All Moments of Discrete and Continuous Arithmetic Averages on Brownian Paths: A Closed Form

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Abstract

This note derives new expressions for the moments of the average of values taken by Wiener paths at an arbitrary number, N, of discrete times. The expressions are closed summations, which entail only the $N-th$ powers of, and the successive differences between, the moments of the lognormal finite dimensional distribution of the process' values at the time of the first averaging. By passing to the limit of the average when the averaging frequency becomes continuous, known forms for the continuous average are generalized by a single expression.

The summands' kernel is itself an expression of some interest which apparently has not previously appeared in the literature. It generalizes the elementary expression for the sum of the geometric sequence of a variable, to an expression for the sum of all products of several variables under the condition that the sum of exponents in each summand is not greater than a specified integer maximum. Proof of the form is given.

1 Introduction and Results

The average of the levels attained by a continuous geometric Brownian motion with specified drift and diffusion is of concern in applications of stochastic processes, particularly in finance. No specification is known for the distribution of the average of a finite number N of values, taken at discrete intervals. An iterative method to obtain the moments of that distribution has been given in [3]. For *continuous* averaging, expressions for the first two moments are commonly known; perhaps the earliest expressions were provided in [1] and more recently in [2].

Knowledge of the finite dimensional distribution of outcomes induced by a geometric Brownian motion is often sufficient for valuation of financial claims. Absent an analytic form, the moments can provide a means of approximating the probability measure; these methods are solutions to the classical "Problem of Moments", first posed by Stieljes. Accordingly, at least in application, establishing the form of the higher moments of a probability law can be deemed tantamount to establishing that law itself.

The results of this note establish a single general expression for the moments of the discrete average of values of a geometric Brownian motion, as defined by the following process.

1.1 The Underlying Process and its Average.

Denote the drift and diffusion terms by r and σ, respectively. Then, specify a geometric Brownian motion, and the finite dimensional distribution of the process values, by:

$$dv_t = rv_t\ dt + \sigma v_t\ dW_t \Leftrightarrow v_t = v_0\ \exp\left\{(r - \sigma^2/2)t + \sigma\sqrt{dt}U\right\}, \quad (1)$$

where W_t denotes a standard Wiener process, and U denotes a unit standard normal variate.

Let Δ_T denote a time interval and assume, for some interval of time, T, that $T/\Delta_T \in \mathbb{Z}^+$.

The expression on the right of (1) is a lognormal density. Let $e_m(\tau)$ denote the $m - th$ moment about zero of the corresponding distribution for time τ. For $m \geq 0$, and for times $\tau = n\Delta_T$, the moments of the distribution are well known to be:

$$e_m(n\Delta_T) = \exp\ \ \{c_m\ n\Delta_T\ ) \quad (2)$$

where c_m, independent of time, is given by:

$$c_m\ = m\left(r - \sigma^2/2\right) + m^2\ \sigma^2/2 \quad (3)$$

The special case of $n = 1$ will be referred to as the *"single-step"*. The single-step random variable will be denoted by $v^* = v_{\Delta_T}$, and its m-th moment by: $\mathbf{E}[{v_{\Delta_T}}^m] \equiv e_m$. Because U in (1) is additively stable, $v_{n\Delta_T}$ can be written as the product of n realizations of v^*, independent and identically distributed. That is:

$$v_n = \prod_{i=1}^{i=n} v_i^* \tag{4}$$

The variable of interest is the arithmetic average of $N = T/\Delta_T$ successive values of the process (1), observed at the end of successive time intervals of length τ. Then, denoting this average as A_N,

$$\begin{aligned} A_N &= \frac{1}{N} \sum_{n=1}^{N} v_n \\ &= \frac{1}{N} \sum_{n=1}^{N} \Pi_n v^* \end{aligned} \tag{5}$$

where $\Pi_n v^*$ describes the product in (4).

