d\mu + \\
&\quad + y^*\mathbf{E}\Big[\langle h^*, g^*\rangle - \int_0^T h^*_t dX^0_t - \int_0^T h^*_t dA^{\widetilde{\mathbb{X}}^b}(h^*)_t - x\Big] \\
&\leq \int U(g^*)d\mu
\end{aligned}
$$

where the last inequality follows from the (23). This proves the optimality of g^*.

Now let $(X^*, c^*) \in \mathcal{A}^*(x)$ be a element that dominates g^* in a sense of (22). By (71) and (23) we have:

$$
\begin{aligned}
x &= \mathbf{E}\Big[\langle g^*, h^*\rangle - \int_0^T h^*_t dX^0_t - \int_0^T h^*_t dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\
&\leq \mathbf{E}\Big[\int_0^T h^*_t c^*_t dt + h^*_T X^*_T - \int_0^T h^*_t dX^0_t - \int_0^T h^*_t dA^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\
&\leq x
\end{aligned}
$$

and this proves that $h^* c^* \bullet \Lambda + h^* X^* = h^* f^* \bullet \Lambda + h^* F^*$ a.s. and

$$
h^* c^* \bullet \Lambda + h^* X^* - h^* \bullet X^0 - h^* \bullet A^{\widetilde{\mathbb{X}}^b}(h^*)
$$

is a **P**-martingale, therefore relation (64) holds.

If $D \in \mathbb{D}^+$ a.s. then $Z^* c^* \bullet \Lambda + Z^* X^* = Z^* f^* \bullet \Lambda + Z^* F^*$ a.s.. and the **P**-supermartingale

$$
\frac{Z^* F^*}{S^0} + \frac{Z^* f^*}{S^0} \bullet \Lambda - \frac{Z^*}{S^0} \bullet X^0 - \frac{Z^*}{S^0} \bullet A^{\widetilde{\mathbb{X}}^b}(Z^*)
$$

is in fact a **P**-martingale and this proves (65).

2c) By the same arguments as in the proof to the sufficient condition of the Lemma 5.1 we get the formulae (66)-(67) for the solution (X^*, c^*) to the primal optimization problem (17)

(3) Since we have

$$
\begin{aligned}
&\langle \mathbf{1}, U(g)\rangle - \langle yh, g\rangle + y\Big(\int_0^T h_t dX^0_t + \int_0^T h_t A^{\widetilde{\mathbb{X}}^b}(h)_t\Big) \\
&\leq \langle \mathbf{1}, \widetilde{U}(yh)\rangle + \int_0^T y h_t dX^0_t + \int_0^T y h_t A^{\widetilde{\mathbb{X}}^b}(h)_t \qquad (72)
\end{aligned}
$$

for all $x \geq v(0)$, $y > 0$, $g \in \mathbb{C}_x^+$, $h \in \mathbb{D}$. Taking expectation in (72) and infimum over $h \in \mathbb{D}$ and using the fact that $v(g) \leq x$, we obtain:

$$u(x) \leq \widetilde{u}(y) + xy, \forall\, x \geq v(0),\, y > 0. \tag{73}$$

Now let us fix any $x \in (v(0), \infty)$ we have

$$\begin{aligned} \widetilde{u}(y^*) &= \int \widetilde{U}(y^* h^*) d\mu + \mathbf{E}\Big[\int_0^T y^* h_t^* dX_t^0 - \int_0^T y^* h_t^* A^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big] \\ &= \int U(g^*) d\mu - y^*\Big(\langle h^*, g^*\rangle - \int_0^T h_t^* dX_t^0 - \int_0^T h_t^* A^{\widetilde{\mathbb{X}}^b}(h^*)_t\Big) \\ &= u(x) + y^* \widetilde{u}'(y^*) \;=\; u(x) - xy^* \end{aligned} \tag{74}$$

From (73), (74) and (72) it follows that (70) holds. Formula (69) now follows from (70) and the general bidual property of the Legendre-transform (see, e.g., Theorem III.12.2 in Rockafellar (1970) [45]). This completes the proof of the Theorem 7.1. □

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