Optimization of Risk Exposure

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Let us start with an analogy. Speed limits help control risks on a road - but what if the same 40 mph speed limit is used for a turnpike, for a city street, and for a school zone? That situation becomes real in finance, where a preset limit for daily, weekly, or monthly losses is widely accepted as a practical method of risk control.

Suppose we decide to keep our daily losses below 3% with some given confidence – let's say 90%. What can go wrong with that approach? The immediate problem that we would face is that markets change all the time, and our 3% daily limit can become either too loose or too tight.

To illustrate that, let us consider two extreme examples.

Imagine that we see no profit opportunities at the moment. In view of that expectation, the 3% limit is indeed too loose: why expose ourselves to useless risk if we expect no payback?

Imagine the opposite extreme: we see a highly probable bet. For instance, we expect the price of stock A either to double tomorrow with a 90% probability or to go 3% down with a 10% probability. Now, the 3% limit looks too tight: why not use a 4:1 margin instead and get a five-fold increase in our wealth, at a moderate price of risking 12% of the capital with a small 10% probability?

These two extreme examples show why maintaining constant risk exposure under changing market conditions can be far from optimal.

A question arises: how should our risk control strategy be modified, to work well under changing market conditions? Qualitatively, the answer is obvious: risk exposure must be adjusted to the expected returns. Quantitatively, the problem has been thoroughly analyzed in the past (see *Merton 1998* for more details), and a formal solution has been found. Let us briefly describe it.

Suppose we have chosen a daily time frame. If we measure risk exposure R as standard deviation of relative changes in our wealth, then the optimal level of risk exposure is

$$R_{opt}=S/a,$$

where the parameter S is the daily Sharpe ratio (the mathematical expectation of daily return on our portfolio over the risk-free rate, divided by the standard deviation of that return), and a is the so-called "relative risk aversion". A large value of a implies that we are risk-averse, and a small value of a implies that we are risk-seeking.

The Sharpe ratio is widely used as a measure of investment performance. It can be shown that in order to maximize the longer-term Sharpe ratio for a series of consecutive trades (for instance, the yearly or quarterly Sharpe ratio for a series of daily trades), we must keep the parameter $a=S/R$ constant throughout that series. In other words, *it is optimal to invest with constant (rather than changing) relative risk aversion*.

What is our risk aversion equal to? Historically, answering that question has been the biggest problem because the definition of risk aversion a in the classical formula for R_{opt} is rather abstract: it contains derivatives of some postulated utility U of our wealth W:

$$a = -\frac{WU''(W)}{U'(W)}$$

Until recently, no practical ways of calculating investor's risk aversion a had been suggested, which is the main reason why practitioners have mostly ignored the formula for optimal risk exposure R_{opt}.

Its practical use was recently made possible by linking risk aversion with the longer-term downside risk characteristic of a portfolio. It has been shown (*Gretchikha 1999*) that the probability of turning our capital *some time in the future* into a fraction $f<1$ of its current size is

$$P=f^{(2a-1)}.$$

This formula enables simple and accurate measurement of our risk aversion a. For example, if we need to be 90% sure ($P=0.1$) that our capital *will never* drop more than 10% below its current value ($f=0.9$), then according to the above formula our risk aversion is $a=11.4$.

Qualitatively, the formula for P simply says that the smaller we bet, the better are our chances of preserving our money. Its mathematical derivation, along with more results regarding the effect that our risk aversion has on our portfolio's behavior, can be found in the original publication (*Gretchikha 1999*), which is also accessible at

http//www.geocities.com/optimalrisk

Formulas for R_{opt} and for P provide an efficient set of tools for practical optimization of risk exposure. Let us consider some typical examples.

Suppose we already determined our risk aversion: $a=11.4$, and want to know how much we should invest in a portfolio with a 0.5% expected daily return over the risk-free rate and a 4% standard daily deviation of its value (it corresponds to the daily Sharpe ratio of $S=0.125$). Using the formula for the optimal risk exposure, we find R_{opt}=1.1%, which means that we should invest about 27% of our money (1.1% / 4% = 0.27).

Next week, daily volatility surges, increasing standard daily deviation of the portfolio's value from 4% to 7%. The formula for the optimal risk exposure now gives us R_{opt}=0.63%, suggesting that we should *reduce* our investment in the portfolio from 27% to 9%.

Two weeks later, a good news increases our estimate of the expected daily return from 0.5% to 7%. Using the formula for the optimal risk exposure R_{opt}, we find that R_{opt} increases from 0.63% to 8.8%, suggesting that we should now *increase* our investment in the portfolio from 9% to 125% of our entire capital (using margin to borrow the additional 25%).

The above examples show how flexible optimal investment behavior must be: no useless risk exposure when expected gains are too small, and confident risk taking when a good investment opportunity appears. In contrast, constant risk exposure uselessly decimates capital when profit opportunities are poor, and shies away from serious profit when an opportunity comes.

We can see now: as soon as we can accurately determine our risk aversion, the good thirty-year-old results of finance theory help us make optimal investment decisions in constantly changing markets. Why not use them?

Literature:

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