1.2 Main Results

The following three propositions are the main results of this note. The first two define equivalent specifications for the moments under discrete averaging, i.e., when the number of intervals for fixed T is finite. The third proposition presents a new specification for the moments under *continuous* averaging. Each is given here, and established, respectively, in the ensuing three sections.

The expressions for the M-th moment entail certain subsets of the first M single-step moments. Also, the first proposition entails summations of powers up to order that depends on N. The following definitions will be used, both to state the propositions, and in their proofs.

Definition 1.1 *Augmented index subset, $\mathcal{U}_k(.)$.*
Let $\tilde{\mathcal{U}}_0 = \varnothing$, and $\tilde{\mathcal{U}}_k(u_1, \ldots, u_k)$ denote a distinct k-subset of the integers $u_j, j \in (1, 2, \ldots, M-1)$, such that:

$$\begin{aligned} &M < u_1 \leq k, \\ &u_1 < u_2 \leq k-1, \\ &\ldots, \\ &u_{j-1} < u_j \leq k+1-j, \\ &\ldots, \\ &u_{k-1} < u_k \leq 1. \end{aligned}$$

Whenever specificity of the values u_j do not effect an argument, the shortened symbol $U_k(.)$ may be written. Define an Augmented index subset by:

$$\mathcal{U}_k(.) \triangleq \tilde{\mathcal{U}}_k(.) \cup \{M\}.$$

Let $\mathcal{U}_{[k]}$ denote the union of all such sets, and $\tilde{\mathcal{U}}_{[k]}$ denote the union of all corresponding k-subsets. It is evident that the elements of $\tilde{\mathcal{U}}_{[k]}$ enumerate the combinations of k non-negative integers chosen from the first $M-1$. Moreover, with:

$$\tilde{\mathcal{U}}^{[M-1]} \triangleq \bigcup_{k=0}^{M-1} \tilde{\mathcal{U}_{[k]}}, \quad \textit{and} \;\; \mathcal{U}^{[M]} \triangleq \bigcup_{k=0}^{M-1} \mathcal{U}_{[k]},$$

$\tilde{\mathcal{U}}^{[M-1]}$ is a power set, so that both $\tilde{\mathcal{U}}^{[M-1]}$ and $\mathcal{U}^{[M]}$ have cardinality 2^{M-1}.
□

Definition 1.2 *Power index set, $\mathcal{I}_k(.)$.*
Let $\mathcal{I}_k(i_0, i_1, \ldots, i_k)$ denote a set of indices, each $i_{(.)} \in \mathbb{Z}^+$. Further, denote the sum of the values of the first j indices by the symbol $\Sigma(j) \equiv \sum_{m=0}^{j} i_m$. The elements $i_{(.)} \in \mathcal{I}_k(.)$ are constrained by $1 \leq i_{(.)} \leq i^*_{(.)}(k)$, with the upper bounds defined by:

$$\begin{aligned} &i^*_0(k) = N - k; \\ &\ldots \\ &i^*_j(k) = N - (k + j) - \Sigma(j-1); \\ &\ldots \\ &i^*_k(k) = N - \Sigma(k-1); \end{aligned}$$

□

Definition 1.3 *Combinatorial Product, $\mathbf{C}^M_k(.)$*
With regard to the elements of an Augmented index subset $\mathcal{U}_k(u_1, \ldots, u_k)$, let $\mathbf{C}^M_k(.)$ denote the product:

$$\mathbf{C}^M_k(\mathcal{U}_k(u_1, \ldots, u_k)) = ({}_M C_{u_1})({}_{u_1} C_{u_2}) \ldots ({}_{u_{k-1}} C_{u_k}).$$

where ${}_n C_r$ denotes the binomial coefficient, "n choose r".
□

Proposition 1.1 *The "Nested Powers" expression for the M-th moment about zero of the arithmetic average of N values of a geometric Brownian motion is given by:*

$$\mathbf{E}[{A_N}^M] = \frac{1}{N^M} \sum_{\forall \mathcal{U}_k(.) \in \mathcal{U}^{[M]}} \mathbf{C}_k^M(\mathcal{U}_k(u_1, \ldots, u_k)) f(\mathcal{U}_k(u_1, \ldots, u_k)),$$

where:

$$f(\mathcal{U}_k(.)) = \sum_{i_0=1}^{i_0^*(k)} {e_M}^{i_0} \sum_{i_1=1}^{i_0^*(k)} {e_{u_1}}^{i_1} \cdots \sum_{i_k=1}^{i_k^*(k)} {e_{u_k}}^{i_k}.$$

Proposition (1.1) is established in Section 2. It formalizes the algorithmic procedure, albeit with different indexing, presented by **TW** in [TW]. This form is neither particularly elegant nor easy to implement, since it depends upon N for resolution.

The next proposition, however, depends only on the power N, and not on sums over powers up to N. It, along with the limiting form which follows from it, are the two main results of this note.

Proposition 1.2 *The "Sum in Highest Powers" form.*
With $u_0 \equiv 0, u_{k+1} \equiv e_M$, and $e_0 \equiv 1$, then every term $f(\mathcal{U}_k(.))$ in Proposition (1.1) can be written as:

$$f(\mathcal{U}_k(.)) = p(\lambda_k; M) \sum_{m=0}^{k+1} \frac{e_{u_m}^N}{P(\lambda_k; m, M)}$$

with

$$p(\mathcal{U}_k(.); M) \equiv \prod_{j=1}^{k+1} e_{i_j},$$

and

$$P(\mathcal{U}_k(.); m, M) \equiv \prod_{\substack{j=0 \\ j \neq m}}^{k+1} (e_{i_m} - e_{i_j})$$

The form of Proposition (1.2) is, in a sense, a generalization of the elementary sum of a of geometric series. Indeed, for $\mathcal{U}_0(.) \triangleq \{M\}$:

$$\begin{aligned} f(\mathcal{U}_0(.)) &= \sum_{i_0=1}^{N} {e_M}^{i_0}, \text{ by Proposition 1.1,} \\ &= e_M \frac{{e_M}^N - 1}{e_M - 1}, \text{ by Proposition 1.2.} \end{aligned}$$

To illustrate the form more generally, consider, for instance, $(\mathcal{U}_1(.) \subset \mathcal{U}^{[3]} : \{1, 3\}$ Then:

$$\begin{aligned} f(\mathcal{U}_1(.)) &= \sum_{i_0=1}^{N-1} {e_3}^{i_0} \sum_{i_1=1}^{N-i_0} {e_1}^{i_1} \\ &= e_1 e_3 \left\{ \frac{{e_3}^N}{(e_3-1)(e_3-e_1)} + \frac{{e_1}^N}{(e_1-e_3)(e_1-1)} \right. \\ &\qquad \left. + \frac{1}{(1-e_3)(1-e_1)} \right\}. \end{aligned}$$

Perhaps somewhat remarkably, the second of these equivalent expressions, generated by Proposition (1.2), does not seem to pre-exist in the literature as an identity of the first, generated from Proposition (1.1). If that is in fact the case, that result is perhaps of some interest in its own right. The equivalence of expressions in Nested Powers form and Sum in Highest Powers form is established in Section 3.

Section 4 advances a general expression for the moments of continuous averaging of the values of the process specified in (1), which have been published to the second order only. Rather than treating the continuous process directly, which is the approach taken in [1] or [2], leading to successively more complicated multiple stochastic integrals, the form for the moments provided by Proposition (1.2) is simply passed to its limit with T fixed. Defining T as a period of unit length does not reduce generality; then, the following Proposition is established.

Proposition 1.3 *Let $\tilde{A}_M$ denote the M-th moment of the continuously averaged levels of a geometric Brownian motion over unit time. Denote each of the M lowest moments of the process' finite dimensional distribution, at* *the end of the period, as* $\mu_m = \exp(c_m)$*, with* c_m *given in (3). Then:*

$$\tilde{A}_M = M! \left\{ \sum_{m=1}^{M} \frac{\mu_m}{c_m \left(\prod_{\substack{j=1 \\ j \neq m}}^{M} (c_m - c_j) \right)} + \frac{(-1)^M}{\prod_{j=1}^{M} c_j} \right\} \tag{6}$$

2 Proof of Proposition (1.1)

For finite N, establishing Proposition (1.1) is greatly simplified by considering the sum, rather than the average, of values. Denote the sum of values by the random variable $S_N \equiv NA_N$. Then the moments of the latter of course obey the relationship:

$$\mathbf{E}[A_N^m] = \frac{1}{N^m}\mathbf{E}[S_N^m]$$

Proposition (1.1) may be established by a resolution of the moments of the sum of values of the process into expressions in the terms of the known single-step moments. This process is well-defined, and terminates with the expression asserted in the Proposition, consisting entirely of values of the single-step moments of all orders $m \leq M$.

The following two lemmas establish the kernel of the resolution process.

Lemma 2.1 *Expansion in Lower Orders.*
Writing $E_m(\eta) \equiv \mathbf{E}[S_m^\eta]$*:*

$$E_m(\eta) = e_m \sum_{u=0}^{m} {}_mC_u \; E_u(\eta - 1) \tag{7}$$

Proof. Expanding the products defined in (5) gives, for the sum:

$$\begin{aligned} S_\eta &= \left\{ v_1^* + (v_1^* v_2^*) + \cdots + (v_1^* v_2^* \ldots v_\eta^*) \right\} \\ &= \left\{ v_1^* \left(1 + v_2^* (1 + v_3^*(1 + \ldots v_{\eta-1}^*(1 + v_\eta^*) \cdots))) \right\} \\ &= v_1^*(1 + S_{\eta-1}) \end{aligned}$$

The single-step random instances are mutual independent; every m-th moment (about zero) of S_η therefore satsifies:

$$\mathbf{E}[S_\eta^m] = e_m \mathbf{E}[(1 + S_{\eta-1})^m]$$

Expanding the binomial and taking expectations over the sum establishes (7). □

Lemma 2.2 *Term-wise Resolution.*

$$E_m(\eta) = \sum_{i=1}^{\eta} e_m^i + \sum_{u=1}^{m-1} {}_mC_u \sum_{i=1}^{\eta-1} e_m^i E_u(\eta - i) \tag{8}$$

Proof. Write (7) as:

$$E_m(\eta) = e_m E_m(\eta - 1) + e_m + e_m \sum_{u=1}^{m-1} {}_mC_u E_u(\eta - 1).$$

Applying (7) to the leading term, $e_m E_m(\eta - 1)$, produces:

$$E_m(\eta) = e_m^2 E_m(\eta - 2) + \sum_{i=1}^{2} e_m^i + \sum_{u=1}^{m-1} {}_mC_u \sum_{i=1}^{2} e_m^i E_u(\eta - i).$$

After a total of $(\eta - 1)$ such subsititutions in the first term, then:

$$E_m(\eta) = e_m^{\eta-1} E_m(1) + \sum_{i=1}^{\eta-1} e_m^i + \sum_{u=1}^{m-1} {}_mC_u \sum_{i=1}^{\eta-1} e_m^i E_u(\eta - i).$$

Since $E_m(1) \equiv e_m$, combining the first two terms establishes (8).
□

Constructive proof of the Nested Powers form is a straightforward application of Term-wise Resolution. In addition to the definitions of index sets given in (1.2), the following two definitions are employed.

Definition 2.1 *Resolved Term of Order k.*
With respect to $\mathcal{U}_k(u_1, \ldots, u_k) \in \tilde{\mathcal{U}}_{[k]}$, an Resolved Term of Order k is defined as:

$$\begin{aligned} f_k(\mathcal{U}_k(.)) &= \sum_{i_0=1}^{i_0^*(k)} e_M^{i_0} \sum_{i_1=1}^{i_0^*(k)} e_{u_1}^{i_1} \cdots \sum_{i_k=1}^{i_k^*(k)} e_{u_k}^{i_k} \\ &\textit{and} \\ F_k &= \sum_{\forall\, \mathcal{U}_k(.)\, \in\, \mathcal{U}_{[k]}} f_k(.) \end{aligned}$$

□

Definition 2.2 *Unresolved Resultant.*
With

$$G_0(M) = \sum_{u_0}^{M} {}_MC_{u_0} E_{u_0}(N-1) \equiv E_M(N-1),$$

the Unresolved Resultant of Stage k is defined by:

$$G_k(M;N) = \sum_{\forall\ \mathcal{U}_k(.)\ \in\ \mathcal{U}_{[k]}} \left\{ \mathbf{C}_k(\mathcal{U}_k(.),M) \sum_{i_0=1}^{i_0^*} e_M^{i_0} \cdots \sum_{i_j=1}^{i_j^*} e_{u_j}^{i_j} \cdots \sum_{i_{k-1}=1}^{i_{k-1}^*} e_{u_{k-1}}^{i_{k-1}} E_{u_k}[N - \Sigma(k-1)] \right\} \quad (9)$$

This expression is "unresolved" because the least significant ("rightmost") summation entails an indeterminant path moment, i.e., $E_{u_k}(.)$. □

These two definitions categorize the two terms that arise in application of (8). With regard to the latter, the following Lemma is self evident and will be stated without formal proof.

Lemma 2.3 $G_k(M;N) \neq 0$ *iff* $\exists\, \mathcal{U}_k(.) \in \mathcal{U}_{[k]} : (u_{k-1} > 1)$.

Theorem 2.1 *The Nested Powers form of Proposition (1.1) attains for the* M*-th moment of* A_N *if:*

i. *Every Unresolved Resultant* $G_k(M;N) = G_{k-1}(M;N) - F_{k-1}$*, and*

ii. $G_M(M;N) = 0$.

Proof.

Clause (*i.*) follows by the sequential application of Lemma 2.2, applied to the least significant summand of each component of $G_{k-1}(M;N)$ in turn. The index on the first term in each application of (8) is u_{k-1}, which corresponds to the element of $\mathcal{U}_{k-1}(u_1,\ldots,u_{k-1})$ of smallest value. It resolves by the lemma to one term, $f_{k-1}(.)$, The sum of these exhausts the index-set elements of $\mathcal{U}_{[k-1]}$, and generates F_{k-1}, with the second term of each application of (8) accumulating to generate $G_{k-1}(M;N)$.

Clause (*ii.*) follows from lemma (2.3). From Definition (1.1),

$$U_{M-1}(u_1,\ldots,u_{M-1}) \stackrel{\Delta}{=} \{M, M-1, \ldots, ..2, 1\},$$

and $U_{[M-1]}$ consists of only this set Augmented index set alone.
□

3 Proof of Proposition (1.2)

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The form $f_k(.)$ in Definition (2.1) is a special case of a product of geometric series of M distinct real numbers. By first establishing the closed form for that general product of summands, Proposition (1.2) will follow directly as a particular case. The following definitions are used in this section.

Let $\mathbf{v}_{[M]}$ denote a set of M distinct non-zero numbers, $\{v_i\} \in \mathbb{R}$, such that $(v_i \neq 0$ and $v_i \neq 1, \forall\ v_i$.

Definition 3.1 *Nested Geometric Series*
Let $V(M,N)$ denote the sum of all products of the form $\left(v_1^{k_1} v_2^{k_2} \dots v_M^{k_M}\right)$, such that:

$$M \leq \sum_{m=1}^{M} k_m \leq N.$$

Then a Nested Geometric Series is defined as:

$$V(M,N) = \sum_{k_1\ =1}^{k_1^*(M)} v_1^{k_1} \cdots \sum_{k_{M-1}\ =1}^{k_{M-1}^*(M)} v_{M-1}^{k_{M-1}} \sum_{k_M\ =1}^{k_M^*(M)} v_M^{k_M}, \tag{10}$$

where the upper bounds [1] *are defined as:*

$$k_m^*(M) = N - (M - m) - \Sigma(m-1),\ \textit{with}\ \Sigma(0) \equiv 0$$

The next definition redefines the functions in Proposition (1.1) more generally.

Definition 3.2 *Sum in Highest Powers.*

[1] It is readily verified that, while the sequence of bounds is dependent upon the ordering, from the left, of the summations, every permutation of the elements of a fixed set of numbers $v_{(.)}$ results in the same product space being defined by the nested form. Regardless of the indexing of the associated summands, the indices will always be written ascending from left to right.

With $v_i \in \mathbf{v}_{[\mathbf{M}]}$, $v_0 \equiv 1$, and $N \in \mathbb{Z}$, define:

$$P(m;M) \equiv \prod_{\substack{j=0\\ j\neq m}}^{M} (v_m - v_j)$$

$$p(M) \equiv \prod_{j=1}^{M} v_j$$

$$t(m,M;N) \equiv p(M)\, \frac{v_m^N}{P(m;M)}.$$

Then a Sum in Highest Powers is defined as:

$$Q(M,N) = \sum_{m=0}^{M} t(m,M;N) \tag{11}$$

For the purposes below, it will be convenient to also define:

$$Q^*(M,N) = \sum_{m=0}^{M} \frac{v_m^N}{P(m;M)}$$

The main result of this section establishes the equivalence of expressions for a Nested Geometric Series and a Sum in Highest Powers. The proof of $V(M,N) \equiv Q(M,N)$ relies upon a boundary result on $Q^*(M,N)$ for $N \to M$, proved in the following lemma.

Lemma 3.1 *$Q^*(M,M) \equiv 1$, and $Q^*(M,M-1) \equiv 0$, $\forall M \geq 1$.*

Proof. It is trivially true that, for $Q^*(1,1)$:

$$\frac{1}{1-v_1} + \frac{v_1}{v_1-1} \equiv 1,$$
$$\frac{1}{1-v_1} + \frac{v_1^0}{v_1-1} \equiv 0.$$

Now induce the value $Z = Q^*(M+1;M+1)$, *i.e.*:

$$Z = \frac{1}{P(0;M+1)} + \cdots + \frac{v_M^{M+1}}{P(M;M+1)} + \frac{v_{M+1}^{M+1}}{P(M+1;M+1)}, \tag{12}$$

under the presumption that:

$$1 = \frac{1}{P(0;M)} + \frac{v_1^M}{P(1;M)} + \cdots + \frac{v_M^M}{P(M;M)}. \tag{13}$$

Subtracting (13) from (12), and dividing the result by the common factor, v_{M+1}, gives:

$$\frac{Z-1}{v_{M+1}} = \frac{1}{P(0;M+1)} + \cdots + \frac{v_M^M}{P(M;M+1)} + \frac{v_{M+1}^M}{P(M+1;M+1)}. \tag{14}$$

Subtracting (14) from (12), dividing the resultant factor $(v_m - 1)$ from every $m-th$ term, and dividing each numerator and denominator by v_1^M gives, presumptively:

$$Z\frac{v_{M+1}-1}{v_{M+1}} + \frac{1}{v_{M+1}} \quad = \quad \frac{1}{\tilde{P}(0;M)} + \cdots + \frac{(v_{M+1}/v_1)^M}{\tilde{P}(M;M)}, \tag{15}$$

where $\tilde{P}(.)$ denotes the function $P(.)$ in Definition (3.2), defined on the numbers $\{\nu_m \equiv v_m/v_1\}$. The right side of (15) satisfies (13), and thus, if the latter holds, then

$$Z\frac{v_{M+1}-1}{v_{M+1}} + \frac{1}{v_{M+1}} = 1 \Leftrightarrow Z = 1.$$

Moreover, (14) implies that, necessarily, $Q(M;M+1) = 0$. [2]
□

The main result can now be established, that is:

Theorem 3.1 *A Nested Geometric Series can be written as a Sum in Highest Powers.*

Proof. The induction principle will be applied to establish the Nested Geometric Series of $\mathbf{v}_{[M+1]} = \{\{v_{M+1}\} \cup \mathbf{v}_{[M]}$. Because the order in which the numbers are taken in the Nested form does not matter, (see note (1)), write, for the convenience of the induction:

$$V(M+1,N) \quad = \quad \sum_{j\ =1}^{N-M} v_{M+1}^j \left\{ \sum_{i_1\ =1}^{k_1^*(M)-j} v_1^{i_1} \cdots \sum_{i_M\ =1}^{k_M^*(M)-j} v_1^{i_M} \right\} \tag{16}$$

[2] It is straightforward to establish that, for every $k \in \{2,\ldots,M-1\}$, $Q(M-k;M) \equiv 0$, by invoking a sequence of algebraic manipulations as applied in the lemma. The result is not required below, and is therefore not explicitly undertaken.

First observing that, for $M = 1$, the theorem states the elementary expression for the sum of a geometric series in a single variable, then the Nested Geometric Series in the braces is presumed to equal to $Q(M, N)$, whence:

$$\begin{aligned}
V(M+1,N) &= \sum_{j\ =1}^{N-M} v_M^j \sum_{m=0}^{M} t(m,M;N) v^{-j} \\
&= \sum_{m=0}^{M} t(m,M;N) \sum_{j=1}^{N-M} (v_{M+1}/v_m)^j \\
&= \sum_{m=0}^{M} t(m,M;N) \left\{ \frac{v_{M+1}^{N-M+1} v_m^{-N+M}}{(v_{M+1}-v_m)} \right\} \\
&\qquad + v_{M+1} \sum_{m=0}^{M} \frac{t(m,M;N)}{(v_m - v_{M+1})} \qquad (17) \\
&= q(M,N) + \sum_{m=0}^{M} t(m, M+1; N), \qquad (18)
\end{aligned}$$

with $q(M, N)$ denoting the first sum in (17), for brevity.

The second term in (18) defines the first M terms of $Q(M+1, N)$; then, to establish the theorem, it is sufficient that $q(M, N)$ equal the "missing" term, *i.e.*, of index $M + 1$, thus completing the expression. Expanding the summands, subsuming terms in the component functions, and clearing powers then gives:

$$\begin{aligned}
q(M,N) &= -p(M+1) v_{M+1}^{N-M} \sum_{m=0}^{M} \frac{v_m^M}{P(m, M+1)} \\
&= -v_{M+1}^{N-M} \sum_{m=0}^{M} t(m, M+1; M) \\
&= v_{M+1}^{N-M} t(M+1, M+1; M). \qquad (19)
\end{aligned}$$

Equation (19) attains by Lemma (3.1), whereupon:

$$v_{M+1}^{N-M} t(M+1, M+1, M) = t(M+1, M+1, N),$$

and (18) can then be written:

$$V(M+1,N) = Q(M+1,N)$$

□

Every one of the 2^{M-1} terms, $f(\mathcal{U}_k(.))$, in Proposition (1.1) is written as a Nested Geometric Series, and thus Theorem (3.1) establishes Proposition (1.2).

4 Proof of Proposition (1.3).

The finite dimensional distribution induced by the continuously averaging the values realized by the process (1) is known to follow a Reciprocal Gamma probability law, *i.e.*, the law of a variate whose inverse has a Gamma distribution. Specifically, in [2], it is proved (with their notation modified to the unit-term epoch invoked below) that, with $\mathbf{G}(.|\alpha,\beta)$ denoting the cumulative ("left tail") distribution of the usual gamma law, then:

$$\begin{aligned} \mathbf{Pr}[A_M^* > a] &= \mathbf{G}(a|\alpha,\beta), \\ \alpha &= 1 + \frac{2r}{\sigma^2}, \\ \beta &= \frac{\sigma^2}{2} \end{aligned} \tag{20}$$

In contrast to the simplicity of structure for the log-normal distribution itself, the moments of the Reciprocal Gamma law have not been expressed in a simple, general, analytic system. The Sum in Highest Powers form, established in Proposition (1.2), however, provides for such an expression, for moments of any order, at least for the special case of the distribution with parameters as in (20). That expression, given as Proposition (1.3), will now be established.

For simplicity, and without loss of generality, fix $T = 1$. Then, averaging over N steps gives $dt = 1/N$, and the single-step moments are written, with c_m given in (2), as:

$$e_m = \exp(c_m\ dt),$$

and the moments of the process' finite dimensional distribution at one period (*e.g.*, the most intuitive and familiar period: one year) are then:

$$\mu_m = e_m^{(1/dt)} = \exp(c_m).$$

Proposition (1.3) will follow immediately from following result.

Lemma 4.1 *Continuous Limits. Let* $u : \mu_1, \mu_2, \ldots, \mu_M$*, with every* $\mu_i \equiv \exp(\chi_i\ dt)$*. For* $Q(.)$ *as in (11), define :*

$$\Psi(M-k, N) = \frac{Q(M-k, N)}{N^M}.$$

Then, for $N = 1/dt$*:*

$$\begin{aligned}
\lim_{dt \to 0} \Psi(M-k, N) &= 0, \ \textit{for } 1 \le k < M, \\
&= \sum_{m=1}^{M} \frac{\mu_m}{\chi_m \left(\prod_{\substack{j=1 \\ j \neq m}}^{M} (\chi_m - \chi_j) \right)} \\
&\quad + \frac{(-1)^M}{\prod_{j=1}^{M} \chi_j}, \ \textit{for } k = 0. \qquad (21)
\end{aligned}$$

Proof. With $1/N = dt$, then, by expanding the functions contained in $Q(.)$ defined in (11):

$$\Psi(M-k, N) = dt^M p(M-k) \sum_{m=0}^{M-k} \frac{\mu_m}{P(m; M-k)}$$

Expanding the components, *e.g.*, $(\mu_m - \mu_j)$, of every factor in each $P(m; M-k)$ in Maclaurin series, gives:

$$(\mu_m - \mu_j) \approxeq (\chi_m - \chi_j)dt + \mathbf{O}(dt^2)$$

Formally, the definition $\mu_0 \equiv 1 \Rightarrow \chi_0 \equiv 0$ provides, in particular, for:

$$\begin{aligned}
P(0; M-k) &\approxeq dt^{M-k} \prod_{j=1}^{M-k} (-\chi_j), \\
P(m; M-k) &\approxeq dt^{M-k} \chi_m \prod_{\substack{j=1 \\ j \neq m}}^{M-k} (\chi_m - \chi_j)
\end{aligned}$$

with all omitted terms in the expansion being at least of order dt^2. Moreover, the function $p(M-k)$ has the form $\exp(\Pi(\chi_{(.)})\ dt)$. Now $\Psi(M-k, N)$ can be expressed as an infinite series in powers of dt with leading power dt^k. Its limit, therefore, vanishes for $k > 1$, and is as given in (21) for $k = 0$.
□

With regard to the expression for the $M-th$ moment in Proposition (1.2) every term which entails a subset of the first M single-step moments will vanish by the lemma. Therefore, the $M-th$ continuously averaged moment, denoted here as $\tilde{A}_M$, is a single sum, as given in Proposition (1.3).

It bears mention again that, for notational simplifity, the drift and diffusion terms are here expressed with the length of the averaging period as the unit. Giving effect to the arbitrary period length presented in, *e.g.*, [2], for the first two moments, direct substitution into (6) replicates those expressions.

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